

Stabilization of a Boussinesq system of KdV–KdV type

Ademir F. Pazoto^a, Lionel Rosier^{b,c,*}

^a*Instituto de Matemática, Universidade Federal do Rio de Janeiro, P.O. Box 68530, CEP 21945-970, Rio de Janeiro, RJ, Brazil*

^b*Centro de Modelamiento Matemático (CMM), Universidad de Chile (UMI CNRS 2807), Avenida Blanco Encalada 2120, Casilla 170-3, Correo 3, Santiago, Chile*

^c*Institut Elie Cartan, UMR 7502 UHP/CNRS/INRIA, B.P. 239, F-54506 Vandœuvre-lès-Nancy Cedex, France*

Abstract

A family of Boussinesq systems has recently been proposed by Bona, Chen, and Saut in [J.L. Bona, M. Chen, J.-C. Saut, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I. Derivation and linear theory, *J. Nonlinear Sci.* 12 (4) (2002) 283–318] to describe the two-way propagation of small-amplitude gravity waves on the surface of water in a canal. In this paper, we investigate the boundary stabilization of the Boussinesq system of KdV–KdV type posed on a bounded domain. We design a two-parameter family of feedback laws for which the solutions issuing from small data are globally defined and exponentially decreasing in the energy space.

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Keywords: Boussinesq system; Boundary control; Stabilization; Exponential stability

1. Introduction

The classical Boussinesq systems were first derived by Boussinesq to describe the two-way propagation of small-amplitude, long wavelength gravity waves on the surface of water in a canal. These systems and their higher-order generalizations also arise when modelling the propagation of long-crested waves on large lakes or on the ocean and in other contexts. In [2] the authors derived a four-parameter family of Boussinesq systems to describe the motion of small-amplitude long waves on the surface of an ideal fluid under the gravity force and in situations where the motion is sensibly two dimensional. More precisely, they studied a family of systems of the form

$$\begin{cases} \eta_t + w_x + (\eta w)_x + aw_{xxx} - b\eta_{xxt} = 0 \\ w_t + \eta_x + ww_x + c\eta_{xxx} - dw_{xxt} = 0, \end{cases} \quad (1)$$

which are all approximations to the same order of the Euler equations [2]. In (1), η is the elevation from the equilibrium position, and $w = w_\theta$ is the horizontal velocity in the flow

at height θh , where h is the undisturbed depth of the liquid. The parameters a, b, c, d , that one might choose in a given modelling situation, are required to fulfill the relations

$$a + b = \frac{1}{2} \left(\theta^2 - \frac{1}{3} \right), \quad c + d = \frac{1}{2} (1 - \theta^2) \geq 0,$$

where $\theta \in [0, 1]$ specifies which horizontal velocity the variable w represents (cf. [2]). Notice that $a + b + c + d = 1/3$.

In mathematical studies, considerations have been mainly given to pure initial value problems and well-posedness results [3]. However, the practical use of the above system and its relatives does not always involve the pure initial value problem. Instead, the initial boundary value problem often comes to the fore.

In [11], a rather complete picture of the control properties of (1) on a periodic domain with a locally supported forcing term was given. According to the values of the four parameters a, b, c, d , the linearized system may be controllable in any positive time, or only in large time, or may not be controllable at all. These results were also extended in [11] to the generic nonlinear system (1), i.e., when all the parameters are different from 0.

In this work we are concerned with the exponential decay of the total energy associated to a Boussinesq system of

* Corresponding author at: Institut Elie Cartan, UMR 7502 UHP/CNRS/INRIA, B.P. 239, F-54506 Vandœuvre-lès-Nancy Cedex, France.

E-mail addresses: ademir@acd.ufRJ.br (A.F. Pazoto), rosier@iecn.u-nancy.fr (L. Rosier).

KdV–KdV type (i.e. $a = c > 0$ and $b = d = 0$) posed on a finite interval $I = (0, L)$

$$\begin{cases} \eta_t + w_x + (\eta w)_x + w_{xxx} = 0 & 0 < x < L, t \geq 0 \\ w_t + \eta_x + w w_x + \eta_{xxx} = 0, & 0 < x < L, t \geq 0 \end{cases} \quad (2)$$

satisfying the boundary conditions

$$\begin{cases} w(0, t) = 0, & w_x(0, t) = \alpha_0 \eta_x(0, t), \\ w_{xx}(0, t) = 0, & t > 0 \\ w(L, t) = \alpha_2 \eta(L, t), & w_x(L, t) = -\alpha_1 \eta_x(L, t), \\ w_{xx}(L, t) = -\alpha_2 \eta_{xx}(L, t), & t > 0 \end{cases} \quad (3)$$

and the initial conditions

$$\begin{cases} \eta(x, 0) = \eta_0(x), & 0 < x < L \\ w(x, 0) = w_0(x), & 0 < x < L. \end{cases} \quad (4)$$

In (3), α_0 , α_1 and α_2 denote some nonnegative real constants. For the sake of simplicity, we have assumed that $a = c = 1$ in using the scaling $t \rightarrow \lambda t$, $x \rightarrow \lambda x$. The KdV–KdV system is expected to admit global solutions on \mathbb{R} , and it also possesses good control properties on the torus [11].

Under the above boundary conditions, if we multiply the first equation by η , the second equation by w and integrate over $(0, L)$, we obtain

$$\begin{aligned} \frac{dE}{dt} &= -\alpha_2 |\eta(L, t)|^2 - \alpha_1 |\eta_x(L, t)|^2 - \alpha_0 |\eta_x(0, t)|^2 \\ &\quad - \frac{1}{3} w^3(L, t) - \int_0^L (\eta w)_x \eta dx, \end{aligned}$$

where $E(t) = \frac{1}{2} \int_0^L (\eta^2 + w^2) dx$ is the total energy associated to (2). This indicates that the boundary conditions play the role of a feedback damping mechanism, at least for the linearized system. Therefore, the following questions arise: Does $E(t) \rightarrow 0$ as $t \rightarrow \infty$? If it is the case, can we give the decay rate? The problem might be easy to solve when the underlying model has a intrinsic dissipative nature. Moreover, in the context of coupled systems, in order to achieve the desired decay property, the damping mechanism has to be designed in an appropriate way in order to capture all the components of the system.

The main result of this paper provides a positive answer for these questions.

Theorem 1.1. *Assume that $\alpha_0 \geq 0$, $\alpha_1 > 0$, and $\alpha_2 = 1$. Then there exist some numbers $\rho > 0$, $C > 0$ and $\mu > 0$ such that for any $(\eta_0, w_0) \in [L^2(I)]^2$ with $\|(\eta_0, w_0)\|_{[L^2(I)]^2} \leq \rho$, the system (2)–(4) admits a unique solution $(\eta, w) \in C(\mathbb{R}^+; [L^2(I)]^2) \cap C(\mathbb{R}^{++}; [H^1(I)]^2) \cap L^2(0, 1; [H^1(I)]^2)$ which fulfills*

$$\|(\eta, w)(t)\|_{[L^2(I)]^2} \leq C e^{-\mu t} \|(\eta_0, w_0)\|_{[L^2(I)]^2} \quad \forall t \geq 0, \quad (5)$$

$$\|(\eta, w)(t)\|_{[H^1(I)]^2} \leq C \frac{e^{-\mu t}}{\sqrt{t}} \|(\eta_0, w_0)\|_{[L^2(I)]^2} \quad \forall t > 0. \quad (6)$$

Let us point out that a quite simple, but not very efficient, boundary damping may be designed as follows. Setting $v = \eta + w$, $u = \eta - w$, we notice that the Boussinesq system (2)

is transformed into a system of two KdV equations coupled through nonlinearities

$$\begin{aligned} v_t + v_x + v_{xxx} + \frac{1}{4}[(v-u)(v+u)]_x \\ + \frac{1}{4}(v-u)(v-u)_x &= 0, \\ u_t - u_x - u_{xxx} + \frac{1}{4}[(v-u)(v+u)]_x \\ - \frac{1}{4}(v-u)(v-u)_x &= 0. \end{aligned}$$

Therefore, the choice of the boundary conditions $v(0) = v(L) = v_x(L) = 0$ together with $u(0) = u(L) = u_x(0) = 0$ yields at once the global well-posedness and the exponential stability of the Boussinesq system in the energy space. However, the exponential stability property holds *only* if L is not a critical length in the sense of [14], that is

$$L \notin \left\{ 2\pi \sqrt{\frac{k^2 + l^2 + kl}{3}}; k, l \in \mathbb{N}^* \right\}.$$

The main interest of [Theorem 1.1](#) rests on the fact that, with the damping mechanism proposed in (3), the stabilization holds for any length of the domain.

The proof of [Theorem 1.1](#) is obtained in two steps: First we study the linearized system to derive some a priori estimates and the exponential decay in the L^2 -norm. We establish the Kato smoothing effect by means of the multiplier method, while the exponential decay is obtained with the aid of some compactness arguments that reduce the issue to a spectral problem (see, for instance, [1,14]). These estimates are then combined to prove the global well-posedness together with the exponential stability of the solutions of the Boussinesq system issued from small initial data. The trick is to combine the decay rate in the H^1 -norm with the Kato smoothing effect to form a single pointwise estimate, and next to apply the contraction mapping theorem in a convenient weighted space.

The stabilization of the KdV equation has been investigated by several authors. First, we can mention the work [6] where a damping mechanism, distributed along all the domain and guaranteeing the mass conservation, was introduced in a periodic domain. In [6] it was shown that the solutions converge exponentially to the averaged constant solution as time tends to infinity (see also [5]). The same was done by Russell and Zhang in [16,18] by means of a damping mechanism with localized support, periodic boundary conditions and small initial data. The same problem has also been addressed by means of a boundary damping with (almost) periodic boundary conditions (see [17]). All the above results, except [6], are local in the sense that only small-amplitude solutions have been shown to decay exponentially; they are essentially linear stability results. For the KdV equation with homogeneous boundary conditions, Rosier [14] proved that the decay of solutions of the linearized system fails for some critical values of the length of the interval $(0, L)$. In order to handle the critical lengths and to have the solutions of the KdV stabilized, Menzala et al. [13] introduced the extra damping term $Bu = a(x)u$

where $a \in L^\infty(0, L)$ and $a(x) \geq a_0 > 0$ a.e. in an open, non-empty subset ω of $(0, L)$ containing a set of the form $(0, \delta) \cup (L - \delta, L)$, $\delta > 0$. Combining multiplier techniques, the so-called compactness-uniqueness argument and Unique Continuation results they concluded that the energy decays exponentially to zero in bounded sets of initial data. Later on, proceeding as in [13], the general case was solved in [12]. This result has also been extended to the generalized KdV model by Rosier and Zhang [15], and Linares and Pazoto [7]. More recently, it was shown in [9] that a very weak amount of additional damping stabilizes the KdV equation. In particular, a damping mechanism dissipating the L^2 -norm as $a(\cdot)$ does is not needed. Dissipating the $H^{-1}(\omega)$ -norm proves to be sufficient. For instance, one can take the damping term $Bu = 1_\omega(-\frac{d^2}{dx^2})^{-1}u$, where 1_ω stands for the characteristic function of the set ω , and $(-\frac{d^2}{dx^2})^{-1}$ denotes the inverse of the Dirichlet Laplacian.

The results described above, except [7,15], were extended with similar statements in [9,10] for a nonlinear coupled system of KdV equations derived by Gear and Grimshaw in [4] as a model to describe strong interactions of long internal gravity waves in stratified fluid.

The paper is outlined as follows: Section 2 is devoted to the proofs of the Kato smoothing effect and of the exponential decay in the L^2 -norm for the linearized system. These estimates are next used in Section 3 to prove the local well-posedness and the exponential stability of the solutions of the Boussinesq system issued from small initial data.

2. Linear estimates

In this section we establish a series of linear estimates used thereafter. To begin with, we apply the classical semigroup theory to the linearized system

$$\eta_t + w_x + w_{xxx} = 0, \quad 0 < x < L, t \geq 0 \quad (7)$$

$$w_t + \eta_x + \eta_{xxx} = 0, \quad 0 < x < L, t \geq 0 \quad (8)$$

with the boundary conditions

$$\begin{cases} w(0, t) = 0, & w_x(0, t) = \alpha_0 \eta_x(0, t), \\ w_{xx}(0, t) = 0, & t > 0 \\ w(L, t) = \alpha_2 \eta(L, t), & w_x(L, t) = -\alpha_1 \eta_x(L, t), \\ w_{xx}(L, t) = -\alpha_2 \eta_{xx}(L, t), & t > 0 \end{cases} \quad (9)$$

and the initial conditions

$$\begin{cases} \eta(x, 0) = \eta_0(x), & 0 < x < L \\ w(x, 0) = w_0(x), & 0 < x < L. \end{cases} \quad (10)$$

Let $X_0 = [L^2(I)]^2$ be endowed with its usual inner product, and let us consider the operator $A : D(A) \subset X_0 \rightarrow X_0$ with domain

$$D(A) = \left\{ (\eta, w) \in [H^3(I)]^2 \mid w(0) = 0, w(L) = \alpha_2 \eta(L), \right. \\ \left. w_x(0) = \alpha_0 \eta_x(0), w_x(L) = -\alpha_1 \eta_x(L), w_{xx}(0) = 0, \right. \\ \left. w_{xx}(L) = -\alpha_2 \eta_{xx}(L) \right\}$$

and defined by $A(\eta, w) = (-w_x - w_{xxx}, -\eta_x - \eta_{xxx})$. Then the following result holds.

Proposition 2.1. *If $\alpha_i \geq 0$ for $i = 1, 2, 3$, then A generates a continuous semigroup of contractions $(S(t))_{t \geq 0}$ in X_0 .*

Proof. Clearly, A is densely defined and closed, so we are done if we prove that A and its adjoint A^* are both dissipative in X_0 . It is readily seen that $A^* : D(A^*) \subset X_0 \rightarrow X_0$ is given by $A^*(\mu, v) = (v_x + v_{xxx}, \mu_x + \mu_{xxx})$ with domain

$$D(A^*) = \left\{ (\mu, v) \in [H^3(I)]^2 \mid v(0) = 0, v(L) = \alpha_2 \mu(L), \right. \\ \left. v_x(0) = -\alpha_0 \mu_x(0), v_x(L) = \alpha_1 \mu_x(L), v_{xx}(0) = 0, \right. \\ \left. v_{xx}(L) = -\alpha_2 \mu_{xx}(L) - 2\alpha_2 \mu(L) \right\}.$$

Pick any $(\eta, w) \in D(A)$. After some integrations by parts, we obtain that

$$(A(\eta, w), (\eta, w))_{X_0} = -\alpha_2 |\eta(L)|^2 \\ - \left\{ \alpha_1 |\eta_x(L)|^2 + \alpha_0 |\eta_x(0)|^2 \right\} \leq 0,$$

which demonstrates that A is a dissipative operator in X_0 . Analogously, we can deduce that for any $(\mu, v) \in D(A^*)$

$$(A^*(\mu, v), (\mu, v))_{X_0} = -\alpha_2 |\mu(L)|^2 \\ - \left\{ \alpha_1 |\mu_x(L)|^2 + \alpha_0 |\mu_x(0)|^2 \right\} \leq 0$$

so that A^* is dissipative, as well. The proof is complete. \blacksquare

The following proposition provides useful estimates for the mild solutions of (7)–(10). The first ones are standard energy estimates, while the last one reveals a Kato smoothing effect.

Proposition 2.2. *Let $(\eta_0, w_0) \in X_0$ and $(\eta, w) = S(\cdot)(\eta_0, w_0)$. Then for any $T > 0$*

$$\int_0^L (|\eta_0(x)|^2 + |w_0(x)|^2) dx \\ - \int_0^L (|\eta(x, T)|^2 + |w(x, T)|^2) dx \\ = 2 \int_0^T \left\{ \alpha_2 |\eta(L, t)|^2 + \alpha_1 |\eta_x(L, t)|^2 \right. \\ \left. + \alpha_0 |\eta_x(0, t)|^2 \right\} dt, \quad (11)$$

$$\frac{T}{2} \int_0^L (|\eta_0(x)|^2 + |w_0(x)|^2) dx \\ = \frac{1}{2} \int_0^L \int_0^T (|\eta|^2 + |w|^2) dt dx \\ + \int_0^T (T - t) \left\{ \alpha_2 |\eta(L, t)|^2 + \alpha_1 |\eta_x(L, t)|^2 \right. \\ \left. + \alpha_0 |\eta_x(0, t)|^2 \right\} dt. \quad (12)$$

If in addition $\alpha_2 = 1$, then $(\eta, w) \in L^2(0, T; [H^1(I)]^2)$ and $\|(\eta, w)\|_{L^2(0, T; [H^1(I)]^2)} \leq C \|(\eta_0, w_0)\|_{X_0}$, where $C = C(T)$ is a positive constant. (13)

Proof. Let C denote a positive constant which may vary from line to line. Pick any $(\eta_0, w_0) \in D(A)$. Multiplying (7) by η , (8) by w , adding the two obtained equations and integrating over $(0, L) \times (0, T)$, we obtain after some integrations by parts (11). The identity may be extended to any initial state $(\eta_0, w_0) \in X_0$ by a density argument. Multiplying (7) by $(T - t)\eta$, (8) by $(T - t)w$, and integrating over $(0, L) \times (0, T)$ we derive (12) in a similar way. Let us proceed to the proof of (13). Multiply (7) by xw , (8) by $x\eta$, integrate over $(0, L) \times (0, T)$, and add the obtained equations. We obtain

$$\begin{aligned} & \int_0^L \int_0^T x(\eta w)_t dx dt + \int_0^L \int_0^T \frac{x}{2}(w^2 + \eta^2)_x dx dt \\ & + \int_0^L \int_0^T x(w w_{xxx} + \eta \eta_{xxx}) dx dt = 0. \end{aligned} \quad (14)$$

After some integrations by parts we obtain

$$\begin{aligned} & \int_0^L \int_0^T x(w w_{xxx} + \eta \eta_{xxx}) dx dt = \frac{3}{2} \int_0^L \int_0^T (w_x^2 + \eta_x^2) dx dt \\ & + (\alpha_1 \alpha_2 - 1) \int_0^T \eta(L, t) \eta_x(L, t) dt \\ & + \int_0^T \eta(0, t) \eta_x(0, t) dt - \frac{L(\alpha_1^2 + 1)}{2} \int_0^T \eta_x(L, t)^2 dt \\ & + L(1 - \alpha_2^2) \int_0^T \eta(L, t) \eta_{xx}(L, t) dt. \end{aligned} \quad (15)$$

By standard inequalities, we may write for any $\delta > 0$

$$\begin{aligned} & \int_0^T \eta(0, t) \eta_x(0, t) dt \leq \int_0^T \left(\frac{\delta}{2} \eta(0, t)^2 + \frac{1}{2\delta} \eta_x(0, t)^2 \right) dt \\ & \leq C\delta \int_0^L \int_0^T (\eta^2 + \eta_x^2) dx dt + (2\delta)^{-1} \int_0^T \eta_x(0, t)^2 dt. \end{aligned}$$

Picking δ so that $C\delta \leq 1/2$, and assuming that $\alpha_2 = 1$, we infer from (11) and (15) that

$$\begin{aligned} & \int_0^L \int_0^T (w_x^2 + \eta_x^2) dx dt \leq \int_0^L \int_0^T x(w w_{xxx} + \eta \eta_{xxx}) dx dt \\ & + C \int_0^L (|\eta_0(x)|^2 + |w_0(x)|^2) dx. \end{aligned} \quad (16)$$

Then (13) follows from (14), (16) and (11). ■

We are in a position to prove the exponential stability of the linearized system.

Theorem 2.3. *Assume that $\alpha_0 \geq 0$, $\alpha_1 > 0$, and that $\alpha_2 = 1$. Then there exist two constants $C_0, \mu_0 > 0$ such that for any $(\eta_0, w_0) \in X_0$, the solution of (7)–(10) satisfies*

$$\|(\eta(t), w(t))\|_{X_0} \leq C_0 e^{-\mu_0 t} \|(\eta_0, w_0)\|_{X_0} \quad \forall t \geq 0. \quad (17)$$

Proof. Using (11) and a classical argument, we only have to prove the following *observability inequality*

$$\begin{aligned} \|(\eta_0, w_0)\|_{X_0}^2 & \leq C \int_0^T \{ |\eta(L, t)|^2 + \alpha_1 |\eta_x(L, t)|^2 \\ & + \alpha_0 |\eta_x(0, t)|^2 \} dt, \end{aligned} \quad (18)$$

where (η, w) denotes the solution to (7)–(10). This is done in three steps.

STEP 1. (COMPACTNESS-UNIQUENESS ARGUMENT)

We argue by contradiction, applying the compactness-uniqueness argument due to E. Zuazua (see [8]). If (18) is false, then we may find a sequence (η_0^n, w_0^n) in X_0 such that

$$\begin{aligned} 1 & = \int_0^L (|\eta_0^n|^2 + |w_0^n|^2) dx > n \int_0^T (|\eta^n(L, t)|^2 \\ & + \alpha_1 |\eta_x^n(L, t)|^2 + \alpha_0 |\eta_x^n(0, t)|^2) dt. \end{aligned} \quad (19)$$

It follows from (13) and (19) that $(\eta^n, w^n) = S(\cdot)(\eta_0^n, w_0^n)$ is bounded in $L^2(0, T; [H^1(I)]^2)$. By (7) and (8), (η_t^n, w_t^n) is bounded in $L^2(0, T; [H^{-2}(I)]^2)$, hence, applying Aubin's lemma, we see that a subsequence of $\{(\eta^n, w^n)\}$, again denoted by $\{(\eta^n, w^n)\}$, converges strongly in $L^2(0, T; X_0)$ towards some pair (η, w) . Using (12) and (19), we see that (η_0^n, w_0^n) is a Cauchy sequence in X_0 , hence for some pair $(\eta_0, w_0) \in X_0$ we have that

$$(\eta_0^n, w_0^n) \rightarrow (\eta_0, w_0) \quad \text{in } X_0.$$

Clearly, $(\eta, w) = S(\cdot)(\eta_0, w_0)$, and we infer from (19) that

$$\eta(L, \cdot) = \alpha_1 \eta_x(L, \cdot) = \alpha_0 \eta_x(0, \cdot) = 0 \quad (20)$$

and that $\|(\eta_0, w_0)\|_{X_0} = 1$.

STEP 2. (REDUCTION TO A SPECTRAL PROBLEM)

We eliminate the time in following the procedure described in [1] for the wave equation and in [14] for the Korteweg–de Vries equation.

Lemma 2.4. *For any $T > 0$, let N_T denote the space of all the (initial states) $(\eta_0, w_0) \in X_0$ for which the solution $(\eta, w) = S(\cdot)(\eta_0, w_0)$ of (7)–(10) satisfies (20). If $N_T \neq \emptyset$ for some $T > 0$, then there exist $\lambda \in \mathbb{C}$ and $(\eta_0, w_0) \in H^3(0, L; \mathbb{C})^2$, with $(\eta_0, w_0) \neq (0, 0)$, such that*

$$\lambda \eta_0 + w_0' + w_0''' = 0 \quad (21)$$

$$\lambda w_0 + \eta_0' + \eta_0''' = 0 \quad (22)$$

$$w_0(0) = w_0'(0) = w_0''(0) = 0 \quad (23)$$

$$w_0(L) = w_0'(L) = 0 \quad (24)$$

$$w_0''(L) = -\eta_0''(L) \quad (25)$$

$$\alpha_0 \eta_0'(0) = \alpha_1 \eta_0'(L) = \eta_0(L) = 0. \quad (26)$$

Proof. The proof is very similar to the one of [14, Lemma 3.4], and so it is omitted. ■

To obtain the contradiction, it remains to prove that a triplet (λ, η_0, w_0) as above does not exist.

STEP 3. (NO NONTRIVIAL SOLUTION FOR THE SPECTRAL PROBLEM)

Lemma 2.5. *Let $\lambda \in \mathbb{C}$ and $(\eta_0, w_0) \in H^3(0, L; \mathbb{C})^2$ fulfilling (21)–(26). Then $\eta_0 = w_0 = 0$.*

Proof. Let us introduce the function $v := \eta_0 + w_0$. Taking the sum of (21) and (22) and using (24)–(26), we see that v fulfills $\lambda v + v' + v''' = 0$ and the (terminal) conditions $v(L) = v'(L) = v''(L) = 0$. (Recall that $\alpha_1 > 0$, hence $\eta'_0(L) = 0$.) It follows that $v \equiv 0$, i.e., $\eta_0 \equiv -w_0$ on $[0, L]$. Thus (21) may be written $-\lambda w_0 + w'_0 + w'''_0 = 0$, and we infer from (23) that $w_0 \equiv 0$. This completes the proofs of Lemma 2.5 and of Theorem 2.3. ■

Remark 2.6. When $\alpha_1 = 0$, the exponential decay (17) fails to be true in general. Indeed, when $L = \frac{\pi}{2} + k\pi$, $k \in \mathbb{N}^*$, we notice that the triplet $(\lambda, \eta_0, w_0) = (0, \sin(x-L), 0)$ solves (21)–(26), so that the energy of the solution issued from (η_0, w_0) is not dissipated. A similar phenomenon was pointed out in [14] for KdV.

For $0 \leq s \leq 3$, let X_s denote the collection of all the functions $(\eta, w) \in [H^s(I)]^2$ satisfying the s -compatibility conditions

$$\begin{aligned} w(0) &= w(L) - \eta(L) = 0 && \text{when } 1/2 < s \leq 3/2 \\ w(0) &= w(L) - \eta(L) = w'(0) - \alpha_0 \eta'(0) = w'(L) + \alpha_1 \eta'(L) \\ &= w''(0) = w''(L) + \eta''(L) = 0 && \text{when } 3/2 < s \leq 3. \end{aligned}$$

X_s is endowed with the Hilbertian norm $\|(\eta, w)\|_{X_s}^2 = \|\eta\|_{H^s(I)}^2 + \|w\|_{H^s(I)}^2$. Using Theorem 2.3 and some interpolation argument, we derive an exponential stability result in each space X_s for $0 \leq s \leq 3$.

Corollary 2.7. *Let α_0, α_1 , and α_2 be as in Theorem 2.3. Then for any $s \in [0, 3]$ there exists a constant $C_s > 0$ such that for any $(\eta_0, w_0) \in X_s$, the solution $(\eta(t), w(t))$ of (7)–(10) belongs to $C(\mathbb{R}^+; X_s)$ and fulfills*

$$\|(\eta(t), w(t))\|_{X_s} \leq C_s e^{-\mu_0 t} \|(\eta_0, w_0)\|_{X_s}. \quad (27)$$

Proof. (27) has already been established for $s = 0$ in Theorem 2.3. Let us proceed to the case $s = 3$. Pick any $U_0 = (\eta_0, w_0) \in X_3 = D(A)$, and write $U(t) = (\eta(t), w(t)) = S(t)U_0$. Let $V(t) = U_t(t) = AU(t)$. Then V is the mild solution of the system

$$\begin{cases} V_t = AV \\ V(0) = AU_0 \in X_0 \end{cases}$$

hence, by Theorem 2.3, the estimate $\|V(t)\|_{X_0} \leq C_0 e^{-\mu_0 t} \|V_0\|_{X_0}$ holds. Since $V(t) = AU(t)$, $V_0 = AU_0$, and the norms $\|U\|_{X_0} + \|AU\|_{X_0}$ and $\|U\|_{X_3}$ are equivalent on X_3 , we conclude that for some constant $C_3 > 0$ we have that

$$\|U(t)\|_{X_3} \leq C_3 e^{-\mu_0 t} \|U_0\|_{X_3}.$$

This proves (27) for $s = 3$. The fact that (27) is still valid for $0 < s < 3$ follows by a standard interpolation argument, since $X_s = [X_0, X_3]_{s/3}$. ■

3. Well-posedness and exponential stability

We now turn our attention to the well-posedness and to the stability properties of (2)–(4). Let $U = (\eta, w)$, $U_0 = (\eta_0, w_0)$

and $N(U) = -((\eta w)', w w')$, where $' = d/dx$. Then Eqs. (2)–(4) may be recast in the following integral form

$$U(t) = S(t)U_0 + \int_0^t S(t-s)N(U(s))ds. \quad (28)$$

Using the Kato smoothing effect established in Proposition 2.2, we first prove that (28) is locally well-posed in the space $X_0 = [L^2(I)]^2$.

Theorem 3.1. *For any $(\eta_0, w_0) \in X_0$, there exists a time $T > 0$ and a unique solution $(\eta, w) \in C([0, T]; X_0) \cap L^2(0, T; X_1)$ of (28).*

Proof. By computations similar to the ones performed in the proof of Proposition 2.2, we obtain that for any $(f, g) \in L^1(0, T; X_0)$, the solution (η, w) of the system

$$\begin{cases} \eta_t + w_x + w_{xxx} = f \\ w_t + \eta_x + \eta_{xxx} = g \end{cases}$$

supplemented with (9) and (10) fulfills

$$\begin{aligned} \sup_{0 \leq t \leq T} \|(\eta, w)(t)\|_{X_0} + \left(\int_0^T \int_0^L (|w_x|^2 + |\eta_x|^2) dx dt \right)^{\frac{1}{2}} \\ \leq C \left(\|(\eta_0, w_0)\|_{X_0} + \int_0^T \|(f, g)\|_{X_0} dx dt \right) \end{aligned} \quad (29)$$

for some constant $C = C(T, L)$ nondecreasing in T . A density argument yields that $(\eta, w) \in L^2(0, T; X_1)$.

Let $U_0 = (\eta_0, w_0)$ be given. To prove the existence of a solution to the integral equation (28), we introduce the map Γ defined by

$$(\Gamma U)(t) = S(t)U_0 + \int_0^t S(t-s)N(U(s))ds.$$

We shall prove that Γ has a fixed point in some ball $B_R(0)$ in the space $E = L^2(0, T; X_1)$, endowed with its natural norm. We need the following

CLAIM 1. There exists a constant $K > 0$ such that

$$\|N(U_1) - N(U_2)\|_{X_0} \leq K(\|U_1\|_{X_1} + \|U_2\|_{X_1})\|U_1 - U_2\|_{X_1} \quad \forall U_1, U_2 \in X_1. \quad (30)$$

The claim follows at once from the following estimate, valid for any $(\eta, w) \in H^1(I) \times H^1(I)$ and some constant $C > 0$

$$\|w\eta'\|_{L^2(I)} \leq \|w\|_{L^\infty(I)} \|\eta'\|_{L^2(I)} \leq C\|w\|_{H^1(I)} \|\eta\|_{H^1(I)}.$$

Let $T > 0, R > 0$ be numbers whose values will be specified later, and let $U \in B_R(0) \subset E$ be given. Then, by Claim 1, $N(U) \in L^1(0, T; X_0)$, hence $\Gamma U \in L^2(0, T; X_1)$ by (29). Moreover

$$\begin{aligned} \|\Gamma U\|_E &\leq \|S(\cdot)U_0\|_E + C \int_0^T \|N(U(s))\|_{X_0} ds \\ &\leq \|S(\cdot)U_0\|_E + CK\|U\|_E^2. \end{aligned}$$

It follows that for $R > 0$ and $T > 0$ small enough, Γ maps $B_R(0)$ into itself. Invoking Claim 1, one can show in a similar way that this mapping contracts if R is small enough. Then by the contraction mapping theorem, there exists a unique solution

$U \in E$ to the fixed-point problem (28). It is easy to see that $U \in C([0, T]; X_0)$. ■

Due to a lack of *a priori* X_0 -estimate, the issue of the global existence of solutions is difficult to address. However, the global existence together with the exponential stability may be established for small initial data. To that end, the Kato smoothing estimate and the exponential decay rate in X_1 are combined into a pointwise (in time) estimate.

Lemma 3.2. *For any $\mu \in (0, \mu_0)$, there exists a constant $C = C(\mu) > 0$ such that for any $U_0 \in X_0$*

$$\|S(t)U_0\|_{X_1} \leq C \frac{e^{-\mu t}}{\sqrt{t}} \|U_0\|_{X_0} \quad \forall t > 0. \quad (31)$$

Proof. Pick any $\mu \in (0, \mu_0)$. Let $u_0 \in X_0$, and set $U(t) := S(t)U_0$ for all $t \geq 0$. By Proposition 2.2 applied with $T = 1$, there exists a constant $C_K > 0$ such that

$$\|U(\cdot)\|_{L^2(0,1;X_1)} \leq C_K \|U_0\|_{X_0}. \quad (32)$$

In particular $U(t) \in X_1$ for almost $t \in (0, 1)$. We may therefore find a sequence $t_n \searrow 0$ such that $U(t_n) \in X_1$ for each n . As $U(t) \in X_1$ for each $t \geq t_n$ by Corollary 2.7, we conclude that $U(t) \in X_1$ for all $t > 0$. On the other hand, by (27), we have that

$$\|U(T)\|_{X_1} \leq C_1 e^{-\mu_0(T-t)} \|U(t)\|_{X_1} \quad \forall T \geq t \quad (33)$$

whenever $U(t) \in X_1$. Pick $T \in (0, 1]$. Integrating in (33) with respect to t on $(0, T)$, we infer that

$$\begin{aligned} [C_1^{-1} \|U(T)\|_{X_1}]^2 &\int_0^T e^{2\mu_0(T-t)} dt \\ &\leq \int_0^T \|U(t)\|_{X_1}^2 dt \leq C_K^2 \|U_0\|_{X_0}^2 \end{aligned}$$

hence

$$\|U(T)\|_{X_1} \leq C_K C_1 \sqrt{\frac{2\mu_0}{e^{2\mu_0 T} - 1}} \|U_0\|_{X_0} \leq \frac{C_K C_1}{\sqrt{T}} \|U_0\|_{X_0}$$

for $0 < T \leq 1$. Therefore

$$\|U(t)\|_{X_1} \leq C_K C_1 e^{\mu} \frac{e^{-\mu t}}{\sqrt{t}} \|U_0\|_{X_0} \quad \forall t \in (0, 1]. \quad (34)$$

(31) follows easily from (33) and (34), since $\mu < \mu_0$. ■

We are in a position to prove the well-posedness and the exponential stability for solutions issued from small initial data in X_1 . Fix a number $\mu \in (0, \mu_0)$, and let us introduce the space

$$F = \{U = (\eta, w) \in C(\mathbb{R}^+; X_1); \|e^{\mu t} U(t)\|_{L^\infty(\mathbb{R}^+; X_1)} < \infty\}$$

endowed with its natural norm.

Theorem 3.3. *There exists a number $r_0 > 0$ such that for any $(\eta_0, w_0) \in X_1$ with $\|(\eta_0, w_0)\|_{X_1} \leq r_0$, the integral equation (28) admits a unique solution $(\eta, w) \in F$.*

Proof. Let $U_0 = (\eta_0, w_0)$ be given with $\|U_0\|_{X_1} \leq r_0$, and let $U(\cdot) = (\eta(\cdot), w(\cdot)) \in F$ be given with $\|U\|_F \leq R$, r_0 and R being chosen later. We define the function ΓU by

$$(\Gamma U)(t) = S(t)U_0 + \int_0^t S(t-s)N(U(s))ds \quad \forall t \geq 0.$$

We shall prove that Γ has a fixed point in the ball $B_R(0) \subset F$ provided that $r_0 > 0$ is small enough. We infer from (29) that $\Gamma U \in C(\mathbb{R}^+; X_0) \cap L^2_{loc}(\mathbb{R}^+; X_1)$ with $(\Gamma U)(0) = U_0$. We claim that $\Gamma U \in F$. Indeed, by (27),

$$\|e^{\mu t} S(t)U_0\|_{X_1} \leq C_1 \|U_0\|_{X_1}$$

and for all $t \geq 0$

$$\begin{aligned} &\left\| e^{\mu t} \int_0^t S(t-s)N(U(s))ds \right\|_{X_1} \\ &\leq e^{\mu t} \int_0^t C \frac{e^{-\mu(t-s)}}{\sqrt{t-s}} \|N(U(s))\|_{X_0} ds \\ &\leq C \int_0^t \frac{e^{\mu s}}{\sqrt{t-s}} K (e^{-\mu s} \|U\|_F)^2 ds \\ &\leq C K \|U\|_F^2 \int_0^t \frac{e^{-\mu(t-s)}}{\sqrt{s}} ds \\ &\leq C K (2 + \mu^{-1}) \|U\|_F^2, \end{aligned}$$

where we used (31), Claim 1, the definition of F , and the calculus estimate

$$\int_0^t \frac{e^{-\mu(t-s)}}{\sqrt{s}} ds \leq \int_0^1 \frac{ds}{\sqrt{s}} + \int_1^{\max(1,t)} e^{-\mu(t-s)} ds \leq 2 + \mu^{-1}.$$

Pick $R > 0$ such that $2CK(2 + \mu^{-1})R \leq \frac{1}{2}$, and r_0 such that $C_1 r_0 = \frac{R}{2}$. Then, for $\|U_0\|_{X_1} \leq r_0$ and $\|U\|_F \leq R$, we obtain that

$$\|e^{\mu t} (\Gamma U)(t)\|_{X_1} \leq C_1 r_0 + C K (2 + \mu^{-1}) R^2 \leq R \quad \forall t \geq 0,$$

hence Γ maps the ball $B_R(0) \subset F$ into itself. Similar computations show that Γ contracts. By the contraction mapping theorem, Γ has a unique fixed point in $B_R(0)$. ■

Let us now complete the proof of Theorem 1.1. Slightly modifying the proof of Theorem 3.1, we obtain that for $T = 1$ there exists a number $\rho > 0$ such that for any $U_0 \in X_0$ with $\|U_0\|_{X_0} \leq \rho$, the integral equation (28) has a unique solution U in the ball $B_R(0) \subset L^2(0, 1; X_1)$, where $R = 2\|S(\cdot)U_0\|_{L^2(0,1;X_1)} \leq C\|U_0\|_{X_0}$. In particular, there exists $t_0 \in (0, 1)$ such that $U(t_0) \in X_1$ and $\|U(t_0)\|_{X_1} \leq R$. If in addition $R \leq r_0$, then we infer from Theorem 3.3 that $U(\cdot)$ may be extended to \mathbb{R}^+ as a solution of (2)–(4) with

$$\begin{aligned} \|U(t)\|_{X_1} &\leq C \|U(t_0)\|_{X_1} e^{-\mu(t-t_0)} \leq C e^{-\mu t} \|U_0\|_{X_0} \\ &\forall t \geq t_0. \end{aligned}$$

Using (29) and Claim 1, we easily establish (5) on $[0, 1]$. The proof of (6) on $(0, 1]$ is essentially the same as for Lemma 3.2.

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