Transversal interface dynamics of a front connecting a stripe pattern to a uniform state

Marcel G. Clerc[1], Daniel Escaff[2] and René Rojas[(a)1]

[1] *Departamento de Física, FCFM, Universidad de Chile, Casilla 487-3, Santiago, Chile,*
[2] *Complex Systems Group, Facultad de Ingeniería, Universidad de los Andes, Av. San Carlos de Apoquindo 2200, Santiago, Chile.*

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Abstract. - Interfaces in two-dimensional systems exhibit unexpected complex dynamical behaviors, the dynamics of a border connecting a stripe pattern and a uniform state is studied. Numerical simulations of a prototype isotropic model, the subcritical Swift-Hohenberg equation, show that this interface has transversal spatial periodic structures, zigzag dynamics and complex coarsening process. Close to a spatial bifurcation, an amended amplitude equation and a one-dimensional interface model allow us to characterize the dynamics exhibited by this interface.

Introduction. – Non equilibrium processes often lead in nature to pattern formation developing from a homogeneous state through the spontaneous breaking of the symmetries present in the system [1]. In recent decades, much effort has been devoted to the study of pattern formation (see review [2] and the references therein) arising in systems such as chemical reactions, gas discharge systems, CO_2 lasers, liquid crystals, hydrodynamic or electroconvective instabilities, and granular matter (see review [3]), to mention a few. A unified description for the dynamics of spatially periodic structures, developed at the onset of a bifurcation, is achieved by means of amplitude equations for the critical modes. Such a description is valid in the case of weak nonlinearities and for a slow spatial and temporal modulation of the base pattern [2]. As an example, the Newell-Whitehead-Segel equation [4] describes the dynamics of a stripe pattern formed in a two-dimensional system. Another ubiquitous phenomenon in nature is the interface dynamics or *front propagation*. The concept of front propagation, emerged in the field of population dynamics [5–7], has gained growing interest in biology, chemistry, physics, and mathematics (See, e.g. [8] and references therein). These interfaces connect two extended states, such as: uniforms states, patterns, oscillatory, standing waves, spatio-temporal chaotic and so forth.

In one-dimensional systems, an interface connecting two uniform stable states, the most favorable state—for instance energetically—invades the other one with a constant and unique speed [9, 10]. This speed is zero, that is the front is motionless, at the Maxwell point [11]. The above picture changes, when one considers an interface connecting a pattern state and a uniform one or two patterns. Due to spatial translation symmetry breaking, the interface is motionless in a range of parameters, *the pinning range* [11–13]. This behavior is called locking phenomenon or pinning effect. In bidimensional dynamical systems, few experimental and theoretical studies have been performed on fronts connecting patterns and uniform states [14–17].

The aim of this letter is to study the dynamical behaviors of a front connecting a stripe pattern to a uniform state. Numerical simulations of a prototype model—the isotropic Swift-Hohenberg equation—show that the locking phenomenon of a flat interface persists. However, this flat interface is nonlinearly transversely unstable, that is, a finite perturbation of this interface leads to the appearance of an initial wave number which is subsequently replaced by zigzag dynamics, which presents a complex coarsening. Increasing the longitudinal interface size, the flat interface exhibits a transversal spatial instability, which originates a periodical structure at the interface. We have termed these interfaces *embroideries*. In order to explain these behaviors in an unified manner, we make use of the amended Newell-Whitehead-Segel equation and prototype one-dimensional model for the interface, which describe

(a) E-mail address: rrojas@dfi.uchile.cl

properly the dynamics observed on the interface connecting a stripe pattern with a uniform state.

The model. – A simple isotropic model which exhibits coexistence between a uniform state and a stripe pattern is (the subcritical Swift-Hohenberg equation [2])

$$\partial_t u = \varepsilon u + \nu u^3 - u^5 - \left(\vec{\nabla}^2 + q^2\right)^2 u, \qquad (1)$$

where $u\,(x,t)$ is an order parameter, ε is the bifurcation parameter, q is the wave number of the stripe pattern, ν the control parameter of the type of bifurcation (supercritical or subcritical), and $\vec{\nabla}^2$ is the Laplacian operator. The model (1) describes the confluence of a stationary and a spatial subcritical bifurcations with reflection symmetry ($u \to -u$), when the parameters scale as $u \sim \varepsilon^{1/4}$, $\nu \sim \varepsilon^{1/2}$, $q \sim \varepsilon^{1/4}$, $\partial_t \sim \varepsilon$ and $\vec{\nabla} \sim \varepsilon^{1/4}$ ($\varepsilon \ll 1$). The above model is often employed in the description of patterns observed in Rayleigh-Benard convection [2]. For small and negative ε and $-9\nu^2/40 \equiv \varepsilon_{sn} < \varepsilon < 0$, the system exhibits coexistence between a uniform state $u(x,y,t) = 0$ and a stripe pattern $u\,(x,y,t) = \sqrt{\nu}\left(\sqrt{2(1+\sqrt{1+40\varepsilon/9\nu})}\cos\left(\vec{q}\cdot\vec{r}\right)\right) + o(\nu^{5/2})$, where $\vec{q}$ is an arbitrary vector with modulus q. For $\varepsilon = \varepsilon_{sn}$ the model has a saddle-node bifurcation that give rises to stable and unstable patterns. For $\varepsilon = 0$ the uniform state becomes unstable. When one considers the above model in one spatial dimension, it is well-known that in the coexistence region ($\varepsilon_{sn} < \varepsilon < 0$) the model exhibits a front connecting a spatially periodic solution and uniform state. For ε close to 0 (ε_{sn}), the pattern (uniform) state invades the uniform (pattern) state, i.e. the system displays front propagation. The front is motionless at the interval $\varepsilon_- < \varepsilon < \varepsilon_+$, the pining range [11]. Note that, in this range, the state which is energetically more favorable does not invade the less favorable one. It is important to note that the interface has two characteristic lengths, the pattern length $2\pi/q$ and the interface size, which is represented by λ in the inset of Fig. **??**. Decreasing ν the interface size increases.

Numerical results. – In two spatial dimension the above scenario changes drastically. If one considers a similar parameters setup ($\varepsilon_{sn} < \varepsilon_- < \varepsilon < \varepsilon_+ < 0$), the flat interface of model (1) exhibits locking phenomenon (cf. Fig. **??**a). However, when the longitudinal interface size λ is increased the interface suffers a supercritical transversal spatial instability and it gives rise to transversal periodic structures. The typical observed structures are depicted in Fig. **??**. We term these periodical structures at the interface *embroidery.* The embroidery size is proportional to the longitudinal interface size. The embroideries are consequence of two facts: the spatial isotropy and the interaction of enveloped variation with the underlying pattern (pinning effect). Seeing that at the interface the stripe pattern can develop in any direction as consequence of the isotropy, however the interaction of enveloped variation with the underlying pattern freezes the growth of these unstable modes, as we shall see with the amended amplitude equation. Based on the 1D theory of front interaction which predicts a family of localized patterns [18], one expects to find a family of localized stripes, that is, the localized stripes are the transversal expansion of 1D localized patterns. The typical stable localized states observed in model (1) are illustrated in Fig. 2. It is important to note that, localized stripe patterns with flat interfaces have been observed in ref. [19] .

Numerically, we observe that in a finite region of parameters the embroidery interface is linearly stable, i.e. this interface with a small initial perturbation evolves to the interface without perturbations. However, when we consider a large perturbation the interface does not evolve to the interface without perturbation. This interface exhibits a nonlinear zig-zag instability, which is characterized by a transversal instability without a well-defined wavelength. In spite of this, initially this instability has a well defined wave number close to $2\pi/q$, later on, the sinusoidal interface becomes an angled line composed of pieces of interface turned with well define angles, zig-*facet or zag-facet.* It is important to note that the flat interface–in the regime of parameters where it is linearly stable—exhibits the same behavior, that is, a finite initial perturbation of the flat interface gives rise to a zig-zag dynamics. In order to illustrate this nonlinear instability, Fig. **??**a shows two embroidery interfaces, one with a initial small perturbation (left interface) and the other with a finite one. After a finite time, the interface with initial small perturbation evolves to an embroidery interface. While the Right-interface develops a zig-zag structure and the pattern propagates over the uniform state (cf. Fig. **??**b). Hence, the flat and embroidery interfaces are nonlinearly transversely unstable.

In the zig-zag interface, two adjacent facets whose orientations are opposite, are connected by a region of strong curvature that we term *a corner.* The dynamics shown by the zig-zag interface consists then in reassembling domains of even orientation, the angle facets staying unchanged, which is a *coarsening dynamics.* This process occurs due to annihilations of corners and without a characteristic length scale. Actually, the averaged domain size increases regularly in time. Simultaneously to this coarsening process, the zigzag interface propagates from one extended state to the other one. Figure 4 depicts the typical coarsening process observed at the interfaces.

The amended amplitude equation. – To explain the dynamics exhibited by the interfaces of model (1) in a unified framework, close to the spatial bifurcation ($\varepsilon \ll 1$, and $\nu \sim \varepsilon^{1/2}$), we can introduce the ansatz

$$u(x,y,t) = a_0 A\left(X = \frac{x}{l_0}, Y = \frac{y}{\sqrt{q l_0}}, \tau = \frac{t}{t_0}\right)e^{iqx} \\ -5\frac{a_0^5|A|^2A^3}{8q^2}e^{i3qx} - \frac{a_0^5A^5}{24q^2}e^{i5qx} + \cdots + c.c., \qquad (2)$$

where $a_0 \equiv \sqrt{3\nu/10}\varepsilon^{1/4}$, $l_0 = 2q\sqrt{10}/3\sqrt{|\varepsilon|}$, $t_0 = 10/9\nu^2|\varepsilon|$, and $A(X,Y,\tau)$ is the envelope of the pattern that describes the front solution (when the envelope is uniform and not null the initial model (1) has an stripe pattern in the x-direction), the "$\cdots$" stands for the high order terms in the envelope A, *c.c.* means complex conjugate, and $\{X,Y,\tau\}$ are slow variables. In this ansatz (2), we consider that q is of order one or larger than the other parameters. Introducing the above ansatz in equation (1), we find the following solvability condition for the envelope (*Amended subcritical Newell-Whitehead-Segel equation*)

$$\partial_\tau A = \left\{\epsilon + |A|^2 - |A|^4 + \left(\partial_X - \frac{i\partial_{YY}}{2q}\right)^2\right\} A + \eta A^3 e^{i\kappa x}, \tag{3}$$

where $\epsilon \equiv -10\varepsilon/9\nu^2$, $\kappa = 2\sqrt{10}q/3\nu\sqrt{\varepsilon}$, and $\eta \equiv 1/3$ for model (1). Using the above scaling the terms inside brackets are order ϵ [20], and the last term is exponentially small which is regularly neglected in the multiple scaling approach. However, to account for the coupling between the large scale envelope and the small scale underlaying the stripe state, we consider η as free parameter and κ a finite number. Notice that consider other solvability conditions for the envelope one can obtain diverse small exponential terms for equation (3) of the form $A^n\bar{A}^m e^{iq(n-1-m)x}$. For the sake of simplicity we have consider the dominant one in A, which correspond to the last term in Eq. (3). Nevertheless, we expect that these entire exponential small terms have qualitatively the same effect in the dynamics. When the term proportional to spatial forcing is zero ($\eta \to 0$), the above model is the Newell-Whitehead-Segel amplitude equation, which have been used deeply to explain the appearance of stripe pattern [2]. Nevertheless, this model does not account for locking phenomenon [11]. The inclusion of spatial forcing terms in the amplitude equations in 1D allows understanding the locking phenomenon, the pinning range and localized structures [12,13,18]. In the extreme limit, $\eta \to 0$, it is straightforward to show that model (3) has a front solution connecting two homogeneous states, 0 and $\left(1+\sqrt{1+4\epsilon}\right)/2$, when $\epsilon < 0$. Which accounts for the connection between the stripe pattern and the uniform state. These two homogeneous states are energetically equivalent at $\epsilon_M = -3/16$—the *Maxwell point*—and it has the form

$$A = \sqrt{\frac{3/4}{1+e^{\pm\sqrt{3/4}(X-P)}}}e^{i\theta}, \tag{4}$$

where P stands for the position of flat interface, and θ is an arbitrary phase. This flat interface is transversally unstable and it gives rise to a complex coarsening process, zigzag instability. Numerically we have computed the growth rate of each mode with wave number k of the flat interface—spectrum. Fig. **??**a depicts this instability and the spectrum of the flat interface. Note that although the numerical simulations have been done at the Maxwell point, the interface propagates from a state that represents the stripe to a uniform one. When η is small the non-null uniform solution becomes a stripe state in the y-direction (cf. Fig. **??**b). The zigzag dynamics exhibited by the interface disappears and is replaced by a transversal periodic structure, embroidery (cf. Fig. **??**b). Hence, these transversal embroidery structures are consequence of the interaction of the spatial forcing—generated by underlying pattern—with the transversal instability of the Newell-Whitehead-Segel model. Changing ϵ the embroidery interface is motionless in a range of parameter of ϵ, the pinning range. Fig. **??**b shows the typical motionless embroidery interface observed in model (3) and the respective spectrum of the flat interface. Increasing η the embroidery interface becomes a flat interface and the amplitude of the stripes increase. Fig. **??**c shows the stable flat interface and its respective spectrum. The flat and embroidery interfaces exhibit by the amended amplitude equation are nonlinear unstable, that is, a finite perturbation of these interfaces give rise a zigzag dynamics. Therefore, the dynamics exhibits by the universal model (3) is similar to those shown by prototype model (1).

Phenomenological 1D model. – The model (3) seems simpler than Eq. (1), however, the analytical description of the interfaces in these model is a thorny task. A standard method to grasp the dynamics exhibited by the interface of the precedent models is to derive a one-dimensional equation for it. This method consists in using as an ansatz, the front solution (5) plus a small correction, that is,

$$A = \left\{\sqrt{\frac{3/4}{1+e^{\pm\sqrt{3/4}(X-P(Y,\tau))}}} + w_0(X-P,Y)\right\}e^{i\theta}, \tag{5}$$

where the continuous parameter P is promoted to a field $(P(Y,\tau))$, w_0 is a small complex correction function, which is order of spatial variation of the position of the interface ($w_0 \sim \partial_{YY}P$) [21–24]. However, we can not use this weakly nonlinear method because the more unstable transversal mode has a finite wave number (q). This type of method requires that unstable modes have small wave number. To understand the mechanism of the different structures observed at the interface, and based on symmetry arguments for the interface [21–24] and on the effect of spatial forcing term [12,13], we propose the following phenomenological equation for the position of the interface (*convective and forced Cahn-Hilliard equation*)

$$\partial_\tau P = \varepsilon P_{YY} + P_Y^2 P_{YY} - P_{YYYY} + \alpha P_Y^2 - \kappa \sin(QP) + \Delta \tag{6}$$

this model has the equilibria $P_n = n\pi/Q$, $n = 0, \pm1, \pm2, \cdots$. The equilibria P_{2m} as function of forcing κ have similar spectrum to those shown in Fig. **??**. For large κ, these equilibria are stable, i.e. this model has a family of stable flat interfaces. Decreasing κ these flat interfaces become unstable and give rise to spatial periodical state, embroideries. Finally, for small $\{\alpha, \kappa\}$ this model exhibit zigzag instability characterized by logarithmic and power law for the coarsening. Hence, the one-dimensional model (6) presents qualitative similar dynamics that those shown by models (1) and (3). In the coexistence region of uniform states, one expects to find stable localized horm solution [7], and then we expect to find localized horm state at the interface. Fig. **??** illustrates the typical stable horm solutions of one-dimensional model (6) and the prototype model (1). The study of the effect of the forcing on zigzag dynamics of above model is in progress.

Summary. – Isotropic systems which have coexistence between stable stripe pattern and uniform states can exhibit interfaces connecting these states. These interfaces present a rich and unexpect transversal dynamics like transversal spatial instability, nonlinear zig-zag instability and localized states. Recently in Ref. [25], it is shown that in anisotropic systems the flat interface linking rolls pattern with uniform one is transversal stable. Hence, the wealthy transversal dynamics is a consequence of the spatial isotropy and the pinning effect.

Although the prototype model under study Eq. (1) is variational—the dynamical evolution of this model has the tendency to minimize its Lyapunov functional—the interface dynamic exhibited by this model is robust. Since, this interface dynamics is well described by the amended amplitude equation, which is valid close to the spatial instability. Hence, we expect that a system that exhibits a coexistence between a stripe pattern and a homogeneous state should be present a rich interface dynamics.

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