Asymptotics at infinity of solutions for p-Laplace equations in exterior domains

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Abstract

Let $1 < p < N$, and u be a nonnegative solution of $-\Delta_p u = f(x,u)$ on $\mathbb{R}^N \setminus \overline{B_1}$ where f behaves like $|x|^{-l}u^q$ near $|x| = \infty$ and $u = 0$, for some constants $q \geq 0$ and $l \in \mathbb{R}$. We obtain asymptotic decay estimates for u. In particular, our results complete the 'sublinear case' $q < p - 1$. A related analysis is carried out for systems like $-\Delta_p u = f(x,v)$, $-\Delta_p v = g(x,u)$, where $p = 2$ corresponds to a Hamiltonian system. In this way we extend and improve some known results of Mitidieri and Pohozaev, Bidaut-Véron and Pohozaev, and other authors. Our proofs use tools such as Harnack inequality, the Maximum Principle, Liouville Theorems and blow-up arguments.

Keywords: p-Laplace; Nonlinear elliptic equation; Entire solution; Asymptotics; Nonlinear system

1. Introduction

Let $N \in \mathbb{N}$, $1 < p < N$, and $f : \overline{B_1^C} \times [0, +\infty) \to [0, +\infty)$, measurable, locally bounded, and continuous in the second variable. Let u be a solution of the following problem in an exterior domain:

$$\textbf{(P)} \begin{cases} u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N \setminus \overline{B_1}), \\ -\Delta_p u = f(x,u), \ u \geq 0 \text{ on } \mathbb{R}^N \setminus \overline{B_1}. \end{cases} \tag{1}$$

We are interested in asymptotic estimates of the solutions near infinity. Such estimates are useful in the study of variational problems associated with (1). They also become important in symmetry proofs of solutions. For instance, the well-known Moving Plane Method and the method of continuous Steiner symmetrization require some information on the decay at infinity (see, for instance, [8,12–14,24–26,5,6,20], and [2]).

Let us first take a look at the radial case, that is, we assume that $u = u(|x|)$ in (1). If $q \neq p - 1$, then the function

$$u(x) = |x|^{-\gamma}, \quad \text{where} \tag{2}$$

$$\gamma := \frac{l - p}{p - 1 - q}, \tag{3}$$

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is a solution of $-\Delta_p u = c_0|x|^{-l}u^q$ with

$$c_0 = |\gamma|^{p-2}\,\frac{\gamma}{p-1-q}\,((N-l)(p-1)-q(N-p))\,.$$

Notice that we have $c_0 > 0$ if $l > p$ and $q < q_l$, or if $l < p$ and $q > q_l$, where q_l is a critical exponent defined by

$$q_l := \frac{(N-l)(p-1)}{N-p}. \tag{4}$$

For the radial case the literature is extensive. Here we mention the work [7], where further references can be found, and the updated and unified approach to nonexistence of a positive solution for quasilinear radial inequalities in [4]. In particular it is known that if $f = f(|x|, t)$ and $f(r, t) \sim r^{-l}t^q$ near $r = +\infty$ and $t = 0$, with either $l > p$ and $0 < q < q_l$, or $l < p$ and $q_l < q$, then we have $u(x) \sim |x|^{-\gamma}$ near $|x| = \infty$ for every *radial* decaying solution of (1), where γ is given by (3). For more and detailed statements, see [7]. Several results for nonradial solutions of (1) are known in the Laplacian case $p = 2$. For instance, see [12–14,24–26]. The situation for the case $p \neq 2$ is more difficult, and some tools which have been successful for the case $p = 2$, such as the Kelvin transform, are no longer available. Estimates can often been achieved by using blow-up arguments together with Liouville theorems (see for instance [9,19]) or by integration techniques which use suitable test functions (see [17,18]). The last technique has also been employed in an important paper of Bidaut-Véron and Pohozaev [1], together with some other tools such as Harnack inequalities and the Strong Maximum Principle. The authors obtain nonexistence results for supersolutions of Eq. (1) in a punctured ball, in an exterior domain, and in a half-space. They also analyze a multipower system associated with a p-Laplace and an m-Laplace operator. In the case of existence, they obtain integral and pointwise estimates of the solutions near the singularity and near infinity, respectively. Some of these results will be mentioned below; see Theorem C, Section 2, and Theorems D–F in Section 3.

In this paper we obtain *pointwise* estimates near infinity for solutions of (1). The main result of Section 2 is Theorem 2, which states that if $0 \le f(x, t) \le c|x|^{-l}$, where $l > p$, then the solution has a finite limit at infinity. Its proof relies on a Harnack inequality for the solution and a Liouville theorem for p-harmonic functions, which is due to Kichenassamy and Véron [11] (see Theorem B, Section 2). Using the Strong Maximum Principle, this leads to the desired asymptotic estimates in Theorem 3. More general nonlinearities f, with $0 \le f(x, t) \le c|x|^{-l}t^q$, are studied in Section 3. First we consider the 'sublinear' case $q < p - 1$, and we obtain pointwise estimates from above in Theorem 4. The proof is based on a Harnack inequality and on Theorem 3, together with an iteration procedure. More restrictions on the solutions and the nonlinearity f are required in the 'superlinear' case, $q > p - 1$, since a Harnack inequality is not available. In the Theorems 5 and 6, we present two situations where a given power decay of u can be improved to the fastest possible decay. Assuming that the nonlinearity f and/or the solution satisfy some limit properties, we obtain in Theorems 7–9 universal boundedness and asymptotic estimates of u. The proofs are based on blow-up arguments, as they appeared in a recent paper Poláčik, Quittner and Souplet [19]. Finally, we apply our methods to a system for the p-Laplace operator in Section 4 where the second nonlinearity satisfies $0 \le g(x, t) \le c|x|^{-l}t^r$. Theorem 10 represents an analogue of Theorem 4 for the sublinear case $qr < (p-1)^2$, and its proof is based on the iteration technique of Section 3. For the superlinear case, there are some partial results which are equivalent to the Liouville Theorem for systems; see [15]. In the general case, we assume the so-called *Lane–Emden conjecture* [19], which establishes that if

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}$$

with $p, q > 1$, then there are no nontrivial solutions of the Lame–Emden system $-\Delta u = v^p,\ -\Delta v = u^q$ in $\mathbb{R}^n$. Using this conjecture, we obtain asymptotic estimates of the solution for a system in the Laplacian case $p = 2$, in Theorems 11 and 12.

Notation. We will write $\Omega^C = \mathbb{R}^N \setminus \overline{\Omega}$ for any domain $\Omega \in \mathbb{R}^N$.

2. First estimates

We first state some known results which will be used in the sequel. Theorem A below is a well-known Harnack inequality for p-Laplacian equations (see for instance [23]).

Theorem A. *Let Ω be a bounded domain, $B : \Omega \times \mathbb{R} \to \mathbb{R}$ a measurable function satisfying*

$$|B(x,z)| \le c_1(1+|z|^{p-1}) \quad \forall (x,z) \in \Omega \times \mathbb{R}$$

and u a nonnegative weak $W^{1,p}$-solution of

$$-\Delta_p u = B(x,u) \quad \text{in } \Omega. \tag{5}$$

Furthermore, assume that $B_{2R}(x_0) \subset \Omega$ for some $x_0 \in \Omega$ and $R > 0$. Then there is a constant $C = C(R, c_1)$ such that

$$\max_{B_R(x_0)} u \le C\left(1 + \inf_{B_R(x_0)} u\right).$$

The following Liouville-type result is due to Kichenassamy and Véron [11].

Theorem B. *Let $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ satisfy $\Delta_p u = 0$ and $|u(x)| \le a|x|^{(p-N)/(p-1)} + b$ on $\mathbb{R}^N \setminus \{0\}$ for some positive constants a, b. Then there exist $\alpha, \beta \in \mathbb{R}$ such that $u(x) = \alpha|x|^{(p-N)/(p-1)} + \beta$ on $\mathbb{R}^N \setminus \{0\}$.*

Now we show a simple auxiliary result.

Lemma 1. *Let $l > p$, $R > 1$ and $W_{R,l}$ be the solution of the following problem:*

$$W_{R,l} \in W^{1,p}_0(B_R \setminus \overline{B_1}),$$
$$-\Delta_p W_{R,l} = |x|^{-l} \quad \text{in } B_R \setminus \overline{B_1}.$$

There exists a constant C, independent of R, such that

$$W_{R,l}(x) \le C \quad \text{in } B_R \setminus \overline{B_1}.$$

Proof. $W_{R,l}$ is radial. Writing $v(|x|) := W_{R,l}(x)$ we have that

$$-(|v'|^{p-2}v'r^{N-1})' = r^{N-1-l}, \quad (1 < r < R),$$

and $v(1) = v(R) = 0$. This implies that there is a number $\rho \in (1, R)$ such that $v(\rho) = \max\{v(r) : 1 < r < R\}$, $v'(r) > 0$ for $1 < r < \rho$, and $v'(r) < 0$ for $\rho < r < R$. Assume that $N > l$. Integrating on (ρ, R) gives us

$$\begin{aligned} v(\rho) &= \int_\rho^R (N-l)^{-1/(p-1)}[t^{1-l} - \rho^{N-l}t^{1-N}]^{1/(p-1)}\,\mathrm{d}t \\ &= (N-l)^{-1/(p-1)}\rho^{(p-l)/(p-1)}J(R/\rho), \end{aligned}$$

where $J(z) := \int_1^z [s^{1-l} - s^{1-N}]^{1/(p-1)}\,\mathrm{d}s$ $(z \ge 1)$. Since J is bounded, the assertion follows in this case.

If $N \le l$ then $-\Delta_p W_{R,l} \le |x|^{-k}$ in $B_R \setminus \overline{B_1}$, where $k = (N+p)/2$. By the Maximum Principle, $W_{R,l}(x) \le W_{R,k}(x)$ in $B_R \setminus \overline{B_1}$, and the assertion follows from the above estimate. □

Now we obtain estimates at infinity if f in (1) does not depend on u. These estimates will have great significance for the analysis of nonlinear equations in Section 3. Notice that Theorems 2 and 3 below are well known and they have been proved for the Laplacian case $p = 2$ by Li and Ni [12] using estimates of the Newtonian potential. Our proof relies on a dilatation argument and on Theorems A and B.

Theorem 2. *Let $l > p$, and let $0 \le u \in W^{1,p}_{\mathrm{loc}}(B_1^C)$ satisfy weakly $0 \le -\Delta_p u \le c_0|x|^{-l}$ in B_1^C, for some constant $c_0 > 0$. Then $\lim_{|x|\to\infty} u(x)$ exists.*

Proof. From imbeddings and elliptic estimates we have that $u \in L^\infty(B_R \setminus B_2)$ for every $R > 2$. For $R \ge 8$, set $u_R(x) := u(Rx)$, $(x \in B^C_{(1/R)})$. Then $0 \le -\Delta_p u_R \le c_0 R^{p-l}|x|^{-l}$ in $B^C_{(1/R)}$. Hence Theorem A tells us that there is a constant $C_1 > 0$, independent of R, such that

$$\sup_B u_R \le C_1\left(1 + \inf_B u_R\right), \quad \text{for any ball } B \subset B_{3/2} \setminus \overline{B_{1/2}}. \tag{6}$$

Assume that u is unbounded. There is a sequence $\{x_n\}$ with $8 \le |x_n| =: R_n$, $n = 1, 2, \ldots$, $\lim_{n\to\infty} R_n = \infty$ such that $\lim_{n\to\infty} u(x_n) = +\infty$. Set $\min\{u(x) : |x| = R_n\} =: c_n$, $n = 1, 2, \ldots$. In view of (6), we have that $\lim_{n\to+\infty} c_n = +\infty$. Let U_n be the solution of the problem

$$\begin{aligned} &-\Delta_p U_n = 0 \quad \text{in } B_{R_n} \setminus \overline{B_1},\\ &U_n = 0 \quad \text{on } \partial B_1, \qquad U_n = c_n \quad \text{on } \partial B_{R_n}, \quad n = 1, 2, \ldots. \end{aligned}$$

By the Strong Maximum Principle, $u \ge U_n$ in $B_{R_n} \setminus \overline{B_1}$. On the other hand we have that

$$U_n(x) = c_n \frac{1 - |x|^{(p-N)/(p-1)}}{1 - R_n^{(p-N)/(p-1)}} \quad \text{in } B_{R_n} \setminus \overline{B_1},$$

which implies that $\lim_{n\to\infty} U_n(x) = +\infty$ on ∂B_2, a contradiction. Hence u is bounded.

By Tolksdorf's regularity result [22], this implies that $u \in C^{1,\alpha}(\overline{B_2^C})$. Let $e = (1, 0, \ldots, 0)$. Choose a sequence $\{x_n\}$ with $8 < |x_n| =: R_n \to \infty$ such that $\lim_{n\to\infty} u(x_n) = \liminf_{|x|\to\infty} u(x) =: u_0$. Define rotations ρ_n about 0 such that $x_n = R_n \rho_n(e)$. Setting $v_n(x) := u(R_n \rho_n(x)) - u(x_n)$, we have that $v_n(e) = 0$, $|v_n(x)| < c$ and $0 \le -\Delta_p v_n(x) \le c_0 (R_n)^{p-l} |x|^{-l}$ on $B^C_{(1/R_n)}$. Let $\varepsilon > 0$, and choose $n_0(\varepsilon) \in \mathbb{N}$ such that $R_n > (1/\varepsilon)$ $\forall n \ge n_0(\varepsilon)$. Since $l > p$, we have

$$|\Delta_p v_n(x)| \le c_0\, \varepsilon^{-l} (R_n)^{p-l} \quad \forall |x| \ge \varepsilon \text{ and } \forall n \ge n_0(\varepsilon).$$

Hence there is a constant k_ε such that $\forall n \ge n_0$

$$\|v_n\|_{C^{1,\alpha}(\overline{B_\varepsilon^C})} \le k_\varepsilon.$$

It follows that there is a function $v \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ such that for a subsequence,

$$\begin{aligned} &v_{n_k} \to v \text{ in } W^{1,p}(B_{(1/\varepsilon)} \setminus \overline{B_\varepsilon}), \quad \text{and} \quad \text{in } C^{1,\alpha}(B_{(1/\varepsilon)} \setminus \overline{B_\varepsilon})\ \forall \varepsilon > 0, \quad \text{as } k \to \infty,\\ &\Delta_p v = 0 \quad \text{and} \quad |v| \le c \text{ on } \mathbb{R}^N \setminus \{0\}, \quad v(e) = 0. \end{aligned}$$

By Theorem B this implies that $v \equiv 0$ on $\mathbb{R}^N \setminus \{0\}$. In particular we find that

$$\lim_{k\to\infty} \sup\{u(x) : |x| = R_{n_k}\} = u_0. \tag{7}$$

Now assume that there is a sequence $\{y_n\}$ with $|y_n| \to \infty$ and a number $\delta > 0$ such that

$$u(y_n) \ge u_0 + \delta. \tag{8}$$

By (7) there exist sequences $\{R_n^+\}$, $\{R_n^-\}$, with $R_n^+ \to \infty$, $R_n^- \to \infty$, $R_n^- < |y_n| < R_n^+$, and such that $u(x) \le u_0 + (\delta/2)$ on $\partial B_{R_n^-} \cup \partial B_{R_n^+}$. Defining $w_n(x) := u(R_n^- x)$ we have that $-\Delta_p w_n(x) \le c_0 (R_n^-)^{p-l} |x|^{-l}$ in $B_{(R_n^+/R_n^-)} \setminus \overline{B_1}$ and $w_n(x) \le u_0 + (\delta/2)$ on $\partial B_1 \cup \partial B_{(R_n^+/R_n^-)}$. Now let W_n be the solution of the following problem:

$$\begin{aligned} &W_n \in W_0^{1,p}(B_{(R_n^+/R_n^-)} \setminus \overline{B_1}),\\ &-\Delta_p W_n = c_0 (R_n^-)^{p-l} |x|^{-l} \quad \text{in } B_{(R_n^+/R_n^-)} \setminus \overline{B_1}. \end{aligned}$$

By the Maximum Principle, $W_n(x) + u_0 + (\delta/2) \ge w_n(x)$ in $B_{(R_n^+/R_n^-)} \setminus \overline{B_1}$, and in view of Lemma 1 we have that

$$\lim_{n\to\infty} \|W_n\|_{L^\infty(B_{(R_n^+/R_n^-)} \setminus B_1)} = 0.$$

This means that $w_n(x) \le u_0 + (3\delta/4)$ in $B_{(R_n^+/R_n^-)} \setminus \overline{B_1}$, if n is large enough, and we have a contradiction. The theorem is proved. $\square$

Using the Maximum Principle we obtain from Theorem 2 the following estimates.

Theorem 3. *Let u be as given in Theorem 2. Then there exist numbers $u_0 \ge 0$ and $c_1 > 0$ such that*

$$|u(x) - u_0| \le \begin{cases} c_1 |x|^{(p-l)/(p-1)} & \text{if } l \in (p, N)\\ c_1 |x|^{(p-N)/(p-1)} (\log|x|)^{1/(p-1)} & \text{if } l = N\\ c_1 |x|^{(p-N)/(p-1)} & \text{if } l > N \end{cases} \quad \text{on } \overline{B_2^C}. \tag{9}$$

In particular, if $l \in (p, N)$, then (9) holds with

$$c_1 = \max\left\{2^{(l-p)/(p-1)}m;\ \left[\frac{c_0}{N-l}\right]^{1/(p-1)} \frac{p-1}{l-p}\right\}, \tag{10}$$

$$\textit{where } m = \sup\{|u(x) - u_0| : |x| = 2\}. \tag{11}$$

Proof. Let $u_0 := \lim_{|x|\to\infty} u(x)$, and m be given by (11). We set, for $|x| \geq 2$,

$$\gamma_l(x) := \begin{cases} (N-l)^{\frac{1}{1-p}} \dfrac{p-1}{l-p} |x|^{\frac{p-l}{p-1}} & \text{if } l \in (p, N) \\ \displaystyle\int_{|x|}^{\infty} t^{\frac{1-N}{p-1}} (\log t)^{\frac{1}{p-1}}\, \mathrm{d}t & \text{if } l = N \\ (l-N)^{\frac{1}{1-p}} \displaystyle\int_{|x|}^{\infty} t^{\frac{1-N}{p-1}} [1 - t^{N-l}]^{\frac{1}{p-1}}\, \mathrm{d}t & \text{if } l > N. \end{cases} \tag{12}$$

Then

$$-\Delta_p \gamma_l = |x|^{-l} \quad \text{on } B_2^C. \tag{13}$$

Let $\varepsilon > 0$, and choose $M > 0$ such that $M^{p-1} \geq c_0$ and $M\gamma_l(x) \geq m$ on ∂B_2. By (13) we have $-\Delta_p(M\gamma_l + \varepsilon) \geq -\Delta_p(u - u_0)$ on B_2^C, $M\gamma_l(x) + \varepsilon > u(x) - u_0$ on ∂B_2, and also on B_R^C for some large enough $R > 2$. By the Maximum Principle this implies that $M\gamma_l(x) + \varepsilon \geq u(x) - u_0$ on B_2^C. Passing to the limit $\varepsilon \to 0$, this shows that $M\gamma_l(x) \geq u(x) - u_0$ on B_2^C. A similar argument gives $-M\gamma_l(x) \leq u(x) - u_0$ on B_2^C. Hence we have that $M\gamma_l(x) \geq |u(x) - u_0|$ on B_2^C. From this the assertions (9) and (10) follow easily. □

For completeness we cite the following result of Bidaut-Véron and Pohozaev [1], which complements our estimates.

Theorem C. *Let $f \in L^1_{\mathrm{loc}}(B_1^C)$, and let $u \in W^{1,p}_{\mathrm{loc}}(B_1^C)$ satisfy $-\Delta_p u = f$, $u \geq 0$, $u \not\equiv 0$ on B_1^C. Furthermore, assume that $f(x) \geq c_0|x|^{-l}$ on $\overline{B_1^C}$ for some numbers $c_0 \geq 0$ and $l \in \mathbb{R}$. Then the following hold:*

(i) *If $c_0 = 0$, then there exists a number $c_1 > 0$ such that*

$$u(x) \geq c_1|x|^{(p-N)/(p-1)} \quad \textit{on } \overline{B_2^C}. \tag{14}$$

(ii) *If $c_0 > 0$, then $l > p$ and there is a constant $c_1 > 0$ such that*

$$u(x) \geq \begin{cases} c_1|x|^{(p-l)/(p-1)} & \textit{if } l \in (p, N) \\ c_1|x|^{(p-N)/(p-1)}(\log|x|)^{1/(p-1)} & \textit{if } l = N \\ c_1|x|^{(p-N)/(p-1)} & \textit{if } l > N \end{cases} \quad \textit{on } \overline{B_2^C}. \tag{15}$$

3. Nonlinear equations

In this section we study Problem **(P)** for when $f(x,t) \sim |x|^{-l}t^q$ near $|x| = \infty$ and $t = 0$, for some constants $q > 0$, and $l \in \mathbb{R}$. Let us first recall some known results of Bidaut-Véron and Pohozaev [1].

Theorem D. *Let u be a solution of* **(P)** *where $f(x,t) \geq c_0|x|^{-l}t^q$ for every $(x,t) \in B_1^C \times [0, +\infty)$, for some $l \in \mathbb{R}$, $q > 0$ and $c_0 > 0$. Then problem* **(P)** *has no nontrivial solution in each one of the following cases:*

(i) *$p - 1 < q \leq q_l$,*
(ii) *$l \leq p$, and $q < p - 1$,*

where q_l is defined by (4).

Theorem E. *Let u be a solution of* **(P)** *where $f(x,t) \geq c_0|x|^{-l}t^q \quad \forall (x,t) \in B_1^C \times [0,+\infty)$, for some numbers $q > p-1$, $l \in \mathbb{R}$, $c_0 > 0$. Then there is a constant $d_0 > 0$ such that*

$$\int_{B_{2R}\setminus B_R} u^q \,\mathrm{d}x \leq d_0 R^{N-q\gamma} \quad \forall R \geq 2, \tag{16}$$

where γ is given by (3).

Theorem F. *Let u be a solution of* **(P)** *where $f(x,t) \geq c_0|x|^{-l}t^q \quad \forall (x,t) \in \overline{B_1^C} \times [0,+\infty)$, for some numbers $q \in (0, p-1)$, $l \in \mathbb{R}$, $c_0 > 0$. Then there is a constant $d_0 > 0$ such that*

$$u(x) \geq d_0|x|^{-\gamma} \quad \text{on } B_2^C, \tag{17}$$

where γ is given by (3).

First we analyze the so-called 'sublinear case' $q \leq p-1$. The proof of the next result uses Theorem 3 together with an iteration procedure.

Theorem 4. *Let u be a solution of* **(P)**, *and let $0 \leq f(x,t) \leq c_0|x|^{-l}t^q \ \ \forall (x,t) \in \overline{B_1^C} \times [0,+\infty)$, for some constants $l > p$, $q > 0$, and $c_0 > 0$, and let γ and q_l be given by* (3) *and* (4), *respectively. Then $\lim_{|x|\to\infty} u(x) =: u_0$ exists. If $u_0 > 0$, then* (9) *holds. If $u_0 = 0$, $l \in (p,N)$, and $q = q_l$, then there exists for every $\varepsilon \in (0, (N-p)/(2p-2))$ a number $C_\varepsilon > 0$ such that*

$$u(x) \leq C_\varepsilon |x|^{\varepsilon-(N-p)/(p-1)} \quad \text{on } B_2^C. \tag{18}$$

If $q < q_l$, then there exists a constant $c_1 > 0$ such that

$$u(x) \leq c_1|x|^{-\gamma} \quad \text{on } B_2^C. \tag{19}$$

Finally, if $u_0 = 0$ and if either $q > q_l$ or $l \geq N$, then there exists a constant $c_2 > 0$ such that

$$u(x) \leq c_2|x|^{-(N-p)/(p-1)} \quad \text{on } B_2^C. \tag{20}$$

Proof. For $R \geq 4$, let $u_R(x) := u(Rx)$ $(x \in B_{(1/R)}^C)$. Then $0 \leq -\Delta_p u_R \leq c_0 R^{p-l}|x|^{-l}u_R^q$ in $B_{(1/R)}^C$. Hence by Theorem A, u_R satisfies (6). Proceeding as in the proof of Theorem 2, this implies that $u(x) \leq M$ in $B_{3/2}^C$, for some $M > 0$, and $u \in C^{1,\alpha}(\overline{B_2^C})$. Since $f(x,u(x)) \leq c_0 M^q|x|^{-l}$, Theorem 2 shows us that $\lim_{|x|\to\infty} u(x) =: u_0$ exists, and (9) follows from Theorem 3.

Now let $u_0 = 0$ and $q \leq q_l$. We define sequences $\{l_n\}$ and $\{d_n\}$ by

$$l_0 := l, \qquad d_0 := M,$$

$$l_{n+1} := l + q\frac{l_n - p}{p-1}, \tag{21}$$

$$d_{n+1} := \max\left\{2^{(l_n-p)/(p-1)}M;\ \left(\frac{c_0 d_n^q}{N-l_n}\right)^{1/(p-1)} \cdot \frac{p-1}{l_n-p}\right\}, \tag{22}$$

$n = 0, 1, \ldots$. This implies

$$\begin{aligned} l_n &= p + (l-p)\sum_{k=0}^{n}\left(\frac{q}{p-1}\right)^k \\ &= -\left(\frac{q}{p-1}\right)^{n+1} \cdot \frac{(l-p)(p-1)}{p-1-q} + \frac{lp-l-pq}{p-1-q}, \end{aligned} \tag{23}$$

that is, $\{l_n\}$ is a *strictly* increasing sequence with

$$\lim_{n\to\infty} l_n = \frac{lp-l-pq}{p-1-q} =: l^*. \tag{24}$$

Since $f(x, u(x)) \le c_0 M^q |x|^{-l}$, by Theorem 3 $u(x) \le d_1 |x|^{(p-l)/(p-1)}$. Hence $f(x, u(x)) \le c_0 (d_1)^q |x|^{-l_1}$ in B_2^C. Applying once again Theorem 3 we find that $u(x) \le d_2 |x|^{(p-l_1)/(p-1)}$. Iterating this procedure we find that

$$u(x) \le d_{n+1} |x|^{(p-l_n)/(p-1)} \quad \text{and}$$
$$f(x, u(x)) \le c_0 (d_{n+1})^q |x|^{-l_{n+1}} \quad \text{on } B_2^C,\ n = 0, 1, \ldots. \tag{25}$$

From (25) we obtain (18) in the case $q = q_l$.

Next assume that $q < q_l$. Then $l^* < N$. This implies that there are constants $k_1, k_2 > 0$ such that

$$d_{n+1} \le \max\left\{k_1;\ k_2 (d_n)^{q/(p-1)}\right\}, \quad n = 0, 1, \ldots, \tag{26}$$

which implies that the sequence $\{d_n\}$ is bounded. Taking $n \to \infty$ in (25), we obtain (19).

Finally, let $u_0 = 0$ and $q > q_l$. Then it follows that $l^* > N$. Hence, there is $n_0 \in \mathbb{N} \setminus \{0\}$ such that $l_n < N \le l_{n_0}$ $\forall n < n_0$. Continuing as before one obtains (25) for $0 \le n < n_0$. Iterating once more time and using Theorem 3 we obtain (20).

A similar argument leads to (20) in the case where $u_0 = 0$ and $l \ge N$. $\square$

Now we analyze problem **(P)** for the 'superlinear case', $q > p - 1$. In the proof of the next two results, Theorems 5 and 6, we employ once again the iteration procedure of the previous proof. This allows us to improve a given slow-power decay of the solution.

Theorem 5. *Let u be a solution of* **(P)**, *and let $f(x, t) \le c_0 |x|^{-p} t^q$ $\forall (x, t) \in \overline{B_1^C} \times [0, +\infty)$, where $q > p - 1$, and $c_0 > 0$. Assume furthermore that*

$$\liminf_{|x| \to \infty} u(x) |x|^m < +\infty, \tag{27}$$

for some positive constant m. Then (20) holds.

Proof. As in the proof of Theorem 4 we obtain that u is bounded and $u \in C^{1,\alpha}(\overline{B_2^C})$.

By our assumption we have that $f(x, u(x)) \le C|x|^{-p-qm}$. If $p + qm \ge N$ then the assertion follows easily as in the previous proof.

Next let $p + qm < N$. We define a sequence $\{l_n\}$ by (21) with $l = p$ and $l_0 = p + qm$. Then

$$l_n = p + \frac{q^{n+1} m}{(p-1)^n}, \quad n = 0, 1, \ldots. \tag{28}$$

This implies that $\{l_n\}$ is strictly increasing with $\lim_{n \to \infty} l_n = +\infty$. Hence there is some $n_0 \in \mathbb{N} \setminus \{0\}$ such that $l_n < N \le l_{n_0}$ $\forall n \in \mathbb{N}$ with $n < n_0$. Similarly to in the proof of Theorem 4 we find that $u(x) \le c|x|^{(p-l_n)/(p-1)}$ in B_2^C for $0 \le n \le n_0 - 1$. In particular, $f(x, u(x)) \le c|x|^{-p-(p-l_{n_0-1})/(p-1)}$ in B_2^C. Since $l_{n_0} \ge N$, the assertion then follows similarly to above. $\square$

Theorem 6. *Let u be a solution of* **(P)**, *and let $f(x, t) \le c_0 |x|^{-l} t^q$ $\forall (x, t) \in \overline{B_1^C} \times [0, +\infty)$, where $q > q_l$, $l < p$ and $c_0 > 0$. Assume furthermore that $u(x) \le d_0 |x|^{-m}$ in B_2^C for some constants $d_0 > 0$ and $m > (p - l)/(q - p + 1)$. Then (20) holds.*

Proof. By hypotheses, $f(x, u(x)) \le c|x|^{-\bar{l}}$ where $\bar{l} = l + mq > p$. Defining a sequence $\{l_n\}$ by (21) with $l_0 = \bar{l}$ we find that

$$l_n = (l - p) \sum_{k=0}^{n} \left(\frac{q}{p-1}\right)^k + p + m \frac{q^{n+1}}{(p-1)^n}, \quad n = 0, 1, \ldots. \tag{29}$$

By our assumptions this implies that $\{l_n\}$ is strictly increasing with $\lim_{n \to \infty} l_n = +\infty$. Hence, there is a number $n_0 \in \mathbb{N} \cup \{0\}$ such that $l_{n_0} \ge N > l_n > p$ $\forall n \in \mathbb{N}$ with $n < n_0$. The assertion then follows as in the proof of Theorem 4. $\square$

In the following we will use a technique which has been employed recently by Poláčik, Quittner and Souplet [19] to obtain decay and singularity estimates for solutions of p-Laplace equations in general domains. The idea is to combine Liouville-type results and blow-up arguments where a central rôle is played by the 'Doubling Lemma' below, which is due to Hu [10].

Theorem G (*Doubling Lemma*). *Let (X, d) be a complete metric space, $\emptyset \neq D \subset \Sigma \subset X$, with Σ closed. Set $\Gamma = \Sigma \setminus D$. Finally, let $M : D \to (0, +\infty)$ be bounded on compact subsets of D and fix a real $k > 0$. If $y \in D$ is such that*

$$M(y)\mathrm{dist}(y, \Gamma) > 2k, \tag{30}$$

then there exists $x \in D$ such that

$$M(x)\mathrm{dist}(x, \Gamma) > 2k, \quad M(x) \geq M(y), \quad \text{and} \tag{31}$$

$$M(z) \leq 2M(x) \quad \forall z \in D \cap \overline{B_X(x, kM^{-1}(x))}. \tag{32}$$

In the next results, we will use the Doubling Lemma with $X = \mathbb{R}^N$, $D = B_2^C$, and $\Sigma = \overline{B_2^C}$. Notice that the remaining results in this section have been already been proved in the special case $l = 0$ in [19]. We emphasize that, unlike the results in the sublinear case $q < p - 1$ above, some limit properties of the nonlinearity f will be needed (see the conditions (33) and (40) below).

First we will show a universal boundedness result (in the case $l = 0$, compare with Theorem 3.1 of [19]).

Theorem 7. *Let $l \leq p$, $p - 1 < r < p^* - 1$, $f : \overline{B_1^C} \times [0, +\infty) \to [0, +\infty)$, locally bounded and continuous in the second variable, and such that*

$$\lim_{|x|\to\infty,\, t\to+\infty} |x|^l t^{-r} f(x, t) =: \lambda \in (0, +\infty). \tag{33}$$

Furthermore, let $u \in L^\infty_{\mathrm{loc}}(B_1^C)$ satisfy

$$-\Delta_p u = f(x, u), \quad u > 0 \quad \text{in } B_1^C,$$

Then $u \in L^\infty(B_2^C)$.

Proof. Assume that $u \notin L^\infty(B_2^C)$. Set $M(x) := u(x)^m/|x|$, where $m := (r - p + 1)/p$. We then find points $y_n \in B_2^C$ such that $M(y_n) > 2n/|y_n|$, $n = 1, 2, \ldots$. By Theorem G this implies that there are points $x_n \in B_2^C$ such that $M(x_n) \geq M(y_n)$,

$$\begin{aligned} &M(x_n) > 2n/|x_n| \quad \text{and} \quad M(x) \leq 2M(x_n) \\ &\quad \text{for all points } x \in B_2^C \text{ with } |x - x_n| \leq n/M(x_n),\ n = 1, 2, \ldots. \end{aligned} \tag{34}$$

Notice that $\lim_{n\to\infty} |x_n| = +\infty$, since $u \in L^\infty_{\mathrm{loc}}(B_1^C)$. Setting

$$y(x, n) := x_n + |x_n|^{l/p}[u(x_n)]^{-m} x, \qquad u_n(x) := \frac{u(y(x, n))}{u(x_n)}, \quad \text{and} \tag{35}$$

$$h_n(x) := \frac{f(y(x, n), u(y(x, n)))}{|x_n|^{-l} u(y(x, n))^r}, \tag{36}$$

we then find that

$$\begin{aligned} &-\Delta_p u_n(x) = h_n(x)[u_n(x)]^r,\ u_n(x) > 0 \quad \text{for } |x| \leq n|x_n|^{1-\frac{l}{p}}, \quad \text{and} \\ &\quad u_n(0) = 1, \quad n = 1, 2, \ldots. \end{aligned}$$

Furthermore, the properties (34) imply that

$$\frac{1}{2} \leq \frac{|y(x, n)|}{|x_n|} \leq \frac{3}{2},$$

which means that there is a constant $C_0 > 0$ such that

$$|h_n(x)| \le C_0, \quad \text{and} \tag{37}$$

$$u_n(x) \le C_0 \quad \forall x \text{ with } |x| \le n|x_n|^{1-(l/p)}, n = 1, 2, \ldots. \tag{38}$$

By using standard elliptic estimates and imbeddings we deduce that some subsequence of u_n converges in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$ to a function $v \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$, $(\alpha \in (0, 1))$, which satisfies $0 \le v \le C_0$, $v(0) = 1$, and $0 \le -\Delta_p v \le C_1$ and for some constant $C_1 > 0$. By the Strong Maximum Principle, $v > 0$ on $\mathbb{R}^N$. Using again (33)–(35) and (35), we find that

$$\lim_{n\to\infty} h_n(x) = \lambda \quad \text{uniformly in } B_R, \ \forall R > 0. \tag{39}$$

Consequently, v is a solution of $-\Delta_p v = \lambda v^r$ on $\mathbb{R}^N$. But this contradicts the Liouville Theorem from Serrin and Zou [21]. □

Theorem 8 below establishes the expected decay (19) for the superlinear case with the additional assumptions that the solution is bounded and it has zero limit at infinity. In the special case $l = 0$, compare with Theorem 3.3 of [19].

Theorem 8. *Let $l < p$, $p - 1 < q < p^* - 1$, $f : \overline{B_1^C} \times [0, +\infty) \to \mathbb{R}$, locally bounded, and such that*

$$\lim_{|x|\to\infty,\, t\to 0} |x|^l t^{-q} f(x, t) =: \mu \in (0, +\infty). \tag{40}$$

Furthermore, let $u \in L^\infty_{\text{loc}}(B_1^C)$ satisfy

$$-\Delta_p u = f(x, u), \ u > 0 \quad \text{in } B_1^C,$$

and

$$\lim_{|x|\to\infty} u(x) = 0. \tag{41}$$

Then (19) holds.

Proof. Assume that (19) does not hold. Let $M(x) := (u(x)|x|^\gamma)^k/|x|$, with $k = (q - p + 1)/p$ and γ given by (3). Applying Theorem G as in the previous proof, we find a sequence $\{x_n\} \subset B_2^C$ with $\lim_{n\to\infty} |x_n| = \infty$, such that (34) holds. Setting

$$z(x, n) := x_n + \frac{x}{M(x_n)}, \qquad u_n(x) := \frac{u(z(x, n))}{u(x_n)}, \quad \text{and}$$
$$k_n(x) := \frac{f(z(x, n), u(z(x, n)))}{|x_n|^{-l} u(z(x, n))^q}, \quad (x \in B_n(0)),$$

we then deduce that

$$-\Delta_p u_n = k_n(x) u_n^q, \ u_n > 0 \quad \text{in } B_n(0), u_n(0) = 1, n = 1, 2, \ldots.$$

Furthermore, the properties (34) and (41) imply that

$$\frac{1}{2} \le \frac{|z(x, n)|}{|x_n|} \le \frac{3}{2} \quad \text{in } B_n(0), \tag{42}$$

which means that

$$\lim_{n\to\infty} k_n(x) = \mu, \quad \text{uniformly in } B_R(0), \ \forall R > 0, \quad \text{and}$$
$$u_n(x) \le C_0 \quad \text{in } B_n(0), \ n = 1, 2, \ldots,$$

for some constant $C_0 > 0$ independent of n. By using standard elliptic estimates and imbedding theorems, we deduce that some subsequence of $\{u_n\}$ converges in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$ to $v \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$ $(\alpha \in (0, 1))$, which satisfies $0 \le v \le C_0$, $v(0) = 1$, and $-\Delta_p v = \mu v^q$ on $\mathbb{R}^N$. But this contradicts the Liouville Theorem from Serrin and Zou [21]. □

The condition (41) can be omitted in the equation $-\Delta_p u = c_0|x|^{-l}u^q$ (compare with Theorem 3.3 of [19] in the special case $l = 0$).

Theorem 9. *Let $1 < p < N$, $l < p$, $p-1 < q < p^*-1$, $c_0 > 0$, and let $u \in L^\infty_{\mathrm{loc}}(B_1^C)$ satisfy*

$$-\Delta_p u = c_0|x|^{-l}u^q,\ u \geq 0 \quad \text{in } B_1^C.$$

Then (19) holds.

Proof. Assume that (19) does not hold. Defining $M(x)$, and $z(x,n)$ as in the previous proof, we find that there is a sequence $x_n \to \infty$ such that (34), (42) and

$$-\Delta_p u_n = c_0\left|\frac{z(x,n)}{|x_n|}\right|^{-l} u_n^q, \quad 0 < u_n \leq C_0, \quad \text{in } B_n(0), \quad \text{and}$$
$$u_n(0) = 1, \quad n = 1, 2, \ldots$$

hold, for some constant $C_0 > 0$, independent of n. Since

$$\lim_{n\to\infty}\left|\frac{z(x,n)}{|x_n|}\right| = 1, \quad \text{uniformly in } B_R(0),\ \forall R > 0,$$

we again obtain a contradiction to the Liouville Theorem of Serrin and Zou [21]. $\square$

4. Hamiltonian systems

In this section we study the following Hamiltonian system:

$$\textbf{(S)}\begin{cases} u, v \in W^{1,p}_{\mathrm{loc}}(B_1^C), \\ -\Delta_p u = f(x,v), \quad -\Delta_p v = g(x,u), \\ u, v \geq 0 \quad \text{on } B_1^C. \end{cases} \tag{43}$$

We will assume that

$$f, g : \overline{B_1^C} \times [0,+\infty) \longrightarrow [0,+\infty)$$

are locally bounded and continuous in the second variable. We are interested in case when $f(x,t) \sim |x|^{-l}t^q$ and $g(x,t) \sim |x|^{-m}t^r$ near $|x| = \infty$ and $t = 0$.

Notice first that if $(p-1)^2 \neq qr$, then the pair $u(x) = |x|^{-\alpha}$, $v(x) = |x|^{-\beta}$ is a solution of **(S)** for $f(x,v) = c_1|x|^{-l}v^q$, $g(x,u) = c_2|x|^{-m}u^r$, where

$$\alpha = \frac{(l-p)(p-1) + (m-p)q}{(p-1)^2 - qr}, \tag{44}$$

$$\beta = \frac{(l-p)r + (m-p)(p-1)}{(p-1)^2 - qr}, \tag{45}$$

$$c_1 = \alpha|\alpha|^{p-2}(N-1-(\alpha+1)(p-1)), \tag{46}$$

$$c_2 = \beta|\beta|^{p-2}(N-1-(\beta+1)(p-1)). \tag{47}$$

Our first result concerns the 'sublinear' case $qr < (p-1)^2$. Again, this proof is based on the iteration procedure for a single equation.

Theorem 10. *Let $u, v \in L^\infty(\mathbb{R}^N)$ be solutions of* **(S)** *where*

$$f(x,t) \leq c_1|x|^{-l}t^q, \qquad g(x,t) \leq c_2|x|^{-m}t^r, \tag{48}$$

for some constants $c_1, c_2 > 0$, $l, m \in \mathbb{R}$, $q, r \geq 0$ satisfying

$$qr < (p-1)^2. \tag{49}$$

Furthermore, let α and β be numbers defined by (44), (45), and satisfying

$$\alpha, \beta \in (0, (N-p)/(p-1)). \tag{50}$$

Then there are constants $c_3, c_4 > 0$ such that

$$u(x) \leq c_3 |x|^{-\alpha}, \qquad v(x) \leq c_4 |x|^{-\beta} \quad \text{on } B_2^C. \tag{51}$$

Proof. From the assumptions (49) and (50) we have that $l < N$, $m < N$, and w.l.o.g. we also may assume that $l > p$. Let $u(x) \leq M$, $v(x) \leq M$ on B_2^C. We then define sequences $\{l_n\}$, $\{d_n\}$, and $\{e_n\}$ by

$$l_1 = l,\ d_1 = \max\left\{2^{(l-p)/(p-1)}M;\ \left(\frac{c_1 M^q}{N-l}\right)^{1/(p-1)} \cdot \frac{p-1}{l-p}\right\},$$

$$l_{n+1} = l + \frac{(m-p)q + rq(l_n - p)/(p-1)}{p-1}, \tag{52}$$

$$\begin{aligned} e_n \ = \ \max\Bigg\{ & 2^{-(m-p)/(p-1)-r(l_n-p)/(p-1)^2}M; \\ & \left(\frac{c_2 (d_n)^r}{N-m-r(l_n-p)/(p-1)}\right)^{1/(p-1)} \cdot \frac{p-1}{m-p+r(l_n-p)/(p-1)}\Bigg\}, \end{aligned} \tag{53}$$

$$d_{n+1} = \max\left\{2^{-(l_{n+1}-p)/(p-1)}M;\ \left(\frac{c_1 (e_n)^q}{N-l_{n+1}}\right)^{1/(p-1)} \cdot \frac{p-1}{l_{n+1}-p}\right\}, \tag{54}$$

$n = 1, 2, \ldots$. Hence

$$l_{n+1} = p + \left(l - p + q\frac{m-p}{p-1}\right)\sum_{k=0}^{n}\left(\frac{qr}{(p-1)^2}\right)^k - q\frac{m-p}{p-1}\left(\frac{qr}{(p-1)^2}\right)^n \tag{55}$$

$n = 0, 1, 2, \ldots$, which means that $\{l_n\}$ is a *strictly* increasing sequence with

$$\lim_{n\to\infty} l_n = p + (p-1)\frac{(l-p)(p-1) + (m-p)q}{(p-1)^2 - qr} =: l^*. \tag{56}$$

Our assumptions show that $l^* \in (p, N)$. Now we apply Theorem 3 iteratively. Since $v(x) \leq M$ on B_2^C, by Theorem 3, $u(x) \leq d_1 |x|^{(p-l)/(p-1)}$ in B_2^C. Hence $-\Delta_p v \leq c_2 |x|^{-m} u^r \leq c_2 (d_1)^r |x|^{-m-r(l-p)/(p-1)}$ in B_2^C. Since $m + r(l-p)/(p-1) \in (p, N)$ we then have that

$$v(x) \leq e_1 |x|^{-[m-p+r(l-p)/(p-1)]/(p-1)} \quad \text{on } B_2^C.$$

This implies $-\Delta_p u \leq c_1 |x|^{-l} v^q \leq c_1 (e_1)^q |x|^{-l_2}$, and then $u(x) \leq d_2 |x|^{\frac{p-l_2}{p-1}}$ on B_2^C. Iterating this, we obtain

$$u(x) \leq d_n |x|^{(p-l_n)/(p-1)} \quad \text{on } B_2^C,\ n = 1, 2, \ldots. \tag{57}$$

Now from (52)–(54) we see that there are positive constants k_1, k_2 such that

$$d_{n+1} \leq \max\{k_1;\ k_2 (d_n)^{qr/(p-1)^2}\}, \quad n = 1, 2, \ldots. \tag{58}$$

This implies that the sequence $\{d_n\}$ is bounded. Then passing to the limit in (57) gives the first inequality in (51). Finally, from this we have $-\Delta_p v \leq c_2 |x|^{-m} u^r \leq c_2 (c_1)^r |x|^{-l-r\alpha}$. In view of Theorem 3 this gives the second inequality in (51). $\square$

The technique of [19] is also applicable in the 'superlinear case' $qr > (p-1)^2$ once Liouville-type results have been established. For instance, one can obtain decay estimates in the Laplacian case $p = 2$ assuming that the so-called Lane–Emden conjecture is true on B_1^C (see e.g. [19,3]). We must notice that for in $\mathbb{R}^N$, this conjecture was proved for the radial case in Mitidieri [15], and some partial results were also obtained for the nonradial case in [16]. Now, we formulate the conjecture for the system (**S**).

Lane–Emden Conjecture. *Assume $p = 2$, $f(x, v) = v^q$, $g(x, u) = u^r$, where $q > 0$, $r > 0$, and*

$$\frac{1}{q+1} + \frac{1}{r+1} > \frac{N-2}{N}. \tag{59}$$

Then the problem **(S)** *does not have any nontrivial solution.*

Theorem 11 below has been proved for the special case $l = m = 0$ in Theorem 4.3 of [19].

Theorem 11. *Assume that the Lane–Emden conjecture is true for* (**S**). *Let $q, r, \alpha, \beta, l, m$ be numbers satisfying $q > 0$, $r > 0$, $qr > 1$ (59),*

$$0 < \alpha = \frac{2 - l + (2 - m)q}{qr - 1} < N - 2, \quad \textit{and} \tag{60}$$

$$0 < \beta = \frac{(2 - l)r + 2 - m}{qr - 1} < N - 2. \tag{61}$$

Furthermore, let $u, v \in L^\infty_{\mathrm{loc}}(B_1^C)$ solve problem **(S)**, *where $p = 2$,*

$$\lim_{|x|\to\infty, t\to 0} \frac{f(x,t)}{|x|^{-l} t^q} =: \lambda \in (0, +\infty), \tag{62}$$

$$\lim_{|x|\to\infty, t\to 0} \frac{g(x,t)}{|x|^{-m} t^r} =: \mu \in (0, +\infty), \quad \textit{and} \tag{63}$$

$$\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0. \tag{64}$$

Then (51) holds.

Proof. Set

$$M(x) := \frac{1}{|x|} \left\{ [u(x)|x|^\alpha]^{\frac{qr-1}{2q+2}} + [v(x)|x|^\beta]^{\frac{qr-1}{2r+2}} \right\}, \quad x \in B_2^C. \tag{65}$$

Assume that the result of the theorem is not true. Using Theorem G we find a sequence $\{x_n\} \subset B_2^C$ such that (34) holds. We set

$$u_n(x) := |x_n|^{\alpha - (2q+2)/(qr-1)} M(x_n)^{-(2q+2)/(qr-1)} u(z(x, n)), \tag{66}$$

$$v_n(x) := |x_n|^{\beta - (2r+2)/(qr-1)} M(x_n)^{-(2r+2)/(qr-1)} v(z(x, n)), \tag{67}$$

where $z(x, n) := x_n + x/M(x_n)$, and $|x| \le n$, $n = 0, 1, \ldots$. Then

$$u_n(0)^{(qr-1)/(2q+2)} + v_n(0)^{(qr-1)/(2r+2)} = 1. \tag{68}$$

Furthermore, condition (34) implies that

$$u_n(x)^{(qr-1)/(2q+2)} + v_n(x)^{(qr-1)/(2r+2)} \le C_0 \quad \forall x \text{ with } |x| \le n, \tag{69}$$

for some number $C_0 > 0$, independent of n. Finally, we have that

$$-\Delta u_n = h_n(x) v_n^q, \tag{70}$$

$$-\Delta v_n = k_n(x) u_n^r, \quad \text{where} \tag{71}$$

$$h_n(x) = \frac{f(z(x,n), v(z(x,n)))}{|x_n|^{-l} v(z(x,n))^q} \quad \text{and} \tag{72}$$

$$k_n(x) = \frac{g(z(x,n), u(z(x,n)))}{|x_n|^{-m} u(z(x,n))^r} \quad \forall x \text{ with } |x| \le n. \tag{73}$$

Now, (42) together with the assumptions (62) and (63) implies that

$$\lim_{n\to\infty} h_n(x) = \lambda, \quad \lim_{n\to\infty} k_n(x) = \mu, \quad \text{uniformly in } B_R(0)\ \forall R > 0. \tag{74}$$

Hence we find subsequences of $\{u_n\}$ and $\{v_n\}$ which converge for every $R > 0$ in $C^{1,\alpha}(B_R)$ ($\alpha \in (0,1)$) to some functions U and V respectively, satisfying

$$\begin{aligned}
&U, V \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N),\\
&-\Delta U = \lambda V^q, \ -\Delta V = \mu U^r, \ U \geq 0, \ V \geq 0, \quad \text{and}\\
&U(x)^{(qr-1)/(2q+2)} + V(x)^{(qr-1)/(2r+2)} \leq C_0 \quad \text{on } \mathbb{R}^N,\\
&u(0)^{(qr-1)/(2q+2)} + v(0)^{(qr-1)/(2r+2)} = 1.
\end{aligned}$$

But this contradicts to the Lane–Emden Conjecture for (**S**). □

The limit condition (64) can be omitted in the case of the system $-\Delta u = c_1|x|^{-l}v^q$, $-\Delta v = c_2|x|^{-m}u^r$. The proof of this result follows the lines of the proof of Theorems 9 and 11 and is left to the reader.

Theorem 12. *Assume the Lane–Emden Conjecture for* (**S**). *Assume, furthermore, that* u, v *are as in Theorem* 11, *except that the condition* (64) *is replaced by*

$$f(x,v) = c_1|x|^{-l}v^q, \qquad g(x,u) = c_2|x|^{-m}u^r, \tag{75}$$

for some positive numbers c_1, c_2. *Then* (51) *holds.*

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