The Toda System and Clustering Interfaces in the Allen–Cahn equation

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Abstract

We consider the Allen–Cahn equation $\varepsilon^2 \Delta u + (1 - u^2)u = 0$ in a bounded, smooth domain Ω in $\mathbb{R}^2$, under zero Neumann boundary conditions, where $\varepsilon > 0$ is a small parameter. Let Γ_0 be a segment contained in Ω, connecting orthogonally the boundary. Under certain nondegeneracy and nonminimality assumptions for Γ_0, satisfied for instance by the short axis in an ellipse, we construct, for any given $N \geq 1$, a solution exhibiting N transition layers whose mutual distances are $O(\varepsilon |\log \varepsilon|)$ and which collapse onto Γ_0 as $\varepsilon \to 0$. Asymptotic location of these interfaces is governed by a Toda-type system and yields in the limit broken lines with an angle at a common height and at main order cutting orthogonally the boundary.

1. Introduction

Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^2$. In this paper we consider the elliptic problem:

$$\varepsilon^2 \Delta u + (1 - u^2)\, u \, = \, 0 \quad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial \nu} \, = \, 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

Here $\varepsilon > 0$ is a small parameter and ν denotes unit outer normal to $\partial\Omega$. Equation (1.1)–(1.2) is known as the Allen–Cahn equation and was introduced in [2] as a model describing the evolution of antiphase boundaries.

Problem (1.1)–(1.2) and its parabolic counterpart have been a subject of extensive research for many years. Its solutions correspond to critical points in $H^1(\Omega)$ of the Allen–Cahn energy,

$$J_\varepsilon(u) = \int_\Omega \left[\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{4\varepsilon}(1 - u^2)^2 \right].$$

We observe that the states $u = \pm 1$ represent both global minimizers of the energy, which in the Allen–Cahn model correspond to two *phases* of a material isolated in the region Ω. We are interested in solutions which *connect* these states, leaving an interface, a narrow region where the transition from one phase to the other takes place. More precisely, we consider solutions u_ε to (1.1)–(1.2) for which $J(u_\varepsilon)$ remains uniformly bounded, which formally entails a uniform bound for the length of the transition region. These solutions approach ± 1 almost everywhere as $\varepsilon \to 0$ while, at least for minimizers, the energy values approach the perimeter of the limiting interface between the regions where values $+1$ and -1 are taken. In the language of Γ-convergence, the Γ-limit of J_ε corresponds precisely to the interface perimeter, see [22,29,38].

In [22] Kohn and Sternberg constructed local minimizers using this fact. Associated with a straight line segment Γ_0 contained in Ω which locally minimizes length among all curves nearby with endpoints lying on $\partial\Omega$, they find a local minimizer u_ε of J_ε with asymptotic interface given by this segment. One has

$$J_\varepsilon(u_\varepsilon) \to c_0|\Gamma_0| \tag{1.3}$$

for some universal constant $c_0 > 0$. Other qualitative properties of minimizers can be found in [5,6,16,35,39].

It is natural to ask for existence and asymptotic description of solutions other than local minimizers. We recall for instance that no local minimizers other than constants are present if the domain is convex [7,30]. On the other hand rich phenomena should be observable even in this case, as heat flow for the Allen–Cahn equation becomes, in the appropriate limit, mean curvature flow of interfaces, see [3,8,9,15,18,21,36,37].

In [17,35,40] the authors have addressed the issue of understanding asymptotic behavior of interfaces for general families of solutions of (1.1)–(1.2) with uniformly bounded energy through a geometric measure theoretical approach. Roughly speaking, they have established that asymptotic interfaces must be locally stationary for the perimeter. More generally, in higher dimensions, they correspond locally to (generalized) minimal hypersurfaces. However, not many concrete cases where such objects can be found have been known. Part of the reason is that, once minimization is not available, traditional variational tools do not yield existence or fine analysis easily.

Along these lines, it is natural to consider a situation like that of Kohn and Sternberg [22] for a critical, not necessarily minimizing segment. We assume in what follows that Ω contains a straight line segment Γ_0 which intersects orthogonally the boundary at exactly two points $P_0, P_1 \in \partial\Omega$.

We assume that Γ_0 is *nondegenerate* in the sense that if $\kappa(P)$, $P \in \partial\Omega$, represents curvature of the boundary, then the following condition holds:

$$\kappa(P_0) + \kappa(P_1) - \kappa(P_0)\kappa(P_1)|\Gamma_0| \neq 0. \tag{1.4}$$

Let us explain the meaning of this condition. After translation and rotation we may assume that Γ_0 is given by

$$\Gamma_0 = \{(x_1, x_2) \mid x_1 = 0, \ 0 < x_2 < \ell\}, \tag{1.5}$$

where $\ell = |\Gamma_0|$. Let us assume that near the endpoints of the segment, $\partial\Omega$ is described as the graph of two smooth functions, respectively, $x_2 = G_0(x_1)$, $x_2 = G_1(x_1)$, with $G_0(0) = 0$, $G_1(0) = \ell$, $G_0'(0) = 0 = G_1'(0)$, so that $G_0''(0) = \kappa(P_0), G_1''(0) = -\kappa(P_1)$. Any curve C^1-close to Γ with endpoints on $\partial\Omega$ can be parameterized as

$$\gamma(t) = (h(t), \, tG_1(h(\ell)) + (\ell - t)G_0(h(0))), \quad t \in [0, \ell],$$

where $h : [0, \ell] \to \mathbb{R}^2$ represents a small function of class C^1. The length of this curve is then given by the functional on h,

$$|\gamma| = \rho(h) = \int_0^\ell \sqrt{[\, G_1(h(\ell)) - G_0(h(0)) \,]^2 + h'(t)^2} \, dt$$

so that in particular its first variation $D\rho(0) = 0$. In fact, criticality of the straight segment means just that it intersects orthogonally the boundary. A second variation of length is then given by the quadratic form

$$D^2\rho(0)[h]^2 = \int_0^\ell h'(t)^2 \, dt + G_1''(0)h(\ell)^2 - G_0''(0)h(0)^2.$$

Nondegeneracy of Γ_0 means precisely that this quadratic form is nondegenerate in its natural space $H^1(0, \ell)$. This is equivalent to the fact that the problem

$$
\begin{aligned}
-h'' &= \lambda h, \quad \text{in } (0, \ell), \\
\kappa(P_0)h(0) + h'(0) &= 0, \\
-\kappa(P_1)h(\ell) + h'(\ell) &= 0,
\end{aligned}
\tag{1.6}
$$

does not have $\lambda = 0$ as an eigenvalue. A direct computation shows that the latter fact is equivalent to condition (1.4). Let us observe that if both curvatures are negative the second variation is positive and we are in the minimizing situation of [22].

The segment Γ_0 separates Ω into two subdomains Ω_-, Ω_+, respectively, to the left and to the right of it. In [24] the second author has established that if (1.4) holds then there exists a solution u_ε to (1.1)–(1.2) exhibiting a transition layer at a distance $O(\varepsilon)$ of Γ_0. More precisely, if s is the signed distance to Γ_0, then these solutions are well approximated by $U(s/\varepsilon)$ where U is the unique solution of the problem

$$
\begin{aligned}
U'' + (1 - U^2)U &= 0, \quad \text{in } \mathbb{R}, \\
U(\pm\infty) &= \pm 1, \quad U(0) = 0.
\end{aligned}
\tag{1.7}
$$

One has for this solution $J_\varepsilon(u_\varepsilon) \to c_0|\Gamma_0|$, just as in the Γ-convergence situation of Kohn and Sternberg (1.3) and

$$\lim_{\varepsilon \to 0} u_\varepsilon = \pm 1 \quad \text{in } \Omega_\pm.$$

The purpose of this paper is to show that in the nonminimizing situation for the segment Γ_0 much richer phenomena are actually present.

Before stating our results, let us discuss a bit further the nondegeneracy condition (1.4). Let us assume that, opposite to the minimizing situation both curvatures

$\kappa(P_0)$, $\kappa(P_1)$, are positive, just as in the case of the short axis or the long axis of an eccentric ellipse. In that case, parallel translation of the segment reduces the length of its portion inside Ω. On the other hand, rotation of the short axis around its center increases length inside Ω, while that of the short axis decreases it: this translates into the fact that the short axis produces a nondegenerate critical point of Morse index one for length (namely only one direction to decrease length is present), while the long axis represents one of Morse index two.

In general, as direct computation of eigenvalues of problem (1.6) shows, when both curvatures are positive, we have that exactly one eigenvalue is negative if

$$\kappa(P_0) + \kappa(P_1) - \kappa(P_0)\kappa(P_1)\,|\Gamma_0| > 0, \tag{1.8}$$

which we call the *short axis situation*, while exactly two of these eigenvalues are negative if

$$\kappa(P_0) + \kappa(P_1) - \kappa(P_0)\kappa(P_1)\,|\Gamma_0| < 0, \tag{1.9}$$

the *long axis situation*. By definition the numbers

$$R_0 = \frac{1}{\kappa(P_0)}, \quad R_1 = \frac{1}{\kappa(P_1)} \tag{1.10}$$

correspond to the radii of the osculating circles to $\partial\Omega$, respectively, at P_0 and P_1, so that condition (1.8) becomes just

$$R_1 + R_2 > \ell. \tag{1.11}$$

As we will see, under this condition and the additional assumption

$$|R_1 - R_2| < \ell, \tag{1.12}$$

which in particular holds for the short axis of an ellipse, there exist solutions with *multiple interfaces*. In fact, given $N \geqq 1$, there is a solution u_ε to problem (1.1)–(1.2) such that

$$J_\varepsilon(u_\varepsilon) \to N c_0 |\Gamma_0| \quad \text{as } \varepsilon \to 0, \tag{1.13}$$

$$\lim_{\varepsilon \to 0} u_\varepsilon = -1 \ \text{ in } \Omega_-, \quad \lim_{\varepsilon \to 0} u_\varepsilon = \begin{cases} 1 & \text{if } N \text{ is odd} \\ -1 & \text{if } N \text{ is even} \end{cases} \ \text{ in } \Omega_+. \tag{1.14}$$

These are solutions exhibiting N transition layers, close and approximately parallel to each other, which eventually collapse onto Γ_0.

An interesting feature of these solutions is the role played by the *Toda system* in the asymptotic location of their multiple interfaces. Let us recall that the Toda system describes dynamics of N particles, arranged on a line and interacting with their neighbors with a force proportional to the exponential of minus their mutual distance. Considering interfaces, we observe that two nearly parallel interfaces *attract*, in the sense that as they get closer to one another, energy decreases proportionally to the exponential of the negative of their mutual distance. On the other hand, because of assumption (1.11), interfaces raise their individual energies

proportionally to their length as they approach horizontally the segment. It is therefore expected that an equilibrium location, maximizing effect due to interactions in parallel motion and minimizing in length should exist. In particular, interfaces in such an equilibrium should prefer to stay parallel since rotation increases length in the short axis situation. This is why the Toda system appears, in which the segment coordinate takes the role of time.

We should take into account that at a closer look, these multiple interfaces should not stay quite as straight lines since they need to arrive orthogonally to the boundary near the upper and lower ends. In reality, analysis of the Toda system under boundary conditions giving account of this orthogonality, shows that interfaces correspond asymptotically to broken lines of the form

$$x_1 = \mu(|x_2 - \gamma_0| + \gamma_1), \quad \mu = O(\varepsilon \log \varepsilon). \tag{1.15}$$

These curves intersect orthogonally the respective osculating circles for P_0 and P_1 precisely if

$$\gamma_0 = \frac{1}{2}(\ell - R_1 + R_0), \quad \gamma_1 = \frac{1}{2}(R_1 + R_0 - \ell). \tag{1.16}$$

Observe that relations (1.11), (1.12) correspond to the facts

$$\gamma_0 \in (0, \ell), \quad \gamma_1 > 0.$$

γ_0 is the x_2-coordinate of the midpoint of the segment between the centers of the two osculating circles, which we thus assume lies on Γ_0. Assuming $\gamma_1 > 0$ implies that the broken lines (1.15) do not touch Γ_0.

Our main result reads as follows.

Theorem 1.1. *Assume that the segment Γ_0 given by (1.5) is such that $\kappa(P_j) > 0$, $j = 0, 1$ and that conditions (1.11) and (1.12) hold. Then for each $N > 1$ and all sufficiently small ε, there exists a solution u_ε to problem (1.1)–(1.2) which satisfies relations (1.13), (1.14), and in a neighborhood of Γ_0 it has the form*

$$u_\varepsilon(x_1, x_2) = o(1) + \sum_{k=1}^{N} (-1)^{k+1} U\left(\frac{x_1 - \varepsilon f_k(x_2)}{\varepsilon}\right) + \begin{cases} 0 \text{ if } N \text{ is odd} \\ -1 \text{ if } N \text{ is even} \end{cases} \tag{1.17}$$

with the functions f_k, $k = 1, \ldots, N$ satisfying the asymptotic expression

$$f_k(x_2) = c_0 |\log \varepsilon| \left(k - \frac{N+1}{2}\right)(|x_2 - \gamma_0| + \gamma_1 + o(1)). \tag{1.18}$$

Here γ_0, γ_1 are the constants given by formula (1.16), $c_0 > 0$ is a universal constant and the quantities $o(1)$ go to zero uniformly as $\varepsilon \to 0$.

We include some pictures for different values of the parameters of the approximate shape of the interfaces in Figs. 1 and 2 below.

We can be much more precise in the way interfaces satisfying expansion (1.18) arise. It turns out that functions f_k solve at main order the Toda system

$$\varepsilon^2 f_k'' = a_0 \left[e^{-\sqrt{2}(f_k - f_{k-1})} - e^{-\sqrt{2}(f_{k+1} - f_k)} \right] \quad \text{in } (0, \ell), \tag{1.19}$$

under boundary conditions

$$f_k'(0) + \kappa(P_0) f_k(0) \; = \; 0 \; = \; f_k'(\ell) - \kappa(P_1) f_k(\ell), \quad k = 1, \dots, N, \quad (1.20)$$

for some universal constant $a_0 > 0$ and with the conventions $f_0 = -\infty$, $f_{N+1} = +\infty$. As we will later justify, this problem has a unique solution, whose expansion as $\varepsilon \to 0$ corresponds to (1.18).

The presence of higher multiplicity interfaces has been first suggested in [3]. In the one-dimensional or radial cases, concentration in the form of clusters of transition layers or spikes has already been observed in various problems in the literature, see [4,10,11,13,27,28,32,33], phenomena in accordance with higher-dimensional multi-spike clustering as predicted in [20]. In particular, multiple radial transition spheres collapsing on the boundary have been found for Allen–Cahn in [28]. The result of Theorem 1.1 seems to be the first of its type in a fully higher dimensional setting. Single-higher dimensional concentration phenomena has been the subject in, among other works, [12,25,26].

A broad literature exists for the Toda system, including various representation formulas for their solutions, we refer the reader to [19,23,31] for results and references. The link found here seems new, and we expect that classical mechanical systems are in agreement with various multiple-curve concentration phenomena arising in elliptic singular perturbations, in a similar way as gravitational or electrostatic interpretation of point concentration is commonly given.

In [34], Allen–Cahn in dimensions three or higher in a compact manifold has been considered, establishing that associated to a nondegenerate minimal hypersurface a solution with a single interface exists, in analogy with the result in [24]. We would expect multiple concentration of codimension one interfaces dimensions two and higher to exist, but with more difficult proofs.

The proof of Theorem 1.1 consists of finding a solution close to an initial approximation, which is essentially the right-hand side of expression (1.17). The correction term satisfies an equation which is solved in two steps: first, the curves f_k are just left as parameters to be determined, and a projected problem is solved which involves a small nonlinear perturbation of a uniformly invertible operator in suitable norms. Second, functions f_k are chosen in such a way that the solution of the projected problem is a full solution. In this second part of the process a small nonlinear, nonlocal perturbation of system (1.19)–(1.20) arises, which is solved thanks to nondegeneracy of the solution of the unperturbed system. We shall develop this scheme in what remains of this paper, in which hypotheses of Theorem 1.1 will always be assumed.

2. First approximation and outline of the proof

2.1. Approximate solution

We first formulate our problem in a conveniently chosen system of coordinates. With no loss of generality, we may assume that the segment Γ_0 satisfies $|\Gamma_0| = 1$ and is given by

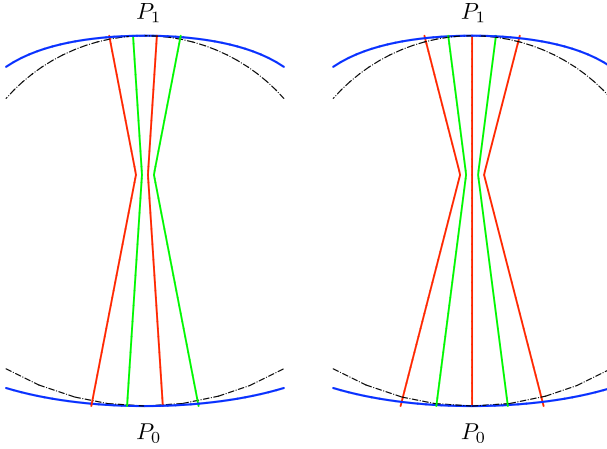$$\Gamma_0 = \{ (x_1, x_2) \mid x_1 = 0, \; 0 < x_2 < 1 \}.$$

Fig. 1. Equilibrium configurations of 4 and 5 interfaces with $\ell = 1$, $R_0 = 3/4$, $R_1 = 1/2$ and $\varepsilon = 1/25$. The dotted lines indicate the osculating circles of the boundary at points P_0 and P_1, respectively
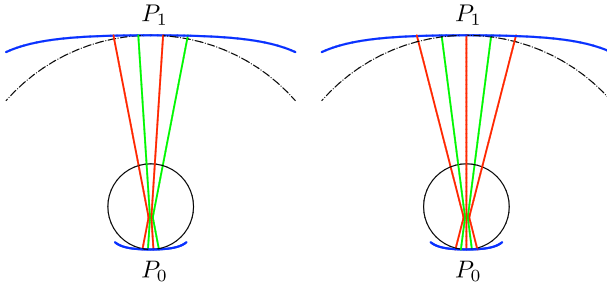

Fig. 2. Equilibrium configurations as in Fig. 1 with $\ell = 1$, $R_0 = 1/5$, $R_1 = 9/10$ and $\varepsilon = 1/25$

We also assume that near the endpoints of the segment, $\partial\Omega$ is described as the graph of two smooth functions, let us say, respectively, $x_2 = G_0(x_1)$, $x_2 = G_1(x_1)$, with

$$G_0(0) = 0, \quad G_1(0) = 1, \quad G_0'(0) = 0 = G_1'(0).$$

Let us consider the scaling $v(y) = u(\varepsilon y)$. Problem (1.1)–(1.2) is thus equivalent to

$$\alpha(v) \equiv \Delta v + f(v) = 0 \quad \text{in } \Omega_\varepsilon, \tag{2.1}$$

$$\beta(v) \equiv \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{2.2}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$. Here and in what follows we denote

$$f(v) = (1 - v^2)v.$$

For some small, fixed number δ_0, we can describe diffeomorphically all points $y \in \Omega_\varepsilon$ with $|y_1| < \delta_0\varepsilon^{-1}$ by means of coordinates (x, z) which straighten the

boundary as follows.

$$x = y_1, \quad z = y_2 - \eta_\sigma(\varepsilon y_2)\,\varepsilon^{-1}G_0(\varepsilon y_1) - \eta_\sigma(1 - \varepsilon y_2)\varepsilon^{-1}[\,G_1(\varepsilon y_1) - 1\,].$$

$$(2.3)$$

Here $\eta_\sigma(s) = \eta(\sigma^{-1}s)$, where η is a smooth cut-off function such that

$$\eta(s) = 1, \quad \text{for } |s| < 1 \quad \text{and} \quad \eta(s) = 0 \quad \text{for} \quad |s| > 2, \qquad (2.4)$$

and $\sigma > 0$ is a small ε-dependent number which for our purposes we take

$$\sigma = \varepsilon^{\frac{1}{8}}.$$

The Laplacian operator in Ω_ε expressed in these new coordinates becomes

$$\Delta_y = \Delta_{x,z} + B,$$

where B is a second order differential operator with small coefficients:

$$B = B_{22}(\varepsilon x, \varepsilon z)\frac{\partial^2}{\partial z^2} + B_{21}(\varepsilon x, \varepsilon z)\frac{\partial^2}{\partial x \partial z} + \varepsilon B_{20}(\varepsilon x, \varepsilon z)\frac{\partial}{\partial z},$$

where B_{ij} are smooth functions with $B_{22}(0, \cdot) = 0$, $B_{21}(0, \cdot) = 0$.

We also have, for $z = 0$,

$$\frac{\partial}{\partial \nu_y} = \frac{-1 + |G_0'(\varepsilon x)|^2}{(1 + |G_0'(\varepsilon x)|^2)^{1/2}}\frac{\partial}{\partial z} + \frac{G_0'(\varepsilon x)}{(1 + |G_0'(\varepsilon x)|^2)^{1/2}}\frac{\partial}{\partial x},$$

with a similar formula near $z = 1/\varepsilon$. Thus we write,

$$\frac{\partial}{\partial \nu_y} = \frac{\partial}{\partial \nu} + B^b,$$

where

$$B_b = b(\varepsilon x, \varepsilon z) \cdot \nabla_{x,z}, \quad l = 0, 1, \quad z = l/\varepsilon,$$

$$\frac{\partial}{\partial \nu} = (-1)^{\ell+1}\frac{\partial}{\partial z}, \quad l = 0, 1, \quad z = l/\varepsilon.$$

Here b is a smooth function with $b(0, \cdot) = 0$. To define the approximate solution we recall that $U = U(s)$ is the heteroclinic solution to (1.7) such that $U(\pm\infty) = \pm 1$. More precisely we have:

$$U(s) - 1 = -A_0 e^{-\sqrt{2}s} + o(e^{-\sqrt{2}s}), \quad \text{as } s \to +\infty,$$
$$U(s) + 1 = A_0 e^{\sqrt{2}s} + o(e^{\sqrt{2}s}), \quad \text{as } s \to -\infty,$$
$$U'(s) = \sqrt{2}A_0 e^{-\sqrt{2}|s|} + o(e^{-\sqrt{2}|s|}), \quad \text{as } |s| \to +\infty,$$

where A_0 is a universal constant.

In the sequel we let $N > 1$ be a fixed positive integer and consider $f_1, \ldots, f_N$ given, arbitrary functions $f_k : (0, 1) \to \mathbb{R}$ which satisfy

$$\| f_k \|_{H^2(0,1)} \leq |\log \varepsilon|^3, \quad f_{k+1}(\zeta) - f_k(\zeta) \geq \sqrt{2} |\log \varepsilon| - 4\sqrt{2} \log |\log \varepsilon|. \tag{2.5}$$

For notational convenience we will also write

$$f_0(\zeta) = -\delta_0/\varepsilon - f_1(\zeta) \quad \text{and} \quad f_{N+1}(\zeta) = \delta_0/\varepsilon - f_N(\zeta).$$

Let us set

$$w_k(x, z) = (-1)^{k+1} U(x - f_k(\varepsilon z)),$$

and define the approximate solution to (2.1)–(2.2) by

$$\mathrm{w}(x, z) = \sum_{k=1}^{N} w_k(x, z) + \frac{1}{2}((-1)^{N+1} - 1), \tag{2.6}$$

so that $\mathrm{w}(x, z) \sim w_k(x, z)$ for $|x - f_k(\varepsilon z)|$ not too large.

2.2. Outline of the proof of Theorem 1.1.

The approximation w defined by formula (3.1) does actually make sense in the infinite strip

$$S = \{(x, z) \,/\, 0 < z < 1/\varepsilon\}.$$

Since the approximation w approaches its limits as $x \to \pm\infty$ at an exponential rate, and the remainder operators B_i and B_b are comparatively small, it is reasonable to believe that the solution of the full problem is *essentially* the same as that of the problem in the whole strip

$$\Delta_{x,z} v + \eta_\delta^\varepsilon B v + f(v) = 0 \quad \text{in } S, \tag{2.7}$$

under boundary conditions

$$\frac{\partial v}{\partial \nu} + \eta_\delta^\varepsilon B^b v = 0 \quad \text{on } \partial S. \tag{2.8}$$

Here and in what follows we denote

$$\eta_\delta^\varepsilon(x, z) := \eta\left(\frac{\varepsilon |x|}{\delta}\right), \tag{2.9}$$

where $\eta(s)$ is a smooth cut-off function as in (2.4), so that

$$\eta_\delta^\varepsilon = 1 \quad \text{for } |x| < \frac{\delta}{\varepsilon}, \quad \eta_\delta^\varepsilon = 0 \quad \text{for } |x| > \frac{2\delta}{\varepsilon}.$$

In fact *the full* original problem in Ω_ε can be reduced to one in the strip by means of a gluing procedure developed in Section 6.

We look for a solution of (2.7)–(2.8) in the form $w = \mathrm{w} + \phi$ where ϕ is globally small compared with w. The equation for ϕ becomes

$$\Delta\phi + f'(\mathrm{w})\phi = -N(\phi) + E \quad \text{in } \mathcal{S}, \tag{2.10}$$

$$\frac{\partial\phi}{\partial\nu} = -\eta_\delta^\varepsilon B^b \phi + E_b \quad \text{on } \partial\mathcal{S}, \tag{2.11}$$

where

$$N(\phi) = f(\mathrm{w} + \phi) - f(\mathrm{w}) - f'(\mathrm{w})\phi + \eta_\delta^\varepsilon B\phi,$$

$$E = \Delta\mathrm{w} + \eta_\delta^\varepsilon B\mathrm{w} + f(\mathrm{w}), \quad E_b = \frac{\partial\mathrm{w}}{\partial\nu} + \eta_\delta^\varepsilon B^b\mathrm{w}.$$

The operator $N(\phi)$ is the sum of a quadratic quantity in small ϕ and a small linear operator in ϕ. The linear operator on the right-hand side of equation (2.11) is also small. Thus if the linear operator defined by the left-hand side of (2.10) were uniformly invertible in suitable norms under the associated Neumann boundary condition, one could recast the problem into a fixed point problem for a contraction mapping, provided of course that the errors E and E_b are small enough in the involved norms. However this uniform invertibility is not expected since there are decaying elements in an *approximate kernel* for the linear operator. Indeed, the functions $w_{j,x} = U_x(x - f_j(\varepsilon z))$ *nearly annihilate* the operator, and unless the functions f_j satisfy very special relations, we do not expect solvability of the original problem by the above means.

Thus we consider the linear operator

$$L(\phi) = \Delta\phi + f'(\mathrm{w})\phi,$$

which we can rewrite as

$$L(\phi) = \Delta - 2\phi + (f'(\mathrm{w}) + 2)\phi = 0.$$

Since $f'(\mathrm{w}) + 2 = O(e^{-\sqrt{2}|x|})$ as $|x| \to \infty$, then the operator can be visualized as a small perturbation at infinity of $\Delta - 2$. Standard elliptic regularity tells us that the problem

$$\Delta\phi - 2\phi = h \quad \text{in } \mathcal{S}, \tag{2.12}$$

$$\frac{\partial\phi}{\partial\nu} = g \quad \text{on, } \partial\mathcal{S} \tag{2.13}$$

has the uniform a priori bound

$$\|\phi\|_{H^2(\mathcal{S})} \leqq C\,[\,\|h\|_{L^2(\mathcal{S})} + \|g\|_{H^1(\mathcal{S})}\,],$$

and a corresponding existence statement. The analogous assertions for the operator L does hold true for the following *projected problem* which *mods out* its above mentioned approximate kernel:

Given functions $h \in L^2(\mathcal{S})$, $g \in H^1(\mathcal{S})$, we consider the problem of finding $\phi \in H^2(\mathcal{S})$ such that for certain functions $c_j \in L^2(0,1)$, $j = 1, \ldots, N$ we have

$$L(\phi) = h + \sum_{j=1}^{N} c_j(\varepsilon z) w_{j,x} \quad \text{in } \mathcal{S}, \tag{2.14}$$

$$\frac{\partial \phi}{\partial v} = g \quad \text{on } \partial \mathcal{S}, \tag{2.15}$$

$$\int_{-\infty}^{\infty} \phi(x,z) w_{j,x}(x,z) \, dx = 0, \quad \forall z \in \left(0, \frac{1}{\varepsilon}\right), \quad j = 1, \ldots, N. \tag{2.16}$$

Proposition 2.1. *There exists a constant $C > 0$, independent of ε and uniform for f_j's satisfying (2.5) such that for all small ε Problem (2.14)–(2.16) has a solution $\phi = T(h,g)$, which defines a linear operator of its arguments and satisfies the estimate*

$$\|\phi\|_{H^2(\mathcal{S})} \leqq C \left[\|h\|_{L^2(\mathcal{S})} + \|g\|_{H^1(\mathcal{S})} \right].$$

We will prove this proposition in Section 3. Using it, we would like to solve via contraction mapping principles the projected nonlinear problem

$$L(\phi) = E - N(\phi) + \sum_{j=1}^{N} c_j(\varepsilon z) w_{j,x} \quad \text{in } \mathcal{S}, \tag{2.17}$$

$$\frac{\partial \phi}{\partial v} = E_b - \eta_\delta^\varepsilon B_b \phi \quad \text{on } \partial \mathcal{S}, \tag{2.18}$$

$$\int_{-\infty}^{\infty} \phi(x,z) w_{j,x}(x,z) \, dx = 0, \quad j = 1, \ldots, N. \tag{2.19}$$

As we will see in Section 4, the inner error E is essentially constituted by quantities carrying $\varepsilon^2 |\log \varepsilon|^q$ times functions exponentially decaying in x from the f_j's. This implies that

$$\|E\|_{L^2(\mathcal{S})} \leqq C \varepsilon^{\frac{3}{2}} |\log \varepsilon|^q,$$

for some $q > 0$, uniformly on functions f_j satisfying (2.5). This error, however, does not match in size with that at the boundary E_b which is much larger: it only carries ε rather than ε^2 as factors, so that we get

$$\|E_b\|_{H^1(\mathcal{S})} \leqq C \varepsilon^{\frac{1}{2}} |\log \varepsilon|^q.$$

It is necessary to improve the approximation in order for both errors of approximation to share size, eliminating the terms of order ε in the expansion of the boundary error. This is achieved in Section 5 by means of two steps: first, boundary conditions are imposed on the functions f_j. It is assumed that

$$f_k'(0) + \kappa(P_0) f_k(0) = 0 = f_k'(1) - \kappa(P_1) f_k(1), \quad k = 1, \ldots, N. \tag{2.20}$$

This assumption eliminates part of the ε-terms in E_b. Second, the remaining terms of order ε are eliminated by building an improvement of approximation of

the form $w + \phi^{**}$ where ϕ^{**}, of size ε, solves certain explicit linear problem and has the property to decay exponentially in z-direction towards the interior of $\mathcal{S}$. This procedure makes the initial problem (2.17)–(2.19) equivalent to a similar one with new errors E and E_b with size $O(\varepsilon^{\frac{3}{2}} |\log \varepsilon|^q)$ and a qualitatively similar operator $N(\phi)$. By contraction mapping principles using Proposition 2.1, we then get a solution ϕ of this variation of (2.17)–(2.19) with

$$\|\phi\|_{H^2(\mathcal{S})} = O(\varepsilon^{\frac{3}{2}}).$$

We shall get then a solution of our original problem if we are able to adjust the functions f_j in such a way that

$$c_j(z) = 0 \quad \forall j = 1, \dots, N.$$

In order to solve the latter system of equations, we simply integrate equation (2.17) against $w_{j,x}$ to get

$$c_j(z) = \int_{-\infty}^{\infty} (\Delta w + \eta_\delta^\varepsilon B w + f(w)) \, w_{x,j} \, \mathrm{d}x \, + O(\varepsilon^{2+\mu})$$

for some $\mu > 0$. These quantities are given at main order in the following lemma.

Lemma 2.1. *The following asymptotic formula holds*

$$\int_{-\infty}^{\infty} (\Delta w + \eta_\delta^\varepsilon B w + f(w)) \, w_{x,k} \, \mathrm{d}x$$
$$= (-1)^k a_0 \left[\varepsilon^2 b_0 f_k'' - e^{-\sqrt{2}(f_k - f_{k-1})} + e^{-\sqrt{2}(f_{k+1} - f_k)} \right] + \mathcal{P}_k, \qquad (2.21)$$

for some positive universal constants a_0, b_0, where

$$\|\mathcal{P}_k\|_{L^2(0,\ell)} \leqq C \varepsilon^{2+\mu}, \quad \text{for some } \mu > 0, \qquad (2.22)$$

uniformly on functions f_k's satisfying conditions (2.5).

We carry our these computations in Section 4.

In this way, the full problem is reduced to solving a system of the form

$$\varepsilon^2 b_0 f_k'' - e^{-\sqrt{2}(f_k - f_{k-1})} + e^{-\sqrt{2}(f_{k+1} - f_k)} = h,$$

under boundary conditions (2.20), where $\|h\|_{L^2(0,\ell)} = O(\varepsilon^{2+\mu})$ and h itself is a nonlinear, nonlocal operator of f. The system for $h \equiv 0$ turns out to have a unique solution which is of size $O(|\log \varepsilon|)$. This solution is nondegenerate in a suitable sense, and the problem is finally solved via a perturbation argument. This part of the process is carried out in Section 7.

In the rest of this paper, we will work out in detail the above outlined scheme.

3. Projected linear theory in the strip

This section will be devoted to the resolution of the projected linear problem (2.14)–(2.16) by proving Proposition 2.1. A first claim we make is that to prove the result it suffices to consider the case $g = 0$, so that we will only need to find the operator $T(h, 0)$. Indeed, let us consider the solution $\phi_0 = \phi_0(g)$ to the problem

$$\Delta \phi_0 - \phi_0 = 0 \quad \text{in } \mathcal{S},$$
$$\frac{\partial \phi_0}{\partial \nu} = g \quad \text{on } \partial \mathcal{S}.$$

From standard elliptic theory, we find that

$$\|\phi_0\|_{H^2(\mathcal{S})} \leqq C \|g\|_{H^1(\mathcal{S})}.$$

On the other hand, we check directly that

$$\tilde{\phi} = \phi - \phi_0$$

satisfies a similar equation, but now for $g = 0$, with Λ replaced by $\tilde{\Lambda}$, where

$$\|\tilde{\Lambda}\|_{H^2(0,1/\varepsilon)} \leqq C[\|\Lambda\|_{H^2(0,1/\varepsilon)} + \|g\|_{H^1(\mathcal{S})}],$$

and with h replaced by $\tilde{h} = \tilde{h}(h, g)$, a linear operator in its argument satisfying

$$\|\tilde{h}\|_{L^2(\mathcal{S})} \leqq C[\|h\|_{L^2(\mathcal{S})} + \|g\|_{H^1(\mathcal{S})}].$$

With the aid of this and the definition of $\tilde{\phi}$, the operator $T(h, g, \Lambda)$ is thus built just from $T(\tilde{h}, 0, \tilde{\Lambda})$, as claimed.

For the proof of the proposition we need the validity of a priori estimates for a simpler problem. Given $h \in L^2(\mathcal{S})$, let us consider the operator

$$L_0(\phi) = \Delta \phi + f'(U(x))\phi$$

and the problem

$$L_0(\phi) = h \quad \text{in } \mathcal{S}, \tag{3.1}$$

$$\frac{\partial \phi}{\partial \nu} = 0 \quad \text{on } \partial \mathcal{S}, \tag{3.2}$$

$$\int_{\mathbb{R}} \phi(x, z) U_x(x) \, dx = \Lambda(z), \quad 0 < z < \frac{1}{\varepsilon} \tag{3.3}$$

where

$$\|\Lambda\|_{H^2(0,1/\varepsilon)} \leqq C. \tag{3.4}$$

Lemma 3.1. *There exists a constant $C > 0$, independent of ε such that solutions of* (3.1)–(3.3) *with Λ satisfying* (3.4) *satisfy the a priori estimate*

$$\|\phi\|_{H^2(\mathcal{S})} \leqq C[\|h\|_{L^2(\mathcal{S})} + \|\Lambda\|_{H^2(0,1/\varepsilon)}].$$

Proof. Assume first that $\Lambda \equiv 0$. Let us consider Fourier series decompositions for h and ϕ of the form

$$\phi(x, z) = \sum_{k=0}^{\infty} \phi_k(x) \cos(\pi k \varepsilon z),$$

$$h(x, z) = \sum_{k=0}^{\infty} h_k(x) \cos(\pi k \varepsilon z).$$

Then we have the validity of the equations

$$-k^2 \varepsilon^2 \phi_k + \mathcal{L}_0(\phi_k) = h_k, \quad x \in \mathbb{R} \tag{3.5}$$

and conditions

$$\int_{-\infty}^{\infty} \phi_k \, U_x \, \mathrm{d}x = 0, \tag{3.6}$$

for all k. We have denoted here

$$\mathcal{L}_0(\phi_k) = \phi_{k,xx} + f'(U(x))\phi_k.$$

Let us consider the bilinear form in $H^1(\mathbb{R})$ associated with the operator $\mathcal{L}_0$, namely

$$B(\psi, \psi) = \int_{\mathbb{R}} [|\psi_x|^2 - f'(U)|\psi|^2] \, \mathrm{d}x.$$

Since (3.6) holds uniformly in k we conclude that

$$C[\|\phi_k\|_{L^2(\mathbb{R})}^2 + \|\phi_{k,x}\|_{L^2(\mathbb{R})}^2] \leqq B(\phi_k, \phi_k) \tag{3.7}$$

for a constant $C > 0$ independent of k. Using this fact and equation (3.5) we find the estimate

$$(1 + k^4 \varepsilon^4)\|\phi_k\|_{L^2(\mathbb{R})}^2 + \|\phi_{k,x}\|_{L^2(\mathbb{R})}^2 \leqq C\|h_k\|_{L^2(\mathbb{R})}^2.$$

In particular, we see from (3.5) that ϕ_k satisfies an equation of the form

$$\phi_{k,xx} - 2\phi_k = \tilde{h}_k, \quad x \in \mathbb{R},$$

where $\|\tilde{h}_k\|_{L^2(\mathbb{R})} \leqq C\|h_k\|_{L^2(\mathbb{R})}$. Hence it follows that additionally we have the estimate

$$\|\phi_{k,xx}\|_{L^2(\mathbb{R})}^2 \leqq C\|h_k\|_{L^2(\mathbb{R})}^2. \tag{3.8}$$

Adding up estimates (3.7), (3.8) in k we conclude that

$$\|D^2\phi\|_{L^2(\mathcal{S})}^2 + \|D\phi\|_{L^2(\mathcal{S})}^2 + \|\phi\|_{L^2(\mathcal{S})}^2 \leqq C\|h\|_{L^2(\mathcal{S})}^2,$$

which ends the proof in the case $\Lambda \equiv 0$. To prove the general case it suffices to apply the above argument with

$$\tilde{\phi} = \phi - \Lambda(z)\frac{U_x(x)}{\int_{\mathbb{R}} U_x^2}.$$

$\square$

We consider now the following problem: given $h \in L^2(\mathcal{S})$, find functions $\phi \in H^2(\mathcal{S})$, $c \in L^2(0, 1)$ such that

$$L(\phi) = h + c(\varepsilon z)U_x \quad \text{in } \mathcal{S}, \tag{3.9}$$

$$\frac{\partial \phi}{\partial v} = g \quad \text{on } \partial \mathcal{S}, \tag{3.10}$$

$$\int_{\mathbb{R}} \phi \, U_x \, dx = \Lambda(z), \quad 0 < z < \frac{1}{\varepsilon}. \tag{3.11}$$

Lemma 3.2. *Problem* (3.9)–(3.11) *possesses a unique solution*

$$\phi = T_0(h, g, \Lambda).$$

Moreover,

$$\|\phi\|_{H^2(\mathcal{S})} \le C[\|h\|_{L^2(\mathcal{S})} + \|\Lambda\|_{H^2(0,1/\varepsilon)} + \|g\|_{H^1(\mathcal{S})}].$$

Proof. We first show that it is sufficient to prove this result for the case $\Lambda \equiv 0$, $g \equiv 0$. To this end, let ϕ_0 be the solution of

$$\Delta\phi_0 - \phi_0 = 0, \quad \text{in } \mathcal{S},$$
$$\frac{\partial \phi_0}{\partial v} = g, \quad \text{on } \partial \mathcal{S},$$

and define

$$\tilde{\phi} = \phi - \phi_0 - \left[\Lambda(z) - \int_{\mathbb{R}} \phi_0 U_x \, dx\right] \frac{U_x}{\|U_x\|^2_{L^2(\mathbb{R})}}.$$

Then $\tilde{\phi}$ satisfies $L(\tilde{\phi}) = \tilde{h} + c(\varepsilon z)U_x$ with homogeneous boundary and orthogonality conditions and the general result will follow.

For existence, we write again

$$h(x, z) = \sum_{k=0}^{\infty} h_k(x) \cos(\pi k \varepsilon z)$$

and consider the problem of finding $\phi_k \in H^1(\mathbb{R})$, and constants c_k, such that

$$-k^2 \varepsilon^2 \phi_k + \mathcal{L}_0(\phi_k) = h_k + c_k U_x \quad x \in \mathbb{R}$$

and

$$\int_{\mathbb{R}} \phi_k \, U_x \, dx = 0.$$

Fredholm's alternative yields that this problem is solvable with the choices

$$c_k = -\frac{\int_{\mathbb{R}} h_k U_x \, dx}{\int_{\mathbb{R}} U_x^2 \, dx}.$$

Observe in particular that

$$\sum_{k=0}^{\infty} |c_k|^2 \leqq C\varepsilon \|h\|_{L^2(\mathcal{S})}^2 . \tag{3.12}$$

Finally define

$$\phi(x, z) = \sum_{k=0}^{\infty} \phi_k(x) \cos(\pi k \varepsilon z),$$

and correspondingly

$$c(\zeta) = \sum_{k=0}^{\infty} c_k \cos(\pi k \zeta) .$$

Estimate (3.12) gives that $c(\varepsilon z)U_x$ has its $L^2(\mathcal{S})$ norm controlled by that of h. The a priori estimates of the previous lemma tell us that the series for ϕ is convergent in $H^2(\mathcal{S})$ and defines a unique solution for the problem with the desired bounds. $\square$

In order to apply the previous result to the resolution of the full problem (2.14)–(2.16), we define first the operator

$$L_j(\phi) = \Delta\phi + f'(w_j)\phi,$$

and consider the following problem

$$L_j(\phi) = h + c_j(\varepsilon z)\, w_{j,x} \quad \text{in } \mathcal{S}, \tag{3.13}$$

$$\frac{\partial \phi}{\partial \nu} = g \quad \text{on } \partial \mathcal{S}, \tag{3.14}$$

$$\int_{\mathbb{R}} \phi w_{j,x}\, \mathrm{d}x = \Lambda(z). \tag{3.15}$$

We have

Lemma 3.3. *Problem* (3.13)–(3.15) *possesses a unique solution*

$$\phi = T_j(h, g, \Lambda).$$

Moreover,

$$\|\phi\|_{H^2(\mathcal{S})} \leqq C[\|h\|_{L^2(\mathcal{S})} + \|\Lambda\|_{H^2(0,1/\varepsilon)} + \|g\|_{H^1(\mathcal{S})}].$$

Proof. We recall that $w_j = (-1)^{j+1}U(x - f_j(\varepsilon z))$. For a function $\xi(x, z)$ defined in $\mathcal{S}$ we denote below

$$\tilde{\xi}(x, z) = \xi(x + f_j(\varepsilon z), z).$$

By direct computation we see that Problem (3.13)–(3.15) is equivalent to

$$\Delta\tilde{\phi} + B_1(\tilde{\phi}) + f'(U)\tilde{\phi} = \tilde{h} + c_j(\varepsilon z)\, U_x \quad \text{in } \mathcal{S},$$

$$\frac{\partial\tilde{\phi}}{\partial\nu} = \tilde{g} + B_2(\tilde{\phi}) \quad \text{on } \partial\mathcal{S},$$

$$\int_{\mathbb{R}} \tilde{\phi}\, U_x \, dx = \Lambda(z),$$

where

$$B_1(\tilde{\phi}) = \varepsilon^2 f_j'(\varepsilon z)^2 \tilde{\phi}_{xx} + \varepsilon^2 f_j''(\varepsilon z)\tilde{\phi}_x + 2\varepsilon f'(\varepsilon z)\tilde{\phi}_{xz},$$

$$B_2(\tilde{\phi}) = \varepsilon f_j'(\varepsilon z)\tilde{\phi}_x.$$

This problem is then equivalent to the fixed point linear problem

$$\tilde{\phi} = T_0(\tilde{h} + B_1(\tilde{\phi}), \tilde{g} + B_2(\tilde{\phi}), \Lambda),$$

where T_0 is the linear operator defined by Lemma 3.2. The linear operators B_1 and B_2 are small in the sense that

$$\|B_1(\tilde{\phi})\|_{L^2(\mathcal{S})} + \|B_2(\tilde{\phi})\|_{H^1(\mathcal{S})} \leqq o(1)\|\tilde{\phi}\|_{H^2(\mathcal{S})},$$

with $o(1) \to 0$ as $\varepsilon \to 0$. From this, unique solvability of the problem and the desired estimate immediately follow. $\quad\square$

Proof of Proposition 2.1. In order to solve for ϕ in Problem (2.14)–(2.16) we assume that $g = 0$, which is sufficient as we have pointed out.

We search for a solution of $\phi = T(h, 0, \Lambda)$ in the form

$$\phi = \sum_{j=1}^{N} \eta_j \phi + \psi, \tag{3.16}$$

where

$$\eta_j(x, z) = \eta_0\left(\frac{x - f_j(\varepsilon z)}{R}\right), \quad R = \frac{\sqrt{2}}{2}|\log\varepsilon|,$$

and η_0 is smooth with $\eta_0(s) = 1$ for $|s| < 1/2$ and $\eta_0(s) = 0$ for $|s| > 5/6$. We will denote

$$\chi = 1 - \sum_{j=1}^{N} \eta_j.$$

It is readily checked that ϕ given by (3.16) solves Problem (2.14)–(2.16) with $g = 0$ if the functions $\phi_j = \eta_j\phi$, ψ satisfy the following linear system of equations.

$$\Delta\phi_j + f'(w_j)\phi_j = h\eta_j + c_j(\varepsilon z)w_{j,x}$$
$$\qquad\qquad - (f'(w) - f'(w_j))\eta_j\phi_j - \eta_j f'(w)\psi \quad \text{in } \mathcal{S}, \quad (3.17)$$

$$\frac{\partial\phi_j}{\partial\nu} = 0 \quad \text{on } \partial\mathcal{S}, \tag{3.18}$$

$$\int_{\mathbb{R}} \phi_j w_{j,x} dx = \tilde{\Lambda}_j, \tag{3.19}$$

where

$$\Lambda_j = \int_{\mathbb{R}} (1 - \eta_j) \phi_j w_{j,x} \, dx - \sum_{k \neq j} \int_{\mathbb{R}} \eta_k \phi_k w_{j,x} \, dx - \int_{\mathbb{R}} \psi w_{j,x},$$

and

$$\Delta \psi + \chi f'(\mathrm{w}) \psi = \chi h + \sum_{j=1}^{N} (1 - \eta_j) c_j(\varepsilon z) w_{j,x}$$

$$- \sum_{j=1}^{N} [2 \nabla \eta_j \cdot \nabla \phi_j + \phi_j \Delta \eta_j], \qquad (3.20)$$

$$\frac{\partial \psi}{\partial \nu} = 0 \quad \text{on } \partial \mathcal{S}. \qquad (3.21)$$

In order to solve this system we will set up a fixed point argument. To this end, assume that function $\tilde{\phi}$ is given and define

$$\tilde{\phi}_j = \tilde{\phi} \eta_j, \quad \tilde{\psi} = \tilde{\phi} - \sum_{j=1}^{N} \tilde{\phi}_j.$$

First we replace ϕ_j, ψ by $\tilde{\phi}_j, \tilde{\psi}$ on the right-hand sides of (3.17), (3.19) and solve (3.17)–(3.19) for each $j = 1, \ldots, n$ using Lemma 3.3. We get the following estimate

$$\|\phi_j\|_{H^2(\mathcal{S})} \leqq C [\|h\|_{L^2(\mathcal{S})} + \|\tilde{\psi}\|_{H^2(\mathcal{S})} + o(1) \sum_{j=1}^{N} \|\tilde{\phi}_j\|_{H^2(\mathcal{S})}. \qquad (3.22)$$

Given $\tilde{\psi}$ we can now find functions $\phi_j = \phi_j(\tilde{\psi})$ which solve (3.17)–(3.19) by a fixed point argument. Next we observe that the norms $\|c_j(\varepsilon z) w_{j,x}\|_{L^2(\mathcal{S})}$ are controlled by $\|h\|_{L^2(\mathcal{S})}$ as it was pointed out in Lemma 3.2 (see (3.12) and the argument that follows). Therefore we can now solve (3.20)–(3.21) for ψ which in addition satisfies

$$\|\psi\|_{H^2(\mathcal{S})} \leqq C \|h\|_{L^2(\mathcal{S})} + o(1) \sum_{j=1}^{N} \|\phi_j(\tilde{\psi})\|_{H^2(\mathcal{S})}. \qquad (3.23)$$

Combining this with (3.22), and applying a fixed point argument again, we get finally a solution to (3.20)–(3.21). This ends the proof. □

4. The inner approximation: size and projections

In what follows the error terms will often involve quantities of the type $O(\varepsilon^p|\log\varepsilon|^q)$. While keeping track of the powers of ε is very important, the logarithmic factors turn out not to play a significant role. Therefore in the rest of the paper we will use q to denote a constant representing a power accompanying $|\log\varepsilon|$. The value of q may change from line to line.

Our first goal is to compute the errors of approximation in a δ_0/ε neighborhood of Γ_ε, namely the quantities

$$E_0 \equiv \alpha(\mathrm{w}) = \Delta_y \mathrm{w} + f(\mathrm{w}), \quad E_{0b} \equiv \beta(\mathrm{w}) = \frac{\partial \mathrm{w}}{\partial \nu_y}.$$

We shall do this in (x, z)-coordinates.

The following result gives account of the size of the inner error of approximation E_0 in the region

$$S_{\delta_0/\varepsilon} = \{|x| < \delta_0/\varepsilon, 0 < z < 1/\varepsilon\},$$

Lemma 4.1. *The following estimate holds:*

$$|E_0(x, z)| \leqq C\varepsilon^2(|f_j''(\varepsilon z)| + |\log\varepsilon|^q)\sum_{j=1}^N e^{-\sqrt{2}|x-f_j(\varepsilon z)|} \quad in \ S_{\delta_0/\varepsilon} \qquad (4.1)$$

uniformly on functions f_j satisfying constraints (2.5). *In particular*

$$\|E_0\|_{L^2(S_{\delta_0/\varepsilon})} \leqq C\varepsilon^{\frac{3}{2}}|\log\varepsilon|^q.$$

Proof. We compute

$$E_0 = (\Delta + B)\mathrm{w} + f(\mathrm{w}) \equiv E_{01} + E_{02}.$$

We have

$$\Delta\mathrm{w} = \sum_{k=1}^N (-1)^{k+1}U''(x - f_k)$$

$$-\varepsilon^2\sum_{k=1}^N (-1)^{k+1}f_k''U'(x - f_k)$$

$$+\varepsilon^2\sum_{k=1}^N (-1)^{k+1}(f_k')^2U''(x - f_k).$$

Then, taking into account Bw we get

$$\Delta w + Bw = \sum_{k=1}^{N} (-1)^{k+1} U''(x - f_k)[1 - \varepsilon B_{21}(\varepsilon x, \varepsilon z) f_k']$$

$$-\varepsilon^2 \sum_{k=1}^{N} (-1)^{k+1} f_k'' U'(x - f_k)[1 + B_{22}(\varepsilon x, \varepsilon z)]$$

$$+\varepsilon^2 \sum_{k=1}^{N} (-1)^{k+1} (f_k')^2 U''(x - f_k)[1 + B_{22}(\varepsilon x, \varepsilon z)]$$

$$-\varepsilon^2 B_{20}(\varepsilon x, \varepsilon z) \sum_{k=1}^{N} (-1)^{k+1} f_k'. U'(x - f_k).$$

We now turn to computing E_{02}. We fix a k, $1 \leqq k \leqq N$ and consider the set

$$A_k = \left\{ (x, z) \mid \frac{f_{k-1}(\varepsilon z) + f_k(\varepsilon z)}{2} \leqq x < \frac{f_{k+1}(\varepsilon z) + f_k(\varepsilon z)}{2} \right\}.$$

For $x \in A_k$ we write

$$f(w) = f(w_k) + [f(w) - f(w_k)]$$

$$= f(w_k) + f'(w_k)(w - w_k) + \frac{1}{2} f''(w_k)(w - w_k)^2 + O(|w - w_k|^3)$$

$$= f(w_k) + f'(w_k)(w - w_k) + \frac{1}{2} f''(w_k)(w - w_k)^2 + \max_{j \neq k} O(e^{-3\sqrt{2}|f_j - x|})$$

$$= \sum_{j=1}^{N} f(w_j) + \left[f'(w_k)(w - w_k) - \sum_{j \neq k} f(w_j) \right]$$

$$+ \frac{1}{2} f''(w_k)(w - w_k)^2 + \max_{j \neq k} O(e^{-3\sqrt{2}|f_j - x|}).$$

It is convenient to introduce the following numbers.

$$\sigma_{kj} = \begin{cases} (-1)^j, & \text{if } j < k, \\ -(-1)^j, & \text{if } j > k. \end{cases}$$

Assuming that N is odd we have

$$f'(w_k)(\mathrm{w} - w_k) - \sum_{j \neq k} f(w_j) = [f'(w_k) - f'((-1)^{k+1})](\mathrm{w} - w_k)$$

$$+ \left[f'((-1)^{k+1})(\mathrm{w} - w_k) - \sum_{j \neq k} f(w_j) \right]$$

$$= 3[1 - U^2(x - f_k(\varepsilon z))](\mathrm{w} - w_k) + f'(1) \left\{ (\mathrm{w} - w_k) - \sum_{j \neq k} [\sigma_{kj} - w_j] \right\}$$

$$- \frac{1}{2} \sum_{j \neq k} f''(\sigma_{kj})[\sigma_{kj} - w_j]^2 + \max_{j \neq k} O(e^{-3\sqrt{2}|f_j - x|})$$

$$= \{3[1 - U^2(x - f_k(\varepsilon z))]\}(\mathrm{w} - w_k) - \frac{1}{2} \sum_{j \neq k} f''(\sigma_{kj})[\sigma_{kj} - w_j]^2$$

$$+ \max_{j \neq k} O(e^{-3\sqrt{2}|f_j - x|}).$$

A similar argument applies when N is even. Summarizing, we get

$$E_{02} = \sum_{j=1}^{N} f(w_j) + \frac{1}{2} f''(\mathrm{w})(\mathrm{w} - w_k)^2 + 3[1 - U^2(x - f_k))](\mathrm{w} - w_k)$$

$$- \sum_{j \neq k} \frac{1}{2} f''(\sigma_{kj})(\sigma_{kj} - w_j)^2 + \max_{j \neq k} O(e^{-3\sqrt{2}|f_j - x|}) \tag{4.2}$$

for $x \in A_k, k = 1, \ldots, N$. It follows then that for $x \in A_k$,

$$E_0 = -\varepsilon \sum_{k=1}^{N} (-1)^{k+1} f_k' U''(x - f_k) B_{21}(\varepsilon x, \varepsilon z)$$

$$- \varepsilon^2 \sum_{k=1}^{N} (-1)^{k+1} f_k'' U'(x - f_k)[1 + B_{22}(\varepsilon x, \varepsilon z)]$$

$$+ \varepsilon^2 \sum_{k=1}^{N} (-1)^{k+1} (f_k')^2 U''(x - f_k)[1 + B_{22}(x, z)]$$

$$- \varepsilon^2 B_2 \sum_{k=1}^{N} (-1)^{k+1} f_k' U'(x - f_k)$$

$$+ \max_{j \neq k} O(e^{-2\sqrt{2}|f_j - x|}).$$

Thus, denoting by $\chi_{A_k}(x)$ the characteristic function of the set A_k, we have

$$E_0(x, z) = \sum_{k=1}^{N} \chi_{A_k}(x) \left[O(\varepsilon^2 |\log \varepsilon|^q) e^{-\sqrt{2}|f_k - x|} + O(1) \max_{j \neq k} e^{-2\sqrt{2}|f_j - x|} \right],$$

and the result of the lemma readily follows. □

As we have mentioned in the outline of the proof, the computation of the projections of the error against the functions $w_{j,x}(x, z)$ is of crucial importance as given by Lemma 2.1. We carry out its proof next.

Proof of Lemma 2.1. Setting $\alpha(w) = \Delta w + \eta_\delta^\varepsilon B w + f(w)$, we have to compute

$$\int_{\mathbb{R}} \alpha(w) U'(x - f_k(\varepsilon z)) \, dx = \left\{ \int_{A_k} + \int_{S \backslash A_k} \right\} \eta_\delta^\varepsilon \alpha(w) U'(x - f_k(\varepsilon z)) \, dx$$
$$= \mathcal{E}_{k1}(\varepsilon z) + \mathcal{E}_{k2}(\varepsilon z).$$

We begin with

$$\mathcal{E}_{1k}(\varepsilon z) = -\varepsilon^2 \int_{A_k} \sum_{j=1}^{N} (-1)^{j+1} f_j'' U'(x - f_j) U'(x - f_k)$$

$$+ \varepsilon^2 \int_{A_k} \sum_{j=1}^{N} (-1)^{j+1} (f_j')^2 U''(x - f_j) U'(x - f_k)$$

$$+ \int_{A_k} (B w) U'(x - f_k)$$

$$+ \int_{A_k} \left[E_2 - \sum_{j=1}^{N} f(w_j) \right] U'(x - f_k) = I_1 + I_2 + I_3 + I_4.$$

Using the asymptotic formula for U' we get

$$I_1 = (-1)^k \varepsilon^2 f_k''(\varepsilon z) \int_{-\infty}^{\infty} (U'(s))^2 \, ds + O(\varepsilon^3) \sum_{j=1}^{N} f_j''(\varepsilon z).$$

Using the fact that $\int_{-\infty}^{\infty} U''(s) U'(s) \, ds = 0$ we get

$$I_2 = O(\varepsilon^3) \sum_{j=1}^{N} (f_j'(\varepsilon z))^2.$$

Now,

$$I_3(\varepsilon z) = \varepsilon^2 f_k' b_\sigma(\varepsilon z) \int_{-\infty}^{\infty} (U'(s))^2 ds + O(\sigma^{-2} \varepsilon^3) \sum_{j=1}^{N} [|f_j|' + (f_j')^2 + |f_j''|],$$

where

$$b_\sigma(\varepsilon z) = -\eta_\sigma(\varepsilon z)G_0''(0) - \eta_\sigma(1 - \varepsilon z)G_1''(0).$$

Now we will evaluate I_4. Using the expressions of the error term E_2 found above we get

$$E_2 - \sum_{j=1}^{N} f(w_j) = \left[\frac{1}{2} f''(w)(w - w_k)^2 - \frac{1}{2} \sum_{j \neq k} f''(\sigma_{kj})(\sigma_{kj} - w_j)^2 \right. \\ \left. + 3(1 - U^2(x - f_k))(w - w_k) \equiv E_{21} + E_{22}. \right.$$

We have

$$E_{21} = \frac{1}{2} f'' \left(w_k + \sum_{j \neq k}[\sigma_{kj} - w_j] \right) \left\{ \sum_{j \neq k}[\sigma_{kj} - w_j] \right\}^2$$

$$- \frac{1}{2} \sum_{j \neq k} f''(\sigma_{kj})[\sigma_{kj} - w_j]^2$$

$$= \frac{1}{2} \sum_{j \neq k}[f''(w_k) - f''(\sigma_{kj})][\sigma_{kj} - w_j]^2 + \max_{j \neq k} O(e^{-2\sqrt{2}|f_j - f_k|})$$

$$= -3 \sum_{j \neq k}[w_k - \sigma_{kj}][\sigma_{kj} - w_j]^2 + \max_{j \neq k} O(e^{-2\sqrt{2}|f_j - f_k|})$$

$$= -3(-1)^{k+1}[U(x - f_k) + 1][1 - U(x - f_{k-1})]^2$$
$$- 3(-1)^{k+1}[U(x - f_k) - 1][1 + U(x - f_{k+1})]^2$$
$$+ \varepsilon^{1/2} \max_{j \neq k} O(e^{-\sqrt{2}|f_j - f_k|}).$$

For fixed k let us consider the following integral

$$\int_{\frac{f_{k-1}+f_k}{2}}^{f_k} [U(x - f_k) + 1][1 - U(x - f_{k-1})]^2 U'(x - f_k) dx$$

$$= \int_{\frac{f_{k-1}+f_k}{2}}^{f_k} O(1)e^{-2\sqrt{2}(x - f_{k-1})} e^{-2\sqrt{2}(f_k - x)} dx$$

$$= O(e^{-2\sqrt{2}(f_k - f_{k-1})}).$$

Similarly,

$$\int_{f_k}^{\frac{f_{k+1}+f_k}{2}} [U(x - f_k) + 1][1 - U(x - f_{k-1})]^2 U'(x - f_k) dx = O(e^{-2\sqrt{2}|f_{k-1} - f_k|}).$$

Therefore we get

$$\int_{A_k} E_{21} U'(x - f_k) dx = \varepsilon^{1/2} \max_{j \neq k} O(e^{-\sqrt{2}|f_j - f_k|}).$$

We will now compute

$$\int_{A_k} E_{22} U'(x - f_k)$$

$$= 3(-1)^{k+1} \int_{A_k} [1 - U(x - f_k)^2](1 - U(x - f_{k-1}))U'(x - f_k)dx$$

$$-3(-1)^{k+1} \int_{A_k} [1 - U(x - f_k)^2](1 + U(x - f_{k+1}))U'(x - f_k)$$

$$+\varepsilon^{1/2} \max_{j \neq k} O(e^{-\sqrt{2}|f_j - f_k|})$$

$$= 3(-1)^k A_0 \int_{\mathbb{R}} a(s) \left[-e^{-\sqrt{2}(f_k - f_{k-1})} + e^{-\sqrt{2}(f_{k+1} - f_k)} \right]$$

$$+\varepsilon^{1/2} \max_{j \neq k} O(e^{-\sqrt{2}|f_j - f_k|}),$$

where we have denoted

$$a(s) = (1 - U^2(s))U'(s)e^{-\sqrt{2}s}.$$

To compute $\mathcal{E}_{k2}(\varepsilon z)$ we notice that for $x \in S \setminus A_k$ we have $U'(x - f_k) = \max_{j \neq k} O(e^{-\frac{\sqrt{2}}{2}|f_j - f_k|})$ and thus we can estimate, using the above notation,

$$\mathcal{E}_{k2}(\varepsilon z) = \varepsilon^{1/2} \max_{j \neq k} O(e^{-\sqrt{2}|f_j - f_k|}) + O(\varepsilon^{1/2}) \sum_{m=1}^{4} I_m.$$

Now, let us define

$$c_1 = 3A_0 \int_{-\infty}^{\infty} (1 - U^2(s))U'(s)e^{-\sqrt{2}s}ds, \quad c_0 = \int_{-\infty}^{\infty} U'(s)^2 ds.$$

Gathering the above estimates, we get the following system for the approximate location of the interfaces:

$$\int_{\mathbb{R}} \alpha(w)U'(x - f_k)dx$$

$$= (-1)^k \left[\varepsilon^2 c_0 f_k''(\varepsilon z) - c_1(e^{-\sqrt{2}(f_k - f_{k-1})} - e^{-\sqrt{2}(f_{k+1} - f_k)}) \right]$$

$$+\mathcal{P}_k(\varepsilon z), \quad k = 1, \ldots, N, \tag{4.3}$$

where, denoting $\zeta = \varepsilon z$, we have

$$\mathcal{P}_k(\zeta) = \varepsilon^2 c_0 f_k' b_\sigma(\zeta) + O(\sigma^{-2}\varepsilon^3) \sum_{j=1}^{N} [|f_j'| + |f_j'|^2 + |f_j''|]$$

$$+\varepsilon^{1/2} \max_{j \neq k} O(e^{-\sqrt{2}|f_j - f_k|}).$$

Now, we observe that

$$\|\mathcal{P}_k\|_{L^2(0,1)} \leq C\varepsilon^{2+\mu}, \quad \text{for some } \mu > 0, \quad k = 1, \ldots, N. \tag{4.4}$$

This last estimate is possible thanks to the fact that we have chosen $\sigma = \varepsilon^{1/8}$. The proof is concluded. $\quad \square$

5. Boundary error and improvement of approximation

Next we compute the accuracy of the approximate solution w on the boundary. Again in (x, z) coordinates we

$$E_{0b}(z = 0) = \frac{-1 + |G_0'(\varepsilon x)|^2}{(1 + |G_0'(\varepsilon x)|^2)^{1/2}} \frac{\partial w}{\partial z} + \frac{G_0'(\varepsilon x)}{(1 + |G_0'(\varepsilon x)|^2)^{1/2}} \frac{\partial w}{\partial x}$$

$$= \varepsilon \sum_{j=1}^{N} [f_j'(0) + G_0''(0) f_j(0)] \frac{\partial w_j}{\partial x}$$

$$+ \varepsilon \sum_{j=1}^{N} G_0''(\varepsilon f_j(0))(x - f_j(0)) \frac{\partial w_j}{\partial x}$$

$$+ \varepsilon^2 \sum_{j=1}^{N} [O(|x - f_j(0)|^2 + O(|f_j(0)|^2)] \frac{\partial w_j}{\partial x}$$

$$+ \varepsilon^3 \sum_{j=1}^{N} [O(|x - f_j(0)|^2 + O(|f_j(0)|^2)] f_j'(0) \frac{\partial w_j}{\partial x}.$$

A similar formula holds for $E_{0b}(z = 1/\varepsilon)$. Thus we see that it is natural to take the following boundary conditions for f_j.

$$\begin{matrix} f_j'(0) + G_0''(0) f_j(0) = 0 \\ f_j'(1) + G_1''(0) f_j(1) = 0 \end{matrix}, \quad j = 1, \ldots, N. \tag{5.1}$$

We shall assume the validity of these conditions in the sequel. We observe that

$$|E_{0b}(z)| \leqq C\varepsilon \sum_{k=1}^{N} |x - f_k(\varepsilon z)| e^{-\sqrt{2}|x - f_k(\varepsilon z)|} \quad \text{for } z = 0, \varepsilon^{-1},$$

which is one order of ε worse in size than E_0. As we have said in the outline of the proof, the discrepancy between the order of approximation in the interior and on the boundary (E_0 and E_{0b}, respectively) makes it necessary to improve the original approximation w and eliminate the $O(\varepsilon)$-part of the error. We will construct an improvement in approximation by first solving the problem

$$\Delta \phi + f'(U(x))\phi = 0 \quad \text{in } \mathcal{S} \tag{5.2}$$

$$\phi_z(x, 0) = -x U_x, \quad \phi_z(x, 1/\varepsilon) = 0 \quad \text{on } \partial \mathcal{S}. \tag{5.3}$$

We need the following result.

Lemma 5.1. *Problem* (5.2)–(5.3) *has a unique solution* $\phi^* \in H^2(\mathcal{S})$ *which is odd in x for each z. Besides, there is a $C > 0$ such that for all small ε,*

$$\|\phi^*\|_{H^2(\mathcal{S})} \leqq C. \tag{5.4}$$

In addition there exist constants $v < 1/4$, $\mu > 0$ and $C > 0$ such that the following estimate holds:

$$|\phi^*(x, z)| + |\nabla \phi^*(x, z)| + |D^2 \phi^*(x, z)| \leq C e^{-[(1-v)\sqrt{2}|x| + \mu z]}. \qquad (5.5)$$

Proof. The existence part of the above lemma as well as estimate (5.4) follow from Lemma 3.2. Indeed, the problem

$$\Delta \phi + f'(U(x))\phi = c(\varepsilon z) U_x \quad \text{in } \mathcal{S} \qquad (5.6)$$

$$\phi_z(x, 0) = -x U_x, \quad \phi_z(x, 1/\varepsilon) = 0, \quad \text{on } \partial \mathcal{S} \qquad (5.7)$$

$$\int_{\mathbb{R}} \phi U_x \, dx = 0 \qquad (5.8)$$

has a unique solution $\phi^* \in H^2(S)$. On the other hand, the fact that $U_x(x)$ is an even function and uniqueness implies that ϕ is odd in x for each z and that $c \equiv 0$. Besides,

$$\|\phi^*\|_{H^2(\mathcal{S})} \leq C \|g\|_{H^1(S)},$$

where g is any H^1-extension of the boundary condition. Let us take for instance

$$g(x, z) = e^{-z} x U_x(x) \eta(2\varepsilon z),$$

with a suitable cutoff function η, in such a way that $\|g\|_{H^1(S)} \leq C$ with C independent of ε. Thus we get

$$\|\phi^*\|_{H^2(\mathcal{S})} \leq C,$$

as desired.

Assuming now (5.4) we will establish the decay estimates (5.5). We observe first that since ϕ^* is an odd function of x therefore we have

$$\int_{-\infty}^{\infty} \phi^*(x, z) U'(x) \, dx = 0$$

hence

$$\int_{-\infty}^{\infty} [|\phi_x^*(x, z)|^2 - f'(U)|\phi(x, z)|^2] \, dx \geq \mu_2 \int_{-\infty}^{\infty} |\phi(x, z)|^2 \, dx, \qquad (5.9)$$

where μ_2 is the second eigenvalue of the operator $L\psi = -\psi'' - f'(U)\psi$ considered in $\mathbb{R}$. It is known that $\mu_2 \geq 2$. Consider function

$$w(z) = \int_{-\infty}^{\infty} |\phi(x, z)|^2 \, dx.$$

From (5.9) it follows that

$$-w_{zz} + \mu_2 w \leq 0$$

and from (5.4) we get that $|w_z(0)| \leqq C$. Clearly we have also $w_z(1/\varepsilon) = 0$ and thus by a comparison argument we get that

$$|w(z)| \leqq Ce^{-\mu z}, \quad \mu \leqq \sqrt{2}.$$

Using local elliptic estimates we then get

$$|\phi^*(x, z)e^{\mu z}| \leqq C, \quad \text{in } \mathcal{S}.$$

From this, passing a suitable barrier we get (5.5). $\square$

We define the approximate solution of

$$\Delta\phi + f'(w)\phi = 0 \quad \text{in } \mathcal{S}$$
$$\phi_z(x, l/\varepsilon) = \varepsilon \sum_j G_l''(0)(x - f_j(l/\varepsilon))\frac{\partial w_j}{\partial x} \quad \text{for } l = 0, 1/\varepsilon,$$

by

$$\phi^{**}(x, z) = \varepsilon \sum_{j=1}^{N} \phi_j(x, z),$$

where

$$\phi_j(x, z) = \left[G_0''(0)\phi^*(x - f_j(\varepsilon z), z) + G_1''(0)\phi^*\left(x - f_j(\varepsilon z), \frac{1}{\varepsilon} - z\right) \right].$$

Our next goal is to prove the following.

Lemma 5.2. *With the notation of the previous section we have the validity of the following fact*

$$|\alpha(w + \phi^{**}) - \alpha(w)| \leq C\varepsilon^3(|f''(\varepsilon z)| + |\log \varepsilon|^q) \sum_{k=1}^{N} e^{-\sqrt{2}|f_k - x|}$$

$$+ \varepsilon^2 \sum_{j=1}^{N} e^{-(1-\nu)\sqrt{2}|f_j - x|}[e^{-\mu z} + e^{-\mu(1/\varepsilon - z)}].$$

Consequently,

$$\|\alpha(w + \phi^{**}) - \alpha(w)\|_{L^2(\mathcal{S})} \leqq C\varepsilon^{\frac{3}{2}+\mu}, \quad \mu > 0. \tag{5.10}$$

Similar decay estimates hold for $\beta(w + \phi^{**})$. In addition, there is an extension of $\beta(w + \phi^{**})$ to the whole strip $\mathcal{S}$ such that

$$\|\beta(w + \phi^{**})\|_{H^1(\mathcal{S})} \leqq C\varepsilon^{\frac{3}{2}}|\log \varepsilon|^q. \tag{5.11}$$

Proof. We have

$$\alpha(w + \phi^{**}) = E_0 + [(\Delta + \eta_\varepsilon^\delta B)\phi^{**} + f'(w)\phi^{**}] + \mathcal{N}(\phi^{**}),$$

where

$$\mathcal{N}(\phi^{**}) = f(w + \phi^{**}) - f(w) - f'(w)\phi^{**}.$$

We fix a k and consider the error in the set A_k, as in the previous section. We write

$$
\begin{aligned}
\Delta\phi^{**} + f'(w)\phi^{**} &= \sum_{j=1}^{N}[\Delta\phi_j + f'(w_j)\phi_j] \\
&+ \sum_{j=1}^{N} f'(w_j)(\phi^{**} - \phi_j) \\
&+ \left[f'(w_k)(w - w_k) - \sum_{j\neq k} f'(w_j) \right]\phi^{**} \\
&+ O(1)|w - w_k|^2\phi^{**} \\
&= \sum_{i=1}^{4} E_{1i}.
\end{aligned}
$$

From the decay estimate (5.5) we get

$$|E_{11}| \leqq C\varepsilon^3 |\log\varepsilon|^q e^{-(1-\nu)\sqrt{2}|f_k - x|},$$

while the term E_{12} is estimated using (5.5) by

$$|E_{12}| \leqq C\varepsilon^2 \max_{j\neq k} e^{-(1-\nu)\sqrt{2}|f_j - x|}[e^{-\mu z} + e^{-\mu(1/\varepsilon - z)}].$$

The remaining terms, including $B\phi^{**}$ and $\mathcal{N}(\phi^{**})$ are easily seen to be smaller then the ones we have just considered. Estimate (5.10) follows immediately and (5.11) is an easy consequence of the construction. This ends the proof. $\square$

6. Reduction to a problem in $\mathcal{S}$

We will now reduce the original problem which is defined in Ω_ε to a problem defined in the strip $\mathcal{S}$. This will be done using a gluing procedure similar to that in [12]. We consider smooth cut-off function $\eta(t)$ where $t \in \mathbb{R}$ such that $\eta(t) = 1$ if $|t| < 1/2$ and $= 0$ if $|t| > 1$.

Denote as well $\eta_\delta^\varepsilon(t) = \eta(\varepsilon t/\delta)$. We define our first global (that is defined in Ω_ε) approximation to be simply

$$
\mathbf{w}(y) = \begin{cases}
\eta_{3\delta}^\varepsilon(y_1)(w + \phi^{**} + 1) - 1, & \text{if } y_1 < 0, \\
\eta_{3\delta}^\varepsilon(y_1)(w + \phi^{**} - (-1)^{N+1}) + (-1)^{N+1}, & \text{if } y_1 > 0.
\end{cases}
$$

There is nothing wrong with considering δ as a quantity approaching zero with ε, however slowly. In fact we fix in the sequel

$$\delta = \varepsilon^{\frac{1}{6}}.$$

We try a solution of (2.1)–(2.2) of the form $v = \mathbf{w} + \tilde{\phi}$. Then

$$\alpha(\mathbf{w} + \tilde{\phi}) = 0, \quad \beta(\mathbf{w} + \tilde{\phi}) = 0$$

if and only if

$$\tilde{L}(\tilde{\phi}) = \tilde{E} + \tilde{N}(\tilde{\phi}) \quad \text{in } \Omega_\varepsilon, \tag{6.1}$$
$$\beta(\tilde{\phi}) = \tilde{E}_b \quad \text{on } \partial\Omega_\varepsilon. \tag{6.2}$$

Here we have denoted

$$\tilde{E} = \alpha(\mathbf{w}), \quad \tilde{L}(\tilde{\phi}) = \Delta_y \tilde{\phi} + f'(\mathbf{w})\tilde{\phi}, \quad \tilde{E}_b = -\beta(\mathbf{w}),$$

and

$$\tilde{N}(\tilde{\phi}) = f(\mathbf{w} + \tilde{\phi}) - f(\mathbf{w}) - f'(\mathbf{w})\phi.$$

We further decompose $\tilde{\phi}$ in the following form:

$$\tilde{\phi} = \eta_{3\delta}^\varepsilon \phi + \psi,$$

where, in coordinates (x, z), we assume that ϕ is defined in the whole strip $\mathcal{S}$, and with slight abuse of notation we call the same way its expression in y coordinates and that in (x, z). Substituting in (6.1) we find

$$\tilde{L}(\eta_{3\delta}^\varepsilon \phi) + \tilde{L}(\psi) = \tilde{E} + \tilde{N}(\eta_{3\delta}^\varepsilon \phi + \psi).$$

We achieve this if the pair (ϕ, ψ) satisfies the following nonlinear coupled system:

$$\tilde{L}(\phi) = \eta_\delta^\varepsilon \tilde{E} + \eta_\delta^\varepsilon \tilde{N}(\phi + \psi) - \eta_\delta^\varepsilon [f'(\mathbf{w}) + 2]\psi, \quad \text{in } \mathcal{S} \tag{6.3}$$
$$\frac{\partial \phi}{\partial \nu} + \eta_{3\delta}^\varepsilon b \cdot \nabla_{x,z}\phi = -\eta_\delta^\varepsilon \tilde{E}_b, \quad \text{on } \partial\mathcal{S}, \tag{6.4}$$

and

$$\Delta_y \psi - (2 - 2(1 - \eta_\delta^\varepsilon)(2 + f'(\mathbf{w})))\psi = (1 - \eta_\delta^\varepsilon)\tilde{E} + 2\nabla \eta_{3\delta}^\varepsilon \nabla \phi$$
$$+ (\Delta \eta_{3\delta}^\varepsilon)\phi + (1 - \eta_\delta^\varepsilon)\tilde{N}(\eta_{3\delta}^\varepsilon \phi + \psi) \tag{6.5}$$
$$\frac{\partial \psi}{\partial \nu_y} = -(1 - \eta_\delta^\varepsilon)\tilde{E}_b - \phi \frac{\partial \eta_{3\delta}^\varepsilon}{\partial \nu_y}, \quad \text{on } \partial\Omega_\varepsilon, \tag{6.6}$$

where ϕ is defined globally on $\mathcal{S}$ and ψ is defined in Ω_ε. Notice that the operator $\tilde{L}$ in the strip $\mathcal{S}$ may be taken as any compatible extension outside the $6\delta/\varepsilon$-neighborhood of the curve.

What we want to do next is to reduce the problem to one in the strip. To do this, we solve, given a small ϕ, Problem (6.5)–(6.6) for ψ.

Since $1 - \mathbf{w}^2 = O(e^{-\gamma\delta/\varepsilon})$ for $|y_1| > \delta\varepsilon^{-1}$, then the problem

$$\Delta_y\psi - 2\psi + (1 - \eta_\delta^\varepsilon)(2 + f'(\mathbf{w}))\psi = h, \quad \text{in } \Omega_\varepsilon, \tag{6.7}$$

$$\frac{\partial\psi}{\partial\nu_y} = g \quad \text{on } \partial\Omega_\varepsilon \tag{6.8}$$

has a unique solution $\psi \in H^2(\Omega_\varepsilon)$ whenever $\|h\|_{L^2(\Omega_\varepsilon)}, \|g\|_{H^1(\Omega_\varepsilon)} < +\infty$. Moreover,

$$\|\psi\|_{L^2(\Omega_\varepsilon)} \leqq C[\|h\|_{L^2(\Omega_\varepsilon)} + \|g\|_{H^1(\Omega_\varepsilon)}].$$

Assume now that $\phi \in H^2(\Omega_\varepsilon)$ satisfies that

$$\|\phi\|_{H^2(\Omega_\varepsilon)} \leqq C\varepsilon^{\frac{3}{2}}. \tag{6.9}$$

Then from Sobolev's embedding

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqq C\varepsilon.$$

Since $\tilde{N}$ has a power-like behavior with power greater than one, a direct application of contraction mapping principles yields that Problem (6.5)–(6.6) has a unique (small) solution $\psi = \psi(\phi)$ with

$$\|\psi(\phi)\|_{H^2(\Omega_\varepsilon)} \leqq C[\|(1 - \eta_\delta^\varepsilon)\tilde{E}\|_{L^2(\Omega_\varepsilon)} + \|(1 - \eta_\delta^\varepsilon)\tilde{E}_b\|_{H^1(\Omega_\varepsilon)}]$$
$$+ C\varepsilon\delta^{-1}\|\phi\|_{H^2(\Omega_\varepsilon)}]. \tag{6.10}$$

The nonlinear operator ψ satisfies a Lipschitz condition of the form

$$\|\psi(\phi_1) - \psi(\phi_2)\|_{H^2(\Omega_\varepsilon)} \leqq C\varepsilon\delta^{-1}[\|\phi_1 - \phi_2\|_{H^2(\Omega_\varepsilon)}. \tag{6.11}$$

The full problem has been reduced to solving the (nonlocal) problem in the infinite strip $\mathcal{S}$

$$L_2(\phi) = \eta_\delta^\varepsilon\tilde{E} + \eta_\delta^\varepsilon\tilde{N}(\phi + \psi(\phi)) - \eta_\delta^\varepsilon[f'(\mathbf{w}) + 2]\psi(\phi) \tag{6.12}$$

for a $\phi \in H^2(\mathcal{S})$ satisfying condition (6.9). Here L_2 denotes a linear operator that coincides with $\tilde{L}$ on the region $|x| < \frac{6\delta}{\varepsilon}$.

We shall define this operator next. The operator $\tilde{L}$ for $|x| < \frac{6\delta}{\varepsilon}$ can be extended in coordinates (x, z) to functions ϕ defined in the entire strip $\mathcal{S}$ as follows:

$$L_2(\phi) = \Delta_{x,z}\phi + \eta_{6\delta}^\varepsilon B(\phi) + f'(\mathbf{w})\phi = L(\phi) + \eta_{6\delta}^\varepsilon B(\phi). \tag{6.13}$$

Rather than solving problem (6.3)–(6.4) directly, we shall do it in steps. We consider the following projected problem in $H^2(\mathcal{S})$: given $\mathbf{f} = (f_1, \ldots, f_N)$ satis-

fying bounds (2.5), find functions $\phi \in H^2(\mathcal{S})$, and $\mathbf{c} = (c_1, \ldots, c_N), c_j \in L^2(0, 1)$ such that

$$L(\phi) = \eta_\delta^\varepsilon \tilde{E} - N_2(\phi) + \sum_{j=1}^{N} c_j(\varepsilon z) w_{j,x} \quad \text{in } \mathcal{S}, \tag{6.14}$$

$$\frac{\partial \phi}{\partial \nu} + \eta_{3\delta}^\varepsilon b \cdot \nabla_{x,z}\phi = -\eta_\delta^\varepsilon \tilde{E}_b, \quad \text{on } \partial\mathcal{S}, \tag{6.15}$$

$$\int_{\mathbb{R}} \phi(x, z) \, w_{j,x}(x, z) \, dx = 0, \quad 0 < z < \frac{1}{\varepsilon}, \quad j = 1, \ldots, N. \tag{6.16}$$

Here

$$N_2(\phi) = \eta_{6\delta}^\varepsilon B(\phi) + \eta_\delta^\varepsilon \tilde{N}(\phi + \psi(\phi)) - \eta_\delta^\varepsilon [f'(\mathbf{w}) + 2]\psi(\phi).$$

We will prove next that this problem has a unique solution whose norm is controlled by the L^2 norm of $\tilde{E}$ and H^1 of the suitable extension of $\tilde{E}_b$. After this has been done, our task is to adjust the parameter $\mathbf{f}$ in such a way that $\mathbf{c}$ is identically zero. As we will see, this turns out to be equivalent to solving a nonlocal, nonlinear second order differential equation for $\mathbf{f}$ Robin boundary conditions. As we will see this system is solvable in a region where the bound (2.5) hold.

6.1. Solving the nonlinear intermediate problem

Next we will solve problem (6.14)–(6.16).

Proposition 6.1. *There exist numbers $D > 0$ such that for all sufficiently small ε and all $\mathbf{f}$ satisfying (2.5) problem (6.14)–(6.16) has a unique solution $\phi = \phi(\mathbf{f})$ which satisfies*

$$\|\phi\|_{H^2(\mathcal{S})} \leqq D\varepsilon^{\frac{3}{2}} |\log \varepsilon|^q,$$

for certain q depending on the accuracy of the approximation by the approximate solution $\mathbf{w}$ only. Besides ϕ depends continuously on $\mathbf{f}$.

Proof. Let $T_{\mathbf{f}}$ be the operator defined by Proposition 2.1, in which its dependence on the chosen $\mathbf{f}$ is emphasized. Let us denote

$$M(\phi) = -\eta_{3\delta}^\varepsilon b \cdot \nabla_{x,z}\phi.$$

Then, given $\mathbf{f}$ the equation (6.14)–(6.16) is equivalent to the fixed point problem for $\phi(\mathbf{f})$:

$$\phi(\mathbf{f}) = T_{\mathbf{f}}(\eta_\delta^\varepsilon \tilde{E}(\mathbf{f}) + N_2(\phi(\mathbf{f})), -\eta_\delta^\varepsilon \tilde{E}_b(\mathbf{f}) + M(\phi(\mathbf{f})), 0) \equiv \mathcal{A}(\phi, \mathbf{f}). \tag{6.17}$$

In the sequel we will not emphasize the dependence on $\mathbf{f}$ whenever it is not necessary.

We will define now the region where the contraction mapping principle applies. We consider the following closed, bounded subset of $H^2(\mathcal{S})$:

$$\mathcal{B} = \left\{ \phi \in H^2(\mathcal{S}) \,\middle|\, \|\phi\|_{H^2(\mathcal{S})} \leq D\varepsilon^{\frac{3}{2}} |\log \varepsilon|^q, \right\},$$

where $q > 0$ is such that

$$\|\tilde{E}\|_{L^2(\mathcal{S})} + \|\tilde{E}_b\|_{H^1(\mathcal{S})} \leqq C\varepsilon^{3/2} |\log \varepsilon|^q.$$

Recall that the existence of such q has already been established in Lemma 5.2.

We claim that there is a constant $D > 0$ such that the map $\mathcal{A}$ defined in (6.17) is a contraction from $\mathcal{B}$ into itself, uniform with respect to $\mathbf{f}$. Given $\tilde{\phi} \in \mathcal{B}$ we denote $\phi = \mathcal{A}(\tilde{\phi}, \mathbf{f})$. First notice that using (6.10) and Lemma 5.2 we get for $\tilde{\phi} \in \mathcal{B}$

$$\|\eta_\delta^\varepsilon \tilde{E} + N_2(\tilde{\phi})\|_{L^2(\mathcal{S})} \leqq C_0 \varepsilon^{3/2} |\log \varepsilon|^q + C\|\tilde{\phi}\|^2_{H^2(\mathcal{S})} \tag{6.18}$$

$$+ C\varepsilon^{1/4} \|\tilde{\phi}\|_{H^2}. \tag{6.19}$$

Using Lemma 5.2, and the fact that $|b(\varepsilon x, \varepsilon z)| \leqq C\varepsilon |x|$ we get for the H^1 extension of $-\eta_\delta^\varepsilon \tilde{E}_b + M(\tilde{\phi})$ (denoted by the same symbol)

$$\| -\eta_\delta^\varepsilon \tilde{E}_b + M(\tilde{\phi})\|_{H^1(\mathcal{S})} \leqq C_1 \varepsilon^{3/2} |\log \varepsilon|^q + C\delta \|\tilde{\phi}\|_{H^2(\mathcal{S})}. \tag{6.20}$$

Since $\delta = \varepsilon^{1/6}$ from (6.19) to (6.20) we get that $\mathcal{A}$ indeed applies $\mathcal{B}$ into itself provided that D is chosen sufficiently large.

Next, let us analyze the Lipschitz character of the nonlinear operator involved in $\mathcal{A}$ for functions in $\mathcal{B}$, namely $N_2(\phi + \psi(\phi))$. For $\phi_1, \phi_2 \in \mathcal{B}$ we have, using (6.10) and (6.11):

$$\|N_2(\phi_1 + \psi(\phi_1)) - N_2(\phi_2 + \psi(\phi_2))\|_{L^2(\mathcal{S})}$$
$$\leqq C\varepsilon^{3/2} |\log \varepsilon|^q \|\phi_1 - \phi_2\|_{H^2(\mathcal{S})}. \tag{6.21}$$

Using this, we readily find that $\mathcal{A}$ is a contraction map in $\mathcal{B}$ and thus show the existence of the fixed point.

A tedious but straightforward analysis of all terms involved in the differential operator and in the error yield that the operator $\mathcal{A}(\phi, \mathbf{f})$ is continuous with respect to $\mathbf{f}$. Indeed, denoting $\phi_i(\mathbf{f}_i) = \mathcal{A}(\tilde{\phi}_i, \mathbf{f}_i), i = 1, 2$, and indicating now the dependence on $\mathbf{f}_i$, let us write:

$$L_{2,\mathbf{f}_1}(\phi(\mathbf{f}_1)) - L_{2,\mathbf{f}_2}(\phi(\mathbf{f}_2)) = L_{2,\mathbf{f}_1}[\phi(\mathbf{f}_1) - \phi(\mathbf{f}_2)]$$
$$+ [f'(\mathbf{w}(\mathbf{f}_1)) - f'(\mathbf{w}(\mathbf{f}_2))]\phi(\mathbf{f}_2),$$

and

$$\eta_\delta^\varepsilon \sum_{j=1}^{N} [c_j(\varepsilon z; \mathbf{f}_1) \, w_{j,x}(\mathbf{f}_1) - c_j(\varepsilon z; \mathbf{f}_2) \, w_{j,x}(\mathbf{f}_2)]$$

$$= \eta_\delta^\varepsilon \sum_{j=1}^{N} [c_j(\varepsilon z; \mathbf{f}_1) - c_j(\varepsilon z; \mathbf{f}_2)] w_{j,x}(\mathbf{f}_1)$$

$$+ \eta_\delta^\varepsilon \sum_{j=1}^{N} c_j(\varepsilon z; \mathbf{f}_2)[w_{j,x}(\mathbf{f}_1) - w_{j,x}(\mathbf{f}_2)]$$

and finally, for each $j = 1, \ldots, N$

$$\int_{\mathbb{R}} [\phi(\mathbf{f}_1) - \phi(\mathbf{f}_2)] w_{j,x}(\mathbf{f}_1) \, dx = - \int_{\mathbb{R}} \phi(\mathbf{f}_2)[w_{j,x}(\mathbf{f}_1) - w_{j,x}(\mathbf{f}_2)] \, dx.$$

Using this decomposition one can estimate $\|\phi(\mathbf{f}_1) - \phi(\mathbf{f}_2)\|_{H^2(\mathcal{S})}$ employing the theory developed in the previous section. Observe that this estimate does not depend on $c_j(\varepsilon z; \mathbf{f}_1) - c_j(\varepsilon z; \mathbf{f}_2)$. A rather lengthy but straightforward computation shows that in fact for fixed ε the fixed point of $\mathcal{A}$, $\phi(\mathbf{f})$ is Lipschitz with respect to $\mathbf{f}$:

$$\|\phi(\mathbf{f}_1) - \phi(\mathbf{f}_2)\|_{H^2(\mathcal{S})} \leqq C\varepsilon^{-1/2} |\log \varepsilon|^q \|\mathbf{f}_1 - \mathbf{f}_2\|_{H^2(0,1)} \qquad (6.22)$$

and thus continuous with respect to $\mathbf{f}$. This ends the proof. $\quad\square$

Clearly a solution to (6.14)–(6.16) will be a solution to (6.1)–(6.2), and consequently yield a solution to our original problem (2.1)–(2.2) if we can find $\mathbf{f}$ such that

$$\mathbf{c}(\mathbf{f}) = 0. \qquad (6.23)$$

As we will see this leads to a small perturbation of a system of N nonlinear ODE's for N. We carry out this argument in the next section.

To prepare for this argument we will examine more closely the dependence of ϕ on $\mathbf{f}$. Notice first that the error term $\tilde{E}$ can be written in the form

$$\tilde{E}(x, z; \mathbf{f}) = \tilde{E}_1(x, z; \mathbf{f}, \mathbf{f}') \cdot (\varepsilon^2 \mathbf{f}'') + \tilde{E}_2(x, z; \mathbf{f}, \mathbf{f}'),$$

that is $\tilde{E}(x, z; \mathbf{f})$ is actually linear as a function of $\varepsilon^2 \mathbf{f}''$. Similarly the error term on the boundary $\beta(\mathbf{w})$ can be written as

$$\beta(\mathbf{w})(\ell) = \beta(x; \mathbf{f})(\ell) = O(\varepsilon^2)\beta_1(x; \mathbf{f}(\ell)) + O(\varepsilon^2)\beta_2(x; \mathbf{f}(\ell)) \cdot (\varepsilon \mathbf{f}'(\ell)),$$

where $\ell = 0, 1/\varepsilon$. Taking advantage of this structure of the error term we can refine estimates in Proposition 6.1 indicating their explicit dependence on $\mathbf{f}''$ as follows.

$$\begin{aligned}
\|\phi(\mathbf{f})\|_{H^2(\mathcal{S})} &\leqq C[\|\tilde{E}_1(\cdot; \mathbf{f}, \mathbf{f}') \cdot (\varepsilon^2 \mathbf{f}'')\|_{L^2(\mathcal{S})} + \|\beta_2(\cdot; \mathbf{f}) \cdot (\varepsilon \mathbf{f}')\|_{H^1(\mathcal{S})}] \\
&\quad + C[\|\tilde{E}_2(\cdot; \mathbf{f}, \mathbf{f}')\|_{L^2(\mathcal{S})} + \|\beta_1(\cdot; \mathbf{f})\|_{H^1(\mathcal{S})}] \\
&\leqq D_1 \varepsilon^{3/2} \|\mathbf{f}''\|_{L^2(0,1)} + D_2 \varepsilon^{3/2} |\log \varepsilon|^q. \qquad (6.24)
\end{aligned}$$

Exploring the linear dependence on $\varepsilon^2 \mathbf{f}''$ further we can refine estimate (6.22) as follows

$$\begin{aligned}
\|\phi(\mathbf{f}_1) - \phi(\mathbf{f}_2)\|_{H^2(\mathcal{S})} &\leqq C\varepsilon^{3/2} |\log \varepsilon|^q \|\mathbf{f}_1 - \mathbf{f}_2\|_{H^2(0,1)} \\
&\quad + C\varepsilon^{-1/2} |\log \varepsilon|^q \|\mathbf{f}_1 - \mathbf{f}_2\|_{H^1(0,1)}. \qquad (6.25)
\end{aligned}$$

7. Location of the interfaces and the Toda system

7.1. Location of the interfaces

By integrating equation (6.14) against $w_{k,x}$ and integrating only in x, we obtain that relations (6.23) are equivalent to the following system of equations for $\mathbf{f}$:

$$\int_{\mathbb{R}} [\eta_\delta^\varepsilon \tilde{E} + N_2(\phi) - L(\phi)] w_{k,x} \, dx = 0, \quad k = 1, \ldots, N. \tag{7.1}$$

We decompose on $|x| < \frac{6\delta}{\varepsilon}$,

$$\tilde{E} = \alpha(\mathrm{w}) + \alpha(\mathrm{w} + \phi^{**}) - \alpha(\mathrm{w}),$$

where

$$\alpha(w) = \Delta w + \eta_{6\delta}^\varepsilon Bw + f(w)$$

and recall that Lemma 2.1 gives us an expansion of the form

$$\int_{\mathbb{R}} \eta_\delta^\varepsilon \alpha(\mathrm{w}) U'(x - f_k) \, dx$$
$$= (-1)^k a_0 \left[\varepsilon^2 b_0 f_k'' - e^{-\sqrt{2}(f_k - f_{k-1})} + e^{-\sqrt{2}(f_{k+1} - f_k)} \right] + \mathcal{P}_k \tag{7.2}$$

for some positive universal constants a_0, b_0, where

$$\|\mathcal{P}_k\|_{L^2(0,1)} \leqq C\varepsilon^{2+\mu}, \quad \text{for some } \mu > 0, \tag{7.3}$$

uniformly on functions f_k's satisfying conditions (2.5).

On the other hand, let us set

$$\mathcal{R}_k(\varepsilon z) = -\int_{\mathbb{R}} (\alpha(\mathrm{w} + \phi^{**}) - \alpha(\mathrm{w})) U'(x - f_k) \, dx.$$

Using Lemma 5.2 we get

$$\mathcal{R}_k(\varepsilon z) = -\int_{\mathbb{R}} (\alpha(\mathrm{w} + \phi^{**}) - \alpha(\mathrm{w})) U'(x - f_k) \, dx$$

that

$$\|\mathcal{R}_k\|_{L^2(0,1)} \leqq C\varepsilon^{2+\mu}. \tag{7.4}$$

Continuing with the terms involved in (7.1), using the contracting nature of $N_2(\phi)$ and Proposition 6.1 we get for

$$\mathcal{Q}_k(\zeta) = \int_{\mathbb{R}} N_2(\phi) U'(x - f_k) \, dx, \quad \zeta = \varepsilon z,$$

a similar estimate

$$\|\mathcal{Q}_k\|_{L^2(0,1)} \leqq C\varepsilon^{2+\mu}, \quad k = 1, \ldots, N. \tag{7.5}$$

We should point out here that by Proposition 6.1 $\mathcal{Q}_k$ is a continuous function of $\mathbf{f}$.

Finally, with $\zeta = \varepsilon z$, we will write

$$\mathcal{S}_k(\zeta) = \int_{\mathbb{R}} L_2(\phi) U'(x - f_k) \, dx = \int_{\mathbb{R}} \phi_{zz} U'(x - f_k)$$

$$+ \int_{\mathbb{R}} B(\phi) U'(x - f_k) \, dx + \int_{\mathbb{R}} \phi[U'_{xx} + f'(\mathbf{w}) U'(x - f_k)] \, dx.$$

It is fairly straightforward to estimate L^2 norms of the functions involved in the definition of $\mathcal{S}_k$. For example, using the orthogonality conditions we can write

$$\mathcal{S}_k^1(\zeta) = \int_{\mathbb{R}} \phi_{zz} U'(x - f_k) = 2\varepsilon f_k' \int_{\mathbb{R}} \phi_z U''(x - f_k) \, dx$$

$$+ \varepsilon^2 \int_{\mathbb{R}} \phi[f_k'' U''(x - f_k) - (f_k')^2 U'''(x - f_k)] \, dx.$$

Using now Proposition 6.1 we get for $\|\mathcal{S}_k^1\|_{L^2(0,1)}$ an estimate similar to (7.5). As before it is also clear that $\mathcal{S}_k$ depends continuously on $\mathbf{f}$.

Let us define

$$\mathcal{N}_k(\zeta; \mathbf{f}, \mathbf{f}', \mathbf{f}'') = -(-1)^k [\mathcal{P}_k + \mathcal{Q}_k + \mathcal{R}_k - \mathcal{S}_k],$$

where $\mathcal{P}_k$ is the quantity in expansion (2.21).

Examining the formulas obtained above we see that $\mathcal{N}_k$ can be decomposed in the following way

$$\mathcal{N}_k(\zeta; \mathbf{f}, \mathbf{f}', \mathbf{f}'') = \mathcal{N}_{k1}(\zeta; \mathbf{f}, \mathbf{f}', \mathbf{f}'') + \mathcal{N}_{k2}(\zeta; \mathbf{f}, \mathbf{f}'),$$

where $\mathcal{N}_{ki}$, $i = 1, 2$ are continuous functions of its arguments. Function $\mathcal{N}_{k1}$ satisfies in addition

$$\|\mathcal{N}_{k1}(\zeta; \mathbf{f}, \mathbf{f}', \mathbf{f}'')\|_{L^2(0,1)} \leqq C\varepsilon^{2+\mu} \tag{7.6}$$

$$\|\mathcal{N}_{k1}(\zeta; \mathbf{f}_1, \mathbf{f}_1', \mathbf{f}_2'') - \mathcal{N}_{k1}(\zeta; \mathbf{f}_2, \mathbf{f}_2', \mathbf{f}_2'')\|_{L^2(0,1)}$$

$$\leqq C\varepsilon^{2+\mu} |\log \varepsilon|^q \|\mathbf{f}_1 - \mathbf{f}_2\|_{H^2(0,1)}, \tag{7.7}$$

and the function $\mathcal{N}_{k2}$ satisfies

$$\|\mathcal{N}_{k2}(\zeta; \mathbf{f}, \mathbf{f}')\|_{L^2(0,1)} \leqq C\varepsilon^{2+\mu}. \tag{7.8}$$

In addition, the functions f_k need to satisfy boundary conditions (5.1). Thus, after obvious algebra, setting $\theta = 2(\zeta - 1/2)$, and using expansion (2.21) in Lemma 2.1 we have the following system to deal with

$$\alpha_0 \sqrt{2} \varepsilon^2 f_k'' - e^{-\sqrt{2}(f_k - f_{k-1})} + e^{-\sqrt{2}(f_{k+1} - f_k)} = \mathcal{N}_k, \quad -1 < \theta < 1 \tag{7.9}$$

$$f_k'(-1) + K_- f_k(-1) = 0, \tag{7.10}$$

$$f_k'(1) + K_+ f_k(1) = 0, \tag{7.11}$$

where $K_- = \frac{1}{2} G_0''(0)$, $K_+ = \frac{1}{2} G_1''(0)$ and $\alpha_0 = c_0/(c_1 \sqrt{2})$.

To solve the nonlinear system (7.9)–(7.11) we will first consider its version with $\mathcal{N}_k$ replaced by a given function h_k. Our goal is to prove the following:

Theorem 7.1. *Assume that the following conditions hold:*

$$K_- > 0, \quad K_+ < 0, \tag{7.12}$$

$$K_- - K_+ + 2K_- K_+ > 0, \tag{7.13}$$

$$\frac{K_- + K_+}{2K_- K_+} \in (-1, 1) \tag{7.14}$$

and let functions h_k be such that

$$\|h_k\|_{L^2(-1,1)} \leqq C\varepsilon^{2+\mu}$$

with some $\mu > 0$. Then, for each sufficiently small ε there exists a unique solution to the system (7.9)–(7.11) *(with h_k replacing $\mathcal{N}_k$) which satisfies*

$$\|f_k\|_{H^2(-1,1)} \leqq C|\log \varepsilon|, \quad k = 1, \dots, N,$$

and

$$f_{k+1}(\theta) - f_k(\theta) > \sqrt{2}|\log \varepsilon| - 2\sqrt{2}\log|\log \varepsilon|,$$
$$c > 0, \quad k = 1, \dots, N - 1.$$

Moreover, there exist $\bar\lambda$, $\bar\lambda = c|\log \varepsilon| + O(\log|\log \varepsilon|)$, $\theta_0 \in (0, 1)$ such that we have the following representation

$$f_{k+1}(\theta) - f_k(\theta) = \frac{1}{\sqrt{2}} \left[q_{k+1}(\bar\lambda(\theta - \theta_0)) - q_k(\bar\lambda(\theta - \theta_0)) - \log \bar\lambda^2 \varepsilon^2 \right]$$
$$+ \varphi_{kk+1}(\theta), \quad k = 1, \dots, N - 1.$$

Functions q_k are explicitly given solutions of the Toda system

$$q_k'' - e^{(q_{k-1} - q_k)} + e^{(q_k - q_{k+1})} = 0, \quad \text{in } \mathbb{R}, \quad k = 1, \dots, N - 1,$$

where we take $q_0 = -\infty$, $q_{N+1} = \infty$, and functions $\varphi_{kk+1}(\theta)$ satisfy

$$\|\varphi_{kk+1}\|_{L^2} \leqq C\varepsilon^{\mu'},$$

with some $\mu' > 0$.

We observe that since we have assumed $|\Gamma_0| = 1$ and

$$K_- = \frac{1}{2}\kappa(P_0), \quad K_+ = -\frac{1}{2}\kappa(P_1),$$

then the assumptions of Theorem 1.1 are precisely equivalent to (7.12)–(7.14).

In what follows we will first outline the proof of the theorem. Next we will prove a special case of two interfaces and then we will consider the general case. Finally we solve our nonlinear problem (7.9)–(7.11) thus concluding the proof of Theorem 1.1.

Let us set

$$f_k = \frac{1}{\sqrt{2}}(\tilde{f}_k - (k-1)\log(\varepsilon^2))$$

so that (7.9)–(7.11) becomes

$$\alpha_0 \tilde{f}_k'' - e^{(\tilde{f}_{k-1}-\tilde{f}_k)} + e^{(\tilde{f}_k - \tilde{f}_{k+1})} = \tilde{h}_k, \quad -1 < \theta < 1, \tag{7.15}$$

$$\tilde{f}_k'(-1) + K_- \tilde{f}_k(-1) = K_-(k-1)\log(\varepsilon^2), \tag{7.16}$$

$$\tilde{f}_k'(1) + K_+ \tilde{f}_k(1) = K_+(k-1)\log(\varepsilon^2), \tag{7.17}$$

where $\tilde{h}_k = \varepsilon^{-2} h_k$. To solve the above problem we will take advantage of the fact that some explicit solutions of the equations (7.15) considered on the whole real line are explicitly known. To this end we introduce two parameters θ_0 and $|\lambda| \gg 1$, define a change of variables $t = \lambda \sqrt{\alpha_0} (\theta - \theta_0)$, and look for a solution to (7.15)–(7.17) in the form:

$$\tilde{f}_k(\theta) = q_k(\bar{\lambda}(\theta - \theta_0)) - (k-1)\log \bar{\lambda}^2,$$

where for convenience we have set $\bar{\lambda} = \lambda \sqrt{\alpha_0}$. To solve

$$q_k'' - e^{(q_{k-1}-q_k)} + e^{(q_k - q_{k+1})} = \bar{\lambda}^{-2} \tilde{h}_k, \quad t^- < t < t^+, \tag{7.18}$$

$$\bar{\lambda} q_k'(t^-) + K_- q_k(t^-) = K_-(k-1)\log(\varepsilon^2 \bar{\lambda}^2), \tag{7.19}$$

$$\bar{\lambda} q_k'(t^+) + K_+ q_k(t^+) = K_+(k-1)\log(\varepsilon^2 \bar{\lambda}^2), \tag{7.20}$$

where $t^- = -\bar{\lambda}(1 + \theta_0)$, $t^+ = \bar{\lambda}(1 - \theta_0)$, we first solve (approximately) the homogeneous problem to determine just two parameters $\theta_0, \bar{\lambda}$ and next we need to determine the corrections ϕ_k. This procedure will be described in detail in the sequel.

7.2. Cluster of two interfaces

In this section we will prove Theorem 7.1 in the special case when $N = 2$. System (7.18)–(7.20) can then be reduced to a single scalar equation for $u = q_1 - q_2$

$$u'' + 2e^u = \bar{\lambda}^{-2} h, \quad t^- < t < t^+, \tag{7.21}$$

$$\bar{\lambda} u'(t^-) + K_- u(t^-) = -K_- \log(\varepsilon^2 \bar{\lambda}^2), \tag{7.22}$$

$$\bar{\lambda} u'(t^+) + K_+ u(t^+) = -K_+ \log(\varepsilon^2 \bar{\lambda}^2). \tag{7.23}$$

The homogeneous version of the equation (7.21) considered on $\mathbb{R}$ has an explicit solution

$$u_0(t) = \log\left(\frac{1}{4\cosh^2(t/2)}\right).$$

It can be seen easily that $u_0(t) < 0$ and also that

$$u_0(t) = -|t| + O(e^{-|t|}), \quad t \to \pm\infty.$$

We will now look for the first approximation of the solution of (7.21)–(7.23) in the form $u(t) = u_0(t)$. We get the following system for $\theta_0, \bar{\lambda}$:

$$\bar{\lambda} u_0'(-\bar{\lambda}(1 + \theta_0)) + K_- u_0(-\bar{\lambda}(1 + \theta_0)) = -K_- \log(\varepsilon^2 \bar{\lambda}^2), \tag{7.24}$$

$$\bar{\lambda} u_0'(\bar{\lambda}(1 - \theta_0)) + K_+ u_0(\bar{\lambda}(1 - \theta_0)) = -K_+ \log(\varepsilon^2 \bar{\lambda}^2). \tag{7.25}$$

This is in fact a nonlinear system for $(\theta_0, \bar{\lambda})$. Although it is in principle possible to find $(\theta_0, \bar{\lambda})$ in such a way that (7.24)–(7.25) are satisfied exactly we will not do it here. Instead, taking into account the asymptotic behavior of u_0, we will look for $(\theta_0, \bar{\lambda})$ that solve the following system

$$
\begin{aligned}
\bar{\lambda}[1 - K_-(1 + \theta_0)] &= -K_- \log(\varepsilon^2 \bar{\lambda}^2), \\
\bar{\lambda}[-1 - K_+(1 - \theta_0)] &= -K_+ \log(\varepsilon^2 \bar{\lambda}^2),
\end{aligned}
\tag{7.26}
$$

which has a solution $(\theta_0, \bar{\lambda})$ such that $\bar{\lambda} = O(|\log \varepsilon|) > 0, \theta_0 \in (-1, 1)$ thanks to (1.4) and our assumption. In fact we have that:

$$
\theta_0 = \frac{K_+ + K_-}{2K_+ K_-}, \quad \bar{\lambda} = \frac{K_+ \log(\varepsilon^2 \bar{\lambda}^2)}{1 + K_+(1 - \theta_0)} = \frac{K_- K_+ \log(\varepsilon^2 \bar{\lambda}^2)}{K_- - K_+ + 2K_- K_+}. \tag{7.27}
$$

Notice that to find $\bar{\lambda}$ in (7.27) we have to solve a simple nonlinear equation. One can show that

$$
\bar{\lambda} = \frac{2K_+}{-1 + K_+(1 - \theta_0)} \log \frac{1}{\varepsilon} + O\left(\log \log \frac{1}{\varepsilon}\right). \tag{7.28}
$$

Now, denoting

$$
\bar{\lambda} u_0'(t) + K_\pm u_0(t) + K_\pm \log(\epsilon^2 \bar{\lambda}^2) = g_\pm, \ \text{for } t = \mp \bar{\lambda}(1 + \pm \theta_0), \ \text{respectively}
$$

we get from (7.28)

$$
|g_\pm| \leqq C \varepsilon^\mu, \tag{7.29}
$$

with some $\mu > 0$. We will seek an exact solution to (7.21)–(7.23) in the form

$$
u(t) = u_0(t) + \varphi(t).
$$

To find φ we will use a fixed point argument and thus we need to study the linearized version of (7.21)–(7.23).

$$
\phi'' + 2e^{u_0}\phi = h, \quad t^- < t < t^+, \tag{7.30}
$$
$$
\bar{\lambda}\phi'(t^-) + K_-\phi(t^-) = g_-, \tag{7.31}
$$
$$
\bar{\lambda}\phi'(t^+) + K_+\phi(t^+) = g_+, \tag{7.32}
$$

with a given function $h \in L^2(t^-, t^+)$ and constants $g_\pm$. The functions

$$
\begin{aligned}
\psi_1(t) &= u_0'(t), \\
\psi_2(t) &= t u_0'(t) + 2
\end{aligned}
$$

form the fundamental set for (7.30)–(7.32) and their Wronskian is actually $W(\psi_1, \psi_2) = 1$. By the variation of constants formula,

$$
\begin{aligned}
\phi(t) = &-\psi_1(t) \int_{-\bar{\lambda}(1+\theta_0)}^t \psi_2(s)h(s)\,\mathrm{d}s + \psi_2(t) \int_{-\bar{\lambda}(1+\theta_0)}^t \psi_1(s)h(s)\,\mathrm{d}s \\
&+ c_1\psi_1(t) + c_2\psi_2(t).
\end{aligned}
$$

The functions ψ_1, ψ_2 satisfy the asymptotic formulas

$$\psi_1(t) = \mp 1 + O(e^{-|t|}), \quad \psi_2(t) = -|t| + O(e^{-|t|}), \quad t \to \pm\infty$$

from which it follows easily

$$\|\phi\|_{L^2} \leqq C\bar{\lambda}^{3/2}[\|h\|_{L^2} + \bar{\lambda}^{-1}|c_1| + |c_2|]. \tag{7.33}$$

To determine the constants c_1, c_2 we need to solve the system

$$c_1[\bar{\lambda}\psi_1' + K_-\psi_1] + c_2[\bar{\lambda}\psi_2' + K_-\psi_2] = g_-, \quad \text{at } t = -\bar{\lambda}(1+\theta_0) \tag{7.34}$$

$$c_1[\bar{\lambda}\psi_1' + K_+\psi_1] + c_2[\bar{\lambda}\psi_2' + K_+\psi_2] = g_+$$

$$+ K_+ \left\{ \psi_1 \int_{-\bar{\lambda}(1+\theta_0)}^{\bar{\lambda}(1+\theta_0)} \psi_2(s)h(s)\,\mathrm{d}s \right.$$

$$\left. - \psi_2 \int_{-\bar{\lambda}(1+\theta_0)}^{\bar{\lambda}(1+\theta_0)} \psi_1(s)h(s)\,\mathrm{d}s \right\},$$

$$\text{at } t = \bar{\lambda}(1+\theta_0). \tag{7.35}$$

This system has a unique solution for $\bar{\lambda} \gg 1$ since the matrix

$$\begin{pmatrix} K_- & 1 - K_-(1+\theta_0) \\ -K_+ & -1 - K_+(1-\theta_0) \end{pmatrix}$$

is nondegenerate thanks to the nondegeneracy condition (1.4). In fact we find

$$|c_1| \leqq C[|g_-| + |g_+| + \bar{\lambda}^{3/2}\|h\|_{L^2}],$$

$$|c_2| \leqq C\bar{\lambda}^{-1}[|g_-| + |g_+| + \bar{\lambda}^{3/2}\|h\|_{L^2}].$$

From (7.33) we get

$$\|\phi\|_{L^2} \leqq C[\bar{\lambda}^2\|h\|_{L^2} + \bar{\lambda}^{1/2}(|g_-| + |g_+|)]. \tag{7.36}$$

By a straightforward argument we get a further estimate

$$\|\phi\|_{\bar{\lambda}} \equiv \|\phi''\|_{L^2} + \|\phi'\|_{L^2} + \|\phi\|_{L^2} \leqq C[\bar{\lambda}^2\|h\|_{L^2} + \bar{\lambda}^{1/2}(|g_-| + |g_+|)]. \tag{7.37}$$

Given (7.29) and assuming in addition that

$$\|h\|_{L^2} \leqq \varepsilon^\mu,$$

it is easy to solve (7.21)–(7.23) using a standard fixed point argument in the set of functions

$$\mathcal{X} = \{\phi \mid \|\phi\|_{\bar{\lambda}} < \varepsilon^{\mu'}\}, \quad \text{with } \mu' < \mu.$$

After this is done one can go back to the original problem solving the following equation for f_1

$$f_1'' = -\bar{\lambda}^2 e^u + \varepsilon^{-2}h_1 + O(e^{-c/\varepsilon})e^{f_1}$$

with the boundary conditions (7.10)–(7.11). The solution to this problem satisfies estimate $\|f_1\|_{H^2} \leqq C|\log \varepsilon|$ as required. The rest of the Theorem 7.1 for $N = 2$ follows now easily.

In this section we will prove Theorem 7.1. This will be done in several steps. We will assume here $N > 2$ since the case $N = 2$ has just been treated above.

For given functions $q_j(t)$, $p_j(t)$, $j = 1, \ldots, N$ such that

$$\sum_{j=1}^{N} q_j = \sum_{j=1}^{N} p_j = 0,$$

we define the Hamiltonian

$$H = \sum_{j=1}^{N} \frac{p_j^2}{2} + V, \quad V = \sum_{j=1}^{N-1} e^{(q_j - q_{j+1})}.$$

From what we have seen above (4.3), to determine the location of the interfaces it is necessary to develop a solvability theory for a Toda system of the form:

$$\frac{dq_j}{dt} = p_j, \tag{7.38}$$

$$\frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}. \tag{7.39}$$

In this section we will often make use of classical results of Konstant [23] and in particular we will use the explicit formula for the solutions of (7.38)–(7.39) (see formula (7.7.10) in [23]). We will first introduce some notation. Given numbers $w_1, \ldots, w_N \in \mathbb{R}$ such that

$$\sum_{k=1}^{N} w_k = 0, \quad \text{and } w_k > w_{k+1}, \quad k = 1, \ldots, N$$

we define

$$\mathbf{w}_0 = \text{diag}\,(w_1, \ldots, w_N).$$

Furthermore, given numbers $g_1, \ldots, g_n \in \mathbb{R}$ such that

$$\prod_{k=1}^{N} g_k = 1, \quad \text{and } g_k > 0, \quad k = 1, \ldots, N$$

we define

$$\mathbf{g}_0 = \text{diag}\,(g_1, \ldots, g_N).$$

The matrices $\mathbf{w}_0$ and $\mathbf{g}_0$ can be parameterized by introducing the following two sets of parameters

$$c_k = w_k - w_{k+1}, \quad d_k = \log g_{k+1} - \log g_k, \quad k = 1, \ldots, N-1. \tag{7.40}$$

We define functions $\Phi_k(\mathbf{g}_0, \mathbf{w}_0; t)$, $t \in \mathbb{R}$, $k = 0, \ldots, N$, by

$$\Phi_0 = \Phi_N \equiv 1$$

$$\Phi_k(\mathbf{g}_0, \mathbf{w}_0; t) =$$
$$(-1)^{k(n-k)} \sum_{1 \leq i_i < \cdots < i_k \leq N} r_{i_1 \ldots i_k}(\mathbf{w}_0) g_{i_1} \ldots g_{i_k} \exp[-t(w_{i_1} + \cdots + w_{i_k})], \quad (7.41)$$

where $r_{i_1 \ldots i_k}(\mathbf{w}_0)$ are rational functions of the entries of the matrix $\mathbf{w}_0$. It is proven in [23] that all solutions of (7.38)–(7.39) are of the form

$$q_j(t) = \log \Phi_{j-1}(\mathbf{g}_0, \mathbf{w}_0; t) - \log \Phi_j(\mathbf{g}_0, \mathbf{w}_0; t), \quad j = 1, \ldots, N. \quad (7.42)$$

We introduce variables $u_k = q_k - q_{k+1}$. In terms of $\mathbf{u} = (u_1, \ldots, u_{N-1})$ system (7.38)–(7.39) becomes

$$\mathbf{u}'' + M e^{\mathbf{u}} = 0, \quad (7.43)$$

where

$$M = \begin{pmatrix} 2 & -1 & 0 \cdots & & 0 \\ -1 & 2 & -1 \cdots & & 0 \\ & & \ddots & & \\ 0 & \cdots & & 2 & -1 \\ 0 & \cdots & & -1 & 2 \end{pmatrix}, \quad e^{\mathbf{u}} = \begin{pmatrix} e^{u_1} \\ \vdots \\ e^{u_{N-1}} \end{pmatrix}.$$

As a consequence of (7.41) all solutions to (7.43) are given by

$$u_j(t) = q_j(t) - q_{j+1}(t) = -2 \log \Phi_j(\mathbf{g}_0, \mathbf{w}_0; t) + \log \Phi_{j-1}(\mathbf{g}_0, \mathbf{w}_0; t)$$
$$+ \log \Phi_{j+1}(\mathbf{g}_0, \mathbf{w}_0; t). \quad (7.44)$$

Our first goal is to prove the following:

Lemma 7.1. *Let $\mathbf{w}_0$ be such that*

$$\left| w_k - (N - k) + \frac{N-1}{2} \right| < \frac{1}{100N^4}, \quad k = 1, \ldots, N. \quad (7.45)$$

Then there holds

$$u_k(t) = \begin{cases} -c_{N-k}t - d_{N-k} + \tau_k^+(\mathbf{c}) + O(e^{-\frac{|t|}{2}}), \text{ as } t \to +\infty, \quad k = 1, \ldots, N-1, \\ c_k t + d_k + \tau_k^-(\mathbf{c}) + O(e^{-\frac{|t|}{2}}), \text{ as } t \to -\infty, \quad k = 1, \ldots, N-1, \end{cases} \quad (7.46)$$

where $\tau_k^{\pm}(\mathbf{c})$ are smooth functions of the vector $\mathbf{c} = (c_1, \ldots, c_{N-1})$.

Proof. Let q_j, $j = 1, \ldots, N$ be a solution of the system (7.38)–(7.39) depending on the (matrix valued) parameters $\mathbf{w}_0$, $\mathbf{g}_0$ and defined in (7.42). We need to study the asymptotic behavior of $\Phi_j(\mathbf{w}_0, \mathbf{g}_0; t)$ as $t \to \pm\infty$ with the entries of $\mathbf{w}_0$ satisfying (7.45) and still undetermined $\mathbf{g}_0$.

Note that inequalities (7.45) together with the condition $\sum_{k=1}^{N} w_k = 0$ imply

$$|c_k - 1| < \frac{1}{50N^3}, \quad k = 1, \ldots, N. \tag{7.47}$$

By (7.41) and (7.47), we get that as $t \to -\infty$

$$\Phi_k = (-1)^{k(N-k)} r_{1\ldots k}(\mathbf{w}_0) g_1 \ldots g_k e^{-(w_1 + \cdots + w_k)t} + O(e^{-|t|(1+k(N-k)/4)}),$$

hence

$$\frac{\Phi_{k+1}\Phi_{k-1}}{\Phi_k^2} = \frac{g_{k+1}r_{1\ldots(k-1)}(\mathbf{w}_0)r_{1\ldots(k+1)}(\mathbf{w}_0)e^{-t}}{g_k r_{1\ldots k}^2(\mathbf{w}_0)}(1 + O(e^{-\frac{|t|}{2}})). \tag{7.48}$$

It follows that as $t \to -\infty$

$$u_k(t) = \log\left(\frac{\Phi_{k+1}\Phi_{k-1}}{\Phi_k^2}\right)$$

$$= -c_k t + \log\left(\frac{g_{k+1}r_{1\ldots(k-1)}(\mathbf{w}_0)r_{1\ldots(k+1)}(\mathbf{w}_0)}{g_k r_{1\ldots k}^2(\mathbf{w}_0)}\right) + O(e^{-\frac{|t|}{2}}) \tag{7.49}$$

$$= c_k t + d_k + \tau_k^-(\mathbf{c}) + O(e^{-\frac{|t|}{2}}), \tag{7.50}$$

where

$$\tau_k^-(\mathbf{c}) = \log\left(\frac{r_{1\ldots(k-1)}(\mathbf{w}_0)r_{1\ldots(k+1)}(\mathbf{w}_0)}{r_{1\ldots k}^2(\mathbf{w}_0)}\right).$$

Similarly, as $t \to +\infty$ we get

$$u_k(t) = \log\left(\frac{\Phi_{k+1}\Phi_{k-1}}{\Phi_k^2}\right)$$

$$= -c_{N-k}t - d_{N-k} + \tau_k^+ + O(e^{-\frac{|t|}{2}}), \tag{7.51}$$

where

$$\tau_k^+(\mathbf{c}) = \log\left(\frac{r_{N+2-k\ldots N}(\mathbf{w}_0)r_{N-k\ldots N}(\mathbf{w}_0)}{r_{N+1-k\ldots N}^2(\mathbf{w}_0)}\right).$$

This ends the proof. □

Next we need to choose (c_k, d_k): let $(\bar{\lambda}, \theta_0)$ be defined at (7.26) as in the case of $N = 2$ and

$$c_k = 1 + \frac{1}{\bar{\lambda}}\bar{c}_k. \tag{7.52}$$

We need to choose u_k such that it satisfies

$$\bar{\lambda}u'_k(t^-) + K_- u_k(t^-) = K_- \log(\varepsilon^2 \bar{\lambda}^2) + O(\varepsilon^\mu), \tag{7.53}$$
$$\bar{\lambda}u'_k(t^+) + K_+ u_k(t^+) = K_+ \log(\varepsilon^2 \bar{\lambda}^2) + O(\varepsilon^\mu), \tag{7.54}$$

for some $\mu > 0$. To this end, we set $(\hat{c}_k, d_k)$ such that

$$-(1 + K_+(1 - \theta_0))\hat{c}_{N-k} - K_+ d_{N-k} + K_+ \tau_k^+ \left(\mathbf{c}_0 + \frac{1}{\bar{\lambda}}\hat{\mathbf{c}}\right) = 0, \tag{7.55}$$

$$(1 - K_-(1 + \theta_0))\hat{c}_k + K_- d_k + K_- \tau_k^- \left(\mathbf{c}_0 + \frac{1}{\bar{\lambda}}\hat{\mathbf{c}}\right) = 0, \tag{7.56}$$

where $\mathbf{c}_0 = (1, \ldots, 1)$, $\hat{\mathbf{c}} = (\hat{c}_1, \ldots, \hat{c}_{N-1})$.

By changing k to $N - k$ in (7.55), we see that (7.55)–(7.56) is equivalent to

$$-(1 + K_+(1 - \theta_0))\hat{c}_k - K_+ d_k + K_+ \tau_{N-k}^+(\mathbf{c}) = 0, k = 1, \ldots, N-1, \tag{7.57}$$
$$(1 - K_-(1 + \theta_0))\hat{c}_k + K_- d_k + K_- \tau_k^-(\mathbf{c}) = 0, k = 1, \ldots, N-1, \tag{7.58}$$

which can be uniquely solved by the implicit function theorem since the matrix

$$\begin{pmatrix} -(1 + K_+(1 - \theta_0)) & -K_+ & 0 & \cdots & & 0 \\ (1 - K_-(1 + \theta_0)) & K_- & 0 & \cdots & & 0 \\ & & & \ddots & & \\ 0 & \cdots & 0 & -(1 + K_+(1 - \theta_0)) & -K_+ \\ 0 & \cdots & 0 & (1 - K_-(1 + \theta_0)) & K_- \end{pmatrix},$$

is again nonsingular thanks to the nondegeneracy condition (1.4).

Then we define the new errors

$$\bar{\lambda}u'_k(t^\pm) + K_\pm u_k(t^-) - K_\pm \log(\varepsilon^2 \bar{\lambda}^2) = g_{k,\pm} \text{ for } t = \mp \bar{\lambda}(1 \pm \theta_0), \text{ respectively.} \tag{7.59}$$

We get from (7.26) and (7.55)–(7.56) that

$$|g_{k,\pm}| \leqq C\varepsilon^\mu, \tag{7.60}$$

with some $\mu > 0$.

The rest of the proof of Theorem 7.1 follows basically the steps of the proof of the special case in the previous section. Thus we have to consider the following linear system

$$\vec{\phi}'' = \begin{pmatrix} 2e^{u_1} & -e^{u_2} & 0\cdots & & 0 \\ -e^{-u_1} & 2e^{u_2} & -e^{u_3}\cdots & & 0 \\ & & \ddots & & \\ 0 & \cdots & 2e^{u_{N-2}} & -e^{u_{N-1}} \\ 0 & \cdots & -e^{u_{N-2}} & 2e^{u_{N-1}} \end{pmatrix} \vec{\phi}^T, \quad \vec{\phi} = (\phi_1, \dots, \phi_{N-1})$$

(7.61)

We first observe that

$$g_j \frac{\partial u_k}{\partial g_j} = \begin{cases} 1, & j = k+1, \quad t \to \infty, \\ -1, & j = k, \quad t \to \infty, \\ 1, & j = N+2-k, \quad t \to -\infty, \\ -1, & j = N+1-k, \quad t \to -\infty, \\ 0, & \text{otherwise.} \end{cases}$$

Hence by a transformation we can find a set of linearly independent solutions to (7.61)

$$\psi_{1k}(t) = \begin{cases} \vec{e}_k, & t \to \infty, \\ -\vec{e}_k, & t \to -\infty. \end{cases}$$

Similarly, considering derivatives of u_k with respect to w_j we can find solutions of (7.61), $\psi_{2k}(t)$, $k = 1, \dots, N-1$ such that

$$\psi_{2k}(t) = \vec{e}_k |t| + O(1).$$

The functions $\{\psi_{1k}(t), \psi_{2k}(t)\}$ form a fundamental set for the system (7.61), whose behavior as $t \to \pm\infty$ is analogous to that of the functions $\psi_1(t)$, $\psi_2(t)$, respectively.

Denoting by $\mathbf{u}_0(t)$ the solution to (7.43) where the parameters $\mathbf{w}_0$ and $\mathbf{g}_0$ satisfying (7.55)–(7.56) and setting $u_k(t) = q_k(t) - q_{k+1}(t)$, $\mathbf{u} = (u_1, \dots, u_{N-1})$, we look for the solution to (7.18)–(7.20) (expressed now in terms of $\mathbf{u}$) in the form

$$\mathbf{u}(t) = \mathbf{u}_0(t) + \vec{\varphi}(t), \quad \vec{\varphi} = (\varphi_1, \dots, \varphi_{N-1}),$$

where the boundary conditions are given at $t = -\bar{\lambda}(1 + \theta_0)$ and $t = \bar{\lambda}(1 - \theta_0)$ with $\bar{\lambda}$ and θ_0 defined in (7.26). By (7.60), $\mathbf{u}_0$ satisfies the required boundary condition up to order $O(\varepsilon^\mu)$, for some $\mu > 0$. Following the scheme in the case $N = 2$, using the fundamental set $\{\psi_{1k}, \psi_{2k}\}$, we can solve for the correction $\vec{\varphi}$. The rest of the proof is an equally straightforward adaptation of the method of the previous section. We leave the details to the reader.

7.4. Conclusion of the proof of Theorem 1.1

To solve system (7.9)–(7.11) we will set up a fixed point argument scheme. To this end let $\tilde{\mathbf{f}} \in H^1(-1, 1)$ be given and such that

$$\|\tilde{\mathbf{f}}\|_{H^1(-1,1)} \leqq |\log \varepsilon|^2$$

and define for each $k = 1, \ldots, N$

$$h_k(\mathbf{f}) = \mathcal{N}_{k1}(\mathbf{f}, \mathbf{f}', \mathbf{f}'') + \mathcal{N}_{k2}(\tilde{\mathbf{f}}, \tilde{\mathbf{f}}').$$

With this $\mathbf{h} = (h_1, \ldots, h_N)$ we can use Theorem 7.1 to solve the (now nonlinear) problem

$$\alpha_0 \sqrt{2} \varepsilon^2 f_k'' - e^{-\sqrt{2}(f_k - f_{k-1})} + e^{-\sqrt{2}(f_{k+1} - f_k)} = h_k(\mathbf{f}), \quad -1 < \theta < 1, \quad (7.62)$$

$$f_k'(-1) + K_- f_k(-1) = 0, \quad (7.63)$$

$$f_k'(1) + K_+ f_k(1) = 0. \quad (7.64)$$

Here it is important that $\mathcal{N}_{k1}$ is a contraction in $H^2(-1, 1)$ and therefore, making use of the theory developed above for the linearized Toda system and employing the Banach Contraction Mapping theorem we find $\mathbf{f}$ given $\tilde{\mathbf{f}}$. This way we define a mapping $\mathcal{T}(\tilde{\mathbf{f}}) = \mathbf{f}$ and the solution of our problem is simply a fixed point of $\mathcal{T}$. Continuity of $\mathcal{N}_{ki}$, $i = 1, 2$, with respect to its parameters and a standard regularity argument allows us to conclude that $\mathcal{T}$ is compact as a mapping from $H^1(-1, 1)$ into itself. Then the Schauder Theorem applies to yield the existence of a fixed point of $\mathcal{T}$ as required. Finally, we should remark that uniqueness of the solutions found for system (7.62)–(7.64) for $h_k = 0$ implies the symmetry fact $f_k = -f_{N-k}$. This symmetry is thus approximately inherited for the inhomogeneous perturbed problem. The construction itself provides the validity of the asymptotic expressions (1.18). This ends the proof of Theorem 1.1. $\quad \square$

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