Relaxation of a Bolza Problem Governed by a Time-Delay Sweeping Process

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Abstract We study in an infinite dimensional Hilbert space a Bolza problem in which the dynamics are given by a time-delay perturbed sweeping process. This is a differential inclusion whose right-hand side involves a normal cone to a moving set, along with a time-delay perturbation. A relaxation result is established from which we deduce a sufficient condition ensuring the existence of an optimal solution.

Keywords Sweeping process • Differential inclusion • Normal cone • Perturbation • Prox-regular set • Optimal control • Sufficient conditions • Young measure • Set-valued map

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1 Introduction

The paper is devoted to a Bolza problem whose dynamic constraint is given by a delay perturbed sweeping process. To state the problem, let H be a real Hilbert space, U a compact metric space, $C\colon [0,T] \rightrightarrows H$ and $\Gamma\colon [0,T] \rightrightarrows U$ two set-valued maps, $\Gamma(\cdot)$ being measurable and compact-valued. Given $\rho \geqslant 0$, one considers the space $\mathcal{C}_H([-\rho, 0])$ of all continuous maps from $[-\rho, 0]$ into H. For $t \in [0,T]$ and $x(\cdot) \in \mathcal{C}_H([-\rho, T])$, one defines a continuous map $x_t(\cdot)$ on $[-\rho, 0]$ by $x_t(s) = x(t+s)$. Let $g\colon [0,T] \times \mathcal{C}_H([-\rho, 0]) \times U \to H$ be a single-valued map, $\zeta(\cdot)$ a measurable selection of Γ, and φ a fixed member of $\mathcal{C}_H([-\rho, 0])$ such that $\varphi(0) \in C(0)$. Let us

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denote by $x^{\zeta}(\cdot)$ the unique solution (under assumptions to be specified below) of the delay perturbed sweeping process

$$\begin{cases} -\dot{x}(t) \in N(C(t), x(t)) + g(t, x_t(\cdot), \zeta(t)) \text{ a.e. } t \in [0, T] \\ x(s) = \varphi(s) \ \forall s \in [-\rho, 0]. \end{cases} \tag{1}$$

Now, given $J\colon [0, T] \times H \times U \to \mathbb{R}$ and $L\colon H \to \mathbb{R}$, we are interested in the existence of an optimal solution for the problem

$$(O.P) \qquad \inf_{\zeta(\cdot)\in S_\Gamma} L\left(x^{\zeta}(T)\right) + \int_0^T J\left(t, x^{\zeta}(t), \zeta(t)\right) dt.$$

In addition to mathematical motivations, this problem is important because some mechanical systems are governed by differential inclusions of the type Eq. 1. The reader is referred to Monteiro Marques [19] and Moreau [21, 22] for details.

Similar problems have been studied in the finite dimensional setting by Jawhar [18] and Castaing et al. [4]. In the latter paper, the authors studied, particularly, variational properties of the value function associated with the problem. They proved in particular that the value function is a viscosity subsolution of an appropriate Hamilton-Jacobi-Bellman equation.

Note that, if $C(t) := H$ for all t, the problem $(O.P)$ reduces to an optimal control problem governed by a differential equation. Indeed, in such a case, $N(C(t), x(t)) = \{0\}$ and then the dynamics are given by a differential equation. Those problems have been extensively studied, particularly in the finite dimensional framework. We refer to Balder [2], Baum [3], Cesari [8, 9], Fleming and Rishel [15], Warga [25] and references therein.

To establish an existence result for the problem $(O.P)$, as usual, we will consider a minimizing sequence and prove its convergence, up to a subsequence, in the set of the Young measures. This leads us to study the following problem.

Let us consider the Lebesgue-measurable set-valued map Σ from $[0, T]$ into the set $\mathcal{M}_+^1(U)$ of all probability measures on $(U, \mathcal{B}(U))$ defined, for each $t \in [0, T]$, by

$$\Sigma(t) := \{P \in \mathcal{M}_+^1(U)\colon\ P(\Gamma(t)) = 1\}.$$

Let S_Σ be the set of all *measurable* selections of Σ. For each $\mu \in S_\Sigma$, let us denote by $x^{\mu}(\cdot)$ the unique solution of the delay perturbed sweeping process

$$\begin{cases} -\dot{x}(t) \in N(C(t), x(t)) + \int_U g(t, x_t(\cdot), u)\mu_t(du) \text{ a.e. } t \in [0, T] \\ x(s) = \varphi(s) \ \forall s \in [-\rho, 0]. \end{cases}$$

Now, we consider the following optimal control problem:

$$(R.P) \qquad \inf_{\mu\in S_\Sigma} L(x^{\mu}(T)) + \int_0^T \int_U J(t, x^{\mu}(t), u)\mu_t(du)dt.$$

The latter problem is called the *relaxed problem* of the problem $(O.P)$.

It turns out that the infimums of the two problems are equal and, further, the infimum of the relaxed problem is attained. This generalizes the result obtained in Edmond and Thibault [14], which is an extension of the ones proved in the finite dimensional setting by Jawhar [18] and Castaing et al. [6] for optimal control problems governed by sweeping processes without delay.

As a consequence of the above result, under a convexity assumption, we prove that the problem $(O.P)$ has an optimal solution. It is known that without such an assumption the problem may have no solution.

The paper is organized as follows: In Section 2 we summarize the notations and some notions that will be used in the paper. Section 3 contains an existence result for delay perturbed sweeping processes that is needed in Section 4 to introduce the Bolza problem to be studied. We recall in Section 5 properties and results for Young measures, which will be used to study the relaxed problem that is presented in Section 6. While Section 7 is concerned with the existence of solutions for the latter problem, Section 8 addresses the existence result for the Bolza problem.

2 Preliminaries

In all the paper $I := [0, T]$ $(0 < T)$ is an interval of $\mathbb{R}$ and H is a real separable Hilbert space whose scalar product will be denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|$.

2.1 Notations

The closed unit ball of H will be denoted by $\mathbb{B}$ and, for $\eta > 0$, $B[0, \eta]$ is the closed ball of radius η centered at 0. For any subset S of H, $\overline{\text{co}}S$ stands for the closed convex hull of S. We will denote by $\mathcal{C}(I, H)$ or $\mathcal{C}_H(I)$ the set of all continuous maps from I to H. The norm of the uniform convergence on $\mathcal{C}(I, H)$ will be denoted by $\| \cdot \|_\infty$. The Lebesgue σ-field of I is denoted by $\mathcal{L}(I)$ and λ denotes the Lebesgue measure. For $p \in [1, +\infty]$, we denote by $L^p(I, H)$ or $L^p_H(I)$ the quotient space of all λ-Bochner measurable maps $g(\cdot) \colon I \to H$ such that $\|g(\cdot)\|$ belongs to $L^p(I, \mathbb{R})$.

2.2 Normal Cones

For the following concepts, the reader is referred to Clarke et al. [10, 11] and Poliquin et al. [23].

Let S be a nonempty closed subset of H and $y \in H$. The distance of y to S, denoted by $d_S(y)$ or $d(y, S)$, is defined by

$$d_S(y) := \inf \left\{ \|y - x\| : x \in S \right\}.$$

One defines the (possibly empty) set of nearest points of y in S by

$$\text{proj}_S(y) := \left\{ x \in S \colon d_S(y) = \|y - x\| \right\}.$$

When $\text{proj}_S(y)$ is a singleton $\{x\}$, we will write $x = \text{proj}_S(y)$.

If $x \in \text{proj}_S(y)$ and $\alpha \geqslant 0$, then the vector $\alpha(y - x)$ is called a proximal normal to S at x. The set of all vectors obtainable in this manner is a cone termed the *proximal normal cone* to S at x. It is denoted by $N^P_S(x)$.

One also defines the *limiting normal cone* (or *Mordukhovich normal cone*, see Mordukhovich and Shao [20]) and the *Clarke normal cone* respectively by

$$N_S^L(x) := \left\{ \xi \in H : \xi_n \stackrel{w}{\rightharpoonup} \xi, \xi_n \in N_S^P(x_n), x_n \stackrel{S}{\to} x \right\}$$

and

$$N_S^C(x) := \overline{\text{co}} N_S^L(x).$$

Here, $\xi_n \stackrel{w}{\rightharpoonup} \xi$ means that the sequence ξ_n converges weakly to ξ, and $x_n \stackrel{S}{\to} x$ means that $x_n \to x$ and $x_n \in S$ for each integer n.

2.3 Prox-Regular Set

For a fixed $r > 0$, the set S is said to be *r-prox-regular* (or *uniformly prox-regular with constant* $\frac{1}{r}$) if, for any $x \in S$ and any $\xi \in N_S^L(x)$ such that $\|\xi\| < 1$, one has $x = \text{proj}_S(x + r\xi)$. Another characterization (see Poliquin et al. [23]) is the following *hypomonotonicity* property: For any $x_i \in S$ $(i = 1, 2)$, the inequality

$$\langle \xi_1 - \xi_2, x_1 - x_2 \rangle \geqslant -\|x_1 - x_2\|^2$$

holds whenever $\xi_i \in N_S^L(x_i) \cap B(0, r)$, where $B(0, r)$ stands for the open ball of radius r centered at 0.

If S is r-prox-regular, then the following holds (see Poliquin et al. [23]):

- for any $x \in S$, all the normal cones defined above coincide. In such a case, they will be denoted by $N_S(x)$ or $N(S, x)$;
- for any $x \in H$ such that $d_S(x) < r$, the set $\text{proj}_S(x)$ is a singleton.

2.4 Standing Assumptions

Let $r > 0$. In all the paper a set-valued map $C(\cdot)$ from $I := [0, T]$ to H will be involved. It is required to satisfy the following assumptions:

(A_1) For each $t \in I$, $C(t)$ is a nonempty closed subset of H which is r-prox-regular;
(A_2) $C(t)$ varies in an *absolutely continuous way*, that is, there exists an absolutely continuous function $v(\cdot) : I \to \mathbb{R}$ such that, for any $y \in H$ and $s, t \in I$,

$$|d(y, C(t)) - d(y, C(s))| \leqslant |v(t) - v(s)|.$$

3 Existence Result for Delay Perturbed Sweeping Processes

Let $\rho \geqslant 0$. Consider the space $\mathcal{C}_0 := \mathcal{C}_H([-\rho, 0])$ endowed with the uniform convergence norm denoted by $\|\cdot\|_{\mathcal{C}_0}$. For $x(\cdot) \in \mathcal{C}_H([-\rho, T])$ and for each $t \in I := [0, T]$, one defines a map $x_t(\cdot) \in \mathcal{C}_0$ by $x_t(s) := x(t + s)$. Let $f : I \times \mathcal{C}_0 \to H$ a single-valued map. Let φ be a fixed member of $\mathcal{C}_0$ such that $\varphi(0) \in C(0)$. We consider the following problem

$$(P_\varphi) \qquad \begin{cases} -\dot{x}(t) \in N(C(t), x(t)) + f(t, x_t(\cdot)) \text{ a.e. } t \in [0, T] \\ x(s) = \varphi(s) \ \forall s \in [-\rho, 0]. \end{cases}$$

One calls *solution* of (P_φ) any map $x(\cdot)\colon [-\rho, T] \to H$ such that:

(i) for any $s \in [-\rho, 0]$, one has $x(s) = \varphi(s)$;
(ii) the restriction $x|_{[0,T]}(\cdot)$ of $x(\cdot)$ is absolutely continuous and its derivative, denoted by $\dot{x}(\cdot)$, satisfies the inclusion

$$-\dot{x}(t) \in N(C(t), x(t)) + f(t, x_t(\cdot)) \text{ a.e. } t \in [0, T].$$

The following theorem, the proof of which is given in Edmond [13], provides an existence result for the delay perturbed sweeping process (P_φ).

Theorem 1 *Let H be a Hilbert space. Assume that $C(\cdot)$ satisfies (A_1), (A_2). Let $f\colon I \times \mathcal{C}_0 \to H$ be a map satisfying:*

(i) *for any $\varphi \in \mathcal{C}_0$, $f(\cdot, \varphi)$ is measurable;*
(ii) *for any $\eta > 0$, there exists a non-negative function $k_\eta(\cdot) \in L^1(I, \mathbb{R})$ such that, for all $\varphi_1, \varphi_2 \in \mathcal{C}_0$ with $\|\varphi_i\|_{\mathcal{C}_0} \leqslant \eta$ $(i = 1, 2)$ and for all $t \in I$,*

$$\| f(t, \varphi_1) - f(t, \varphi_2)\| \leqslant k_\eta(t)\|\varphi_1 - \varphi_2\|_{\mathcal{C}_0};$$

(iii) *there exists a non-negative function $\beta(\cdot) \in L^1(I, \mathbb{R})$ such that, for all $t \in I$ and for all $\varphi \in \mathcal{C}_0$,*

$$\| f(t, \varphi)\| \leqslant \beta(t)(1 + \|\varphi\|_{\mathcal{C}_0}).$$

Then, for any $\varphi \in \mathcal{C}_0$ with $\varphi(0) \in C(0)$, the problem (P_φ) has one and only one solution $x(\cdot)$, which satisfies, for

$$l := \|\varphi\|_{\mathcal{C}_0} + \exp\left\{2\int_0^T \beta(\tau)d\tau\right\}\int_0^T [2(1 + \|\varphi\|_{\mathcal{C}_0})\beta(s) + |\dot{v}(s)|]ds,$$

$$\| f(t, x_t(\cdot))\| \leqslant (1 + l)\beta(t) \text{ a.e. } t \in I$$

and

$$\|\dot{x}(t) + f(t, x_t(\cdot))\| \leqslant (1 + l)\beta(t) + |\dot{v}(t)| \text{ a.e. } t \in I.$$

4 The Bolza Problem

In this section we consider an optimal control problem governed by a delay perturbed sweeping process. We are interested in the existence of solutions for this problem in the infinite dimensional setting. A problem of the same kind has been studied by Jawhar [18] for perturbed sweeping processes without delay in the case $H = \mathbb{R}^n$.

Let U be a compact metric space and let $\Gamma\colon [0, T] \rightrightarrows U$ be a measurable set-valued map taking nonempty compact values. Denote by B_0 the closed unit ball of $\mathcal{C}_0$. Let $g\colon I \times \mathcal{C}_0 \times U \to H$ be a map satisfying:

(A_3) for any $t \in I$, $g(t, \cdot, \cdot)$ is continuous on $\mathcal{C}_0 \times U$;
(A_4) for each $(\varphi, u) \in \mathcal{C}_0 \times U$, $g(\cdot, \varphi, u)$ is λ-measurable on I;
(A_5) for every $\eta > 0$, there exists a non-negative function $k_\eta(\cdot) \in L^1(I, \mathbb{R})$ such that, for all $t \in I$ and for all $\varphi_1, \varphi_2 \in \eta B_0$,

$$\|g(t, \varphi_1, u) - g(t, \varphi_2, u)\| \leqslant k_\eta(t)\|\varphi_1 - \varphi_2\|_{\mathcal{C}_0};$$

(A_6) there exists a non-negative function $\beta(\cdot) \in L^1(I, \mathbb{R})$ such that, for all $(t, \varphi, u) \in I \times \mathcal{C}_0 \times U$, one has

$$\|g(t, \varphi, u)\| \leqslant \beta(t)(1 + \|\varphi\|_{\mathcal{C}_0}).$$

Let φ be a fixed member of $\mathcal{C}_0$ such that $\varphi(0) \in C(0)$. Let S_Γ be the set of all measurable selections (up to almost everywhere equality) of Γ, which are called original controls.

According to Theorem 1, for each $\zeta(\cdot) \in S_\Gamma$, the delay perturbed sweeping process

$$\begin{cases} -\dot{x}(t) \in N(C(t), x(t)) + g(t, x_t(\cdot), \zeta(t)) \text{ a.e. } t \in [0, T] \\ x(s) = \varphi(s) \ \forall s \in [-\rho, 0] \end{cases} \tag{2}$$

has a unique solution denoted by $x^\zeta(\cdot)$.

Given two maps $J\colon [0, T] \times H \times U \to \mathbb{R}$ and $L\colon H \to \mathbb{R}$ satisfying appropriate conditions, we are interested in the existence of an optimal solution for the problem

$$(O.P) \qquad \inf_{\zeta(\cdot) \in S_\Gamma} L(x^\zeta(T)) + \int_0^T J(t, x^\zeta(t), \zeta(t))dt.$$

To find a solution of this problem, as usual, we will consider a minimizing sequence. This approach will lead us to deal with Young measures, which are the object of the following section.

5 Young Measures

In this section we recall the concept of Young measures and provide some important results that is needed later. For ampler details, we refer the reader to Balder [1], Castaing, Raynaud de Fitte, and Valadier [5], Jawhar [17], and Valadier [24].

Let $(S, \mathcal{S}, \sigma)$ be a *complete* measure space with a non-negative *finite* measure σ and let V be a *complete separable metric* space. $\mathcal{B}(V)$ being the Borel *sigma*-field, one denotes by $\mathcal{Y}(S, \sigma, V)$ the set of all positive measures ν on $(S \times V, \mathcal{S} \otimes \mathcal{B}(V))$ whose projections on S (that is, their images by the map $(s, v) \mapsto s$) equal σ. Equivalently, $\nu \in \mathcal{Y}(S, \sigma, V)$ if and only if, for all $E \in \mathcal{S}$, one has $\nu(E \times V) = \sigma(E)$. The members of $\mathcal{Y}(S, \sigma, V)$ are called *Young measures*, in reference to the pioneering work of Young [26].

On the other hand, let $\mathcal{M}_+^1(V)$ be the set of all probability measures on $(V, \mathcal{B}(V))$. Following Castaing et al. [5], we denote by $\mathcal{Y}_{\text{dis}}(S, \sigma, V)$ the set of all maps $\mu\colon S \to \mathcal{M}_+^1(V)$ (up to σ-almost everywhere equality) that are λ-measurable in the sense that, for any $B \in \mathcal{B}(V)$, the function $s \mapsto \mu_s(B)$ is $\mathcal{S}$-measurable.

Remark 1 In Jawhar [17], the set $\mathcal{Y}_{\text{dis}}(S, \sigma, V)$ is denoted by $\mathcal{R}(S, \sigma, V)$ and any of its members is called *transition probability measure* on $S \times V$.

Let us recall that if $\mu \in \mathcal{Y}_{\text{dis}}(S, \sigma, V)$, $A \in \mathcal{S} \otimes \mathcal{B}(V)$, and if $\mathbb{1}_A$ is the characteristic function of A (that is, $\mathbb{1}_A(w) = 1$ if $w \in A$ and 0 otherwise), then the function $s \mapsto \int_V \mathbb{1}_A(s, v)\mu_s(dv)$ is $\mathcal{S}$-measurable on S and the set function ν defined by

$$\nu(A) = \int_S \int_V \mathbb{1}_A(s, v)\mu_s(dv)\, \sigma(ds) \tag{3}$$

for all $A \in \mathcal{S} \otimes \mathcal{B}(V)$ is a Young measure on $S \times V$. Accordingly, any member of $\mathcal{Y}_{\text{dis}}(S, \sigma, V)$ is called a *disintegrable Young measure*.

Conversely, under the above assumptions on S and V, any Young measure on $S \times V$ is associated with some $\mu \in \mathcal{Y}_{\text{dis}}(S, \sigma, V)$ in the way above (see Valadier [24]).

Remark 2

1) If ν is the Young measure corresponding to the member $\mu \in \mathcal{Y}_{\text{dis}}(S, \sigma, V)$, i.e, the Young measure defined by Eq. 3, then, for any function $\psi\colon S \times V \to \mathbb{R} \cup \{-\infty, +\infty\}$ which is $\mathcal{S} \otimes \mathcal{B}(V)$-measurable and non-negative (resp. ν-integrable), the function $s \mapsto \int_V \psi(s, v)\mu_s(dv)$ is σ-measurable (resp. σ-integrable) and one has

$$\int_{S\times V} \psi \, d\nu = \int_S \int_V \psi(s, v)\mu_s(dv)\,\sigma(ds).$$

2) If ν is a Young measure associated with some $\mu \in \mathcal{Y}_{\text{dis}}(S, \sigma, V)$ we will make no distinction between ν and μ, that is, for all $s \in S$, we will write ν_s instead of μ_s.

3) Any $\mathcal{S}$-measurable map $u(\cdot)\colon S \to V$ defines a Young measure on $S \times V$ termed the *Young measure associated* with $u(\cdot)$. This is the Young measure corresponding to the member μ of $\mathcal{Y}_{\text{dis}}(S, \sigma, V)$ defined by $\mu_s := \delta_{u(s)}$, where $\delta_{u(s)}$ is the Dirac mass at the point $u(s)$, i.e, for any $B \in \mathcal{B}(V)$, $\delta_{u(s)}(B) = 1$ if $u(s) \in B$ and 0 otherwise.

5.1 Caratheodory Integrand and Narrow Convergence

One calls *integrand* any function $\psi\colon S \times V \to \mathbb{R} \cup \{-\infty, +\infty\}$ that is $\mathcal{S} \otimes \mathcal{B}(V)$-measurable. An integrand is said to be *of Caratheodory* type if, for any $s \in S$, the partial function $\psi(s, \cdot)$ is continuous and takes finite values on V. An integrand ψ is said to be L^1*-bounded* if there exists some non-negative function $\gamma \in L^1_{\mathbb{R}}(S, \sigma)$ such that $|\psi(s, v)| \leqslant \gamma(s)$ for all $(s, v) \in S \times V$.

The set $\mathcal{Y}(S, \sigma, V)$ will be endowed with the *narrow topology* (see Balder [1] and Valadier [24]), called *stable topology* in Castaing et al. [5]. Recall that a sequence (ν^n) converges to ν in $\mathcal{Y}(S, \sigma, V)$ if, for any L^1-bounded Caratheodory integrand ψ,

$$\lim_n \int_{S\times V} \psi \, d\nu^n = \int_{S\times V} \psi \, d\nu. \tag{4}$$

On the other hand, one says that a sequence (μ^n) converges in $\mathcal{Y}_{\text{dis}}(S, \sigma, V)$ to μ if the sequence of the corresponding Young measures converges in $\mathcal{Y}(S, \sigma, V)$. It amounts to saying that, for any L^1-bounded Caratheodory integrand ψ,

$$\lim_n \int_S \int_V \psi(s, v)\, \mu^n_s(dv)\,\sigma(ds) = \int_S \int_V \psi(s, v)\, \mu_s(dv)\,\sigma(ds).$$

5.2 Some Important Results

We recall that $I := [0, T]$ and λ is the Lebesgue measure on I.

We will need the following results which are proved in a more general setting in Valadier [24] (see also Castaing et al. [5]).

Proposition 1 *Let* $h_n(\cdot), h(\cdot) \in \mathcal{C}(I, H)$ *(*$n \geqslant 1$*) and* $\mu^n, \mu \in \mathcal{Y}_{\text{dis}}(I, \lambda, V)$*. Assume that* $(h_n(\cdot))$ *converges uniformly to* $h(\cdot)$ *and* (μ^n) *converges to* μ *in* $\mathcal{Y}_{\text{dis}}(I, \lambda, V)$*. Let* $\theta^n \in \mathcal{Y}(I, \lambda, H \times V)$ *be defined by* $\theta^n_t := \delta_{h_n(t)} \otimes \mu^n_t$*. Then,* θ^n *converges in* $\mathcal{Y}(I, \lambda, H \times V)$ *to the Young measure* $\theta \in \mathcal{Y}(I, \lambda, H \times V)$ *defined by* $\theta_t := \delta_{h(t)} \otimes \mu_t$*.*

Before stating a second result, let us recall that a sequence of functions $(f_n(\cdot))$ is said to be *uniformly integrable* in $L^1(I, \mathbb{R})$ if it is bounded in $L^1(I, \mathbb{R})$ and

$$\lim_{\lambda(A) \to 0} \sup_n \int_A |f_n(t)|\, dt = 0.$$

Proposition 2 *Let* $u_n(\cdot)\colon I \to V$ *(*$n \geqslant 1$*) be measurable maps. Assume that the sequence of the associated Young measures* (ν^n) *(that is,* $\nu^n_t := \delta_{u_n(t)}$*) converges to* ν *in* $\mathcal{Y}(I, \lambda, V)$*. Let* $\psi : I \times V \to \mathbb{R}$ *be a Caratheodory integrand. Assume that the sequence* $(\psi(\cdot, u_n(\cdot)))_n$ *is uniformly integrable in* $L^1(I, \mathbb{R})$*. Then,* ψ *is* ν*-integrable and*

$$\int_{I \times V} \psi \, d\nu = \lim_n \int_I \psi(t, u_n(t))\, dt.$$

The following result is also useful (see Balder [1], Jawhar [17], and Valadier [24]).

Proposition 3 *If* V *is a compact metric space, then any sequence in* $\mathcal{Y}_{\text{dis}}(I, \lambda, V)$ *has a subsequence which converges in* $\mathcal{Y}_{\text{dis}}(I, \lambda, V)$*.*

6 The Relaxed Problem

To establish the existence result for our Bolza problem $(O.P)$ (see Section 4), it is convenient to consider another optimal control problem called *relaxed problem*. We will prove that the latter has an optimal solution and that its optimal value is equal to the infimum in the problem $(O.P)$.

Let us consider the set-valued map $\Sigma(\cdot)$ defined on I by

$$\Sigma(t) := \left\{ P \in \mathcal{M}^1_+(U)\colon P(\Gamma(t)) = 1 \right\}.$$

Denote by S_Σ the set of all λ-measurable selections (up to almost everywhere equality) of Σ. The set S_Σ, whose members are called *relaxed controls*, is nonempty. In fact, the following result holds (see Castaing and Valadier [7] and Jawhar [17]).

Proposition 4 *Let* $\Gamma\colon [0, T] \rightrightarrows U$ *be a* λ*-measurable set-valued map with nonempty compact values. Consider the set-valued map* $\Sigma(\cdot)$ *defined on* $[0, T]$ *by*

$$\Sigma(t) := \left\{ P \in \mathcal{M}^1_+(U)\colon P(\Gamma(t)) = 1 \right\}.$$

Then the set S_Σ *is nonempty and sequentially closed in* $\mathcal{Y}_{\text{dis}}(I, \lambda, U)$*.*

Now, for each $\mu \in S_\Sigma$, consider the following delay perturbed sweeping process:

$$(DSP(\mu)) \qquad \begin{cases} -\dot{x}(t) \in N(C(t), x(t)) + \int_U g(t, x_t(\cdot), u)\mu_t(du) \text{ a.e. } t \in [0, T] \\ x(s) = \varphi(s) \;\forall s \in [-\rho, 0]. \end{cases}$$

According to Remark 2 (1), the map

$$h_\mu(t,\varphi) := \int_U g(t,\varphi,u)\mu_t(du)$$

is separately scalarly λ-measurable on I and thus separately measurable (see [12]). Moreover, thanks to the assumptions on g, we have

- for every $\eta > 0$, for all $t \in I$ and for all $\varphi_1, \varphi_2 \in \eta B_0$,

$$\|h_\mu(t,\varphi_1) - h_\mu(t,\varphi_2)\| \leqslant k_\eta(t)\|\varphi_1 - \varphi_2\|;$$

- for all $(t,\varphi) \in I \times \mathcal{C}_0$,

$$\|h_\mu(t,\varphi)\| \leqslant \beta(t)(1 + \|\varphi\|_{\mathcal{C}_0}). \tag{5}$$

Consequently, by Theorem 1, for any $\mu \in S_\Sigma$, the delay perturbed sweeping process $(DSP(\mu))$ has one and only one solution, which will be denoted by $x^\mu(\cdot)$.

Therefore, we may consider the following optimal control problem, which is called the relaxed problem:

$$(R.P) \qquad \inf_{\mu \in S_\Sigma} L(x^\mu(T)) + \int_0^T \int_U J(t, x^\mu(t), u)\mu_t(du)dt.$$

7 Existence of Solutions for the Relaxed Problem

In this section we prove that the problem $(R.P)$ has a solution, that is, the infimum is attained.

From now on, we assume that the functions J and L satisfy the following assumptions:

(A_7) for each $t \in I$, the partial function $J(t,\cdot,\cdot)$ is continuous on $H \times U$;
(A_8) for any sequence $(x_n(\cdot))$ bounded in $(C([0,T],H), \|\cdot\|_\infty)$ and for any sequence $(\zeta_n(\cdot))$ in S_Γ, the sequence $(J(\cdot, x_n(\cdot), \zeta_n(\cdot)))$ is *uniformly integrable* in $L^1([0,T],\mathbb{R})$;
(A_9) the function L is continuous on H.

This uniform integrability assumption entails in particular that, for any $\zeta \in S_\Gamma$, the function $J(\cdot, x_\zeta(\cdot), \zeta(\cdot))$ is λ-integrable on $[0,T]$ and hence the integral $\int_0^T J(t, x_\zeta(t), \zeta(t))dt$ appearing in the problem $(O.P)$ is well defined in $\mathbb{R}$.

We are going to prove that, for any $\mu \in S_\Sigma$, the integral involved in the problem $(R.P)$ is also well defined. To do this, we will use the following lemma which is proved in Edmond and Thibault [14].

Lemma 1 *Let $h_n(\cdot), h_\infty(\cdot)\colon I \to H$ $(n \geqslant 1)$ be λ-measurable maps and let $\nu^n, \nu^\infty \in \mathcal{Y}(I,\lambda,U)$. Assume that (ν^n) converges to ν^∞ in $\mathcal{Y}(I,\lambda,U)$ and $(h_n(t))$ converges weakly in H to $h_\infty(t)$ for all $t \in I$. Let $\theta^n, \theta^\infty \in \mathcal{Y}(I,\lambda,H \times U)$ be defined by $\theta^n_t := \delta_{h_n(t)} \otimes \nu^n_t$ and $\theta^\infty_t := \delta_{h_\infty(t)} \otimes \nu^\infty_t$. Let $\Phi\colon I \times (H \times U) \to \mathbb{R}$ be an integrand such that, for any $t \in I$, $\Phi(t,\cdot,\cdot)$ is sequentially continuous on $H^w \times U$, where H^w denotes* *the space H endowed with the weak topology. Assume further that the measurable function* $t \mapsto \sup_{(n,u)\in(\mathbb{N}\cup\{\infty\})\times U} |\Phi(t, h_n(t), u)|$ *is* λ*-integrable on* I*. Then,*

$$\lim_{n\to\infty} \int_{I\times H\times U} \Phi \, d\theta^n = \int_{I\times H\times U} \Phi \, d\theta.$$

We will also need the following lemma which is a consequence of Gronwall's lemma.

Lemma 2 *Let* $I = [0, T]$ *and let* $(\eta_n(\cdot))$ *be a sequence of non-negative continuous functions defined on* I*,* (α_n) *a sequence of real numbers, and* $\beta(\cdot) \in L^1(I, \mathbb{R}^+)$*. Assume that* $\lim_n \alpha_n = 0$ *and, for all* n*,*

$$\eta_n(t) \leqslant \int_0^t \beta(s)\eta_n(s)\, ds + \alpha_n.$$

Then, for all $t \in [0, T]$*,*

$$\lim_n \eta_n(t) = 0.$$

Thanks to the two foregoing lemmas, we can prove the following result, which is interesting of its own.

Theorem 2 *Let* S_Σ *be endowed with the topology induced by the narrow topology of* $\mathcal{Y}(I, \lambda, U)$ *and* $\mathcal{C}([0, T], H)$ *be equipped with the uniform convergence topology. Under the assumptions* (A_1)*-*(A_6)*, the map which associates with each* $\mu \in S_\Sigma$ *the unique solution* $x^\mu(\cdot)$ *of the delay perturbed sweeping process* $(DSP(\mu))$ *is sequentially continuous.*

Proof We are going to adapt the proof of Proposition 7 in Edmond and Thibault [14]. Let us fix $\mu \in S_\Sigma$. Let (μ^n) be a sequence in S_Σ converging in $\mathcal{Y}([0, T], \lambda, U)$ to μ. We are going to prove that the sequence $(x^{\mu^n}(\cdot))$ converges uniformly in $\mathcal{C}([0, T], H)$ to $x^\mu(\cdot)$.

We recall that, for each n, $x^{\mu^n}(\cdot)$ denotes the unique solution of the delay perturbed sweeping process

$$\begin{cases} -\dot{x}(t) \in N(C(t), x(t)) + \int_U g(t, x_t(\cdot), u)\mu_t^n(du) \text{ a.e. } t \in [0, T] \\ x(s) = \varphi(s) \ \forall s \in [-\rho, 0]. \end{cases} \tag{6}$$

Let us set, for each $(t, \varphi) \in [0, T] \times \mathcal{C}_0$,

$$h_n(t, \varphi) := \int_U g(t, \varphi, u)\mu_t^n(du)$$

and

$$M := \|\varphi\|_{\mathcal{C}_0} + \exp\left\{2\int_0^T \beta(s)ds\right\} \int_0^T \left[2\beta(s)(1 + \|\varphi\|_{\mathcal{C}_0}) + |\dot{v}(s)|\right] ds.$$

Theorem 1 ensures that, for almost all $t \in [0, T]$,

$$\|\dot{x}^{\mu^n}(t) + h_n\left(t, x_t^{\mu^n}(\cdot)\right)\| \leqslant \alpha(t) := (1+M)\beta(t) + |\dot{v}(t)|, \tag{7}$$

$$\|h_n\left(t, x_t^{\mu^n}(\cdot)\right)\| \leqslant (1+M)\beta(t), \tag{8}$$

and then

$$\|\dot{x}^{\mu^n}(t))\| \leqslant 2(1+M)\beta(t) + |\dot{v}(t)| \text{ a.e. } t \in [0, T]. \tag{9}$$

Recall also that $x^\mu(\cdot)$ is the unique solution of the delay perturbed sweeping process

$$\begin{cases} -\dot{x}(t) \in N(C(t), x(t)) + \int_U g(t, x_t(\cdot), u)\mu_t(du) \text{ a.e. } t \in [0, T] \\ x(s) = \varphi(s) \ \forall s \in [-\rho, 0]. \end{cases} \tag{10}$$

To show that $(x^{\mu^n}(\cdot))$ converges uniformly in $\mathcal{C}([0, T], H)$ to $x^\mu(\cdot)$, we will prove that any subsequence of $(x^{\mu^n}(\cdot))$ has a subsequence converging uniformly in $\mathcal{C}([0, T], H)$ to $x^\mu(\cdot)$.

Let us fix a subsequence of $(x^{\mu^n}(\cdot))$, still denoted by $(x^{\mu^n}(\cdot))$. Taking Eq. 9 into account, we may suppose, without loss of generality, that the corresponding subsequence $(\dot{x}^{\mu^n}(\cdot))$ converges weakly in $L^1([0, T], H)$ to some map $a(\cdot) \in L^1([0, T], H)$. It follows that, for any $t \in [0, T]$,

$$\int_0^t \dot{x}^{\mu^n}(s)\, ds \to \int_0^t a(s)\, ds \text{ weakly in } H.$$

Considering the map $w(\cdot) \in \mathcal{C}([0, T], H)$ defined by

$$w(t) := \varphi(0) + \int_0^t a(s)\, ds,$$

one has, for all $t \in [0, T]$,

$$x^{\mu^n}(t) \to w(t) \text{ weakly in } H.$$

We are going to prove that the subsequence $(x^{\mu^n}(\cdot))$ converges uniformly to $x^\mu(\cdot)$. Applying Theorem 1 to Eq. 10, we have, with

$$h(t, \varphi) := \int_U g(t, \varphi, u)\mu_t(du),$$

for almost all $t \in [0, T]$,

$$\|\dot{x}^\mu(t) + h\left(t, x_t^\mu(\cdot)\right)\| \leqslant \alpha(t) = (1+M)\beta(t) + |\dot{v}(t)|$$

and

$$\|h(t, x_t^\mu(\cdot))\| \leqslant (1+M)\beta(t). \tag{11}$$

Thanks to the hypomonotonicity of the normal cone, one has, for all n and for almost all $t \in [0, T]$,

$$\left\langle \dot{x}^{\mu^n}(t) + h_n\left(t, x_t^{\mu^n}(\cdot)\right) - \dot{x}^\mu(t) - h\left(t, x_t^\mu(\cdot)\right), x^{\mu^n}(t) - x^\mu(t) \right\rangle$$

$$\leqslant \frac{\alpha(t)}{r}\|x^{\mu^n}(t) - x^\mu(t)\|^2$$

and then

$$\left\langle \dot{x}^{\mu^n}(t) - \dot{x}^{\mu}(t), x^{\mu^n}(t) - x^{\mu}(t) \right\rangle \leqslant \frac{\alpha(t)}{r} \|x^{\mu^n}(t) - x^{\mu}(t)\|^2 + \\ + \left\langle h_n\left(t, x_t^{\mu^n}(\cdot)\right) - h\left(t, x_t^{\mu}(\cdot)\right), x^{\mu}(t) - x^{\mu^n}(t) \right\rangle.$$

It results that

$$\frac{d}{dt}\left(\|x^{\mu^n}(t) - x^{\mu}(t)\|^2\right) \leqslant \frac{2\alpha(t)}{r} \|x^{\mu^n}(t) - x^{\mu}(t)\|^2 + \\ + 2\left\langle h_n\left(t, x_t^{\mu^n}(\cdot)\right) - h\left(t, x_t^{\mu}(\cdot)\right), x^{\mu}(t) - x^{\mu^n}(t) \right\rangle.$$

Let us write

$$\left\langle h_n\left(t, x_t^{\mu^n}(\cdot)\right) - h\left(t, x_t^{\mu}(\cdot)\right), x^{\mu}(t) - x^{\mu^n}(t) \right\rangle \\ = \left\langle h_n\left(t, x_t^{\mu^n}(\cdot)\right) - h_n\left(t, x_t^{\mu}(\cdot)\right), x^{\mu}(t) - x^{\mu^n}(t) \right\rangle + \\ + \left\langle h_n\left(t, x_t^{\mu}(\cdot)\right) - h\left(t, x_t^{\mu}(\cdot)\right), x^{\mu}(t) - x^{\mu^n}(t) \right\rangle.$$

Owing to Eq. 9 and the fact that $x^{\mu}(\cdot)$ belongs to $\mathcal{C}([0, T], H)$, there exists some $\eta > 0$ such that, for all n and for all $t \in [0, T]$,

$$x_t^{\mu^n}(\cdot), x_t^{\mu}(\cdot) \in \eta B_0.$$

Taking the assumptions on g into account, there exists a non-negative function $k(\cdot) \in L^1([0, T], \mathbb{R})$ such that, for any n and for any $t \in [0, T]$, $h_n(t, \cdot)$ is $k(t)$-Lipschitz on ηB_0. It follows that, for all n and for almost all $t \in [0, T]$,

$$\frac{d}{dt}\left(\|x^{\mu^n}(t) - x^{\mu}(t)\|^2\right) \leqslant \gamma(t) \|x^{\mu^n}(\cdot) - x^{\mu}(\cdot)\|^2_{\mathcal{C}_H([0,t])} + 2\gamma_n(t), \tag{12}$$

where

$$\gamma(t) := 2\left(\frac{\alpha(t)}{r} + k(t)\right)$$

and

$$\gamma_n(t) := \left\langle h_n\left(t, x_t^{\mu}(\cdot)\right) - h\left(t, x_t^{\mu}(\cdot)\right), x^{\mu}(t) - x^{\mu^n}(t) \right\rangle. \tag{13}$$

In the following we use the fact that the map $t \mapsto \|x^{\mu^n}(\cdot) - x^{\mu}(\cdot)\|_{\mathcal{C}_H([0,t])}$ is continuous. Integrating Eq. 12 on $[0, t]$, it follows that

$$\|x^{\mu^n}(t) - x^{\mu}(t)\|^2 \leqslant \int_0^t \gamma(s) \|x^{\mu^n}(\cdot) - x^{\mu}(\cdot)\|^2_{\mathcal{C}_H([0,s])} ds + 2\int_0^t \gamma_n(s) ds.$$

The last inequality being true for all $t \in [0, T]$, we deduce that, for any $t \in [0, T]$,

$$\|x^{\mu^n}(\cdot) - x^{\mu}(\cdot)\|^2_{\mathcal{C}_H([0,t])} \leqslant \int_0^t \gamma(s) \|x^{\mu^n}(\cdot) - x^{\mu}(\cdot)\|^2_{\mathcal{C}_H([0,s])} ds + 2\int_0^{\tau_n} \gamma_n(s) ds, \tag{14}$$

where $\tau_n \in [0, T]$ is such that

$$\int_0^{\tau_n} \gamma_n(s) ds = \sup_{\tau \in [0, T]} \int_0^{\tau} \gamma_n(s) ds.$$

We are going to prove that

$$\lim_n \int_0^{\tau_n} \gamma_n(s) ds = 0.$$

We may suppose without loss of generality that τ_n converges to some $\tau \in [0, T]$. Note that, by Eq. 5, for all n and for all $t \in [0, T]$,

$$|\gamma_n(t)| \leqslant 4\eta(1+\eta)\beta(t). \tag{15}$$

Writing

$$\int_0^{\tau_n} \gamma_n(s) ds = \int_0^{\tau_n} \gamma_n(s) ds - \int_0^{\tau} \gamma_n(s) ds + \int_0^{\tau} \gamma_n(s) ds,$$

we deduce that

$$\left| \int_0^{\tau_n} \gamma_n(s) ds \right| \leqslant 4\eta(1+\eta) \left| \int_\tau^{\tau_n} \beta(s) ds \right| + \left| \int_0^{\tau} \gamma_n(s) ds \right|.$$

Straightforwardly, $\lim_n \int_\tau^{\tau_n} \beta(s) ds = 0$. It remains to prove that

$$\lim_n \int_0^{\tau} \gamma_n(t)\, dt = 0.$$

According to the definitions of h_n and h, we have

$$\begin{aligned} \int_0^{\tau} \gamma_n(t)\, dt \; &= \int_0^{\tau} \int_U \left\langle g\left(t, x_t^{\mu}(\cdot), u\right), x^{\mu}(t) - x^{\mu^n}(t) \right\rangle \mu_t^n(du) dt - \\ &- \int_0^{\tau} \int_U \left\langle g\left(t, x_t^{\mu}(\cdot), u\right), x^{\mu}(t) - x^{\mu^n}(t) \right\rangle \mu_t(du) dt. \end{aligned}$$

Let us define, for $(t, x, u) \in [0, T] \times H \times U$,

$$\Phi(t, x, u) := \left\langle g\left(t, x_t^{\mu}(\cdot), u\right), x^{\mu}(t) - x \right\rangle \mathbb{1}_{[0,\tau]}(t).$$

Recalling that H^w denotes the space H endowed with the weak topology, it is obvious that, for any $t \in [0, T]$, the function $\Phi(t, \cdot, \cdot)$ is sequentially continuous on $H^w \times U$. Moreover, setting $x^{\mu^\infty}(\cdot) := w(\cdot)$, one has, for all $(t, n, u) \in [0, T] \times (\mathbb{N} \cup \{\infty\}) \times U$,

$$\left| \Phi\left(t, x^{\mu^n}(t), u\right) \right| \leqslant 2\eta(1+\eta)\beta(t). \tag{16}$$

Let us consider the Young measures $\theta^n, \rho^n, \theta \in \mathcal{Y}([0, T], \lambda, H \times U)$ defined by

$$\theta_t^n := \delta_{x^{\mu^n}(t)} \otimes \mu_t^n, \;\; \rho_t^n := \delta_{x^{\mu^n}(t)} \otimes \mu_t, \;\text{ and } \theta_t := \delta_{w(t)} \otimes \mu_t.$$

We can write

$$\int_0^{\tau} \gamma_n(t)\, dt = \int_{[0,T]\times H \times U} \Phi\, d\theta^n - \int_{[0,T]\times H \times U} \Phi\, d\rho^n,$$

and then Lemma 1 yields

$$\lim_{n\to\infty} \int_0^{\tau} \gamma_n(t)\, dt = \int_{[0,T]\times H \times U} \Phi\, d\theta - \int_{[0,T]\times H \times U} \Phi\, d\theta = 0.$$

Therefore,

$$\lim_n \int_0^{\tau_n} \gamma_n(s)ds = 0.$$

Finally, applying Lemma 2 to the inequality Eq. 14, we obtain that the subsequence $(x^{\mu^n}(\cdot))$ converges uniformly to $x^\mu(\cdot)$ in $\mathcal{C}([0, T], H)$. This ends the proof. □

Now, we can prove that the function $t \mapsto \int_U J(t, x^\mu(t), u)\mu_t(du)$ is λ-integrable. The proof is exactly the same as the part B of the proof of Proposition 7 in Edmond and Thibault [14], but we will write it for completeness.

Proposition 5 *Let $\mu \in S_\Sigma$. Then, under the assumptions (A_1)–(A_8), the function $t \mapsto \int_U J(t, x^\mu(t), u)\mu_t(du)$ belongs to $L^1([0, T], \mathbb{R})$. Moreover, for any sequence $(\zeta_n(\cdot))$ in S_Γ such that the sequence of the associated Young measures converges in $\mathcal{Y}(I, \lambda, U)$ to μ, one has*

$$\int_0^T \int_U J(t, x^\mu(t), u)\mu_t(du)dt = \lim_n \int_0^T J(t, x^{\zeta_n}(t), \zeta_n(t))dt.$$

Proof Let us fix $\mu \in S_\Sigma$. Fix any sequence $(\zeta_n(\cdot))$ in S_Γ such that the sequence of the associated Young measures converges in $\mathcal{Y}([0, T], \lambda, U)$ to μ. According to Theorem 2, the sequence $(x^{\zeta_n}(\cdot))$ converges uniformly in $\mathcal{C}([0, T], H)$ to $x^\mu(\cdot)$. For each n, let us define the Young measure $\theta^n \in \mathcal{Y}(I, \lambda, H \times U)$ and the map $u_n(\cdot) : [0, T] \to H \times U$ by

$$\theta_t^n := \delta_{x^{\zeta_n}(t)} \otimes \delta_{\zeta_n(t)} \text{ and } u_n(t) := (x^{\zeta_n}(t), \zeta_n(t)).$$

It is easily seen that $\theta_t^n = \delta_{u_n(t)}$. On the other hand, according to Proposition 1, the sequence (θ^n) converges in $\mathcal{Y}([0, T], \lambda, H \times U)$ to the Young measure θ defined by $\theta_t := \delta_{x^\mu(t)} \otimes \mu_t$.

As the sequence $(J(\cdot, u_n(\cdot)))$ is uniformly integrable by assumptions, we obtain, by Proposition 2, that J is θ-integrable and

$$\lim_n \int_0^T J(t, x^{\zeta_n}(t), \zeta_n(t))\, dt = \int_{[0,T]\times H\times U} J d\theta.$$

Now, taking Remark 2 into account, the function

$$t \mapsto \int_{H\times U} J(t, x, u)\delta_{x^\mu(t)} \otimes \mu_t(d(x, u)) = \int_U J(t, x^\mu(t), u)\mu_t(du)$$

is λ-integrable and satisfies

$$\int_{[0,T]\times H\times U} J d\theta = \int_0^T \int_U J(t, x^\mu(t), u)\mu_t(du)dt.$$

As a result,

$$\lim_n \int_0^T J(t, x^{\zeta_n}(t), \zeta_n(t))\, dt = \int_0^T \int_U J(t, x^\mu(t), u)\mu_t(du)dt,$$

which completes the proof. □

Now we are going to prove that the relaxed problem $(R.P)$ has a solution. This is the object of the theorem below.

First, let us recall the following density result in a form that directly follows from that in Castaing et al. [6] (see Proposition 3.2 in [6]).

Lemma 3 *Let $\mu \in S_\Sigma$. Then, there exists a sequence $(\zeta_n(\cdot))$ in S_Γ such that the sequence of the associated Young measures (μ^n), that is, $\mu_t^n := \delta_{\zeta_n(t)}$, converges in $\mathcal{Y}([0,T], \lambda, U)$ to μ.*

Theorem 3 *Under the assumptions (A_1)–(A_9), the control problem $(R.P)$ has an optimal solution. Furthermore, one has*

$$\min (R.P) = \inf (O.P).$$

Proof According to Lemma 3, for any $\mu \in S_\Sigma$, there exists a sequence $\zeta_n(\cdot) \in S_\Gamma$ such that the associated Young measures converges in $\mathcal{Y}(I, \lambda, U)$ to μ. Thus, Theorem 2 and Proposition 5 entail that

$$\lim_n L(x^{\zeta_n}(T)) + \int_0^T J(t, x^{\zeta_n}(t), \zeta_n(t))\, dt = L(x^\mu(T)) + \int_0^T \int_U J(t, x^\mu(t), u)\mu_t(du)dt.$$

As it is obvious that, for each n,

$$L(x^{\zeta_n}(T)) + \int_0^T J(t, x^{\zeta_n}(t), \zeta_n(t))\, dt \geqslant \inf (O.P),$$

one has

$$L(x^\mu(T)) + \int_0^T \int_U J(t, x^\mu(t), u)\mu_t(du)dt \geqslant \inf (O.P),$$

and thus

$$\inf (R.P) \geqslant \inf (O.P).$$

The reverse inequality being always true, it results that

$$\inf (R.P) = \inf (O.P).$$

Let us prove that $\inf (R.P)$ is a minimum. Let $(\zeta_n(\cdot))$ be a minimizing sequence of $(O.P)$, that is, $\zeta_n(\cdot) \in S_\Gamma$ for each n and

$$\inf (O.P) = \lim_n L(x^{\zeta_n}(T)) + \int_0^T J(t, x^{\zeta_n}(t), \zeta_n(t))dt. \tag{17}$$

Consider, for each n, the map $\nu^n \colon [0,T] \to \mathcal{M}_+^1(U)$ defined by $\nu_t^n := \delta_{\zeta_n(t)}$. Since $\nu^n \in S_\Sigma$ obviously, by Proposition 3 and Proposition 4, we may suppose, without loss of generality, that the sequence ν^n converges in $\mathcal{Y}_{\text{dis}}([0,T], \lambda, U)$ to some $\nu \in S_\Sigma$. Thanks to Proposition 5 and Theorem 2, one has

$$\lim_n L(x^{\zeta_n}(T)) + \int_0^T J(t, x^{\zeta_n}(t), \zeta_n(t))dt = L(x^\nu(T)) + \int_0^T \int_U J(t, x^\nu(t), u)\nu_t(du)dt,$$

Whence

$$\inf (O.P) = L(x^{\nu}(T)) + \int_0^T \int_U J(t, x^{\nu}(t), u)\nu_t(du)dt.$$

As, moreover, $\inf (R.P) = \inf (O.P)$, it follows that $\inf (R.P)$ is attained at ν, and

$$\min (R.P) = \inf (O.P).$$

The proof is then complete. □

8 Existence of Solutions for the Bolza Problem

This section gives an existence result for the Bolza problem $(O.P)$ under a convexity assumption. To be precise, we will suppose that the following holds:

(A_{10}) For any $t \in [0, T]$, for any $x \in H$, and for any $\varphi \in \mathcal{C}_H([-\rho, 0])$, the set

$$G(t, x, \varphi) := \{(J(t, x, u), g(t, \varphi, u)) : u \in \Gamma(t)\}$$

is convex.

This assumption is often used to prove existence results for classical optimal control problems. It is worth noting that convexity assumption is unavoidable to establish existence results for general optimal control problems. Indeed, there are examples (see Fleming and Rishel [15]) showing that, without an appropriate convexity assumption, those problems may have no solution.

Theorem 4 *Under the assumptions* (A_1)–(A_{10})*, the Bolza problem*

$$(O.P) \qquad \inf_{\zeta(\cdot)\in S_\Gamma} L(x^{\zeta}(T)) + \int_0^T J(t, x^{\zeta}(t), \zeta(t))dt,$$

where x^{ζ} *is the unique solution of the differential inclusion*

$$\begin{cases} -\dot{x}(t) \in N(C(t), x(t)) + g(t, x_t(\cdot), \zeta(t)) \text{ a.e. } t \in [0, T] \\ x(s) = \varphi(s) \ \forall s \in [-\rho, 0], \end{cases}$$

has an optimal solution.

Proof According to Theorem 3, we have, for some $\mu \in S_\Gamma$,

$$\inf (O.P) = L(x^{\mu}(T)) + \int_0^T \int_U J(t, x^{\mu}(t), u)\mu_t(du)dt, \tag{18}$$

where the map $x^{\mu}(\cdot)$ satisfies

$$\begin{cases} -\dot{x}^{\mu}(t) \in N(C(t), x^{\mu}(t)) + \int_U g\left(t, x_t^{\mu}(\cdot), u\right) \mu_t(du) \text{ a.e. } t \in [0, T] \\ x^{\mu}(s) = \varphi(s) \ \forall s \in [-\rho, 0]. \end{cases}$$

The assumption (A_{10}) ensures that, for any $t \in [0, T]$,

$$\left(\int_U J(t, x^{\mu}(t), u)\mu_t(du), \int_U g\left(t, x_t^{\mu}(\cdot), u\right) \mu_t(du)\right) \in G\left(t, x^{\mu}(t), x_t^{\mu}(\cdot)\right). \tag{19}$$

Let us set

$$\psi(t) := \left(\int_U J(t, x^\mu(t), u)\mu_t(du), \int_U g\left(t, x_t^\mu(\cdot), u\right)\mu_t(du)\right)$$

and consider the set-valued map Ψ: $[0, T] \rightrightarrows U$ defined by

$$\Psi(t) := \left\{u \in U : u \in \Gamma(t) \text{ and } \psi(t) = \left(J(t, x^\mu(t), u), g\left(t, x_t^\mu(\cdot), u\right)\right)\right\}.$$

Thanks to Eq. 19, the set $\Psi(t)$ is nonempty for every $t \in [0, T]$. Moreover, Ψ is closed-valued and it is not difficult to prove that its graph is $\mathcal{L}(I) \otimes \mathcal{B}(U)$-measurable. Consequently, Ψ is measurable and thus has at least one measurable selection (see Castaing and Valadier [7]). In other words, there exists a measurable map ξ: $[0, T] \to U$ such that $\xi(t) \in \Psi(t)$ for almost all $t \in [0, T]$. Obviously $\xi \in S_\Gamma$,

$$\inf(O.P) = L(x^\mu(T)) + \int_0^T J(t, x^\mu(t), \xi(t))dt,$$

and

$$\begin{cases} -\dot{x}^\mu(t) \in N(C(t), x^\mu(t)) + g\left(t, x_t^\mu(\cdot), \xi(t)\right) \text{ a.e. } t \in [0, T] \\ x^\mu(s) = \varphi(s) \; \forall s \in [-\rho, 0]. \end{cases}$$

Since the above differential inclusion has, for ξ fixed in S_Γ, a unique solution $x^\xi(\cdot)$, we conclude that $x^\mu = x^\xi$. This implies that the infimum of the problem $(O.P)$ is attained at ξ, which ends the proof. □

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