A COUNTEREXAMPLE TO A CONJECTURE BY DE GIORGI IN LARGE DIMENSIONS

MANUEL DEL PINO, MICHAŁ KOWALCZYK, AND JUNCHENG WEI

Abstract. We consider the Allen-Cahn equation

$$\Delta u + u(1-u^2) = 0 \quad \text{in } \mathbb{R}^N .$$

A celebrated conjecture by E. De Giorgi (1978) states that if u is a bounded solution to this problem such that $\partial_{x_N} u > 0$, then the level sets $\{u = \lambda\}$, $\lambda \in \mathbb{R}$, must be hyperplanes *at least* if $N \le 8$. We construct a family of solutions which shows that this statement does not hold true for $N \ge 9$.

Résumé. Un contre-exemple à la conjecture de De Giorgi en grandes dimensions. Nous considérons l'équation d'Allen-Cahn

$$\Delta u + u(1-u^2) = 0 \quad \text{dans } \mathbb{R}^N .$$

Une conjecture célèbre de E. De Giorgi (1978) affirme que si u est une solution bornée de ce problème telle que $\partial_{x_N} u > 0$, alors les ensembles de niveau $\{u = \lambda\}$, $\lambda \in \mathbb{R}$, sont des hyperplans *au moins* si $N \le 8$. Nous contruisons une famille de solutions qui montre que cette conjecture n'est pas vraie pour $N \ge 9$.

1. Version Française abrégée

Considérons l'équation d'Allen-Cahn

$$\Delta u + (1-u^2)u = 0 \quad \text{dans } \mathbb{R}^N. \tag{1.1}$$

E. De Giorgi [6] formula en 1978 la conjecture célèbre suivante:

(DG) *Soit u une solution bornée de l'équation* (1.1) *telle que $\partial_{x_N} u > 0$. Alors les ensembles de niveau $\{u = \lambda\}$ sont des hyperplans,* au moins *pour des dimensions $N \le 8$.*

La conjecture de De Giorgi a été complètement établie en dimension $N = 2$ par Ghoussoub et Gui [12], et en dimension $N = 3$ par Ambrosio et Cabré [1]. Savin [19] a démontré sa validité pour $4 \le N \le 8$ sous une condition additionnelle, peu contraignante. Un contre-exemple à (DG) en dimension $N \ge 9$ était attendu depuis longtemps, mais la question était resté ouverte. Dans cette note, nous en établissons un. La conjecture (DG) est analogue au *théorème de Bernstein* pour les graphes minimaux qui, dans sa forme la plus générale due à Simons [21], affirme que toute surface minimale dans $\mathbb{R}^N$, qui est aussi le graphe d'une fonction de $N-1$ variables, est un hyperplan si $N \le 8$. Bombieri, De Giorgi et Giusti [4] ont démontré que ce résultat devient faux en dimension $N \ge 9$, en construisant une solution non-triviale du problème

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}} \right) = 0 \quad \text{dans } \mathbb{R}^8 \tag{1.2}$$

par la méthode des sur- et sous-solutions. Ecrivons

$$x' = (x_1, \dots , x_8) \in \mathbb{R}^8, \quad u = \sqrt{x_1^2 + \cdots + x_4^2}, \quad v = \sqrt{x_5^2 + \cdots + x_8^2}.$$

La solution BDG est de la forme $F(x') = F(u, v)$ avec la propriété de symétrie $F(u, v) = -F(v, u)$ si $u \ge v$. De plus, nous pouvons montrer que F devient asymptotique à une fonction homogène de degré 3 qui s'annulle sur le cône $u = v$. Soit $\Gamma = \{x_9 = F(x')\}$ le graphe minimal (BDG) ainsi construit. Considérons pour $\alpha > 0$ sa dilatation $\Gamma_\alpha = \alpha^{-1}\Gamma$, qui est aussi un graphe minimal. Notre principal résultat est le théorème suivant.

Théorème 1. *Soit $N \geq 9$. Pour tout $\alpha > 0$ suffisamment petit, il existe une solution bornée $u_\alpha(x)$ de l'équation* (1.1) *telle que $u_\alpha(0) = 0$, $\partial_{x_N} u_\alpha(x) > 0$ pour tout $x \in \mathbb{R}^N$, et $|u_\alpha(x)| \to 1$ quand* $\operatorname{dist}(x, \Gamma_\alpha) \to +\infty$, *uniformément pour tout $\alpha > 0$ petit.*

La preuve fournit des informations très précises sur u_α. Si $t = t(y)$ dénote un choix de distance signée au graphe Γ_α, alors pour un nombre $\delta > 0$ fixé, petit, notre solution se comporte comme $u_\alpha(x) \sim w(t), \quad \text{si } |t| < \frac{\delta}{\alpha}$ avec $w(t) = \tanh\left(\frac{t}{\sqrt{2}}\right)$, solution uni-dimensionnelle hétérocline de (1.1). La preuve est basés sur une forme infini-dimensionelle de la réduction de Lyapunov-Schmidt. Considérons des coordonnées de Fermi pour décrire des points proches de Γ_α, $x = y + z\nu_\alpha(y), \quad y \in \Gamma_\alpha, \quad |z| < \frac{\delta}{\alpha}$ où ν_α est le vecteur unitaire normal à Γ_α pour lequel $\nu_{\alpha 9} > 0$. Nous choisissons alors comme première approximation $\mathtt{w}(x) := w(z + h(\alpha y))$ où h est une fonction régulière, petite, sur $\Gamma = \Gamma_1$, à déterminer. En cherchant une solution de la forme $\mathtt{w} + \phi$, il s'avère que le problème se réduit essentiellement à

$$\Delta_{\Gamma_\alpha}\phi + \partial_{zz}\phi + f'(w(z))\phi + E + N(\phi) = 0 \quad \text{dans } \Gamma_\alpha \times \mathbb{R}$$

où $S(\mathtt{w}) = \Delta\mathtt{w} + f(\mathtt{w})$, $E = \chi_{|z|<\alpha^{-1}\delta}\, S(\mathtt{w})$, $N(\phi) = f(\mathtt{w}+\phi) - f(\mathtt{w}) - f'(\mathtt{w})\phi + B(\phi)$, $f(\mathtt{w}) = \mathtt{w}(1-\mathtt{w}^2)$, et $B(\phi)$ est un opérateur linéaire du second ordre avec des coefficients petits, tronqués pour $|z| > \delta\alpha^{-1}$. Plutôt que de résoudre directement le problème précédent, nous en considérons une version projetée:

$$\mathcal{L}(\phi) := \Delta_{\Gamma_\alpha}\phi + \partial_{zz}\phi + f'(w(z))\phi = -E - N(\phi) + c(y)w'(z) \quad \text{dans } \Gamma_\alpha \times \mathbb{R} \tag{1.3}$$

$$\int \phi(y,z)w'(z)\, dz \;=\; 0 \quad \text{pour tout} \quad y \in \Gamma_\alpha \tag{1.4}$$

Une solution de ce problème peut être trouvée de sorte à respecter la taille et le taux de décroissance de l'erreur E, qui est, grossièrement, de l'ordre de $r(\alpha y)^{-3}e^{-|z|}$. Ceci est établi de manière précise à l'aide d'une théorie linéaire pour le problème projeté dans des normes de Sobolev à poids et en utilisant un principe de contraction. Pour finir, h est déterminée de sorte que $c(y) \equiv 0$. Nous avons $c(y)\int w'^2 dz = \int (E + N(\phi))w'\, dz$ et réduisons la question à une EDP non-linéaire (non-locale) sur Γ de la forme

$$\mathcal{J}(h) := \Delta_\Gamma h + |A|^2 h = O(\alpha)r(y)^{-3} + M_\alpha(h) \quad \text{dans } \Gamma, \quad h = 0 \quad \text{sur } \Gamma \cap [u = v], \tag{1.5}$$

où $M(h)$ est un opérateur petit qui contient des termes non-locaux. Une théorie de solvabilité pour l'opérateur de Jacobi dans des espaces de Sobolev à poids est alors établie, avec l'ingrédient crucial qui consiste en la présence de barrières explicites pour les inégalités impliquant l'opérateur linéaire défini précédemment. En utilisant cette théorie, le problème (1.5) est finallement résolu au moyen du principe de l'application contractante. La propriété de monotonie découle facilement du principe du maximum.

2. Introduction and statement of main result

This paper deals with entire bounded solutions of the Allen-Cahn equation

$$\Delta u + (1-u^2)u = 0 \quad \text{in } \mathbb{R}^N. \tag{2.1}$$

E. De Giorgi [6] formulated in 1978 the following celebrated conjecture:

(DG) *Let u be a bounded solution of equation* (2.1) *such that $\partial_{x_N} u > 0$. Then the level sets $\{u = \lambda\}$ are hyperplanes, at least for dimension $N \leq 8$.*

Equivalently, u depends on just one Euclidean variable so that it must have the form $u(x) = \tanh\left((x-p)\cdot b/\sqrt{2}\right)$, for some $p, b \in \mathbb{R}^N$ with $|b| = 1$ and $b_N > 0$.

Equation (2.1) arises in the gradient theory of phase transitions by Cahn-Hilliard and Allen-Cahn, in connection with the energy functional

$$J_\varepsilon(u) = \frac{\varepsilon}{2}\int |\nabla u|^2 + \frac{1}{4\varepsilon}\int (1-u^2)^2, \quad \varepsilon > 0, \tag{2.2}$$

which has been extensively studied by Γ-convergence methods [14, 15, 16, 17]. It is known that a family of local minimizers u_ε with uniformly bounded energy approaches as $\varepsilon \to 0^+$ in L^1-loc sense a function of the form $\chi_S - \chi_{S^c}$ where ∂S is a set with minimal perimeter, so that the limiting interface between the *stable phases* $u = 1$ and $u = -1$, is expected to approach a minimal hypersurface. If u is a solution of (2.1) , then its scalings $u(x/\varepsilon)$ are local minimizers of J_ε (on bounded sets), whose level sets are graphs. This connection led De Giorgi to formulate his conjecture, as a parallel with *Bernstein's theorem* for minimal graphs which in its most general form, due to Simons [21], states that any minimal hypersurface in $\mathbb{R}^N$, which is also a graph of a function of $N-1$ variables, must be a hyperplane if $N \le 8$. Bombieri, De Giorgi and Giusti [4] proved that this fact is false in dimension $N \ge 9$. De Giorgi conjecture has been fully established in dimensions $N = 2$ by Ghoussoub and Gui [12] and for $N = 3$ by Ambrosio and Cabré [1]. Savin [19] proved its validity for $4 \le N \le 8$ under the mild additional assumption

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1 \quad \text{for all} \quad x'. \tag{2.3}$$

Condition (2.3) is related to the so-called Gibbons' Conjecture:

Gibbons' Conjecture: Let u be a bounded solution of equation (2.1) *satisfying*

$$\lim_{x_N \to \pm\infty} u(x^{'}, x_N) = \pm 1, \ \textit{uniformly in } x'. \tag{2.4}$$

Then the level sets $\{u = \lambda\}$ are all hyperplanes.

Gibbons' Conjecture has been proved in all dimensions with different methods by Farina [11], Barlow, Bass and Gui [2], and Berestycki, Hamel, and Monneau [3]. If the uniformity in (2.4) is dropped, a counterexample can be built using the method by Pacard and the authors in [9].

A counterexample to (DG) in dimension $N \ge 9$ has long been believed to exist, but the question has remained open. Partial progress in this direction has been achieved by Jerison and Monneau [13] and by Cabré and Terra [5],

In this note we sketch the construction of a bounded solution of equation (2.1) which is monotone in one direction, whose level sets are not hyperplanes when $N \ge 9$. This example "disproves" De Giorgi conjecture in dimension $N \ge 9$. It suffices to do such a construction in dimension $N = 9$ by extending the solution as a constant dependence in the remaining variables. Our starting point is the existence of non-trivial smooth, entire solutions of the minimal surface equation

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^8 \tag{2.5}$$

as found in [4] by means of the super-subsolution method. Let us write

$$x' = (x_1, \dots, x_8) \in \mathbb{R}^8, \quad u = \sqrt{x_1^2 + \cdots + x_4^2}, \quad v = \sqrt{x_5^2 + \cdots + x_8^2}.$$

The BDG solution has the form $F(x') = F(u, v)$ with the symmetry property $F(u, v) = -F(v, u)$ if $u \ge v$. A refinement of the analysis in [4] leads to a precise description of the asymptotic behavior of F:

Proposition 2.1. *Introducing polar coordinates $u = r\cos\theta$, $v = r\sin\theta$, there exists an explicit, smooth positive function $g(\theta)$, $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$ with $g(\frac{\pi}{4}) = 0$, $g'(\frac{\pi}{2}) = 0$ such that if we set $F_0(x') = r^3 g(\theta)$, then for some $\sigma > 0$ we have that for $v > u$, the BDG-solution can be assumed to satisfy*

$$F_0(x') \le F(x') \le F_0(x) + |x'|^{-\sigma} \min\{F_0(x'), 1\} \quad \textit{for all} \quad |x'| \gg 1.$$

Let $\Gamma = \{x_9 = F(x')\}$ be the minimal BDG graph predicted above and let us consider for $\alpha > 0$ its dilation $\Gamma_\alpha = \{x_9 = \alpha^{-1} F(\alpha x')\}$, which is also a minimal graph.

Our main result states as follows:

Theorem 1. *Let $N \geq 9$. For all $\alpha > 0$ sufficiently small there exists a bounded solution $u_\alpha(x)$ of equation* (2.1) *such that $u_\alpha(0) = 0$,*

$$\partial_{x_N} u_\alpha(x) > 0, \quad \textit{for all} \quad x \in \mathbb{R}^N,$$

and

$$|u_\alpha(x)| \ \to \ 1, \qquad \textit{as} \ \ \operatorname{dist}(x, \Gamma_\alpha) \ \to \ +\infty, \tag{2.6}$$

uniformly in small $\alpha > 0$.

Property (2.6) implies that the 0 level set of u_α lies inside the region dist $(y, \Gamma_\alpha) < R$ for some fixed $R > 0$ and all small α, and hence it cannot be a hyperplane. Moreover, it can be shown that u_α inherits the symmetry of Γ_α, and hence u_α has value 0 and the origin. Much more accurate information on the solution will be drawn from the proof. The idea is simple. If $t(y)$ denotes a choice of signed distance to the graph Γ_α then, for a small fixed number $\delta > 0$, our solution looks like

$$u_\alpha(x) \sim \tanh\left(\frac{t}{\sqrt{2}}\right) \quad \text{if } |t| < \frac{\delta}{\alpha}.$$

We should mention that Allen-Cahn with a small parameter in a compact manifold was established to have concentration phenomena around nondegenerate minimal submanifolds, as found in [18]. Our proof is based on an infinite-dimensional form of Lyapunov Schmidt reduction, close in spirit to that in [7, 8, 10, 9].

3. Sketch of the proof of Theorem 1

Let us consider Fermi coordinates to describe points near Γ_α,

$$x = y + z\nu_\alpha(y), \quad y \in \Gamma_\alpha, \quad |z| < \frac{\delta}{\alpha},$$

where ν_α is the unit normal to Γ_α for which $\nu_{\alpha 9} > 0$. Observe that $\nu_\alpha(y) = \nu_1(\alpha y)$. It is natural to consider the function $w(z)$ as a good first approximation to a solution of (2.1) whose zero level set equals Γ_α which is of course far from a hyperplane. Rather than taking this as our approximation we choose $\mathtt{w}(x) := w(z + h(\alpha y))$, where h is a smooth, small function on $\Gamma = \Gamma_1$, to be determined. It is natural to consider Laplace's operator in $\mathbb{R}^9$ expressed in these coordinates,

$$\Delta = \partial_z^2 + \Delta_{\Gamma_{\alpha,z}} - H_{\alpha,z}\partial_z.$$

Here

$$\Gamma_{\alpha,z} = \left\{ x \in \mathbb{R}^9 \mid d(x, \Gamma_\alpha) = z \right\}, \quad |z| < \frac{\delta}{\alpha},$$

$\Delta_{\Gamma_{\alpha,z}}$ is its Laplace-Beltrami operator and $H_{\alpha,z}$ its mean curvature. Let us remark that if $\tilde{p}(y) = p(\alpha y)$ then $\Delta_{\Gamma_{\alpha,z}}\tilde{p}\,(y) = \Delta_{\Gamma_{1,\alpha z}} p\,(\alpha y)$, besides $H_{\alpha,z}(y) = \alpha H_{1,\alpha z}(\alpha y)$. Since Γ_α is a minimal surface, we get that $H_{\alpha,0}(y) = 0$. We can also expand

$$H_{\alpha,z}(y) = \alpha z H_1(\alpha y) + \alpha^2 z^2 H_2(\alpha y) + \alpha^3 z^3 H_3(\alpha y) + \dots,$$

where $H_1 = |A|^2$, the square of the Euclidean norm of the second fundamental form on Γ. We actually have decay in the coefficients H_j of the form $H_j(y) = O(r^{-j-1})$, which follows from curvature estimates for minimal graphs due to L. Simon [20]. Here and henceforth we denote

$$r(y) = |x'| \quad \text{for} \quad y = (x', F(x')).$$

Besides, we assume (a priori) in h a size $r^2|D_\Gamma^2 h| + |h| \sim \alpha r^{-1}$, which gets a posteriori justified as h at last is chosen as the solution of certain fixed problem that yields this size. We expand

$$S(\mathtt{w}) := \Delta\mathtt{w} + f(\mathtt{w}) = \alpha^2(\Delta_\Gamma h + |A|^2 h)(\alpha y) w_z + \alpha^2 |A|^2 z w_z + O(\alpha^3) r^{-3}(\alpha y) e^{-\sigma|z|}$$

where the last term is a smaller order operator which depends on first and second derivatives of h. The term $\alpha^2|A|^2 z w_z$ can be easily eliminated by an improvement of approximation, adding a suitable (explicit) small term to $\mathtt{w}$. By simplicity we keep the same notation for this new approximation and the error.

We write in this region $u = \mathtt{w} + \phi$, so that the equation for ϕ reads at the main order

$$\Delta_{\Gamma_\alpha}\phi + \partial_{zz}\phi + f'(w(z))\phi + E + N(\phi) = 0, \quad \text{in } \Gamma_\alpha \times \mathbb{R},$$

where

$$E = \chi_{|z|<\alpha^{-1}\delta}\, S(\mathtt{w}), \quad N(\phi) = f(\mathtt{w}+\phi) - f(\mathtt{w}) - f'(\mathtt{w})\phi + B(\phi), \quad f(\mathtt{w}) = \mathtt{w}(1-\mathtt{w}^2) \tag{3.7}$$

and $B(\phi)$ is a second order linear operator with "small" coefficients, also cut-off for $|z| > \delta\alpha^{-1}$. This problem is of course not equivalent in entire space to the original one, but a *gluing reduction* procedure, in the same line as those used in [7, 8, 9], makes it *essentially so*, to the expense of including small size operators and perturbing N by a an exponentially small (in α) nonlocal operator in ϕ which can be neglected for the remaining of the argument. Rather than solving the above problem directly we consider a projected version of it:

$$\mathcal{L}(\phi) := \Delta_{\Gamma_\alpha}\phi + \partial_{zz}\phi + f'(w(z))\phi = -E - N(\phi) + c(y)w'(z), \quad \text{in } \Gamma_\alpha \times \mathbb{R}, \tag{3.8}$$

$$\int_{\mathbb{R}} \phi(y,z)w'(z)\, dz \;=\; 0, \quad \text{for all} \quad y \in \Gamma_\alpha. \tag{3.9}$$

A solution to this problem can be found in such a way that it respects the size and decay rates of the error E, which is roughly of the order $\sim r(\alpha y)^{-2}e^{-|z|}$. Indeed, let us consider the norm

$$\|g\|_{p,\sigma,\nu} = \sup_x e^{\sigma|z|} r(\alpha y)^{\mu} \|g\|_{L^p(B(x,1))}, \quad x = y + z\nu_\alpha(y) \tag{3.10}$$

and the auxiliary linear problem

$$\mathcal{L}(\phi) = g(y,z) + c(y)w'(z), \quad \text{in } \Gamma_\alpha \times \mathbb{R}, \quad \int_{\mathbb{R}} \phi(y,z)w'(z)\, dz \;=\; 0 \quad \text{for all} \quad y \in \Gamma_\alpha. \tag{3.11}$$

We show that there is a unique solution ϕ to the above linear problem with the property that

$$\|\phi\|_* := \|D^2\phi\|_{p,\sigma,\nu} + \|D\phi\|_{\infty,\sigma,\nu} + \|D\phi\|_{\infty,\sigma,\nu} \le C\|g\|_{p,\sigma,\nu}, \tag{3.12}$$

where C does not depend on α small, provided that p is fixed large enough. As a conclusion, a direct application of contraction mapping principle yields the desired result: A unique small solution $\phi = \phi(h)$ of problem (3.8)-(3.9) exists. Its size is controlled by that of E in the sense of the above norms with $\nu = 3$ (after explicit improvement of the approximation is done by adding terms of $O(\alpha^2)$). We will of course have a solution to the original problem (2.1) if we manage to get $c(y) = 0$. This can indeed be achieved by conveniently adjusting the perturbation h. An equation for this h is obtained by multiplying the above relation by $w'(z)$ and integrating in z-variable: we get

$$c(y)\int_{\mathbb{R}} w'^2 dz = \int_{\mathbb{R}} (E + N(\phi))w'\, dz.$$

This relation gets reduced to a (nonlocal) nonlinear PDE in Γ of the form

$$\mathcal{J}(h) := \Delta_\Gamma h + |A|^2 h = O(\alpha)r(y)^{-3} + M_\alpha(h) \quad \text{in } \Gamma, \quad h = 0 \quad \text{on } \Gamma \cap \{u = v\}, \tag{3.13}$$

where $M(h)$ is a small operator which includes nonlocal terms. The linear operator $\mathcal{J}(h)$ is nothing but the Jacobi operator of Γ, which in turn is the linearization of its mean curvature operator in the direction of perturbations of the form $h(y)\nu(y)$. The symmetries of the BDG minimal surface allow us to assume the boundary condition on h. These boundary conditions are interpreted as eliminating translation effect of the manifold, yielding a form of nondegeneracy: the only decaying Jacobi field of this operator is $h = \frac{1}{\sqrt{1+|\nabla F|^2}}$ which is positive, hence it does not satisfy the boundary conditions. A solvability theory for the Jacobi operator in weighted Sobolev norms, similar to (3.10), (3.12), suitably adapted to problem (3.13) is then built up. In this setting, it is crucial the presence of explicit barriers for inequalities of the type $-\mathcal{J}(\bar{h}) \leq r^{-\mu}$ (roughly with growth $\bar{h} \sim r^{2-\mu}$). Using this theory, problem (3.13) is finally solved by means of contraction mapping principle. Finally, the solution built this way is easily seen to satisfy $\partial_{x_9} u > 0$, using the linear equation satisfied by this function, $\Delta p + f'(u)p = 0$, the fact that it is positive near Γ_α and maximum principle. Full details are provided in the paper [10].

Acknowledgments: The first author has been partly supported by research grants Fondecyt 1070389 and FONDAP, Chile. The second author has been supported by Fondecyt grant 1050311, Nucleus Millennium grant P04-069-F, and FONDAP, Chile. The research of the third author is partially supported by an Earmarked Grant from RGC of Hong Kong and a Direct Grant from CUHK.

References

[1] L. Ambrosio and X. Cabré, *Entire solutions of semilinear elliptic equations in R^3 and a conjecture of De Giorgi,* Journal Amer. Math. Soc. 13 (2000), 725–739.
[2] M.T. Barlow, R.F. Bass and C. Gui, *The Liouville property and a conjecture of De Giorgi*, Comm. Pure Appl. Math. 53(8)(2000), 1007-1038.
[3] H. Berestycki, F. Hamel and R. Monneau, *One-dimensional symmetry of bounded entire solutions of some elliptic equations*, Duke Math. J. 103(2000), 375-396.
[4] E. Bombieri, E. De Giorgi, E. Giusti, *Minimal cones and the Bernstein problem.* Invent. Math. 7 1969 243–268.
[5] X. Cabré, J. Terra *Saddle-shaped solutions of bistable diffusion equations in all of R^{2m}.* Preprint 2008.
[6] E. De Giorgi, *Convergence problems for functionals and operators,* Proc. Int. Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), 131–188, Pitagora, Bologna (1979).
[7] M. del Pino, M. Kowalczyk, J. Wei, *Concentration on curves for nonlinear Schrdinger equations.* Comm. Pure Appl. Math. 60 (2007), no. 1, 113–146.
[8] M. del Pino, M. Kowalczyk, J. Wei, *The Toda system and clustering interfaces in the Allen-Cahn equation.* To appear in Arch. Rational Mech. Anal.
[9] M. del Pino, M. Kowalczyk, F. Pacard, J. Wei, *The Toda system and multiple-end solutions of autonomous planar elliptic problems.* Preprint 2007.
[10] M. del Pino, M. Kowalczyk, J. Wei, *On De Giorgi conjecture in dimansion $N \geq 9$.* Preprint 2008.
[11] A. Farina, *Symmetry for solutions of semilinear elliptic equations in $\mathbb{R}^N$ and related conjectures*, Ricerche Mat. 48(suppl.) (1999), 129-154.
[12] N. Ghoussoub and C. Gui, *On a conjecture of De Giorgi and some related problems,* Math. Ann. 311 (1998), 481-491.
[13] D. Jerison and R. Monneau, *Towards a counter-example to a conjecture of De Giorgi in high dimensions,* Ann. Mat. Pura Appl. 183 (2004), 439–467.
[14] R. V. Kohn and P. Sternberg, *Local minimizers and singular perturbations,* Proc. Roy. Soc. Edinburgh Sect. A, 11 (1989), pp. 69–84.
[15] L. Modica, *Convergence to minimal surfaces problem and global solutions of* $\Delta u = 2(u^3 - u)$. Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), pp. 223–244, Pitagora, Bologna, (1979).
[16] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion.* Arch. Rational Mech. Anal. 98 (1987), no. 2, 123–142.
[17] L. Modica, S. Mortola, *Un esempio di Γ-convergenza.* Boll. Unione Mat. Ital. Sez. B 14, 285–299 (1977).
[18] F. Pacard and M. Ritoré, *From the constant mean curvature hypersurfaces to the gradient theory of phase transitions,* J. Differential Geom. 64 (2003), no. 3, 356–423.
[19] O. Savin, *Regularity of flat level sets in phase transitions.* To appear in Ann. of Math.
[20] L. Simon, *Entire solutions of the minimal surface equation*, J. Differential Geometry 30 (1989), 643–688
[21] J. Simons, *Minimal varieties in riemannian manifolds.* Ann. of Math. (2) 88 1968 62–105.

M. del Pino - Departamento de Ingeniería Matemática and CMM, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile.
E-mail address: `delpino@dim.uchile.cl`

M. Kowalczyk -Kent State University, Department of Mathematical Sciences, Kent, OH 44242 and Departamento de Ingeniería Matemática, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile.
E-mail address: `kowalcyk@math.kent.edu`

J. Wei - Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: `wei@math.cuhk.edu.hk`