Covering by squares

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Abstract

In this paper we introduce the "do not touch" condition for squares in the discrete plane. We say that two squares "do not touch" if they do not share any vertex or any segment of an edge. Using this condition we define a covering of the discrete plane, the covering can be strong or weak, regular or non-regular. For simplicity, in this article, we will restrict our attention to regular coverings, i.e., only a size is allowed for the squares and all the squares have the same number of adjacent squares. We establish minimal conditions for the existence of a weak or strong regular covering of the discrete plane, and we give a bound for the number of adjacent squares with respect to the size of the squares in the regular covering.

Keywords: Discrete geometry; Covering; Tiling

1. Introduction

The tiling problem was introduced by Hao Wang in [5] and since then, many different versions of it have been studied (Tiling with poliominoes [4], rotation tiling [2,3]). In the tiling problem the basic idea is to cover the plane or a part of the plane without any overlap or gap, using tiles that can have different forms, also two tiles in a tiling are adjacent if they share an edge. In this work we are interested in a related problem, in this case, the tiles are squares and the overlap of tiles is allowed, in fact we say that two squares are adjacent in the covering if they overlap and do not share any vertex or segment of an edge. Also we study "weak coverings", this is an infinite set of squares, where all the squares are connected and every point in the plane is "near" a square, i.e., in this type of covering we allow overlaps and gaps, but the size of the gap is bounded.

In Section 2, we introduce the "do not touch" condition and set-up notation. In Section 3, we discuss the cardinality of a finite set of squares that we call overlap, where the squares do not touch each other and the intersection is pairwise not empty. We prove that the cardinality of an overlap depends on the size of the smallest square in the overlap, also

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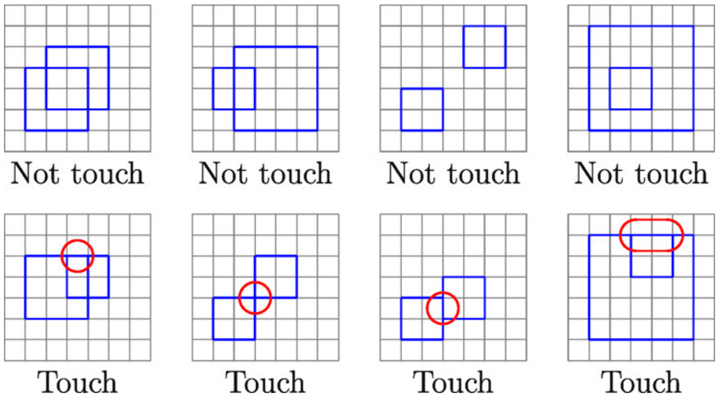

Fig. 1. Examples of squares touching and not touching another square.

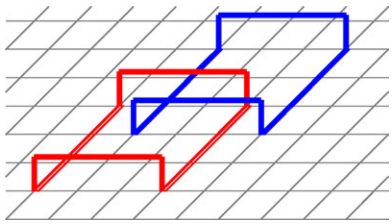
Fig. 2. Representation of a square with horizontal edges in an upper plane.

we prove that this dependence is $O(n)$ (where n is the size of the smallest square), but if we do not use the "do not touch" condition this dependence is $O(n^2)$.

Section 4 is devoted to the study of coverings of the discrete plane. We give minimal conditions of a strong covering and a weak covering to exist, these conditions are related with the size of squares and the number of adjacent squares. Also, we establish the maximum intersection number for a regular covering with respect to the size of the squares.

2. Notation and definitions

We denote by $S = \begin{bmatrix} i \\ j \end{bmatrix}_n$, the square $S \subseteq \mathbb{Z}^2$ with lower left vertex in $\binom{i}{j}$ and width n.

$$S = \begin{bmatrix} i \\ j \end{bmatrix}_n = \left\{ \begin{pmatrix} i+l \\ j+m \end{pmatrix} \in \mathbb{Z}^2 : l, m = 0, \ldots, n \right\}.$$

For a given square $S = \begin{bmatrix} i \\ j \end{bmatrix}_n$, we call:

- Interior points of S the elements of $\mathrm{Int}(S) = \left\{ \binom{i+l}{j+m} : l, m = 1, \ldots, n-1 \right\}$,
- Frontier points of S the elements of $\mathrm{Fr}(S) = S \setminus \mathrm{Int}(S)$,
- Vertices of S the elements of $\mathrm{Ver}(S) = \left\{ \binom{i}{j}, \binom{i+n}{j}, \binom{i}{j+n}, \binom{i+n}{j+n} \right\}$,
- $\mathrm{Fr}_*(S) = \mathrm{Fr}(S) \setminus \mathrm{Ver}(S)$.

Definition 1. Let S and S' be two squares in $\mathbb{Z}^2$, S *touches* S' if either $\mathrm{Ver}(S) \cap \mathrm{Fr}(S') \neq \emptyset$ or $\mathrm{Ver}(S') \cap \mathrm{Fr}(S) \neq \emptyset$, i.e. there is a square that has a vertex in the frontier of the other square.

As seen in Fig. 1 two squares touch if they share any vertex or segment of an edge. Also, Fig. 2 shows two squares with horizontal edges in an upper plane than vertical edges, we observe that if these two squares do not touch each other, the represented squares hold the "do not touch" condition that we have introduced in this article.

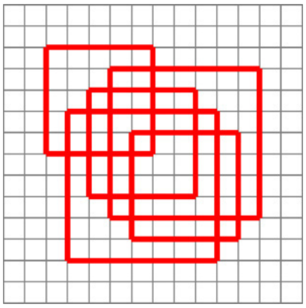

Fig. 3. Example of overlap.

Definition 2. An *overlap of squares* $\mathcal{O}$ is a set of squares in the discrete plane satisfying:

(1) $\forall S, S' \in \mathcal{O}, S \nsubseteq S'$.
(2) $\forall S, S' \in \mathcal{O}$, S does not touch S'.
(3) $\forall S, S' \in \mathcal{O}, S \cap S' \neq \emptyset$.

Definition 3. We call a set of squares $\mathcal{C}$, a *strong covering* such that:

(1) $\forall S, S' \in \mathcal{C}, S \nsubseteq S'$.
(2) $\forall S, S' \in \mathcal{C}$, S does not touch S'.
(3) $\forall \binom{i}{j} \in \mathbb{Z}^2, \exists S \in \mathcal{C}$, such that $\binom{i}{j} \in \mathrm{Int}(S)$.

Definition 4. We call a set of squares $\mathcal{C}^*$, a *k-weak covering* ($k \in \mathbb{N}$) such that:

(1) $\forall S, S' \in \mathcal{C}^*, S \nsubseteq S'$.
(2) $\forall S, S' \in \mathcal{C}^*$, S does not touch S'.
(3) $\forall \binom{i}{j} \in \mathbb{Z}^2, \exists S \in \mathcal{C}^*$ such that $\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]_{2k} \cap \mathrm{Int}(S) \neq \emptyset$.
(4) All squares are connected, i.e. $\forall S_0, S_m \in \mathcal{C}^*, \exists S_1 \dots S_{m-1} \in \mathcal{C}^*$, such that $S_l \cap S_{l-1} \neq \emptyset, \forall l = 1, \dots, m$.

Notice that the squares in a strong covering are connected, and hence, a strong covering is a weak covering too. In fact a strong covering is a 0-weak covering. Also, for $k > k'$ if $\mathcal{C}$ is a k'-weak covering then $\mathcal{C}$ is a k-weak covering.

Definition 5. The *intersection number* of a square S in a covering (strong or weak) is the number of squares in the covering that intersect S.

Definition 6. The *intersection number of a covering* (strong or weak) is the maximum intersection number of the squares in the covering. If the maximum does not exist, we say that the intersection number of the covering is infinity.

Definition 7. A *regular strong covering* (*regular weak covering* resp.) is a strong covering (weak covering resp.) where every square has the same width and the same intersection number.

Notice that a periodic set of squares in the plane can be described by two vectors $\binom{x_1}{y_1}, \binom{x_2}{y_2} \in \mathbb{Z}^2$ and a finite set of squares $S_l, l = 1, \dots, k$ such that $S_l \cap P\left(\binom{x_1}{y_1}, \binom{x_2}{y_2}\right) \neq \emptyset$, where $P\left(\binom{x_1}{y_1}, \binom{x_2}{y_2}\right)$ is a parallelogram generated by the vectors $\binom{x_1}{y_1}$ and $\binom{x_2}{y_2}$. This representation is not unique.

We say $\Pi_{\binom{x_1}{y_1}, \binom{x_2}{y_2}}\binom{a}{b}$ is the projection of $\binom{a}{b}$ on $P\left(\binom{x_1}{y_1}, \binom{x_2}{y_2}\right)$, if:

$$\Pi_{\binom{x_1}{y_1}, \binom{x_2}{y_2}}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} - \left(\lfloor \alpha \rfloor \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \lfloor \beta \rfloor \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right)$$

where $\binom{a}{b} = \alpha\binom{x_1}{y_1} + \beta\binom{x_2}{y_2}$.

The sides of the squares S_l in the parallelogram are the sides of S_l and their projections.

We define a set $A \subseteq \mathbb{Z}^2$, such that $\forall \binom{i}{j} \in A$, $\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]_n \cap P\left(\binom{x_1}{y_1}, \binom{x_2}{y_2}\right) \neq \emptyset$,

$$\left[A; \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right]_n = \left\{ \begin{bmatrix} i_* \\ j_* \end{bmatrix}_n : \exists \begin{pmatrix} i \\ j \end{pmatrix} \in A, \Pi_{\binom{x_1}{y_1}, \binom{x_2}{y_2}} \begin{pmatrix} i_* \\ j_* \end{pmatrix} = \Pi_{\binom{x_1}{y_1}, \binom{x_2}{y_2}} \begin{pmatrix} i \\ j \end{pmatrix} \right\}.$$

The set A is called the generator set of the covering.

3. Cardinality of an overlap

In this section we prove that the maximum cardinality of an overlap is lineal with respect to the size of the smallest square. It is easy to see that if we do not consider the "do not touch" condition this cardinality is $(n + 1)^2$ where n is the size of the smallest square. An example is the set $\left\{ \left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]_n : i, j = 0, \ldots, n \right\}$.

Fact 8. *Given S and S', two squares in $\mathbb{Z}^2$ such that $S \cap S' \neq \emptyset$ and S does not touch S' then* $|\operatorname{Fr}_*(S) \cap \operatorname{Fr}_*(S')| = 2$.

Proof. Without loss of generality, let $S = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]_n$ and $S' = \left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]_{n'}$ where $0 < i < n$. Thus, we consider two possibilities:

(1) $\binom{i}{j} \in \operatorname{Int}(S) \Longrightarrow 0 < i, j < n$ and $n' > \max\{n - i, n - j\}$. Hence

$$\operatorname{Fr}_*(S) \cap \operatorname{Fr}_*(S') = \left\{ \begin{pmatrix} i \\ n \end{pmatrix}, \begin{pmatrix} n \\ j \end{pmatrix} \right\}$$

(2) $\binom{i}{j} \notin \operatorname{Int}(S) \Longrightarrow 0 < i < n$ and $-n' < j < 0$. Hence,

$$\operatorname{Fr}(S)_* \cap \operatorname{Fr}_*(S') = \begin{cases} \left\{ \begin{pmatrix} i \\ 0 \end{pmatrix}, \begin{pmatrix} i \\ n \end{pmatrix} \right\} & \text{if } j + n' > n \\ \left\{ \begin{pmatrix} i \\ 0 \end{pmatrix}, \begin{pmatrix} n \\ j + n' \end{pmatrix} \right\} & \text{otherwise.} \quad \square \end{cases}$$

The main result of this section is Proposition 9, to prove this proposition we use Fact 8 which gives a bound for the cardinality of the overlap.

Proposition 9. *The maximum cardinality of an overlap of squares $\mathcal{O}$ containing a given square S_0 of width n is $2n - 1$.*

Proof. If S and S_0 are two squares in the same overlap, then by Fact 8, the intersection between their frontiers has two points. These points do not belong to the frontier of another square S' because, in this case, S' touches either S or S_0. Since $|\operatorname{Fr}_*(S_0)| = 4(n - 1)$, there exists a maximum of $2(n - 1)$ squares intersecting S_0. Now we show a construction which reaches this bound.

Without loss of generality we consider $S_0 = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]_n$, and the other elements of $\mathcal{O}$ are:

$$S_i = \begin{bmatrix} -i \\ i \end{bmatrix}_{n_i}, \qquad S'_i = \begin{bmatrix} i \\ -i \end{bmatrix}_{n_i}, \qquad i = 1, \ldots, n - 1$$

where $n_1 = n + 2$ and,

$$n_i = \min\{n_* > n_{i-1} : S_i \text{ do not touch the squares } S_j \text{ or } S'_j,\ j = 1, \ldots, i - 1\}.$$

It is easy to verify that the squares so defined satisfy the constraints of an overlap of squares. $\square$

Examples of overlaps of maximum cardinality are shown in Fig. 4.

4. Strong and weak coverings

As seen in the previous section, the cardinality of an overlap is finite, then it covers only a limited part of the discrete plane. Now, we treat some aspects of strong and weak coverings of the discrete plane. We will restrict our attention to regular coverings, but we will give a few comments about the non-regular case.

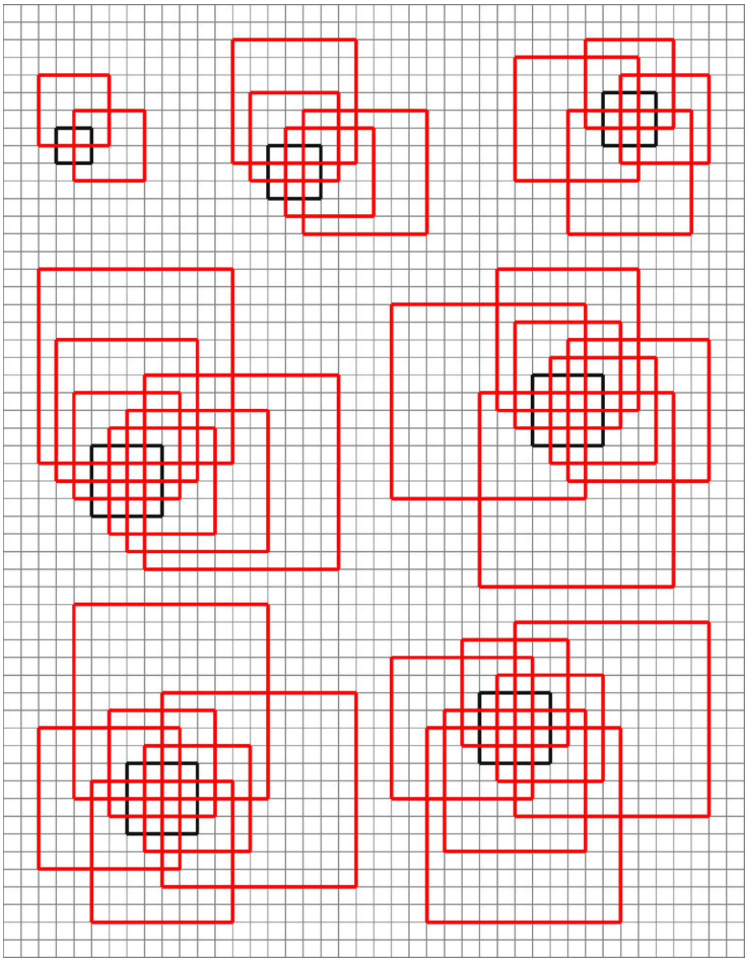

Fig. 4. Examples of overlaps of maximum cardinality.

4.1. Maximum intersection number in a covering

The main result of this section is Theorem 13, this theorem provides a bound for the intersection number of a regular covering with respect to the sizes of squares in the covering. In the non-regular case this bound does not exist because the sizes of squares in the covering are not bounded, and if we bound the sizes of squares, the maximum intersection number will be the maximum intersection number for the largest square in the covering. In first place, we give some previous results that we will use in the proof of the main result of this section.

Lemma 10. *Given a square S of width $n > 2$, let R_S be a set of squares of width n such that:*

(1) $\forall S' \in R_S$, $S \cap S' \neq \emptyset$,
(2) $\forall S' \in R_S$, *S' does not touch S,*
(3) $\forall S', S'' \in R_S$, *S' does not touch S''.*

Then the maximum cardinality of R_S is $2(n-1)$.

Proof. Notice that R_S is not an overlap because the squares in R_S do not intersect necessarily.

By Fact 8 every square in R_S intersects S in two points belonging to $\mathrm{Fr}_*(S)$, and $|\,\mathrm{Fr}_*(S)| = 4(n-1)$ then there is a maximum of $2(n-1)$ squares. If $S = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_n$, an example of R_S reaching the maximum cardinality is the set: $\begin{bmatrix} n-1 \\ n-1 \end{bmatrix}_n$, $\begin{bmatrix} -(n-1) \\ -(n-1) \end{bmatrix}_n$, $\begin{bmatrix} -n+k \\ 1+k \end{bmatrix}_n$, $k = 1, \ldots, n-2$ and $\begin{bmatrix} n-k \\ -(1+k) \end{bmatrix}_n$, $k = 1, \ldots, n-2$ (see Fig. 5). $\square$

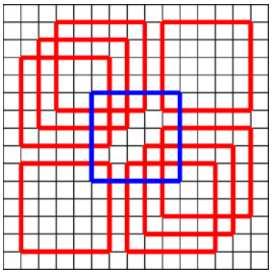

Fig. 5. R_S of maximum cardinality.

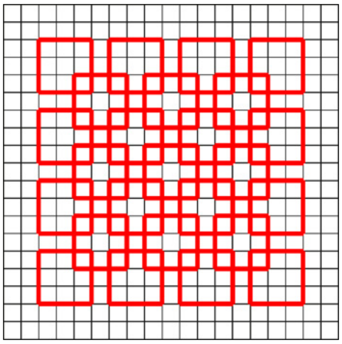

Fig. 6. 1-weak covering of width 3 and intersection number 4.

Fact 11. *Let* $S = \left[\binom{i}{j}\right]_n$ *be a square in a regular covering intersected by* $S_* = \left[\binom{i_*}{j_*}\right]_n$, *such that either* $|i - i_*| = 1$ *or* $|j - j_*| = 1$ *then the intersection number of* $\left[\binom{i}{j}\right]_n$ *is lesser than* $2(n-1)$.

Proof. To have the maximum intersection number for all, every line and every column must have two squares with a vertex on it. If either $|i - i_*| = 1$ or $|j - j_*| = 1$ there will be either a line or a column which can be used only by one square. $\square$

Fact 12. *There does not exist a regular covering of width even n with an intersection number* $2(n-1)$.

Proof. If we have a regular covering every square in the covering must intersect the same number of squares, then if the intersection number of the covering is $2(n-1)$ we cannot have two squares with the condition $|i - i_*| = 1$ or $|j - j_*| = 1$, then for every square in the covering the odd lines must contain upper vertices of the squares intersecting it and even lines have to contain lower vertices. Since n is even the line $n-1$ must have upper vertices, then this square holds the condition $|i - i_*| = 1$ or $|j - j_*| = 1$ and it is not possible to have an intersection number $2(n-1)$. $\square$

Theorem 13. *Let n be the size of the squares in a regular covering (strong or weak), then the intersection number of the covering is lesser than or equal to:* $2(n-1)$ *if n is odd and* $2(n-1)-1$ *if n is even.*

Proof. If *n* is odd. We will prove that the covering $R_{n,2(n-1)}$ is a regular covering with the intersection number $2(n-1)$. In the case $n = 3$, it is a 1-weak covering and for $n \geq 5$, it is a strong covering.

$$R_{n,2(n-1)} = \left[\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}; \begin{pmatrix}n+1\\0\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\right]_n.$$

For the case $n = 3$ it is enough to see the Fig. 6 to prove that it is a 1-weak covering with an intersection number four.

For the case $n \geq 5$, we see that every point in the plane is in the interior of a square. It is enough to see that it is true for the points in $P\left(\binom{n+1}{0}, \binom{2}{2}\right) = \left\{\binom{i}{j} / j = 0, 1 \text{ and } i = j, \ldots, j+n\right\}$.

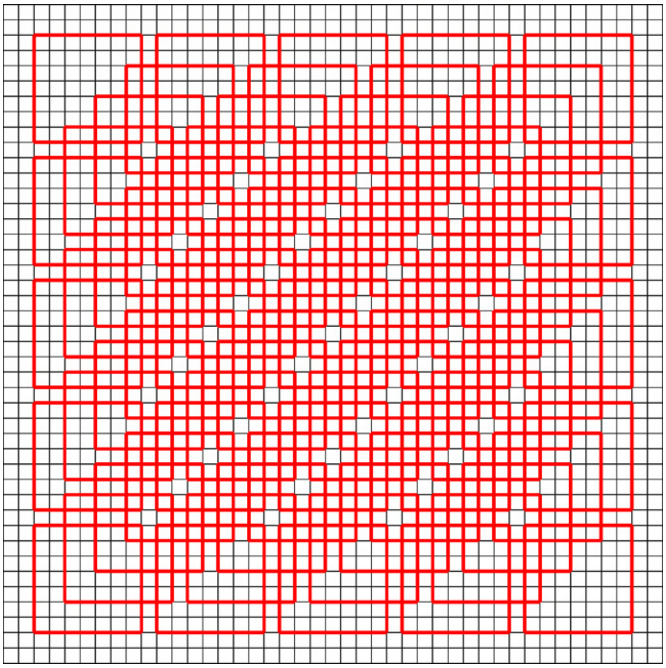

Fig. 7. Covering of maximum cardinality for n odd.

- If $j = 0$ and $0 \le i \le n-3 \Longrightarrow \binom{i}{j} \in \operatorname{Int}\begin{bmatrix}-2\\-2\end{bmatrix}_n$,
- If $j = 0$ and $n-2 \le i \le n \Longrightarrow \binom{i}{j} \in \operatorname{Int}\begin{bmatrix}n-3\\-4\end{bmatrix}_n$,
- if $j = 1$ and $1 \le i \le n-1 \Longrightarrow \binom{i}{j} \in \operatorname{Int}\begin{bmatrix}0\\0\end{bmatrix}_n$,
- if $j = 1$ and $n \le i \le n+1 \Longrightarrow \binom{i}{j} \in \operatorname{Int}\begin{bmatrix}n-1\\-2\end{bmatrix}_n$.

To prove that the intersection number of every square is $2(n-1)$ it is enough to prove that $\begin{bmatrix}0\\0\end{bmatrix}_n$ intersects $2(n-1)$ squares. If we consider the generator squares it is easy to see that the squares intersecting $\begin{bmatrix}0\\0\end{bmatrix}_n$ are:

$$\begin{bmatrix}2i-a\\2i-b\end{bmatrix}_n, \quad a, b \in \{0, n+1\} \text{ and } i = 1, \dots, \frac{n-1}{2}.$$

The vertices of squares intersecting $\begin{bmatrix}0\\0\end{bmatrix}_n$ are in the points:

$$\begin{pmatrix}2i-a\\2i-b\end{pmatrix}, \quad a, b \in \{0, 1\} \text{ and } i = 1, \dots, \frac{n-1}{2}$$

which are interior points of $\begin{bmatrix}0\\0\end{bmatrix}_n$. Hence, they do not touch $\begin{bmatrix}0\\0\end{bmatrix}_n$.

If *n* is even. We will show a construction with an intersection number $2(n-1)-1$, this construction is a 1-weak covering in the case $n = 4$ and a strong covering for $n \ge 6$. Let the set:

$$A = \left\{ \begin{pmatrix}2i+l(n-1)\\2i+l(n-1)\end{pmatrix} \Big/ i = 0, \dots, \frac{n}{2}-2, l = 0, 1, 2, 3 \right\} \cup \left\{ \begin{pmatrix}4n-5\\4n-5\end{pmatrix}, \begin{pmatrix}4n-4\\4n-4\end{pmatrix} \right\}$$

and the covering,

$$R_{n,2n-3} = \left[A; \begin{pmatrix}4n-2\\4n-2\end{pmatrix}, \begin{pmatrix}2n\\n-1\end{pmatrix} \right]_n.$$

We give the proof for the case $n \ge 6$ because for the case $n = 4$ it is straightforward from Fig. 8.

In the first place we observe that the squares of the generator set have their lower left corner on the diagonal $\binom{k}{k}$, $k \in \mathbb{Z}$, also the first vector defining the parallelogram is $\binom{4n-2}{4n-2}$ then we can say that the covering is a series of copies

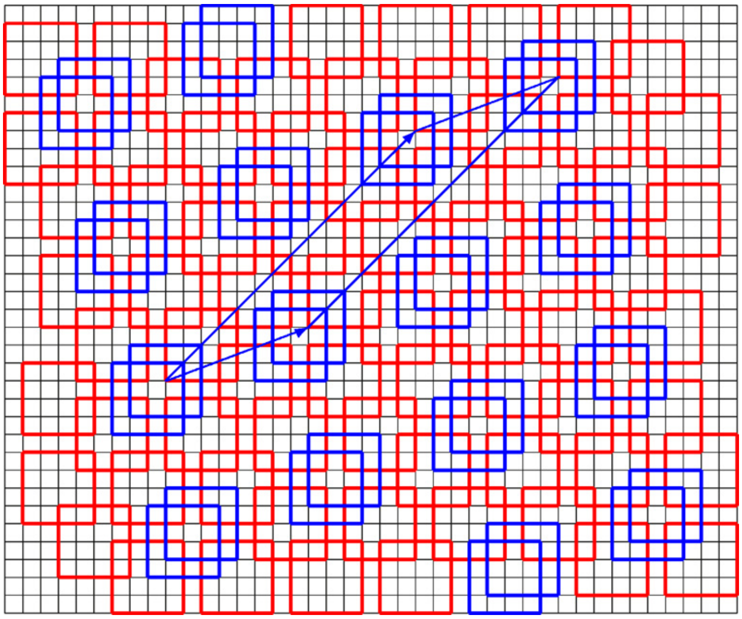

Fig. 8. 1-weak regular covering of width 4 and intersection number 5.

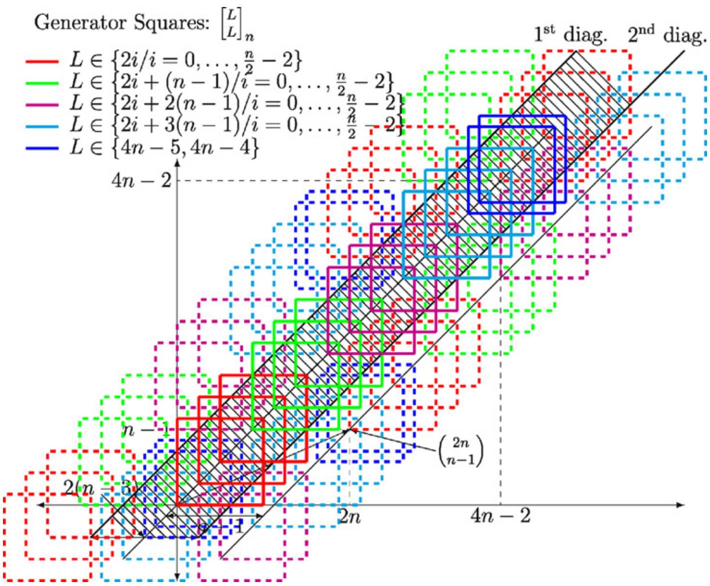

Fig. 9. Covering.

of the squares in the diagonal $\binom{k}{k}$, $k \in \mathbb{Z}$, then every square in the covering has its lower left corner in a point on a diagonal $\binom{k}{k} + c\binom{n+1}{0}$, for some k and c in $\mathbb{Z}$. In this way, we call first diagonal to all the squares with lower left corner in $\binom{k}{k}$, $k \in \mathbb{Z}$ and second diagonal the set of squares with lower left corner in $\binom{k}{k} + \binom{n+1}{0}$, $k \in \mathbb{Z}$.

To see that every point of the plane is covered by a square of the covering we observe that every diagonal covers a strip of wide $2(n-3)$ and the distance between two adjacent diagonals is $n+1$ (as seen in Fig. 9) then for $n > 7$ every point of the plane is covered. For the case $n = 6$ the proof is in Fig. 10.

To verify the intersection number and whether the squares in the covering touch another square, we study the squares in the generator set, because every square in the covering is a copy of one of them.

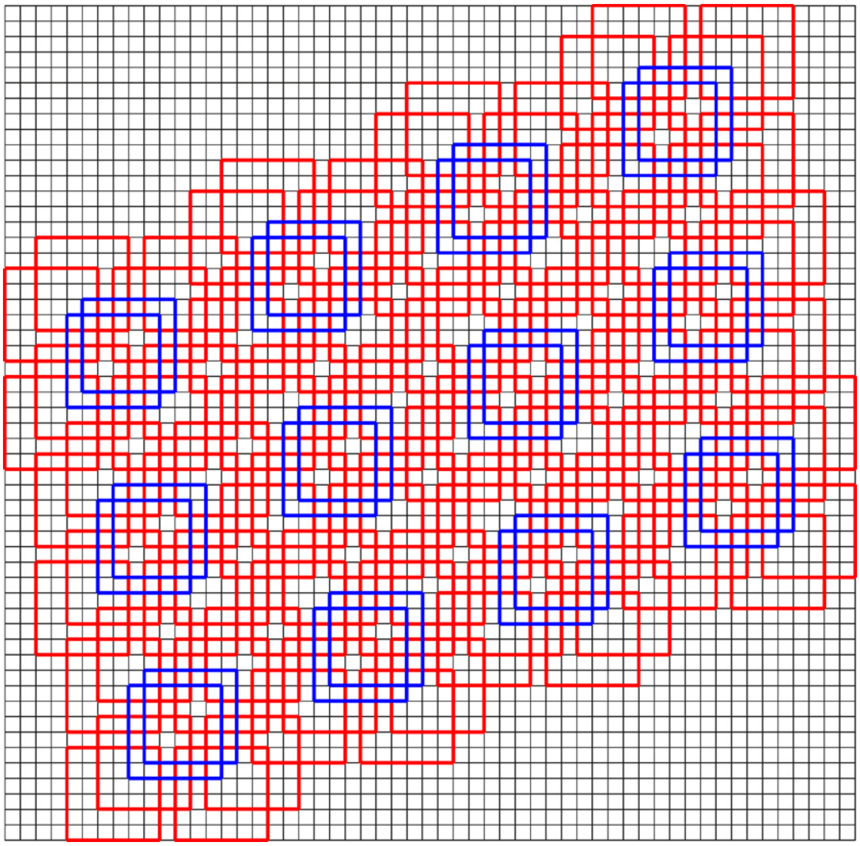

Fig. 10. Strong covering of width 6 and intersection number 9.

The projection on the parallelogram $P\left(\binom{4n-2}{4n-2}, \binom{2n}{n-1}\right)$ of the lower left corners of generator squares are:

$$\binom{0}{0}, \binom{2}{2}, \ldots, \binom{n-4}{n-4}$$
$$\binom{n-1}{n-1}, \binom{n+1}{n+1}, \ldots, \binom{2n-5}{2n-5}$$
$$\binom{2n-2}{2n-2}, \binom{2n}{2n}, \ldots, \binom{3n-6}{3n-6}$$
$$\binom{3n-3}{3n-3}, \binom{3n-1}{3n-1}, \ldots, \binom{4n-7}{4n-7}$$
$$\binom{4n-5}{4n-5}, \binom{4n-4}{4n-4}$$

and upper right corners are:

$$\binom{n}{n}, \binom{n+2}{n+2}, \ldots, \binom{2n-4}{2n-4}$$
$$\binom{2n-1}{2n-1}, \binom{2n+1}{2n+1}, \ldots, \binom{3n-5}{3n-5}$$
$$\binom{3n-2}{3n-2}, \binom{3n}{3n}, \ldots, \binom{4n-6}{4n-6}$$

$$\binom{4n-3}{4n-3}, \binom{1}{1}, \binom{3}{3}, \ldots, \binom{n-5}{n-5}$$
$$\binom{n-3}{n-3}, \binom{n-2}{n-2}.$$

We observe that all corners have different coordinates, then there are no squares touching in the same diagonal. We have seen that squares in the second diagonal have their lower left corner in $\binom{n+1}{0} + \binom{k}{k}$ then, if a square in the second diagonal touches a square in the first diagonal it implies $L - k \in \{1, n\}$, where $\begin{bmatrix} L \\ L \end{bmatrix}_n$ is a generator square. It is enough to prove $L-k \notin \{1, n\}$ for $\begin{bmatrix} L \\ L \end{bmatrix}_n$ a generator square, then the values of $k \in \mathbb{Z} \cap [-n, 4n-2[$ such that $\begin{bmatrix} n+1+k \\ k \end{bmatrix}_n$ belongs to the covering are:

$$\begin{aligned}
&n-1, n+1, \ldots, 2n-5\\
&2n-2, 2n, \ldots, 3n-6\\
&3n-3, 3n-1, \ldots, 4n-7\\
&-(n-1), -(n-3), \ldots, -5\\
&4n-4, -2, 0, \ldots, n-4\\
&n-4, n-3.
\end{aligned}$$

From these values we conclude that there are no squares touching each other.

Finally, we prove that every square intersects $2n-3$ squares in the covering. Given there are no squares touching we will study the intersection number of every square in the generator set. We classify the squares in the generator set into five groups:

(1) $S = \begin{bmatrix} 2i \\ 2i \end{bmatrix}_n$, $i = 0, \ldots, \frac{n}{2} - 2$.
(2) $S = \begin{bmatrix} 2i+(n-1) \\ 2i+(n-1) \end{bmatrix}_n$, $i = 0, \ldots, \frac{n}{2} - 2$.
(3) $S = \begin{bmatrix} 2i+2(n-1) \\ 2i+2(n-1) \end{bmatrix}_n$, $i = 0, \ldots, \frac{n}{2} - 2$.
(4) $S = \begin{bmatrix} 2i+3(n-1) \\ 2i+3(n-1) \end{bmatrix}_n$, $i = 0, \ldots, \frac{n}{2} - 2$.
(5) $S = \begin{bmatrix} 4n-5 \\ 4n-5 \end{bmatrix}_n$ and $S = \begin{bmatrix} 4n-4 \\ 4n-4 \end{bmatrix}_n$.

For every group we give a table that shows the squares intersecting these types of squares. The squares intersecting are described by their generator square, the vertex located in the interior of the intersected square (LL: Lower left, LR: Lower right, UL: Upper left and UR: Upper right) and the number of these types of squares intersecting.

- If $S = \begin{bmatrix} 2i \\ 2i \end{bmatrix}_n$, $i = 0, \ldots, \frac{n}{2} - 2$, then S is intersected by:

Generator square		**vertex**	**number**
$\begin{bmatrix} 2j \\ 2j \end{bmatrix}_n$,	$j = i+1, \ldots, \frac{n}{2} - 2$	LL	$\frac{n}{2}-i-2$
$\begin{bmatrix} 2j+(n-1) \\ 2j+(n-1) \end{bmatrix}_n$,	$j = i+1, \ldots, \frac{n}{2} - 2$	LR	$\frac{n}{2}-i-2$
$\begin{bmatrix} 2j \\ 2j \end{bmatrix}_n$,	$j = 0, \ldots, i-1$	UR	i
$\begin{bmatrix} 2j+3(n-1) \\ 2j+3(n-1) \end{bmatrix}_n$,	$j = 1, \ldots, i$	LR	i
$\begin{bmatrix} 2j+(n-1) \\ 2j+(n-1) \end{bmatrix}_n$,	$j = 0, \ldots, i$	LL	$i+1$
$\begin{bmatrix} 2j+2(n-1) \\ 2j+2(n-1) \end{bmatrix}_n$,	$j = 0, \ldots, i$	LR	$i+1$
$\begin{bmatrix} 2j+3(n-1) \\ 2j+3(n-1) \end{bmatrix}_n$,	$j = i+1, \ldots, \frac{n}{2} - 2$	UR	$\frac{n}{2}-i-2$
$\begin{bmatrix} 2j+2(n-1) \\ 2j+2(n-1) \end{bmatrix}_n$,	$j = i+1, \ldots, \frac{n}{2} - 2$	UL	$\frac{n}{2}-i-2$
$\begin{bmatrix} 4n-5 \\ 4n-5 \end{bmatrix}_n$, $\begin{bmatrix} 4n-4 \\ 4n-4 \end{bmatrix}_n$		UR	2
$\begin{bmatrix} 3n-3 \\ 3n-3 \end{bmatrix}_n$		UL	1
Total			$2n-3$

- If $S = \begin{bmatrix}2i+(n-1)\\2i+(n-1)\end{bmatrix}_n$, $i = 0, \dots, \frac{n}{2} - 2$, then S is intersected by:

Generator square	**vertex**	**number**
$\begin{bmatrix}2j+(n-1)\\2j+(n-1)\end{bmatrix}_n$, $j = i+1, \dots, \frac{n}{2} - 2$	LL	$\frac{n}{2}-i-2$
$\begin{bmatrix}2j+2(n-1)\\2j+2(n-1)\end{bmatrix}_n$, $j = i+1, \dots, \frac{n}{2} - 2$	LR	$\frac{n}{2}-i-2$
$\begin{bmatrix}2j+(n-1)\\2j+(n-1)\end{bmatrix}_n$, $j = 0, \dots, i-1$	UR	i
$\begin{bmatrix}2j\\2j\end{bmatrix}_n$, $j = 0, \dots, i-1$	LR	i
$\begin{bmatrix}2j+2(n-1)\\2j+2(n-1)\end{bmatrix}_n$, $j = 0, \dots, i$	LL	$i+1$
$\begin{bmatrix}2j+3(n-1)\\2j+3(n-1)\end{bmatrix}_n$, $j = 0, \dots, i$	LR	$i+1$
$\begin{bmatrix}2j\\2j\end{bmatrix}_n$, $j = i, \dots, \frac{n}{2} - 2$	UR	$\frac{n}{2}-i-1$
$\begin{bmatrix}2j\\2j\end{bmatrix}_n$, $j = i+1, \dots, \frac{n}{2} - 2$	UL	$\frac{n}{2}-i-2$
$\begin{bmatrix}4n-5\\4n-5\end{bmatrix}_n$, $\begin{bmatrix}4n-4\\4n-4\end{bmatrix}_n$	UL	2
Total		$2n-3$

- If $S = \begin{bmatrix}2i+2(n-1)\\2i+2(n-1)\end{bmatrix}_n$, $i = 0, \dots, \frac{n}{2} - 2$, then S is intersected by:

Generator square	**vertex**	**number**
$\begin{bmatrix}2j+2(n-1)\\2j+2(n-1)\end{bmatrix}_n$, $j = i+1, \dots, \frac{n}{2} - 2$	LL	$\frac{n}{2}-i-2$
$\begin{bmatrix}2j+3(n-1)\\2j+3(n-1)\end{bmatrix}_n$, $j = i+1, \dots, \frac{n}{2} - 2$	LR	$\frac{n}{2}-i-2$
$\begin{bmatrix}2j+2(n-1)\\2j+2(n-1)\end{bmatrix}_n$, $j = 0, \dots, i-1$	UR	i
$\begin{bmatrix}2j+(n-1)\\2j+(n-1)\end{bmatrix}_n$, $j = 0, \dots, i-1$	LR	i
$\begin{bmatrix}2j+3(n-1)\\2j+3(n-1)\end{bmatrix}_n$, $j = 0, \dots, i$	LL	$i+1$
$\begin{bmatrix}2j\\2j\end{bmatrix}_n$, $j = 0, \dots, i-1$	LR	i
$\begin{bmatrix}4n-5\\4n-5\end{bmatrix}_n$, $\begin{bmatrix}4n-4\\4n-4\end{bmatrix}_n$	LR	2
$\begin{bmatrix}2j+(n-1)\\2j+(n-1)\end{bmatrix}_n$, $j = i, \dots, \frac{n}{2} - 2$	UR	$\frac{n}{2}-i-1$
$\begin{bmatrix}2j\\2j\end{bmatrix}_n$, $j = i, \dots, \frac{n}{2} - 2$	UL	$\frac{n}{2}-i-1$
Total		$2n-3$

- If $S = \begin{bmatrix}2i+3(n-1)\\2i+3(n-1)\end{bmatrix}_n$, $i = 0, \dots, \frac{n}{2} - 2$, then S is intersected by:

Generator square	**vertex**	**number**
$\begin{bmatrix}2j+3(n-1)\\2j+3(n-1)\end{bmatrix}_n$, $j = i+1, \dots, \frac{n}{2} - 2$	LL	$\frac{n}{2}-i-2$
$\begin{bmatrix}2j\\2j\end{bmatrix}_n$, $j = i, \dots, \frac{n}{2} - 3$	LR	$\frac{n}{2}-i-2$
$\begin{bmatrix}2j+3(n-1)\\2j+3(n-1)\end{bmatrix}_n$, $j = 0, \dots, i-1$	UR	i
$\begin{bmatrix}2j+2(n-1)\\2j+2(n-1)\end{bmatrix}_n$, $j = 0, \dots, i-1$	LR	i
$\begin{bmatrix}2j\\2j\end{bmatrix}_n$, $j = 0, \dots, i-1$	LL	i
$\begin{bmatrix}2j+(n-1)\\2j+(n-1)\end{bmatrix}_n$, $j = 0, \dots, i-1$	LR	i
$\begin{bmatrix}2j+2(n-1)\\2j+2(n-1)\end{bmatrix}_n$, $j = i, \dots, \frac{n}{2} - 2$	UR	$\frac{n}{2}-i-1$
$\begin{bmatrix}2j+(n-1)\\2j+(n-1)\end{bmatrix}_n$, $j = i, \dots, \frac{n}{2} - 2$	UL	$\frac{n}{2}-i-1$
$\begin{bmatrix}4n-5\\4n-5\end{bmatrix}_n$, $\begin{bmatrix}4n-4\\4n-4\end{bmatrix}_n$	UR	2
$\begin{bmatrix}3n-3\\3n-3\end{bmatrix}_n$	UR	1
Total		$2n-3$

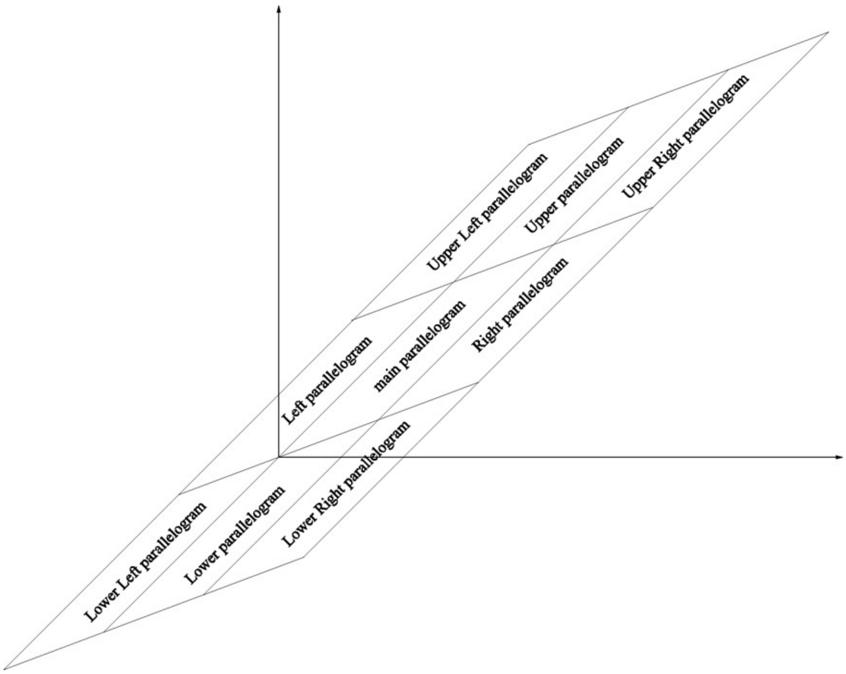

Fig. 11. Scheme of parallelograms of the covering.

- If $S = \begin{bmatrix} 4n-5 \\ 4n-5 \end{bmatrix}_n$ (or $S = \begin{bmatrix} 4n-4 \\ 4n-4 \end{bmatrix}_n$ resp.), then S is intersected by:

Generator square	**vertex**	**number**
$\begin{bmatrix} 2j \\ 2j \end{bmatrix}_n$, $\quad j = 0, \dots, \frac{n}{2} - 2$	LL	$\frac{n}{2} - 1$
$\begin{bmatrix} 2j+(n-1) \\ 2j+(n-1) \end{bmatrix}_n$, $\quad j = 0, \dots, \frac{n}{2} - 2$	LR	$\frac{n}{2} - 1$
$\begin{bmatrix} 2j+3(n-1) \\ 2j+3(n-1) \end{bmatrix}_n$, $j = 0, \dots, \frac{n}{2} - 2$	UR	$\frac{n}{2} - 1$
$\begin{bmatrix} 2j+2(n-1) \\ 2j+2(n-1) \end{bmatrix}_n$, $j = 0, \dots, \frac{n}{2} - 2$	UL	$\frac{n}{2} - 1$
$\begin{bmatrix} 4n-5 \\ 4n-5 \end{bmatrix}_n$ (resp. $\begin{bmatrix} 4n-4 \\ 4n-4 \end{bmatrix}_n$)	LL (resp. UR)	1
Total		$2n - 3$

□

Remark. If we replace in the generator set the squares:

$$\left\{ \begin{pmatrix} 4n-5 \\ 4n-5 \end{pmatrix}, \begin{pmatrix} 4n-4 \\ 4n-4 \end{pmatrix} \right\}$$

by

$$\left\{ \begin{pmatrix} 4n-5 \\ 4n-4 \end{pmatrix}, \begin{pmatrix} 4n-4 \\ 4n-5 \end{pmatrix} \right\}$$

we obtain a strong covering of squares of width n and intersection number $2n - 3$. We observe that the squares in the covering intersecting $\begin{bmatrix} 4n-5 \\ 4n-4 \end{bmatrix}_n$, $\begin{bmatrix} 4n-4 \\ 4n-5 \end{bmatrix}_n$ are the same squares intersecting $\begin{bmatrix} 4n-4 \\ 4n-4 \end{bmatrix}_n$, $\begin{bmatrix} 4n-5 \\ 4n-5 \end{bmatrix}_n$. Then, we can choose one of two pairs of squares to use in every copy of the parallelogram. In this way we can have a non-periodic regular covering. We also see that the number of feasible covering is not countable.

Theorem 14. *Let n be the maximum width of a square in a covering, the intersection number of the covering is lesser than or equal to $2(n-1)$ if n is odd and $2(n-1) - 1$ if n is even.*

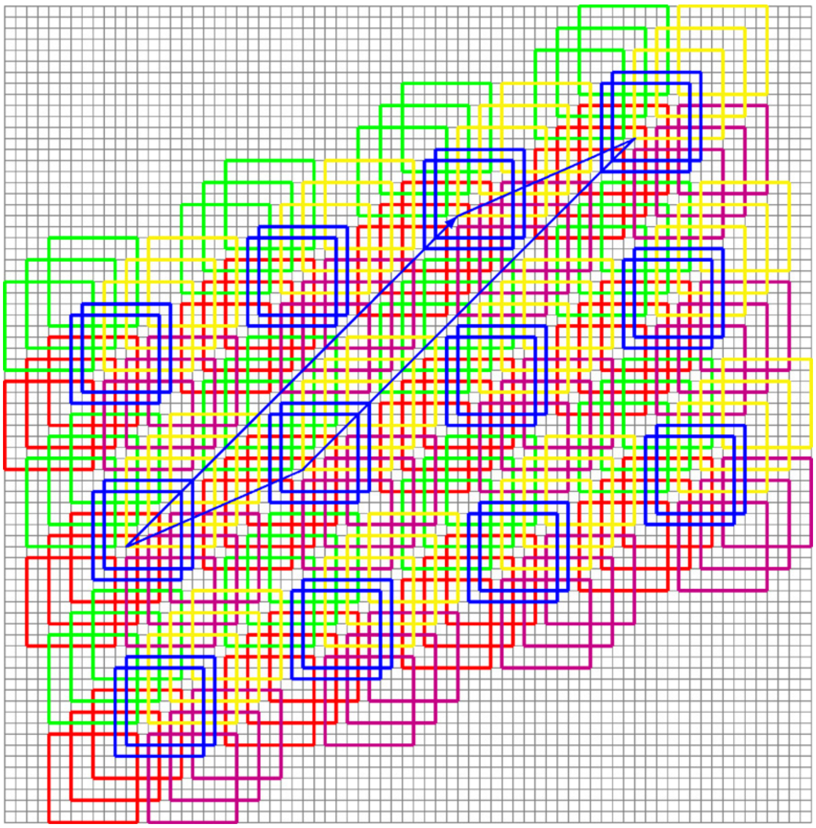

Fig. 12. Maximal intersection number for a regular covering of width n even.

The proof is straightforward from the above results.

4.2. Minimal conditions for a covering

We look for the minimal conditions for the existence of a weak covering and a strong covering. We see that in the case of weak covering these conditions are not very restrictive. On the other hand, in the case of strong covering we find the minimal conditions for the regular case. We prove that the minimum intersection number is equal to 6 and, we obtain that the minimum size of squares is 5. For simplicity, the non-regular case has not been treated in this article, however we conjecture that the minimum intersection number for this case is six as in the regular case.

Fact 15. *Given S and S', two squares with the same width and such that S does not touch S'. If $S \cap S' \neq \emptyset$ then $\mathrm{Int}(S) \cap \mathrm{Ver}(S')$ is a singleton.*

Proof. Without loss of generality, we consider $S = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_n$, $S' = \begin{bmatrix} i \\ j \end{bmatrix}_n$ with $0 < i < n$, $-n < j < n$ and $j \neq 0$.

If $-n < j < 0$ the only vertex in $\mathrm{Int}(S)$ is $\binom{i}{j+n}$ and if $0 < j < n$ the only vertex in $\mathrm{Int}(S)$ is $\binom{i}{j}$. $\square$

Observe that:

(1) In Fact 15, if we do not have the "do not touch" condition then $|\,\mathrm{Int}(S) \cap \mathrm{Ver}(S')| \leq 1$ (see Fig. 13(a)).
(2) In Fact 15, if S and S' do not touch and do not have the same width then $|\,\mathrm{Int}(S) \cap \mathrm{Ver}(S')| \leq 2$ (see Fig. 13(b)).

Theorem 16. *The squares in a regular weak covering have a width at least 3 and their intersection numbers are at least 2.*

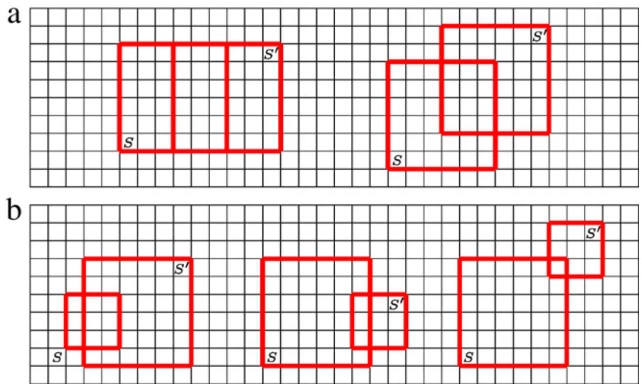

Fig. 13. (a) At left, $|\operatorname{Int}(S) \cap \operatorname{Ver}(S')| = 0$ and at right $|\operatorname{Int}(S) \cap \operatorname{Ver}(S')| = 1$. (b) At left, $|\operatorname{Int}(S) \cap \operatorname{Ver}(S')| = 0$, at center $|\operatorname{Int}(S) \cap \operatorname{Ver}(S')| = 2$ and at right $|\operatorname{Int}(S) \cap \operatorname{Ver}(S')| = 1$.

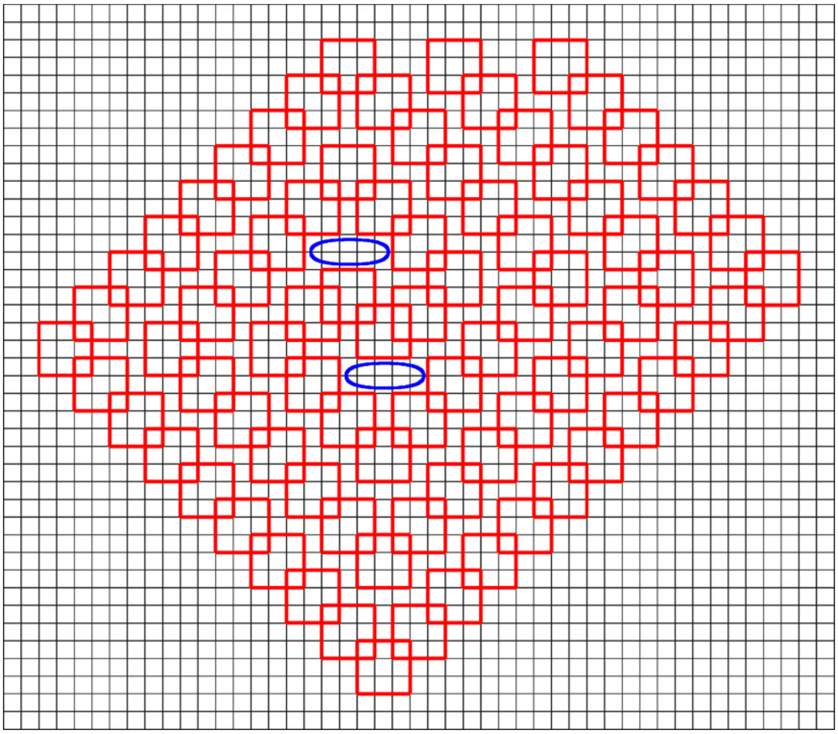

Fig. 14. Minimal 2-weak covering.

Proof. If the intersection number of the covering is one, the only possible configuration is two squares with a not empty intersection and that does not cover the plane. Then the intersection number of a weak covering has to be at least 2.

Assuming that squares of the weak covering have width 2, all squares S of the covering has only one interior point. Then, if there exist two squares S' and S'' of width 2 intersecting S, without touching it, by Fact 15 $\operatorname{Int}(S) \cap \operatorname{Ver}(S') = \operatorname{Int}(S) \cap \operatorname{Ver}(S'')$, i.e. S' touches S'', which is contradictory to the definition of weak covering.

Fig. 14 shows a construction of a 2-weak covering with intersection number 2 and squares of width 3. We can see that there is only a finite number of points that are not in the frontier or interior of a square in the covering. □

Theorem 17. *The squares in a regular strong covering have a width at least* 5 *and its intersection number is at least* 6*.*

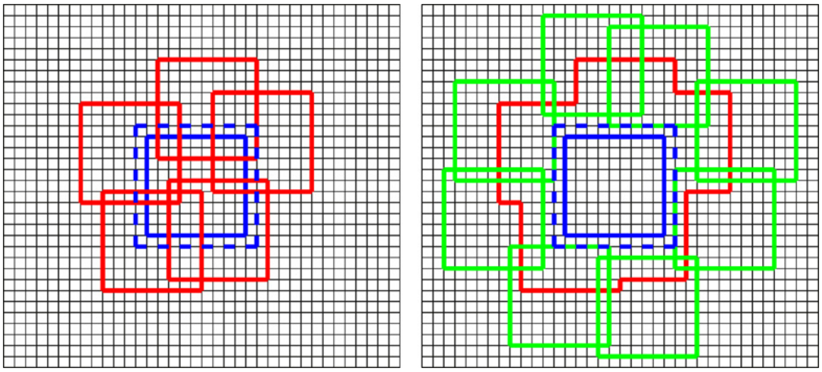

Fig. 15. Minimum number of squares around S_0.

The proof of this theorem is by contradiction, for every intersection number lesser than 6, we try to construct the covering and we see that is not possible.

We prove some previous results.

Fact 18. *Let S' and S'' be two squares covering a side of S in a regular strong covering. Then there exists $S''' \neq S$ in the covering such that $S''' \cap S' \neq \emptyset$ and $S''' \cap S'' \neq \emptyset$.*

Proof. We have that $\mathrm{Fr}_*(S') \cap \mathrm{Fr}_*(S'') = \left\{\binom{i_1}{j_1}, \binom{i_2}{j_2}\right\}$. We can see that any square S_* covering one of these points is such that $S_* \cap S' \neq \emptyset$ and $S_* \cap S'' \neq \emptyset$. Since S' and S'' cover a side of S, S can cover only one of these points, for example $\binom{i_1}{j_1}$, then there necessarily exists a square S''' covering $\binom{i_2}{j_2}$, hence $S''' \cap S' \neq \emptyset$ and $S''' \cap S'' \neq \emptyset$. □

Lemma 19. *Let $P = \bigcup_{i=0}^{k} S_i$ where S_i are squares such that $S_1, \ldots, S_k$ cover S_0 in a regular strong covering. If D_j, $j = 1, \ldots, m$ are squares covering the external frontier of P and $D_j \cap S_0 = \emptyset$, $\forall j = 1, \ldots, m$, then $m \geq 8$.*

Proof. Since the squares D_j do not intersect S_0, they have to cover at least a distance $n + 2$ in the four sides of S_0. On the other hand, since the width of squares is n, this distance can be covered by a minimum of two different squares by side, then the minimum of squares covering this surface is 8, as shown in Fig. 15. □

Lemma 20 is the crucial result that allows us to find the minimal conditions of a regular strong covering, the basic idea of the proof is to show that it is impossible to have an intersection number lesser than six in a regular strong covering.

Lemma 20. *There does not exist a regular strong covering with intersection number $m \leq 5$.*

Proof. ***m* ≤ 3**. By Fact 15 every square in a strong covering has its vertices in the interior of distinct squares, so $m \geq 4$.

***m* = 4**. We prove this result by contradiction. For a given square S_0 in a strong covering, let $S_1, \ldots, S_4$ be the squares covering their frontiers, we prove that one of the squares S_i intersects three others squares among the previous ones and it has two vertices uncovered. Hence, S_i has at least an intersection number five, which is a contradiction.

Without loss of generality, let $S_0 = \begin{bmatrix}0\\0\end{bmatrix}_n$ and $S_k = \begin{bmatrix}i_k\\j_k\end{bmatrix}_n$, $k = 1, 2, 3, 4$ such that:

- $-(n-1) < i_1 < 0, \quad -(n-1) < j_1 < 0$
- $-(n-1) < i_2 < 0, \quad 0 < j_2 < j_1 + n \quad \Longrightarrow S_1 \cap S_2 \neq \emptyset$
- $0 < i_3 < i_2 + n, \quad 0 < j_3 < n \quad \Longrightarrow S_2 \cap S_3 \neq \emptyset$
- $0 < i_4 < i_1 + n, \quad j_3 - n < j_4 < 0 \quad \Longrightarrow S_3 \cap S_4 \neq \emptyset \wedge S_1 \cap S_4 \neq \emptyset.$

From these conditions, we obtain that each S_k, $k = 1, 2, 3, 4$ intersects three squares. Thus to obtain a cover with intersection number $m = 4$, we can add only one square intersecting each S_k. We prove that there exists S_k, $k = 1, 2, 3, 4$, such that S_k has two vertices not covered, then it is impossible to have a strong covering with 4 intersections.

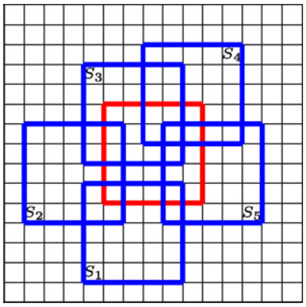

Fig. 16. Covering of square S_0 by 5 squares.

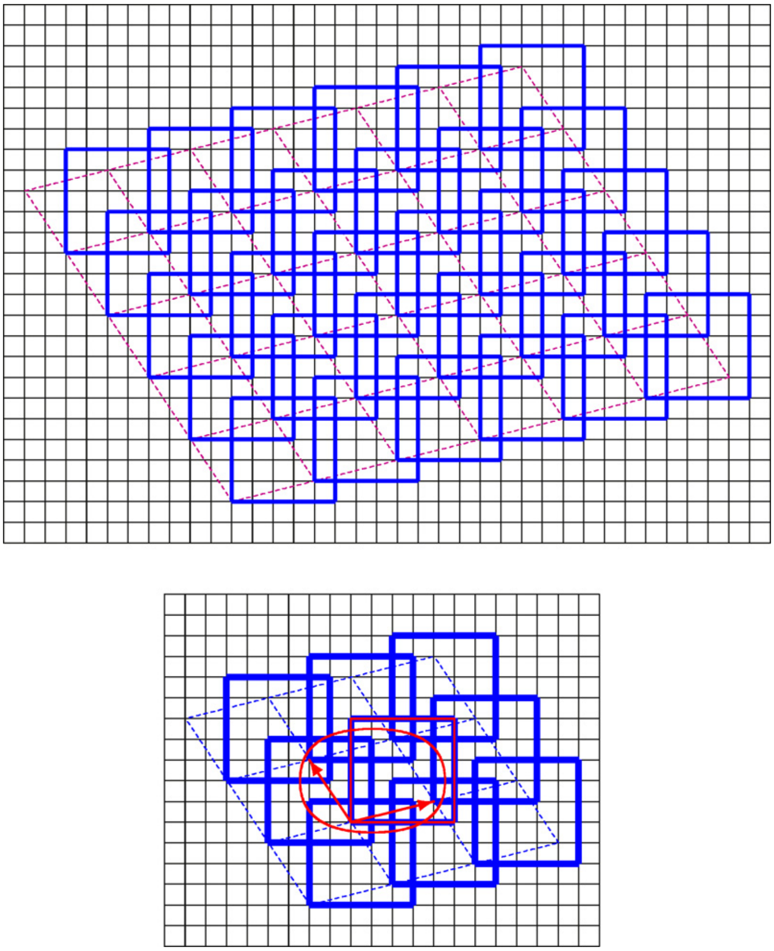

Fig. 17. Periodic regular strong covering.

From necessary conditions, the vertices $\binom{i_1}{j_1}$ of S_1 and $\binom{i_2}{j_2+n}$ of S_2 are not covered. To cover the vertex $\binom{i_2}{j_2}$ of S_2 the only square which we can use is S_1, and in this case the vertex $\binom{i_1}{j_1+n}$ remains uncovered, and if we cover the vertex $\binom{i_1}{j_1+n}$, the vertex $\binom{i_2}{j_2}$ will be uncovered.

$\boldsymbol{m = 5}$. For a given square $S_0 = \begin{bmatrix}0\\0\end{bmatrix}_n$ in a strong covering, let $S_1, \ldots, S_5$ be the squares covering its frontiers. Since each square S_k covers a vertex of S_0, there exists a vertex of S_0 which is covered by two squares S_k. Without loss of generality, we consider two squares S_1 and S_2 covering the vertex $\binom{0}{0}$ and S_3, S_4 and S_5 covering the other vertices clockwise (as shown Fig. 16).

Then, we obtain the conditions:

$$
\begin{array}{lll}
\bullet \; -(n-1) < i_1 < 0, & -(n-1) < j_1 < 0 & \\
\bullet \; -(n-1) < i_2 < 0, & -(n-1) < j_2 < 0 & \Rightarrow S_1 \cap S_2 \neq \emptyset \\
\bullet \; -(n-1) < i_3 < 0, & 0 < j_3 < j_* + n & \Rightarrow (S_1 \cup S_2) \cap S_3 \neq \emptyset \\
\bullet \; 0 < i_4 < i_2 + n, & 0 < j_4 < n & \Rightarrow S_3 \cap S_4 \neq \emptyset \\
\bullet \; 0 < i_5 < i_* + n, & j_3 - n < j_5 < 0 & \Rightarrow S_4 \cap S_5 \neq \emptyset \text{ and} \\
 & & \quad (S_1 \cup S_2) \cap S_5 \neq \emptyset
\end{array}
\tag{1}
$$

where $j_* = \max\{j_1, j_2\}$ and $i_* = \max\{i_1, i_2\}$.

According to the above conditions (1) we have that S_3, S_4 and S_5 are intersected by three squares. Then their coverings can have only two extra squares. Furthermore, S_1 and S_2 are intersected at least by two squares, and the union of both ones intersects other two squares (S_3 and S_5), then the covering of the union of S_1 and S_2 has at most four other squares. Hence, the covering of the union of all S_i has at most ten squares. Applying the Fact 18 to the intersections between S_3 and S_4, S_4 and S_5, S_1 or S_2 with S_3 and S_1 or S_2 with S_5, we reduce the number of squares in the covering of $\bigcup_{i=0}^{5} S_i$ to six, which is a contradiction by Lemma 19. Therefore, there does not exist a regular strong covering with intersection number five. $\square$

Now we give the proof of Theorem 17.

Proof. We use the fact that there does not exist a regular strong covering with intersection numbers 2, 3, 4 or 5. To have a regular covering with intersection number six, by Proposition 13 we need a square of width at least five.

Now, we show a construction of a strong covering with intersection number 6 and squares of width $n \geq 5$. Let

$$R_{n,6} = \left[\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} ; \begin{pmatrix} n-1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ n-2 \end{pmatrix} \right]_n .$$

The projections of the square sides in the parallelogram are in this case:

$$
\begin{array}{ll}
\begin{pmatrix} i \\ 0 \end{pmatrix} = \binom{i-2}{n-2} \quad +0\binom{n-1}{1} - 1\binom{-2}{n-2} & i = 1, \ldots, n-1 \\
\begin{pmatrix} n \\ j \end{pmatrix} = \binom{-1}{j+n-3} +1\binom{n-1}{1} - 1\binom{-2}{n-2} & j = 0, 1 \\
\begin{pmatrix} n \\ j \end{pmatrix} = \binom{1}{j-1} \quad +1\binom{n-1}{1} + 0\binom{-2}{n-2} & j = 2, \ldots, n-1 \\
\begin{pmatrix} i \\ n \end{pmatrix} = \binom{i-n+3}{1} +1\binom{n-1}{1} + 1\binom{-2}{n-2} & i = n-3, n-2, n-1 \\
\begin{pmatrix} i \\ n \end{pmatrix} = \binom{i+2}{2} \quad +0\binom{n-1}{1} + 1\binom{-2}{n-2} & i = 0, \ldots, n-4 \\
\begin{pmatrix} 0 \\ j \end{pmatrix} = \binom{2}{j-n+2} +0\binom{n-1}{1} + 1\binom{-2}{n-2} & j = n-1, n \\
\begin{pmatrix} 0 \\ j \end{pmatrix} = \binom{0}{j} \quad +0\binom{n-1}{1} + 0\binom{-2}{n-2} & j = 0, \ldots, n-2.
\end{array}
$$

On account of the above conditions we observe that the projections of the horizontal sides of the square do not intersect among themselves, and the same property holds for the projections of the vertical sides.

Now, we prove that every point of the discrete plane belongs to the interior of a square of the covering, that is:

$$\left(\forall \begin{pmatrix} i \\ j \end{pmatrix} \in \mathbb{Z}^2 \right), (\exists l, k \in \mathbb{Z}) : \begin{pmatrix} i \\ j \end{pmatrix} \in \operatorname{Int} \begin{bmatrix} k(n-1) - 2l \\ k + l(n-2) \end{bmatrix}_n .$$

By periodicity this is equivalent to:

$$\left(\forall \begin{pmatrix} i' \\ j' \end{pmatrix} \in P\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) \right), (\exists l, k \in \mathbb{Z}) : \begin{pmatrix} i' \\ j' \end{pmatrix} \in \operatorname{Int} \begin{bmatrix} k(n-1) - 2l \\ k + l(n-2) \end{bmatrix}_n .$$

- If $i' < 1$ then $\binom{i'}{j'} \in \mathrm{Int}\begin{bmatrix}-(n-1)\\-1\end{bmatrix}_n$.
- If $i' \geq 1$ then $\binom{i'}{j'} \in \mathrm{Int}\begin{bmatrix}0\\0\end{bmatrix}_n$.

Finally, we study the intersection number of every square in the covering. It is enough to count the number of intersections of the square $\begin{bmatrix}0\\0\end{bmatrix}_n$. We can observe that the only squares intersecting $\begin{bmatrix}0\\0\end{bmatrix}_n$ are:

$$\left\{\begin{bmatrix}n-1\\1\end{bmatrix}_n, \begin{bmatrix}-(n-1)\\-1\end{bmatrix}_n, \begin{bmatrix}-2\\n-2\end{bmatrix}_n, \begin{bmatrix}2\\-(n-2)\end{bmatrix}_n, \begin{bmatrix}n-3\\n-1\end{bmatrix}_n, \begin{bmatrix}-(n-3)\\-(n-1)\end{bmatrix}_n\right\}. \quad \square$$

This construction is not unique, it is not difficult to find out a lot of other regular strong coverings with intersection number 6.

5. Other related problems

One question still unanswered is whether non-regular coverings have a behavior similar to regular coverings. For example if we study the minimal intersection number it is easy to see that for a weak covering we need an intersection number two in the regular and non-regular cases. But for strong covering this is an open question, we conjecture that the minimal intersection number is six as in the regular case.

Also we could study the coverings from the point of view of how many times every point of the discrete plane is covered and study the regularity of the covering from this point of view. This problem could be seen as a generalization of the tiling problem because in the tiling problem we have to cover each point by only one tile and in this problem we can have an arbitrary number of tiles covering every point.

For further reading

Figs. 3, 7, 12 and 17, [1]

Acknowledgements

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