Renormalized energy of interacting Ginzburg–Landau vortex filaments

Manuel Del Pino and Michał Kowalczyk

Abstract

We consider the Ginzbug–Landau energy in a cylinder in $\mathbb{R}^3$, and a canonical approximation for critical points with an assembly of $n \geqslant 2$ periodic vortex lines near the axis of the cylinder. We find a formula for the energy which, up to a large additive constant and to leading order, is the action functional of the n-body problem with a logarithmic potential in $\mathbb{R}^2$, the axis variable playing the role of time. A special family of rotating helicoidal critical points of the functional is found to be non-degenerate up to the invariances of the problem, and therefore persistent under small perturbations. Our analysis suggests the presence of very complex stationary configurations for vortex filaments, potentially also involving intersecting filaments.

1. *Introduction*

Let Ω be a domain in $\mathbb{R}^N$, $N \geqslant 2$. For a small parameter $\varepsilon > 0$, we consider the Ginzburg–Landau functional

$$J_\varepsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2, \quad u \in H^1(\Omega, \mathbb{C}), \tag{1.1}$$

with critical points corresponding to complex-valued solutions of the boundary value problem

$$\varepsilon^2 \Delta u + u(1-|u|^2) = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

For $N = 2, 3$, functional (1.1) is often regarded as a model for the energy arising in the standard Ginzburg–Landau theory of superconductivity [**14**] when no external applied magnetic field is present. In that setting, the complex-valued state of the system u corresponds to a critical point of J_ε in which $|u|^2$ represents the density of the superconductive property of the sample Ω (Cooper pairs of electrons). The function u is expected to stay away from zero as $\varepsilon \to 0$, except on a lower-dimensional zero set, *the vortex set*, corresponding to a region where superconductivity is not present. The construction and asymptotic description of critical points of this energy have been extensively studied following the work by Bethuel, Brezis and Helein [**6**]. Among other results, they established the existence of a family of global minimizers of J_ε under the further constraint that $u = g$ on $\partial\Omega$, where g is a smooth function with values into S^1. When $n := \deg(g, \partial\Omega) > 0$, it is found in [**6**] that u_ε has exactly n zeros (vortices) of local degree one, which approach, up to subsequences, n distinct points ξ_j. Moreover,

$$u_\varepsilon(x) \longrightarrow e^{i\varphi(x,\xi)} \prod_{j=1}^{n} \frac{x-\xi_j}{|x-\xi_j|} =: w(x,\xi), \tag{1.4}$$

where products are understood in the complex sense and $\varphi(x, \xi)$ is the unique real-valued harmonic function such that $w(x, \xi) = g(x)$ on $\partial\Omega$. Besides, ξ globally minimizes a *renormalized* *energy*, $W(\xi)$, characterized as the limit

$$W(\xi) \equiv \lim_{\rho \to 0} \left[\int_{\Omega \setminus \bigcup_{j=1}^{n} B_\rho(\xi_j)} |\nabla_x w|^2 \, dx - k\pi \log \frac{1}{\rho} \right], \tag{1.5}$$

for which an explicit expression in terms of Green's functions is found in [6]. Actually there are critical points of J_ε with the behavior (1.4) developing vortices at other critical points of W_g; see for instance [5, 22, 23, 26, 34–36]. The gradient flow of the renormalized energy also drives, in an appropriate sense, the dynamics of vortices by heat flow; see for instance [16, 19, 24, 39].

In reality, the behavior of these solutions near the core of degree 1 vortices is also understood [30, 34, 36, 43]. A better approximation than $w(x, \xi)$ in (1.4) is actually

$$w_\varepsilon(x, \xi) = e^{i\varphi(x)} \prod_{j=1}^{n} w_\varepsilon(x - \xi_j), \quad w_\varepsilon(x) = w\left(\frac{x}{\varepsilon}\right), \tag{1.6}$$

where $w(x)$ is the standard degree $+1$ solution of the equation

$$\Delta w + (1 - |w|^2) w = 0 \quad \text{in } \mathbb{R}^2,$$

$w(x) = U(r)e^{i\theta}$, where r, θ designate the usual polar coordinates and $U(r)$ is the unique solution of the problem

$$\begin{gathered} U'' + \frac{U'}{r} - \frac{U}{r^2} + (1 - |U|^2)U = 0 \qquad \text{in } (0, \infty), \\ U(0) = 0, \quad U(+\infty) = 1. \end{gathered} \tag{1.7}$$

For zero Neumann boundary data, a renormalized energy playing a similar role is present for J_ε; see [18, 19, 36, 40–42]. Evaluating the energy of the approximation $w_\varepsilon(x, \xi)$, where in the Neumann case φ is understood to be a harmonic and such that w_ε satisfies the Neumann boundary condition, we find that

$$J_\varepsilon(w_\varepsilon(\cdot, \xi)) = k\pi \log \frac{1}{\varepsilon} + W(\xi) + c_0 + \mathrm{O}(\varepsilon), \tag{1.8}$$

where c_0 is a constant. In this way, if for a certain ξ the function $w_\varepsilon(x, \xi)$ is approximating an actual critical point of J_ε, we expect that this is in correspondence with a critical point of W. For the zero Neumann case, an expression for W is given by

$$W(\xi) = \pi \sum_{i \neq j} G(\xi_i, \xi_j) - \pi \sum_{j=1}^{k} H(\xi_j, \xi_j) + b(\xi),$$

where $G(x, y)$ is the Green's function of $-\Delta$ in Ω with Dirichlet boundary conditions, H is its regular part, $H(x, y) = G(x, y) + (1/2\pi) \log |x - y|$, and b is a smooth function in $\bar{\Omega}^k$ that is zero if the domain is simply connected. In this expression we recognize the vortex interaction term $G(\xi_i, \xi_j)$, which raises the energy to $+\infty$ if any two vortices get very close, while $-H(\xi_j, \xi_j)$ pulls the energy to $-\infty$ if the point approaches the boundary. If Ω is an annular domain, this intuitively yields an equilibrium, min-max situation for points as distant as possible from one another, as well as from the boundary. It is proved in [36] that an n-vortex solution, an actual critical point of J_ε, exists for any $n \geqslant 1$, with vortices associated to such an equilibrium.

The purpose of this paper is to address the issue of understanding the mechanism analogous to renormalized energy which drives the energy equilibrium of *multiple vortex lines* in three dimensions. In $\mathbb{R}^3$, the concentration of zero sets of solutions with properly bounded energy is no longer expected to take place at points but, rather, along one-dimensional sets. Of course the simplest way of obtaining a single vortex line is by considering, for $x' \in \mathbb{R}^2$, $z \in \mathbb{R}$, the solution

$$w_\varepsilon(x', z) = w\left(\frac{x' - \xi}{\varepsilon}\right), \quad \xi \in \mathbb{R}^2,$$

which exhibits a vortex filament of sectional degree one along the line $x' = \xi$. If this line intersects the domain Ω exactly on a segment Γ then we easily compute that

$$J_\varepsilon(w_\varepsilon) = \pi|\Gamma| \log \frac{1}{\varepsilon} + \mathrm{O}(1),$$

where $|\Gamma|$ is the length of the segment. Montero, Sternberg and Ziemer [**31**] proved that, associated to a segment that orthogonally intersects the boundary that strictly minimizes the length of curves with endpoints on the boundary, there indeed exists a local minimizer of J_ε in $H^1(\Omega)$ at this energy level. Their construction is based on Γ-convergence methods developed in [**3**, **17**]. See also [**2**, **15**] for related results. The asymptotic vortex set for critical points with energy $\mathrm{O}(\log \varepsilon)$ in dimensions $N = 3$ or higher has been analysed via geometric measure theory tools in [**4**, **7**, **27**], mostly under Dirichlet boundary conditions: they turn out to be, in a generalized sense, minimal submanifolds of dimension $N - 2$ or less. In the context of rotating Bose–Einstein condensates, a related variational problem for concentration on curves has been investigated in [**1**].

Rather puzzlingly, the existence result in [**31**] asserts that for each $n \geqslant 1$ there exists a local minimizer u_ε of J_ε with energy at leading order given by $n\pi|\Gamma| \log(1/\varepsilon)$. One would speculate that these solutions actually exhibit multiple vortex lines with degree one collapsing onto the segment, rather than, say, a single line with a higher degree, since, as has been known for some time, two-dimensional higher-degree vortices are in general unstable.

In this paper we shall derive an expression for the renormalized energy of an assembly of n vortex lines. Such a formula should take into account the short-range effect due to the repulsive interaction of two-dimensional vortices of the same degree, and the long-range tendency of each individual filament to shorten its total length. As we will see, achieving a balance between these two effects requires the vortex lines to be allowed to approach each other as $\varepsilon \to 0$. This aspect of the problem is quite similar to a 'clustering of interfaces' phenomenon recently discovered for the Allen–Cahn equation in [**37**]. To clearly isolate the features of the problem that make it possible for multiple filaments to exist, we shall restrict ourselves to a very simple geometric situation: that of an infinite cylinder, where we look for solutions of problem (1.2)–(1.3), which are additionally periodic in the axial direction. In what follows, we fix a number $R > 0$ and consider the cylinder in $\mathbb{R}^3$: equations on lines 126, 136.

$$\mathcal{C} = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} \;/\; |x| < R\}.$$

Given $\ell > 0$, we are interested in the boundary value problem

$$\varepsilon^2 \Delta u + u(1 - |u|^2) = 0 \quad \text{in } \mathcal{C}, \tag{1.9}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\mathcal{C}, \tag{1.10}$$

$$u(x, z + \ell) = u(x, z) \quad \text{for all } (x, z) \in \mathcal{C}. \tag{1.11}$$

Let $H^1_\ell(\mathcal{C}, \mathbb{C})$ be the space of all complex-valued functions u defined on $\mathcal{C}$ with $u(x, z) = u(x, z + \ell)$ for all (x, z) which are in $H^1(\Omega, \mathbb{C})$, where Ω is the periodic cell

$$\Omega = \{(x, z) \in \mathcal{C} \;/\; 0 < z < \ell\;\}.$$

Solutions of this problem correspond to critical points in $H^1_\ell(\mathcal{C}, \mathbb{C})$ of the energy J_ε in (1.1). Let us consider the space $H^1_\ell(\mathbb{R}, \mathbb{R}^2)$ of locally H^1, $\mathbb{R}^2$-valued functions f with $f(z + \ell) = f(z)$.

We want to set up a family of approximate solutions to problem (1.9)–(1.11) in exact analogy with $w_\varepsilon(x, \xi)$ in (1.6), now in the three-dimensional case, where the parameters are chosen to be n vortex lines, represented by functions $f_j \in H^1_\ell(\mathbb{R}, \mathbb{R}^2)$, $j = 1, \ldots, n$. Thus we set

$$w_\varepsilon(x, z, \mathbf{f}) = e^{i\varphi(x,z,\mathbf{f})} \prod_{j=1}^{n} w_\varepsilon(x - f_j(z)), \tag{1.12}$$

where $\varphi = \sum_{j=1}^n \varphi_j$ and φ_j is the unique solution to the problem

$$\Delta\varphi_j = 0 \quad \text{in } \mathcal{C}, \tag{1.13}$$

$$\frac{\partial \varphi_j}{\partial \nu} = -\frac{\partial \arg(x - f_j(z))}{\partial \nu} \quad \text{on } \partial\mathcal{C}, \tag{1.14}$$

$$\varphi_j(x,z) = \varphi_j(x, z+\ell) \quad \text{for all } (x,z) \in \mathcal{C}, \tag{1.15}$$

$$\int_\Omega \varphi_j = 0, \tag{1.16}$$

such that, in particular, w_ε satisfies the zero Neumann boundary condition on $\partial\mathcal{C}$. The mean value on $|x| = R$ of the right-hand side of (1.14) is actually zero for each z, such that this problem is indeed solvable. The functions (1.12) represent a family of approximate solutions to (1.2)–(1.3) parametrized by $\mathbf{f} \in H^1_\ell(\mathbb{R}, \mathbb{R}^2)^n$, which exhibit the n ℓ-periodic vortex lines $x = f_j(z)$, $z \in \mathbb{R}$.

Let us consider the functional

$$\mathbf{f} \in H^1_\ell(\mathbb{R}, \mathbb{R}^2)^n \longmapsto J_\varepsilon(w_\varepsilon(\cdot, \mathbf{f})). \tag{1.17}$$

If an actual critical point of J_ε in $H^1_\ell(\mathcal{C}, \mathbb{C})$ lies close to one particular $w_\varepsilon(\cdot, \mathbf{f}_0)$, it is then natural to expect that $\mathbf{f}_0$ is approximately stationary for this functional. Thus, it is important to understand its dependence on f through an expansion similar to (1.8). To this end, we will rewrite $\mathbf{f}$ in the convenient form

$$\mathbf{f}(z) = \frac{\tilde{\mathbf{f}}(z)}{\sqrt{|\log\varepsilon|}}, \tag{1.18}$$

and assume the following constraints on $\tilde{\mathbf{f}}$: there is a (large) fixed positive number M such that for all j, and all $k \neq j$, we have

$$\|\tilde{f}_j\|_{H^1(0,\ell)} \leqslant M, \quad |\tilde{f}_k(z) - \tilde{f}_j(z)| \geqslant M^{-1} \quad \text{for all } z \in (0,\ell). \tag{1.19}$$

Our main result reads as follows.

THEOREM 1.1. *The following asymptotic formula holds*:

$$J_\varepsilon\left(w_\varepsilon\left(\cdot, \frac{1}{\sqrt{|\log\varepsilon|}}\tilde{\mathbf{f}}\right)\right) = n\pi\ell\log\frac{1}{\varepsilon} + \mathcal{I}_0(\tilde{\mathbf{f}}) + c_\varepsilon + \frac{1}{\sqrt{|\log\varepsilon|}}\Theta_\varepsilon(\tilde{\mathbf{f}}), \tag{1.20}$$

where

$$\mathcal{I}_0[\tilde{\mathbf{f}}] = \frac{1}{2}\pi\sum_{j=1}^n \int_0^\ell |\tilde{f}_j'|^2\,dz - \pi\sum_{k\neq j}\int_0^\ell \log|\tilde{f}_k - \tilde{f}_j|\,dz, \tag{1.21}$$

$c_\varepsilon = c_1\log|\log\varepsilon| + c_2$ for certain constants c_1, c_2, and the functions $\Theta_\varepsilon(\tilde{\mathbf{f}})$, $D\Theta_\varepsilon(\tilde{\mathbf{f}})$ and $D^2\Theta_\varepsilon(\tilde{\mathbf{f}})$ are uniformly bounded for $\tilde{\mathbf{f}}$ in $H^1_\ell(\mathbb{R}, \mathbb{R}^2)^n$ satisfying constraints (1.19) and all small $\varepsilon > 0$.

The functional $\mathcal{I}_0$ is nothing but the action functional associated to the n-body problem for a logarithmic gravitational potential in $\mathbb{R}^2$ if the third spatial variable z is interpreted as time. Its Euler–Lagrange equation becomes

$$\tilde{f}_k'' + 2\sum_{j\neq k}\frac{\tilde{f}_k - \tilde{f}_j}{|\tilde{f}_k - \tilde{f}_j|^2} = 0, \quad k = 1, \ldots, n. \tag{1.22}$$

Thus, one may conjecture that there exists a critical point of J_ε of the form (1.12), associated to an ℓ-periodic solution $\tilde{\mathbf{f}}_0$ of (1.22). Such a critical point would have n vortex lines with degree 1, close to the curves

$$x' = \frac{1}{\sqrt{|\log \varepsilon|}} \tilde{f}_{j0}(z), \quad j = 1, \dots, n,$$

which collapse, at a rather slow rate $O(1/\sqrt{|\log \varepsilon|})$, onto the z-axis. Proving such a result is a challenging problem. Its analog in the two-dimensional case, already known for renormalized energy of points, requires considerable analysis. A simpler problem with a similar feature is that of finding solutions with multiple interfaces in the Allen–Cahn equation in $\mathbb{R}^2$, which have been rigorously built in [**37**]. In that problem the location of the interfaces is determined by the *Toda system*, describing (in its mechanical interpretation) an assembly of particles on a line with an interaction of exponential forces between the nearest neighbors.

It is of course reasonable to ask for simple solutions of system (1.22) which have a chance to represent vortex filaments. This system appears in a different fluid-dynamical setting, and special helicoidal solutions were analysed for stability of the associated Schrödinger flow by Klein, Majda and Damodaran [**21**], and by Kenig, Ponce and Vega [**20**]. We prove in § 2 that in our setting there are indeed critical points of the functional

$$\mathcal{I}_\varepsilon(\tilde{\mathbf{f}}) := J_\varepsilon \left(w_\varepsilon \left(\cdot, \frac{1}{\sqrt{|\log \varepsilon|}} \tilde{\mathbf{f}} \right) \right) \tag{1.23}$$

with this type of pattern. This makes it natural to conjecture then that, also, a critical point of J_ε close to $w_\varepsilon(\cdot, (1/\sqrt{|\log \varepsilon|})\tilde{\mathbf{f}}^n))$ indeed exists. We postpone the proof of Theorem 1.1 to § 3.

2. *Existence of helicoidal vortex-lines*

For each $n > 1$, we see that the array $\tilde{\mathbf{f}}^n$ with components in $\mathbb{R}^2$, again identified with $\mathbb{C}$, given by

$$\tilde{f}^n_k(z) = R_n e^{2\pi i z/\ell} e^{2\pi i (k-1)/n}, \quad k = 1, \dots, n,$$
$$R_n = \frac{\ell\sqrt{n-1}}{2\pi} \tag{2.1}$$

is an ℓ-periodic solution of system (1.22), the Euler–Lagrange equation for the functional $\mathcal{I}_0$. These solutions can be simply described as an assembly of vortex lines located at the vertices of a regular n-polygon and twisting around one another once over one period. Let us consider the functional $\mathcal{I}_\varepsilon(\tilde{\mathbf{f}})$ defined in (1.23), which according to Theorem 1.1 lies close in the C^2-sense (up to an additive constant) to $\mathcal{I}_0$ uniformly on functions $\tilde{\mathbf{f}}$ that satisfy constraints (1.19). Thus, it is expected that if $\tilde{\mathbf{f}}$ is (in a suitable sense) a non-degenerate critical point of $\mathcal{I}_0$, then there exists a critical point of $\mathcal{I}_\varepsilon$, $\tilde{\mathbf{f}}_\varepsilon \approx \tilde{\mathbf{f}}$. The detailed statement is as follows.

THEOREM 2.1. *There exists a critical point $\tilde{\mathbf{f}}^n_\varepsilon$ in $H^1_\ell(\mathbb{R}, \mathbb{R}^2)^n$ of the functional $\mathcal{I}_\varepsilon$ given by (1.23), such that*

$$\tilde{\mathbf{f}}^n_\varepsilon = \tilde{\mathbf{f}}^n + o(1)$$

with $o(1) \to 0$ in H^1_ℓ-sense as $\varepsilon \to 0$.

Proof. As we will argue next, non-degeneracy of $\tilde{\mathbf{f}}^n$ holds once we restrict the functional, taking into account its natural invariances. If we restrict the space to the linear constraint $\operatorname{Re} \int_0^\ell \tilde{\mathbf{f}}^{n\prime} \bar{\tilde{f}}\, dz = 0$ then a critical point of $\mathcal{I}_\varepsilon$ under this constraint will be such that

$$D\mathcal{I}_\varepsilon(\tilde{\mathbf{f}}^n)[\mathbf{g}] = \lambda \operatorname{Re} \int_0^\ell \tilde{\mathbf{f}}^{n\prime} \bar{\mathbf{g}}\, dz$$

for some Lagrange multiplier λ. However, since the functional is invariant under translations of its argument in the z-variable, we see that $D\mathcal{I}_\varepsilon(\tilde{\mathbf{f}})[\tilde{\mathbf{f}}'] = 0$, and hence a constrained critical point close to $\mathbf{f}^n$ is a full critical point. On the other hand, let us consider this functional restricted to the subspace of functions $\tilde{\mathbf{f}}$ with n-polygonal symmetry, namely such that

$$\tilde{f}_k(z) = \tilde{f}_1(z)e^{2\pi i(k-1)/n}, \quad k = 1, \dots, n.$$

Critical points of the restriction of $\mathcal{I}_\varepsilon$ to this space are unconstrained critical points because of the natural rotation invariance of this functional inherited from the symmetries of the problem. It is thus sufficient to analyse the non-degeneracy of the above critical points of $\mathcal{I}_0$ only for perturbations subject to this space which are, in addition, L^2-orthogonal to $\tilde{\mathbf{f}}^n$. This amounts to considering the eigenvalue problem for the linearization of system (1.22) around $\tilde{\mathbf{f}}^n$,

$$\mathbb{L}\phi = -\mu\phi, \tag{2.2}$$

where $\phi = (\phi_1, \dots, \phi_n)$ and

$$(\mathbb{L}\phi)_k = \phi_k'' + 2\sum_{j\neq k} \frac{\phi_k - \phi_j}{|\tilde{f}_k^n - \tilde{f}_j^n|^2} - 4\sum_{j\neq k} \frac{(\tilde{f}_k^n - \tilde{f}_j^n)\mathrm{Re}\,[(\phi_k - \phi_j)\overline{(\tilde{f}_k^n - \tilde{f}_j^n)}]}{|\tilde{f}_k^n - \tilde{f}_j^n|^4}, \tag{2.3}$$

subject to ϕ of the form

$$\phi_k = \varphi e^{2\pi i(k-1)/n}, \quad k = 1, \dots, n, \tag{2.4}$$

where φ is an ℓ-periodic function, such that $\phi = (\phi_1, \dots, \phi_n)$ in addition satisfies

$$\mathrm{Re}\int_0^\ell \mathbf{f}^{n\prime}\bar{\phi}\,dz = 0. \tag{2.5}$$

Substituting ϕ of this form into (2.2) for any k yields the single equation for φ,

$$\varphi'' + 2\varphi\sum_{j\neq k} \frac{1 - e^{2\pi i(j-k)/n}}{|\tilde{f}_k^n - \tilde{f}_j^n|^2} - 4\sum_{j\neq k} \frac{h_{kj}^n\mathrm{Re}\,[\varphi\overline{g_{kj}^n}]}{|\tilde{f}_k^n - \tilde{f}_j^n|^4} = -\mu\varphi, \tag{2.6}$$

where

$$h_{kj}^n = (\tilde{f}_k^n - \tilde{f}_j^n)e^{-2\pi i(k-1)/n}, \quad g_{kj}^n = (\tilde{f}_k^n - \tilde{f}_j^n)(e^{-2\pi i(k-1)/n} - e^{-2\pi i(j-1)/n}).$$

We notice that in general (that is, without assuming (2.4)), this eigenvalue problem is rather delicate, as the stability analysis of these solutions in [**20**, **21**] shows.

Writing φ in (2.6) in the form

$$\varphi = iR_n e^{2\pi iz/\ell}\psi, \quad \psi = \psi_1 + i\psi_2, \tag{2.7}$$

with ψ_1 and ψ_2 real-valued, we get the following eigenvalue problem to solve:

$$\psi_1'' - 2m\psi_2' = -\mu\psi_1, \tag{2.8}$$

$$\psi_2'' + 2m\psi_1' - 2m^2\psi_2 = -\mu\psi_2, \tag{2.9}$$

where we have denoted $m = 2\pi/\ell$. The pair (ψ_1, ψ_2) will be an ℓ-periodic solution of this system if and only if it corresponds to the real or imaginary part of a $\mathbb{C}^2$-valued solution of the form

$$e^{2\pi i\beta z/\ell}(1, \alpha), \quad \beta \in \mathbb{Z},\ \alpha \in \mathbb{C}.$$

From (2.8)–(2.9) we get the following system for α, μ:

$$-\beta^2 - 2i\alpha\beta = -\frac{\mu}{m^2},$$

$$-\alpha\beta^2 + 2i\beta - 2\alpha = -\frac{\mu}{m^2},$$

and hence, given $\beta \in \mathbb{Z}$, we have two solutions $\mu_{\beta,j}$, $j = 1, 2$:

$$\mu_{\beta,1} = \left(\frac{2\pi}{\ell}\right)^2 [\beta^2 + 1 + \sqrt{4\beta^2 + 1}], \tag{2.10}$$

$$\mu_{\beta,2} = \left(\frac{2\pi}{\ell}\right)^2 [\beta^2 + 1 - \sqrt{4\beta^2 + 1}]. \tag{2.11}$$

As functions of β, the eigenvalues $\mu_{\beta,1}$ are easily seen to be positive for each $\beta \in \mathbb{Z}$. On the other hand $\mu_{\beta,2} = 0$ if and only if $\beta = 0$ or $\beta = \sqrt{2}$. In the latter case β is not an integer, while in the former the eigenfunction corresponds to $\psi_1 = 1$, $\psi_2 = 0$. However, this is incompatible with the orthogonality relation (2.5). In either case, $\mu_{\beta,2} \neq 0$, and hence (constrained to the invariances assumed) the critical point $\tilde{\mathbf{f}}^n$ of $\mathcal{I}_0$ is non-degenerate. C^2-closeness thus implies the presence of a critical point of $\mathcal{I}_\varepsilon$ close to it. □

Remark 2.1. We point out that these helicoidal solutions had been previously analysed for vortex filaments derived formally from some limit in Navier–Stokes equations; see [**20**, **21**]. In that context, it has been established that the solutions $\tilde{\mathbf{f}}^n$ are actually *stable* (for perturbations in $H^1(\mathbb{R})^n$) for $n = 2, 3$ for the Hamiltonian flow associated to $\mathcal{I}_0$, namely as a stationary state of the Shrödinger equation

$$-i\tilde{f}_{kt} = \tilde{f}_k'' + 2\sum_{j\neq k} \frac{\tilde{f}_k - \tilde{f}_j}{|\tilde{f}_k - \tilde{f}_j|^2}, \quad k = 1, \dots, n.$$

It is interesting to mention that this is exactly the system arising formally from the Gross–Pitaievski equation

$$-iu_t = \Delta u + \frac{(1 - |u|^2)u}{\varepsilon^2}$$

when substituting the evolution of an n-vortex line array of the form (1.12). A single helicoidal traveling vortex line was built rigorously in [**12**]. If, instead, heat flow is considered, then these helicoidal patterns should be unstable, since the linearization being analysed always has a negative eigenvalue. Analyses of evolution of vortex lines by heat flow are contained in [**8**, **25**, **28**, **38**].

Remark 2.2. The simple solutions considered here are of course *non-colliding*. On the other hand, system (1.22) does have colliding periodic solutions. Indeed, if we consider for instance the case $n = 2$, and allow $f_1 = -f_2 = \rho(z)$ with ρ real-valued, then the equation resulting for ρ is just

$$\rho'' + \frac{2}{\rho} = 0$$

such that along the solutions,

$$\frac{1}{2}|\rho'|^2 + 2\log\rho = E$$

for a constant E. Direct integration then yields the existence of a unique value of E such that $\rho(0) = 0$, $\rho'(\ell/2) = 0$. Even reflections provide a weak, ℓ-periodic solution of the problem with finite energy. It would be interesting to understand whether these *colliding vortex lines* correspond in some sense to solutions of the Ginzburg–Landau equation. Even at the formal level this problem has yet to be understood. Notice that a cross-section for each fixed z exhibits two vortices of degree one when the filaments are 'well' separated and a vortex of degree two when they merge.

In general, because of the analogy with the n-body problem, besides the helicoidal filaments the functional $\mathcal{I}_0$ is expected to have quite an exotic set of critical points; see for instance [**11**, **32**]. Describing them in general and proving that they are non-degenerate and do not have collisions seems a difficult problem. Some recent developments in the analysis of the action functional under symmetries (see [**13**, **29**]) could be applicable here. The real challenge from the point of view of the original Ginzburg–Landau problem is to show that the critical points of $\mathcal{I}$ give rise to critical points of J_ε. This would mean carrying out a program similar to that for the planar vortex problem that began with the formal argument in [**33**], which later led to rigorous results starting with [**6**]. The method used in [**37**] to treat the phenomenon of clustering of interfaces in the Allen–Cahn equation could be one possible way to justify the formal results of the present paper. The main step in [**37**] is the analysis of the linearized Allen–Cahn operator. It seems then that in the context of Ginzburg–Landau equation one would have to start with analogous results.

3. *Asymptotic expansion of the renormalized energy*

In this section we will prove Theorem 1.1. In what follows we will often use various asymptotic formulae for the solution of (1.7). They can be found, for example, in [**10**].

We consider an arrangement of n vortex filaments $\mathbf{f} = (f_1, \dots, f_n)$ with $f_k \in H^1_\ell(\mathbb{R}, \mathbb{R}^2)$, $k = 1, \dots, n$, satisfying the estimates

$$|f_k(z)| \leqslant \frac{M}{|\log\varepsilon|^{1/2}} \quad \forall z \in (0, \ell), \tag{3.1}$$

$$|f_k(z) - f_j(z)| \geqslant \frac{M^{-1}}{|\log\varepsilon|^{1/2}} \quad \forall k \neq j,\ z \in (0, \ell) \tag{3.2}$$

for a given positive number M. The main ingredient in the proof of Theorem 1.1 is the following result.

LEMMA 3.1. *There is a constant c depending only on n and ℓ such that for any $\mathbf{f} \in H^1_\ell(\mathbb{R})^n$ satisfying (3.1)–(3.2) we have*

$$J_\varepsilon(w_\varepsilon(\cdot, \mathbf{f})) = n\pi\ell\log\frac{1}{\varepsilon} + c + \mathcal{I}(\mathbf{f}),$$

where

$$\mathcal{I}(\mathbf{f}) = \frac{1}{2}\pi\log\frac{1}{\varepsilon}\int_0^\ell |\mathbf{f}'|^2\,dz - 2\pi\sum_{k<m}\int_0^\ell \log|f_k - f_m|\,dz + \int_0^\ell N(\mathbf{f})\,dz, \tag{3.3}$$

with

$$\int_0^\ell |N(\mathbf{f})|\,dz \leqslant C[|\log\varepsilon|^{1/2}\|\mathbf{f}\|^2_{H^1_\ell(\mathbb{R})^n} + \|\mathbf{f}\|_{H^1_\ell(\mathbb{R})^n} + \varepsilon|\log\varepsilon|^p]$$
$$+C\int_0^\ell (\varepsilon|\log\varepsilon|^p + |\mathbf{f}'|^2)\max_{k\neq m}\big|\log|f_k - f_m|\big|.$$

Here C and p are constants independent of ε and $\mathbf{f}$.

Proof. We will introduce the following notation:

$$w^* = U^* e^{i(\Theta+\varphi)}, \quad \text{where} \ \ U^* = |\mathrm{w}^*|, \quad \varphi = \sum_{k=1}^n \varphi_k, \quad \Theta = \sum_{k=1}^n \arg(x - f_k(z)).$$

For brevity we will also denote

$$U_k = U\left(\frac{|x - f_k(z)|}{\varepsilon}\right), \quad \theta_k = \arg(x - f_k(z)), \quad U_{\check{k}} = \prod_{j \neq k} U_j.$$

With this notation we get

$$\begin{aligned}\int_{\mathcal{C}} |\nabla w^*|^2 &= \int_{\mathcal{C}} |\nabla(\Theta + \varphi)|^2 |U^*|^2 + \int_{\mathcal{C}} |\nabla U^*|^2 \\ &= I[\mathbf{f}] + \int_{\mathcal{C}} |\nabla U^*|^2,\end{aligned}$$

where we have made use of the definition of φ and, in particular, the fact that $\partial(\Theta + \varphi)/\partial\nu = 0$ on $\partial\mathcal{T}$. Let us set

$$\begin{aligned}I[\mathbf{f}] &= \int_{\mathcal{C}} |\nabla\Theta|^2 |U^*|^2 + 2\int_{\mathcal{C}} \nabla\Theta \cdot \nabla\varphi |U^*|^2 + \int_{\mathcal{C}} |\nabla\varphi|^2 |U^*|^2 \\ &= I_1[\mathbf{f}] + I_2[\mathbf{f}] + I_3[\mathbf{f}]. \end{aligned} \tag{3.4}$$

The main term in the expansion of J_ε turns out to be the one related to $I_1[\mathbf{f}]$. We will now calculate its approximate formula. Observe that

$$\begin{aligned}I_1[\mathbf{f}] &= \sum_{k,m} \int_{\mathcal{C}} \nabla_x^\perp \log|x - f_k(z)| \cdot \nabla_x^\perp \log|x - f_m(z)| |U^*|^2 \\ &\quad + \sum_{k,m} \int_{\mathcal{C}} (f_k'(z) \cdot \nabla_x^\perp \log|x - f_k(z)|)(f_m'(z) \cdot \nabla_x^\perp \log|x - f_m(z)|)|U^*|^2 \\ &= \sum_{k,m} A_{km} + \sum_{k,m} B_{km}, \end{aligned} \tag{3.5}$$

where we have denoted $\nabla_x^\perp = (-\partial/\partial x_2, \partial/\partial x_1)$. In computing A_{km}, B_{km} we will consider two cases.

Case 1. For $k = m$, we have

$$A_{kk} = \int_{\mathcal{C}} |\nabla_x \log|x - f_k(z)||^2 |U^*|^2 = \int_0^\ell dz \int_{\Omega_k(z)} |\nabla_x \log|x - f_k(z)||^2 |U^*|^2,$$

where $\Omega(z) = \{x \mid (x, z) \in \mathcal{C}\}$. Let us denote $\Gamma(x - \xi) = \log|x - \xi|$. Integrating by parts we get

$$\begin{aligned}&\int_{\Omega(z)} \big|\nabla_x \log|x - f_k(z)|\big|^2 |U^*|^2 \big|\nabla_x \log|x - f_k(z)|\big|^2 |U^*|^2 \\ &\quad = \int_{\partial\Omega(z)} \Gamma(x - f_k(z)) \frac{\partial\Gamma(x - f_k(z))}{\partial\nu} |U^*|^2 \, dS \\ &\qquad - \int_{\Omega(z)} \Gamma(x - f_k(z)) \Delta\Gamma(x - f_k(z)) (U^*)^2 \, dx \\ &\qquad - \int_{\Omega(z)} \Gamma(x - f_k(z)) \nabla\Gamma(x - f_k(z)) \nabla(U^*)^2 \, dx. \end{aligned} \tag{3.6}$$

We have

$$\int_{\partial\Omega(z)} \Gamma(x - f_k(z)) \frac{\partial\Gamma(x - f_k(z))}{\partial\nu} |U^*|^2 \, dS = 2\pi \log R + \mathrm{O}(|f_k(z)|).$$

The second integral in (3.6) is equal to 0. The last integral can be decomposed as follows:

$$\begin{aligned}
&-\int_{\Omega(z)} \Gamma(x-f_k(z))\nabla\Gamma(x-f_k(z))\nabla(U^*)^2 \\
&\quad = -2\int_{\Omega(z)} U^* U_{\check{k}}\Gamma(x-f_k(z))\nabla\Gamma(x-f_k(z))\cdot\nabla U_k\,dx \\
&\qquad - 2\sum_{j\neq k}\int_{\Omega(z)} U^* U_{\check{j}}\Gamma(x-f_k(z))\nabla\Gamma(x-f_k(z))\cdot\nabla U_j\,dx.
\end{aligned}$$

Changing variables, we have

$$\xi = \frac{x-f_k(z)}{\varepsilon}, \tag{3.7}$$

and denoting the image of $\Omega(z)$ under this change of variables by $\Omega_\varepsilon(z)$, we get

$$\begin{aligned}
&-2\int_{\Omega(z)} U^* U_{\check{k}}\Gamma(x-f_k(z))\nabla\Gamma(x-f_k(z))\cdot\nabla U_k\,dx \\
&\quad = -2\int_{\Omega_\varepsilon(z)} U_{\check{k}}^2(\varepsilon\xi+f_k)\log|\varepsilon\xi|\frac{UU'}{|\xi|}\,d\xi \\
&\quad = -2\int_{\Omega_\varepsilon(z)} \log|\varepsilon\xi|\frac{UU'}{|\xi|}\,d\xi - 2\int_{\Omega_\varepsilon(z)}[U_{\check{k}}^2(\varepsilon\xi+f_k)-1]\log|\varepsilon\xi|\frac{UU'}{|\xi|}\,d\xi \\
&\quad = 2\pi\log\frac{1}{\varepsilon} - 4\pi\int_0^\infty UU'\log r\,dr + \mathrm{O}(\varepsilon|\log\varepsilon|^p),
\end{aligned}$$

for $\mathbf{f}$ satisfying (3.1)–(3.2), with some $p>0$. Finally, we get

$$A_{kk} = 2\pi\ell\log\frac{1}{\varepsilon} - 4\pi\ell\int_0^\infty UU'\log r\,dr + 2\pi\ell\log R + \int_0^\ell Q_1(\mathbf{f}), \tag{3.8}$$

where

$$\int_0^\ell |Q_1(\mathbf{f})| \leqslant C(\|\mathbf{f}\|_{L^2_\ell(\mathbb{R})} + \varepsilon|\log\varepsilon|^p).$$

To compute B_{kk} we will write

$$\begin{aligned}
B_{kk} &= \int_{\mathcal{C}} [f_k'\cdot\nabla_x^\perp\Gamma(x-f_k)]^2(U^*)^2 \\
&= \int_0^\ell dz\int_{\Omega(z)}[f_k'\cdot\nabla_x^\perp\Gamma(x-f_k)]^2(U^*)^2\,dx,
\end{aligned}$$

and then decompose, denoting the unit tangent of $\partial\Omega(z)$ by τ,

$$\int_{\Omega(z)}[f_k'\cdot\nabla_x^\perp\Gamma(x-f_k)]^2(U^*)^2\,dx \tag{3.9}$$

$$\begin{aligned}
&= \int_{\partial\Omega(z)}\Gamma(x-f_k)[f_k'\cdot\nabla_x^\perp\Gamma(x-f_k)(U^*)^2](f_k'\cdot\tau)\,dS \\
&\quad - \int_{\Omega(z)}\Gamma(x-f_k)f_k'\cdot\nabla_x^\perp[f_k'\cdot\nabla_x^\perp\Gamma(x-f_k)](U^*)^2\,dx \\
&\quad - 2\int_{\Omega(z)}\Gamma(x-f_k)U^*[f_k'\cdot\nabla_x^\perp\Gamma(x-f_k)]f_k'\cdot\nabla_x^\perp U^*.
\end{aligned} \tag{3.10}$$

For the first integral above we get

$$\int_{\partial\Omega(z)}\Gamma(x-f_k)[f_k'\cdot\nabla_x^\perp\Gamma(x-f_k)(U^*)^2](f_k'\cdot\tau)\,dS = |f_k'|^2[2\pi\log R + \mathrm{O}(|f_k|)].$$

Denoting $C_k(z) = \{|x| < R - 2|f_k(z)|\}$, we can write the second integral as

$$\begin{aligned}
&\int_{\Omega(z)} \Gamma(x-f_k) f_k' \cdot \nabla_x^\perp [f_k' \cdot \nabla_x^\perp \Gamma(x-f_k)](U^*)^2 \, dx \\
&\quad = \int_{C_k(z)} \Gamma(x-f_k) f_k' \cdot \nabla_x^\perp [f_k' \cdot \nabla_x^\perp \Gamma(x-f_k)] U_k^2 \, dx \\
&\qquad + \int_{\Omega(z)\setminus C_k(z)} \Gamma(x-f_k) f_k' \cdot \nabla_x^\perp [f_k' \cdot \nabla_x^\perp \Gamma(x-f_k)] U_k^2 \, dx \\
&\qquad + \int_{\Omega(z)} \Gamma(x-f_k) f_k' \cdot \nabla_x^\perp [f_k' \cdot \nabla_x^\perp \Gamma(x-f_k)] U_k^2 (U_{\check{k}}^2 - 1) \, dx.
\end{aligned} \tag{3.11}$$

Changing variables as in (3.7) we get, by a direct calculation,

$$\begin{aligned}
&\int_{C_k(z)} \Gamma(x-f_k) f_k' \cdot \nabla_x^\perp [f_k' \cdot \nabla_x^\perp \Gamma(x-f_k)] U_k^2 \, dx \\
&\quad = \int_{|\xi|<(R-2|f_k(z)|)/\varepsilon} U^2 \log|\varepsilon\xi| f_k' \cdot \nabla_\xi^\perp [f_k' \cdot \nabla_\xi^\perp \log|\xi|] \, d\xi \\
&\quad = 0.
\end{aligned}$$

Denoting by $\Omega_\varepsilon(z)$ the image of $\Omega(z)$ under the change of variables (3.7) and by $C_{k\varepsilon}(z)$ the set $\{|\xi| < (R - 2|f_k(z)|)/\varepsilon\}$, we obtain

$$\begin{aligned}
&\int_{\Omega(z)\setminus C_k(z)} \Gamma(x-f_k) f_k' \cdot \nabla_x^\perp [f_k' \cdot \nabla_x^\perp \Gamma(x-f_k)] U_k^2 \, dx \\
&\quad = \int_{\Omega_\varepsilon(z)\setminus C_{k\varepsilon}(z)} U^2 \log|\varepsilon\xi| f_k' \cdot \nabla_\xi^\perp [f_k' \cdot \nabla_\xi^\perp \log|\xi|] \, d\xi \\
&\quad = Q_2(f_k),
\end{aligned} \tag{3.12}$$

where $Q_2(f_k)$ is a smooth function of f_k, f_k' such that

$$|Q_2(f_k)| \leqslant C|f_k'|^2 |f_k|. \tag{3.13}$$

Finally, for $\mathbf{f} \in H_\ell^1(\mathbb{R})$ satisfying (3.1)–(3.2), the last integral in (3.11) is estimated as follows:

$$\begin{aligned}
&\int_{\Omega(z)} \Gamma(x-f_k) f_k' \cdot \nabla_x^\perp [f_k' \cdot \nabla_x^\perp \Gamma(x-f_k)] U_k^2 (U_{\check{k}}^2 - 1) \, dx \\
&\quad = \int_{B(f_k,\delta_\varepsilon)} \Gamma(x-f_k) f_k' \cdot \nabla_x^\perp [f_k' \cdot \nabla_x^\perp \Gamma(x-f_k)] U_k^2 (U_{\check{k}}^2 - 1) \\
&\qquad + \sum_{l\neq k} \int_{B(f_l,\delta_\varepsilon)} \Gamma(x-f_k) f_k' \cdot \nabla_x^\perp [f_k' \cdot \nabla_x^\perp \Gamma(x-f_k)] U_k^2 (U_{\check{k}}^2 - 1) \\
&\qquad + \int_{\Omega(z)\setminus \cup B(f_l,\delta_\varepsilon)} \Gamma(x-f_k) f_k' \cdot \nabla_x^\perp [f_k' \cdot \nabla_x^\perp \Gamma(x-f_k)] U_k^2 (U_{\check{k}}^2 - 1) \\
&\quad = \mathrm{O}(\varepsilon|\log\varepsilon|^p)|f_k'|^2,
\end{aligned}$$

where $\delta_\varepsilon = 1/|\log\varepsilon|^4$.

Now we will deal with the last integral in (3.10). We have

$$\begin{aligned}
&-2\int_{\Omega(z)}\Gamma(x-f_k)U^*[f_k'\cdot\nabla_x^\perp\Gamma(x-f_k)]f_k'\cdot\nabla_x^\perp U^*\\
&\quad=-2\varepsilon^{-1}\int_{\Omega(z)}\Gamma(x-f_k)\frac{[f_k'\cdot(x-f_k)^\perp]^2}{|x-f_k|^3}U_k'U_k\\
&\qquad-2\varepsilon^{-1}\int_{\Omega(z)}\Gamma(x-f_k)\frac{[f_k'\cdot(x-f_k)^\perp]^2}{|x-f_k|^3}U_k'U_k[U_{\tilde{k}}^2-1]\\
&\qquad-2\varepsilon^{-1}\sum_{j\neq k}\int_{\Omega(z)}\Gamma(x-f_k)\left\{\frac{[f_k'\cdot(x-f_k)^\perp]}{|x-f_k|^2}\frac{f_k'\cdot(x-f_j)^\perp}{|x-f_j|}\right\}U_j'U_jU_{\tilde{j}}^2\\
&\quad=\left\{\pi\log\frac{1}{\varepsilon}-2\pi\int_0^\infty UU'\log r\,dr\right\}|f_k'|^2+Q_3(\mathbf{f})\mathrm{O}(\varepsilon|\log\varepsilon|^p),
\end{aligned}$$

for some $p>0$, where Q_3 is a smooth function of $\mathbf{f},\mathbf{f}'$ such that for $\mathbf{f}$ satisfying (3.1)–(3.2) we have:

$$|Q_3(\mathbf{f})|\leqslant C|\mathbf{f}'|^2. \tag{3.14}$$

Summarizing the above we get

$$B_{kk}=\pi\ell\log\frac{1}{\varepsilon}\int_0^\ell|f_k'|^2\,dz+\int_0^\ell Q_4(\mathbf{f})\,dz, \tag{3.15}$$

where

$$\int_0^\ell|Q_4(\mathbf{f})|\leqslant C\|\mathbf{f}'\|^2_{L^2_\ell(\mathbb{R})}[1+\|\mathbf{f}\|_{H^1_\ell(\mathbb{R})}+\mathrm{O}(\varepsilon|\log\varepsilon|^p)].$$

Continuing now the calculation of (3.5), we consider Case 2: $k\neq m$.

Case 2. Fixing $k\neq m$, we get

$$\begin{aligned}
A_{km}&=\int_{\mathcal{C}}|U^*|^2\nabla_x^\perp\Gamma(x-f_k)\cdot\nabla_x^\perp\Gamma(x-f_m(z))\\
&=\int_0^\ell dz\int_{\partial\Omega(z)}|U^*|^2\Gamma(x-f_k)\frac{\partial\Gamma(x-f_m)}{\partial\nu}\,dS\\
&\quad-\int_0^\ell dz\int_{\Omega(z)}\Gamma(x-f_k)\nabla\Gamma(x-f_m)\cdot\nabla|U^*|^2\,dx.
\end{aligned}$$

The first integral above is estimated in a similar manner as in Case 1, namely

$$\int_{\partial\Omega(z)}|U^*|^2\Gamma(x-f_k)\frac{\partial\Gamma(x-f_m)}{\partial\nu}\,dS=2\pi\log R+\mathrm{O}(|\mathbf{f}|). \tag{3.16}$$

The second integral is decomposed as follows:

$$\begin{aligned}
&-\int_{\Omega(z)}\Gamma(x-f_k)\nabla\Gamma(x-f_m)\cdot\nabla|U^*|^2\,dx\\
&\quad=-\int_{\Omega(z)}\Gamma(x-f_k)\nabla\Gamma(x-f_m)\cdot\nabla U_m^2\,dx\\
&\qquad-\int_{\Omega(z)}\Gamma(x-f_k)\nabla\Gamma(x-f_m)\cdot\nabla(|U^*|^2-U_m^2)\,dx.
\end{aligned} \tag{3.17}$$

We have, using the change of variables (3.7),

$$
\begin{aligned}
&-\int_{\Omega(z)} \Gamma(x-f_k)\nabla\Gamma(x-f_m)\cdot\nabla U_m^2\,dx\\
&\quad= -2\log|f_k-f_m|\int_{\Omega_\varepsilon(z)}\frac{UU'}{|\xi|}\,d\xi - 2\int_{\Omega_\varepsilon(z)}\log\left(\frac{|\varepsilon\xi+f_m-f_k|}{|f_m-f_k|}\right)\frac{UU'}{|\xi|}\,d\xi\\
&\quad= -2\pi\log|f_k-f_m| + \mathrm{O}(\varepsilon|\log\varepsilon|^p).
\end{aligned} \tag{3.18}
$$

The last integral in (3.17) is estimated as follows:

$$
\begin{aligned}
&-\int_{\Omega(z)} \Gamma(x-f_k)\nabla\Gamma(x-f_m)\cdot\nabla(|U^*|^2-U_m^2)\,dx\\
&\quad= -2\int_{\Omega(z)}[U_m(U_{\check{m}}^2-1)]\Gamma(x-f_k)\nabla\Gamma(x-f_m)\cdot\nabla U_m\,dx\\
&\qquad -2\int_{\Omega(z)} U_m^2 U_{\check{m}}\Gamma(x-f_k)\nabla\Gamma(x-f_m)\cdot\nabla U_{\check{m}}\,dx\\
&\quad= \mathrm{O}(\varepsilon|\log\varepsilon|^p).
\end{aligned}
$$

Summarizing (3.15)–(3.18), we get

$$
A_{km} = -2\pi\int_0^\ell \log\left(\frac{|f_k-f_m|}{R}\right) + \int_0^\ell Q_5(\mathbf{f}), \tag{3.19}
$$

where

$$
\int_0^\ell |Q_5(\mathbf{f})| \leqslant C(\|\mathbf{f}\|_{L^2_\ell(\mathbb{R})} + \varepsilon|\log\varepsilon|^p).
$$

Now we will deal with B_{km}, $k\neq m$. We have

$$
B_{km} = \int_0^\ell dz\int_{\Omega(z)} |U^*|^2[(f_k')^\perp\cdot\nabla_x\Gamma(x-f_k)][(f_m')^\perp\cdot\nabla_x\Gamma(x-f_m)]\,dx.
$$

Integrating by parts we get

$$
\begin{aligned}
&\int_{\Omega(z)} |U^*|^2[(f_k')^\perp\cdot\nabla_x\Gamma(x-f_k)][(f_m')^\perp\cdot\nabla_x\Gamma(x-f_m)]\,dx\\
&\quad= \int_{\partial\Omega(z)} |U^*|^2\Gamma(x-f_k)[(f_m')^\perp\cdot\nabla_x\Gamma(x-f_m)](f_k')^\perp\cdot\nu\,dS\\
&\qquad -\int_{\Omega(z)} |U^*|^2\Gamma(x-f_k)(f_k')^\perp\cdot D_x^2\Gamma(x-f_m)\cdot(f_m')^\perp\,dx\\
&\qquad -\int_{\Omega(z)} \Gamma(x-f_k)[(f_m')^\perp\cdot\nabla_x\Gamma(x-f_m)](f_k')^\perp\cdot\nabla_x|U^*|^2\,dx.
\end{aligned} \tag{3.20}
$$

The first integral in (3.20) is estimated in a similar way as in (3.16) except for the extra factor, which is of the order $\mathrm{O}(|\mathbf{f}'|^2)$. Thus we get

$$
\int_0^\ell dz\int_{\partial\Omega(z)} |U^*|^2\Gamma(x-f_k)[(f_m')^\perp\cdot\nabla_x\Gamma(x-f_m)](f_k')^\perp\cdot\nu\,dS = \int_0^\ell Q_6(\mathbf{f})\,dz, \tag{3.21}
$$

where

$$
\int_0^\ell |Q_6(\mathbf{f})| \leqslant C\|\mathbf{f}'\|^2_{L^2_\ell(\mathbb{R})}.
$$

To estimate the second integral in (3.20) we decompose the domain of integration $\Omega(z)$ into three subsets as follows:

$$
E_1 = \{|x-f_m|\leqslant\delta_\varepsilon\},\quad E_2 = \{|x-f_k|\leqslant\delta_\varepsilon\},\quad E_3 = \Omega(z)\setminus(E_1\cup E_2),
$$

where $\delta_\varepsilon = 1/|\log\varepsilon|^4$. In E_1 we will write

$$\log|x-f_k| = \log\left|\frac{x-f_m}{|f_m-f_k|} + \frac{f_m-f_k}{|f_m-f_k|}\right| + \log|f_m-f_k|.$$

Notice that

$$\left|\log\left|\frac{x-f_m}{|f_m-f_k|} + \frac{f_m-f_k}{|f_m-f_k|}\right|\right| \leqslant C\frac{|x-f_m|}{|f_m-f_k|}.$$

Using this we can estimate

$$\begin{aligned}
&\left|\int_{E_1} |U^*|^2\Gamma(x-f_k)(f_k')^\perp \cdot D_x^2\Gamma(x-f_m)\cdot(f_m')^\perp\,dx\right| \\
&\quad\leqslant \left|\int_{E_1} |U^*|^2\log|f_m-f_k|(f_k')^\perp \cdot D_x^2\Gamma(x-f_m)\cdot(f_m')^\perp\,dx\right| \\
&\qquad + C\int_{E_1}|U^*|^2\frac{|x-f_m|}{|f_m-f_k|}|(f_k')^\perp \cdot D_x^2\Gamma(x-f_m)\cdot(f_m')^\perp|\,dx \\
&\quad\leqslant C|\mathbf{f}'|^2\frac{\delta_\varepsilon}{|f_m-f_k|} \\
&\quad\leqslant C\frac{|\mathbf{f}'|^2}{|\log\varepsilon|^2}.
\end{aligned}$$

To estimate the integral over E_2 we write

$$\begin{aligned}
&\left|\int_{E_2} |U^*|^2\Gamma(x-f_k)(f_k')^\perp \cdot D_x^2\Gamma(x-f_m)\cdot(f_m')^\perp\,dx\right| \\
&\quad\leqslant C\frac{|\mathbf{f}'|^2}{|f_m-f_k|^2}\int_{E_2}\big|\log|x-f_k|\big|\,dx \\
&\quad\leqslant C|\mathbf{f}'|^2\frac{\delta_\varepsilon^2|\log\delta_\varepsilon|}{|f_m-f_k|^2} \\
&\quad\leqslant C\frac{|\mathbf{f}'|^2}{|\log\varepsilon|^2}.
\end{aligned}$$

Finally in E_3 we have

$$\begin{aligned}
&\left|\int_{E_3} |U^*|^2\Gamma(x-f_k)(f_k')^\perp \cdot D_x^2\Gamma(x-f_m)\cdot(f_m')^\perp\,dx\right| \\
&\quad\leqslant C|\log\delta_\varepsilon||\mathbf{f}'|^2\int_{\{|x-f_m|>\delta_\varepsilon\}}|D_x^2\Gamma(x-f_m)|\,dx \\
&\quad\leqslant C|\log\delta_\varepsilon||\mathbf{f}'|^2\int_{\delta_\varepsilon/\varepsilon}^{2R/\varepsilon}\frac{dr}{r} \\
&\quad\leqslant C\log^2|\log\varepsilon||\mathbf{f}'|^2.
\end{aligned}$$

Thus for functions $\mathbf{f}\in H^1_\ell(\mathbb{R})^n$ satisfying (3.1)–(3.2) we get

$$\begin{aligned}
&\int_0^\ell dz\int_{\Omega(z)}|U^*|^2\Gamma(x-f_k)(f_k')^\perp \cdot D_x^2\Gamma(x-f_m)\cdot(f_m')^\perp\,dx \\
&\quad = \mathrm{O}(\log^2|\log\varepsilon|)\int_0^\ell Q_7(\mathbf{f})\,dz, \qquad (3.22)
\end{aligned}$$

where

$$\int_0^\ell|Q_7(\mathbf{f})|\leqslant C\|\mathbf{f}'\|^2_{L^2_\ell(\mathbb{R})}.$$

Finally, to find the asymptotic formula for the last integral in (3.20) we use the same approach as in (3.17)–(3.19) to get

$$\int_0^\ell dz \int_{\Omega(z)} \Gamma(x-f_k)[(f_m')^\perp \cdot \nabla_x \Gamma(x-f_m)](f_k')^\perp \cdot \nabla_x |U^*|^2\, dx$$
$$= \int_0^\ell P(\mathbf{f}) \log|f_k - f_m|\, dz, \tag{3.23}$$

where

$$|P(\mathbf{f})| \leqslant C(|\mathbf{f}'|^2 + \mathrm{O}(\varepsilon|\log\varepsilon|^p)). \tag{3.24}$$

Summarizing, we have

$$B_{km} = \mathrm{O}(\log|\log\varepsilon|) \int_0^\ell Q_8(\mathbf{f})\, dz + \int_0^\ell P(\mathbf{f}) \log|f_k - f_m|\, dz, \tag{3.25}$$

where

$$\int_0^\ell |Q_8(\mathbf{f})| \leqslant C\|\mathbf{f}'\|^2_{L^2_\ell(\mathbb{R})}.$$

Let us go back to the expression for I_1 in (3.5). We have just shown that

$$I_1[\mathbf{f}] = 2n\pi\ell \log\frac{1}{\varepsilon} + c_0 + \pi\log\frac{1}{\varepsilon}\int_0^\ell |\mathbf{f}'|^2\, dz - 4\pi \sum_{k<m} \int_0^\ell \log|f_k - f_m|\, dz + \int_0^\ell N_1(\mathbf{f}), \tag{3.26}$$

where the non-linear term $N_1(\mathbf{f})$ satisfies

$$\int_0^\ell |N_1(\mathbf{f})|\, dz \leqslant C\Big[\|\mathbf{f}\|_{L^2_\ell(\mathbb{R})} + \mathrm{O}(\log|\log\varepsilon|)\|\mathbf{f}'\|^2_{H^1_\ell(\mathbb{R})}$$
$$+ \int_0^\ell P(\mathbf{f}) \max_{k\neq m} \big|\log|f_k - f_m|\big|\, dz + \mathrm{O}(\varepsilon|\log\varepsilon|^p)\Big], \tag{3.27}$$

with $P(\mathbf{f})$ satisfying (3.24). From our argument it is seen that $I_1[\mathbf{f}]$ is a well-defined functional for all functions $\mathbf{f}$ satisfying (3.1)–(3.2). Let us also observe that

$$J_\varepsilon\left(w_\varepsilon\left(\cdot; \frac{1}{\sqrt{|\log\varepsilon|}}\tilde{\mathbf{f}}\right)\right) = \frac{1}{2} I_1[\mathbf{f}] + \dots.$$

We will now consider functionals $I_2[\mathbf{f}]$ and $I_3[\mathbf{f}]$ defined in (3.4). We want to show that they are also well defined in $H^1_\ell(\mathbb{R})^n$. To this end, given $f \in H^1_\ell(\mathbb{R})$ let us examine the function $\varphi(x,z;f)$ which is the unique solution of (1.13)–(1.16). We have

$$\begin{aligned}
I_2[\mathbf{f}] &= 2\int_{\mathcal{C}} \nabla\Theta\cdot\nabla\varphi|U^*|^2 = 2\sum_{k=1}^n \int_{\mathcal{C}} \nabla\theta_k\cdot\nabla\varphi|U^*|^2 \\
&= 2\sum_{k=1}^n \int_0^\ell dz \int_{\partial\Omega(z)} \varphi|U^*|^2 \frac{\partial\theta_k}{\partial\nu}\, dS \\
&\quad + 2\sum_{k=1}^n \int_0^\ell dz \int_{\Omega(z)} \varphi\nabla_x\theta_k\cdot\nabla_x|U^*|^2\, dx \\
&\quad + 2\sum_{k=1}^n \int_0^\ell dz \int_{\Omega(z)} \varphi_z \frac{(x-f_k)^\perp\cdot f_k'}{|x-f_k|^2}|U^*|^2\, dx \\
&= 2\sum_{k=1}^n \int_0^\ell C_k(z)\, dz + 2\sum_{k=1}^n \int_0^\ell D_k(z)\, dz + 2\sum_{k=1}^n \int_0^\ell E_k(z)\, dz.
\end{aligned}$$

Let us recall that φ_k satisfies

$$\Delta\varphi_k = 0 \quad \text{in } \mathcal{C},$$
$$\frac{\partial \varphi_k}{\partial \nu} = -\frac{\partial \theta_k}{\partial \nu} \quad \text{in } \partial\mathcal{C} \cap \{|x| = R\},$$
$$\int_\Omega \varphi_k = 0,$$

and that we have denoted

$$\varphi = \sum_{k=1}^n \varphi_k.$$

Standard elliptic estimates imply that

$$\|\varphi\|_{H^2_\ell(\mathcal{C})} \leqslant C\|\mathbf{f}\|_{H^1_\ell(\mathbb{R})^n}, \tag{3.28}$$

since on $\partial\mathcal{C} \cap \{|x| = R\}$,

$$\frac{\partial \theta_k}{\partial \nu} = \frac{(x - f_k)^\perp}{|x - f_k|^2} \cdot \frac{x}{R} = -\frac{f_k^\perp \cdot x}{R|x - f_k|^2}. \tag{3.29}$$

Then we get

$$\left|\int_0^\ell |C_k(z)|\, dz\right| \leqslant C\|\mathbf{f}\|^2_{H^1_\ell}\, dz. \tag{3.30}$$

To estimate $D_k(z)$ let us notice that for each fixed $m = 1, \ldots, n$ we have

$$\int_{\Omega(z)} \varphi_m \nabla_x \theta_k \cdot \nabla_x |U^*|^2\, dx = \sum_{j \neq k} \int_{\Omega(z)} \varphi_m U_j^2 \nabla_x \theta_k \cdot \nabla_x U_j^2\, dx.$$

Using (3.29) we obtain

$$\begin{aligned}
\left|\int_{\Omega(z)} \varphi_m U_j^2 \nabla\theta_k \cdot \nabla U_j^2\, dx\right| &\leqslant C\|\varphi_m\|_{H^2(\mathcal{C})} \int_{\Omega(z)} \frac{|\nabla U_j^2| U_j^2}{|x - f_k|}\, dx \\
&\leqslant C\varepsilon |\log\varepsilon|^p \|\mathbf{f}\|_{H^1_\ell(\mathbb{R})},
\end{aligned}$$

and hence

$$\left|\int_0^\ell D_k(z)\, dz\right| \leqslant C\varepsilon |\log\varepsilon|^p \|\mathbf{f}\|_{H^1_\ell(\mathbb{R})}. \tag{3.31}$$

Finally, we have

$$\begin{aligned}
\left|\int_0^\ell E_k(z)\, dz\right| &\leqslant \|\varphi\|_{H^1(\mathcal{C})} \left(\int_{\mathcal{C}} \frac{|f_k'|^2 |U^*|^4}{|x - f_k|^2}\right)^{1/2} \\
&\leqslant C|\log\varepsilon|^{1/2} \|\varphi\|_{H^1(\mathcal{C})} \|f_k\|_{H^1_\ell(\mathbb{R})} \\
&\leqslant C|\log\varepsilon|^{1/2} \|\mathbf{f}\|^2_{H^1_\ell(\mathbb{R})}.
\end{aligned} \tag{3.32}$$

Combining (3.30)–(3.32) we find that $I_2[\mathbf{f}]$ is well defined and

$$|I_2[\mathbf{f}]| \leqslant C|\log\varepsilon|^{1/2} \|\mathbf{f}\|^2_{H^1_\ell(\mathbb{R})} + C\varepsilon|\log\varepsilon|^p \|\mathbf{f}\|_{H^1_\ell(\mathbb{R})}. \tag{3.33}$$

Finally, we have

$$|I_3[\mathbf{f}]| = \left|\int_{\mathcal{C}} |\nabla\varphi|^2 |U^*|^2\right| \leqslant C\|\mathbf{f}\|^2_{H^1_\ell(\mathbb{R})}. \tag{3.34}$$

To estimate the rest of the functional J_ε we set

$$I_4[\mathbf{f}] = \frac{1}{2}\int_{\mathcal{C}} |\nabla U^*|^2 + \frac{1}{4\varepsilon^2}\int_{\mathcal{C}} (1 - |U^*|^2)^2.$$

It is a matter of rather straightforward calculations to show that

$$I_4[\mathbf{f}] = \tilde{c}_1 + \mathrm{O}(\|\mathbf{f}\|_{H^1_\ell(\mathbb{R})}),$$

where $\tilde{c}_1$ is a constant independent of $\mathbf{f}$. Combining estimates for $I_j[\mathbf{f}]$, $j = 1, \ldots, 4$, ends the proof. $\square$

Proof of Theorem 1.1. The C^0-bound for Θ_ε in formula (1.20) is an immediate consequence of Lemma 3.1. The estimates for $D\Theta_\varepsilon$ and $D^2\Theta_\varepsilon$ follow from asymptotic expressions for $D\mathcal{I}(\mathbf{f})$ and $D^2\mathcal{I}(\mathbf{f})$ where $\mathcal{I}$ is the functional defined by formula (3.3). On the one hand, we find that for any test function $h \in H^1_\ell(\mathbb{R}, \mathbb{R}^2)^n$ the following asymptotic formula holds:

$$\begin{aligned} D\mathcal{I}(\mathbf{f})(\mathbf{h}) = \pi \log \frac{1}{\varepsilon} \int_0^\ell \mathbf{f}' \cdot h' - 2\pi \sum_{m \neq k} \int_0^\ell \frac{(f_k - f_m) \cdot (h_k - h_m)}{|f_k - f_m|^2} \\ + R(\mathbf{f})[\mathbf{h}], \end{aligned} \tag{3.35}$$

where

$$R(\mathbf{f})[\mathbf{h}] = \int_0^\ell \mathcal{F}_1(\mathbf{f}, \mathbf{f}') \cdot \mathbf{h}_k + \int_0^\ell \mathcal{F}_2(\mathbf{f}, \mathbf{f}') \cdot \mathbf{h}',$$

with $\mathcal{F}_j(\mathbf{f}, \mathbf{f}')$ $j = 1, 2$, satisfying the pointwise estimates

$$|\mathcal{F}_1(\mathbf{f}, \mathbf{f}')| \leqslant C[1 + \log^2 |\log \varepsilon| |\mathbf{f}'|^2 + \max_{k \neq j} \{|f_k - f_j|^{-1}\} |\mathbf{f}'|^2]. \tag{3.36}$$

and

$$|\mathcal{F}_{2k}(\mathbf{f}, \mathbf{f}')| \leqslant C\big[1 + \log^2 |\log \varepsilon|\, |\mathbf{f}|\, |\mathbf{f}'| + \max_{k \neq j} \big\{\big| \log |f_k - f_j| \big|\big\} |\mathbf{f}'|^2\big]. \tag{3.37}$$

From here we find that the desired C^0-estimate for $D\Theta_\varepsilon(\mathbf{f})$ follows at once. Similarly, we obtain that

$$\begin{aligned} D^2\mathcal{I}[\mathbf{f}](\mathbf{h}, \mathbf{h}) = \pi \log \frac{1}{\varepsilon} \|\mathbf{h}'\|^2_{L^2(0,\ell)^n} \\ -2\pi \sum_{k=1}^n \sum_{m \neq k} \left\{ \int_0^\ell \frac{|h_k|^2}{|f_k - f_m|^2} + 2 \int_0^\ell \frac{\left\{ \mathrm{Re}\left[(f_k - f_m) \overline{(h_k - h_m)} \right] \right\}^2}{|f_k - f_m|^4} \right\} \\ + S(\mathbf{f})(\mathbf{h}, \mathbf{h}), \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} |S(\mathbf{f})(\mathbf{h}, \mathbf{h})| \leqslant C\sqrt{|\log \varepsilon|} \|\mathbf{f}\|^2_{H^1_\ell(\mathbb{R})^n} \|\mathbf{h}\|^2_{H^1_\ell(\mathbb{R})^n} \\ + \frac{C}{\sqrt{|\log \varepsilon|}} \int_0^\ell |\mathbf{f}'|^2 |\mathbf{h}|^2 \max_{j \neq l} \{|f_j - f_l|^{-2}\}. \end{aligned}$$

This gives the desired estimate on the second derivative of Θ_ε in Theorem 1.1. The estimates above also involve simple elliptic bounds for the first and second derivatives of the phase function $\varphi(\cdot, \mathbf{f})$ with respect to $\mathbf{f}$ in H^1_ℓ. All these computations are rather lengthy, but they proceed along the same lines as those in the proof of the previous lemma, so we omit the details here. $\square$

Acknowledgement. The work of the first author has been partly supported by Chilean research grants Fondecyt 1070389 and FONDAP. The second author has been partially supported by Fondecyt 1050311, FONDAP and Nucleus Millennium grant P04-069-F.

References

1. A. Aftalion and R. L. Jerrard, 'Properties of a single vortex solution in a rotating Bose–Einstein condensate', *C. R. Math. Acad. Sci. Paris* 336 (2003) 713–718.
2. S. Alama, L. Bronsard and A. Montero, 'On the Ginzburg–Landau model of a superconducting ball in a uniform field', *Ann. Inst. H. Poincaré Anal. Non Linéaire* 23 (2006) 237–267.
3. G. Alberti, S. Baldo and G. Orlandi, 'Functions with prescribed singularities', *J. Eur. Math. Soc.* (*JEMS*) 5 (2003) 275–311.
4. G. Alberti, S.Baldo and G. Orlandi, 'Variational convergence for functionals of Ginzburg–Landau type', *Indiana Univ. Math. J.* 54 (2005) 1411–1472.
5. L. Almeida and F. Bethuel, 'Topological methods for the Ginzburg–Landau equations', *J. Math. Pures Appl.* (9) 77 (1998) 1–49.
6. F. Bethuel, H. Brezis and F. Helein, *Ginzburg–Landau vortices*, Progress in Nonlinear Differential Equations and their Applications 13 (Birkhäuser Boston, Boston, MA, 1994).
7. F. Bethuel, H. Brezis and G. Orlandi, 'Asymptotics for the Ginzburg–Landau equation in arbitrary dimensions', *J. Funct. Anal.* 186 (2001) 432–520.
8. F. Bethuel, G. Orlandi and D. Smets, 'Convergence of the parabolic Ginzburg–Landau equation to motion by mean curvature', *Ann. of Math.* (2) 163 (2006) 37–163.
9. K.-C. Chen, 'Action-minimizing orbits in the parallelogram four-body problem with equal masses', *Arch. Ration. Mech. Anal.* 158 (2001) 293–318.
10. X. Chen, C. M. Elliott and T. Qi, 'Shooting method for vortex solutions of a complex-valued Ginzburg–Landau equation', *Proc. Roy. Soc. Edinburgh Sect. A* 124 (1994) 1075–1088.
11. A. Chenciner and R. Montgomery, 'A remarkable periodic solution of the three body problem in the case of equal masses', *Ann. of Math.* 152 (1999) 881–901.
12. D. Chiron, 'Vortex helices for the Gross–Pitaevskii equation', *J. Math. Pures Appl.* 84 (2005) 1555–1647.
13. D. L. Ferrario and S. Terracini, 'On the existence of collisionless equivariant minimizers for the classical n-body problem', *Invent. Math.* 155 (2004) 305–362.
14. V. L. Ginzburg and L. D. Landau, 'On the theory of superconductivity', *J. Exp. Theor. Phys.* 20 (1950) 1064–1082.
15. R. Jerrard, A. Montero and P. Sternberg, 'Local minimizers of the Ginzburg–Landau energy with magnetic field in three dimensions', *Comm. Math. Phys.* 249 (2004) 549–577.
16. R. L. Jerrard and H.M. Soner, 'Dynamics of Ginzburg–Landau vortices', *Arch. Ration. Mech. Anal.* 142 (1998) 99–125.
17. J. Jerrard and H. Soner, 'The Jacobian and the Ginzburg–Landau energy', *Calc. Var. Partial Differential Equations* 14 (2002) 151–191.
18. S. Jimbo and Y. Morita, 'Stability of nonconstant steady-state solutions to a Ginzburg–Landau equation in higher space dimensions', *Nonlinear Anal.* 22 (1994) 753–770.
19. S. Jimbo and Y. Morita, 'Vortex dynamics for the Ginzburg–Landau equation with Neumann condition', *Methods Appl. Anal.* 8 (2001) 451–477.
20. C. E. Kenig, G. Ponce and L. Vega, 'On the interaction of nearly parallel vortiex filaments', *Comm. Math. Phys.* 243 (2003) 471–483.
21. R. Klein, A. Majda and K. Damodaran, 'Simplified equations for the interactions of nearly parallel vortex filaments', *J. Fluid. Mech.* 288 (1995) 201–248.
22. F. H. Lin, 'Solutions of Ginzburg–Landau equations and critical points of the renormalized energy', *Ann. Inst. H. Poincare Anal. Non Lineaire* 12 (1995) 599–622.
23. F. H. Lin, 'Mixed vortex-antivortex solutions of Ginzburg–Landau equations', *Arch. Ration. Mech. Anal.* 133 (1995) 103–127.
24. F. H. Lin, 'Some dynamical properties of Ginzburg–Landau vortices', *Comm. Pure Appl. Math.* 49 (1996) 323–359.
25. T. C. Lin, 'Rigorous and generalized derivation of vortex line dynamics in superfluids and superconductors', *SIAM J. Appl. Math.* 60 (2000) 1099–1110.
26. F. H. Lin and T.C. Lin, 'Minimax solutions of the Ginzburg–Landau equations', *Selecta Math.* (*N.S.*) 3 (1997) 99–113.
27. F. H. Lin and T. Riviere, 'Complex Ginzburg–Landau equations in high dimensions and codimension two area minimizing currents', *J. Eur. Math. Soc.* (*JEMS*) 1 (1999) 237–311.
28. F. H. Lin and T. Riviere, 'A quantization property for moving line vortices', *Comm. Pure Appl. Math.* 54 (2001) 826–850.
29. C. Marchal, 'How the method of minimization of action avoids singularities', *Celestial Mech. Dynam. Astronom.* 83 (2002) 325–353.
30. P. Mironescu, 'On the stability of radial solutions of the Ginzburg–Landau equation', *J. Funct. Anal.* 130 (1995) 334–344.
31. A. Montero, P. Sternberg and W. P. Ziemer, 'Local minimizers with vortices in the Ginzburg–Landau system in three dimensions', *Comm. Pure Appl. Math.* 57 (2004) 99–125.
32. C. Moore, 'Braids in classical dynamics', *Phys. Rev. Lett.* 70 (1993) 3675–3679.
33. J. Neu, 'Vortices in complex scalar fields', *Phys. D* 43 (1990) 385–406.
34. F. Pacard and T. Riviere, *Linear and nonlinear aspects of vortices: The Ginzburg–Landau model.* Progress in Nonlinear Differential Equations and their Applications 39 (Birkhauser Boston, Boston, MA, 2000).

35. M. del Pino and P. Felmer, 'Local minimizers for the Ginzburg–Landau energy', *Math. Z.* 225 (1997) 671–684.
36. M. del Pino, M. Kowalczyk and M. Musso, 'Variational reduction for Ginzburg–Landau vortices', *J. Funct. Anal.* 239 (2006) 497–541.
37. M. del Pino, M. Kowalczyk and J. Wei, 'The Toda system and clustering interfaces in the Allen–Cahn equation', *Arch. Ration Mech. Anal.*, to appear.
38. L. M. Pismen and J. Rubinstein, 'Motion of vortex lines in the Ginzburg–Landau model', *Phys. D* 47 (1991) 353–360.
39. E. Sandier and S. Serfaty, 'Gamma-convergence of gradient flows with applications to Ginzburg–Landau', *Comm. Pure Appl. Math.* LVII (2004) 1627–1672.
40. S. Serfaty, 'Local minimizers for the Ginzburg–Landau energy near critical magnetic field. I', *Comm. Contemp. Math.* 1 (1999) 213–254.
41. S. Serfaty, 'Stable configurations in superconductivity: uniqueness, multiplicity, and vortex-nucleation', *Arch. Ration. Mech. Anal.* 149 (1999) 329–365.
42. S. Serfaty, 'Stability in 2D Ginzburg–Landau passes to the limit', *Indiana Univ. Math. J.* 54 (2005) 199–221.
43. I. Shafrir, 'Remarks on solutions of $-\Delta u = (1 - |u|^2)u$ in $\mathbb{R}^2$', *C. R. Acad. Sci. Paris Ser. I Math.* 318 (1994) 327–331.
44. M. Struwe, 'On the asymptotic behavior of minimizers of the Ginzburg–Landau model in 2 dimensions', *Differential Integral Equations* 7 (1994) 1613–1624.

Manuel del Pino and Michał Kowalczyk
Departamento de Ingeniería Matemática and CMM
Universidad de Chile
Casilla 170 Correo 3
Santiago
Chile

delpino@dim.uchile.cl
kowalczy@dim.uchile.cl