Uniqueness of radially symmetric positive solutions for $-\Delta u + u = u^p$ in an annulus

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Abstract

In this article we prove that the semi-linear elliptic partial differential equation

$$-\Delta u + u = u^p \quad \text{in } \Omega,$$

$$u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

possesses a unique positive radially symmetric solution. Here $p > 1$ and Ω is the annulus $\{x \in \mathbf{R}^N \mid a < |x| < b\}$, with $N \geqslant 2$, $0 < a < b \leqslant \infty$. We also show the positive solution is non-degenerate.

Keywords: Non-linear elliptic equation; Radially symmetric solutions; Non-degeneracy

1. Introduction

In this article we study the uniqueness of radially symmetric positive solution of the semi-linear elliptic problem

$$-\Delta u + u = u^p \quad \text{in } \Omega, \tag{1}$$

$$u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

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where Ω is the annulus $\{x \in \mathbf{R}^N \mid a < |x| < b\}$ with $0 < a < b \leqslant \infty$, $N \in \mathbf{N}$ and $p > 1$. When $b = \infty$ the boundary condition is interpreted as $\lim_{|x| \to \infty} u(x) = 0$.

The problem of uniqueness for (1)–(2) has been recently solved in the case $N \geqslant 3$ by Tang [18], but it seems that the case $N = 2$ still is not known. It is the purpose of this article to give a proof of uniqueness in the case $N = 2$ for a more general version of (1)–(2). Our approach can also be applied for the case $N \geqslant 3$, providing in this way of another proof of the result by Tang.

The problem of uniqueness for positive solutions of (1)–(2), but when Ω is the ball $\{x \in \mathbf{R}^N \mid |x| < b\}$ or the entire space, has been solved by Kwong [11] after a long history and contributions of many authors, among whom we mention Coffman [3,4], Peletier and Serrin [16], Mc Leod and Serrin [14] and Mc Leod [15]. After the result of Kwong there have been many extensions and refinements of this theorem and we cannot quote them all, but we would like to mention the work by Clemons and Jones [2], Erbe and Tang [5,6], Kabeya and Tanaka [9].

The problem in the annulus has a shorter history, starting with the work by Coffman [4], following with Yadava [19,20] and concluding with Tang [18], for the case $N \geqslant 3$. We refer to [18] for a detailed discussion. Worth mentioning here are the contributions Kwong and Zhang [13] and Kwong and Li [12] for related questions on the annulus.

Now we present our results in a precise way. First we write our theorem for $N = 2$, which is our main result.

Theorem 1.1. *For $N = 2$ and $0 < a < b \leqslant \infty$, the Dirichlet problem* (1)–(2) *has a unique radially symmetric positive solution. Moreover the unique solutions is non-degenerate in the space of H^1-radially symmetric functions.*

For radially symmetric solutions, writing $r = |x|$, Eqs. (1)–(2) becomes

$$u'' + \frac{N-1}{r} u' + u^p - u = 0 \quad \text{in } (a, b), \tag{3}$$

$$u(r) > 0, \quad r \in (a, b), \qquad u(a) = 0, \qquad u(b) = 0, \tag{4}$$

where $'$ means derivative with respect to r. As we mentioned before we will consider a more general equation, namely

$$u'' + \frac{\nu}{r} u' + u^p - V(r) u = 0 \quad \text{in } (a, b), \tag{5}$$

$$u(r) > 0, \quad r \in (a, b), \qquad u(a) = 0, \qquad u(b) = 0, \tag{6}$$

where $\nu \geqslant 0$, $p \in (1, \infty)$ and $V(r) \in C^1([a, b], \mathbf{R})$ satisfies the following conditions:

(V1) $0 < \inf V(r) \leqslant \sup V(r) < \infty$.

(V2) If $b < \infty$, then

$$U(r) = V'(r) r^3 + \beta V(r) r^2 + (\beta - 2) L$$

is either:

(i) negative in (a, b),

(ii) positive in (a, b) or

(iii) $U(a) < 0$ and changes sign only once in (a, b).

(V2′) If $b = \infty$, then U satisfies (ii) or (iii) and in both cases $\liminf_{r\to\infty} U(r) > 0$.

Here

$$\alpha = \frac{2\nu}{p+3}, \quad \beta = (p-1)\alpha \quad \text{and} \quad L = \alpha(\nu - 1 - \alpha). \tag{7}$$

We prove the following theorem

Theorem 1.2. *Assume that $\nu \geqslant 0$, $p \in (1, \infty)$ and V satisfies* (V1), (V2) *if $b < \infty$ and* (V1), (V2′) *if $b = \infty$. Then* (5)–(6) *possesses at most one positive solution. Moreover the unique positive solution is non-degenerate in the space of H^1-radially symmetric functions provided $b < \infty$ or $\nu > 0$.*

Remark 1.3.

(i) For any $N \in \mathbf{N}$ and $p \in (1, \infty)$ we have that $\nu = N - 1$ and $V(r) \equiv 1$ satisfies (V1)–(V2) if $b < \infty$ and (V1), (V2′) if $b = \infty$. Thus Theorem 1.2 is applicable to (1)–(2) in any dimension.

(ii) We observe that when $b = \infty$, (V2)(i) is incompatible with (V1). In fact, if (V2)(i) holds, then for $\varepsilon \in (0, \beta)$ and r large $V'(r) < -(\beta - \varepsilon)\frac{V(r)}{r}$ holds and thus $V(r) \leqslant \frac{C}{r^{\beta-\varepsilon}}$.

We write the result for the whole range of $\nu \in [0, \infty)$, since this is of interest in some applications, as in the work by Felmer and Martínez [7]. There, the parameter ν is a homotopy variable in $[0, N - 1]$ and the uniqueness result is needed to apply the Nehari method in the construction of highly oscillatory solutions for a singularly perturbed problem.

It is clear that Theorem 1.1 follows directly from Theorem 1.2. The proof of these theorems is based on the ideas developed by Kabeya and Tanaka in [9] for the problem in the entire space. First we proceed to prove that the solution is unique by a contradiction argument, assuming there are more than one solution. In doing so, we first characterize two possible positive solutions by the number of crossing points and then we use an energy analysis to get a contradiction. Second, we prove the non-degeneracy of the positive solution by analyzing its Morse index as a critical point of a perturbed functional which has the mountain pass geometry.

We end this introduction with some words about the existence of positive solutions of (3)–(4). One possible approach to existence is the variational method through the mountain pass theorem. Special mention deserves the case $b = \infty$, where some extra compactness arguments have to be used, based on the Strauss' compactness embedding of $H^1_{\mathrm{rad}}(\mathbf{R}^N) = \{u \in H^1_0(\mathbf{R}^N) \mid u(x) = u(|x|)\}$ in $L^r(\mathbf{R}^N)$, where $2 < r < \frac{2N}{N-2}$ for $N \geqslant 3$, $2 < r < \infty$ for $N = 2$. See Strauss [17] and Berestycki and Lions [1]. In this unbounded situation we have to exclude $\nu = 0$, where problem (3)–(4) does not have a solution.

2. Uniqueness of radially symmetric positive solutions

In this section we consider the uniqueness part of Theorem 1.2, that is, we prove that (5)–(6) has at most one positive solution. In case $b = \infty$, we change the boundary condition $u(b) = 0$ by $\lim_{r\to\infty} u(r) = 0$. Under this boundary condition, using hypothesis (V1) and comparison arguments it can be proved that $u(r)$, $|u'(r)|$, $|u''(r)| \leqslant e^{-Cr}$, for some $C > 0$ and all r large.

We assume, for contradiction, that (5)–(6) possesses two distinct positive solutions $u_1(r)$ and $u_2(r)$, and we consider the number of points of intersection between them, that is, $N(u_1, u_2) = \#\{r \in (a, b) \mid u_1(r) = u_2(r)\}$.

We assume (V1)–(V2) or (V1)–(V2′) throughout this paper.

Proposition 2.1. *Suppose that* (5)–(6) *has two distinct solutions $u_1(r)$ and $u_2(r)$ such that $u_1'(a) < u_2'(a)$. Then there exists a radially symmetric positive solution $u_3(r)$ of* (5)–(6) *such that*

$$u_3'(a) \geqslant u_2'(a) \quad \textit{and} \quad N(u_1, u_3) \leqslant 1.$$

Proof. The proof of this proposition is given in Appendix A of [9], we include it here for completeness. We use a shooting argument, so we consider the initial value problem:

$$u'' + \frac{\nu}{r}u' + u^p - V(r)u = 0,$$
$$u(a) = 0, \quad u'(a) = \alpha > 0.$$

We denote by $u(r;\alpha)$ the solution of this equation, and we notice that $u(r;\alpha)$ varies continuously as a function of α. We assume $N(u_1, u_2) \geqslant 2$, because in the contrary, we just take $u_3(r) = u_2(r)$, as the desired solution. We set $\alpha_1 = u_1'(0) < \alpha_2 = u_2'(0)$.

We start with $\alpha = \alpha_2$ and increase α progressively. We keep track the position of the first and second intersection point of $u(r;\alpha)$ and $u_1(r)$, denoting them by $\sigma_1(\alpha)$ and $\sigma_2(\alpha)$, respectively. We see that $\sigma_1(\alpha), \sigma_2(\alpha) \in (a, b)$ for α close to α_2 and

$$u(r;\alpha) > 0 \quad \text{in } \big(a, \sigma_2(\alpha)\big).$$

On the other hand, since $p > 1$ we can prove that for large $\bar{\alpha} > \alpha_2$, the solution $u(r;\bar{\alpha})$ is the solution of the Dirichlet boundary value problem for (5) in (a, r_0), for r_0 close to a. Thus, $u(r_0, \bar{\alpha}) = 0$ and $\#\{r \in (a, r_0) \mid u(r;\bar{\alpha}) = u_1(r)\} = 1$.

By continuity of the point of intersection $\sigma_2(\alpha)$, we can find $\alpha_3 \in (\alpha_2, \bar{\alpha})$ such that for α close to α_3 and $\alpha < \alpha_3$ we have $\sigma_2(\alpha) < b$ and $\sigma_2(\alpha_3) = b$. We conclude that $u_3(r) = u(r;\alpha_3)$ is the desired solution. □

In the rest of the section, we assume that $u_1(r)$ and $u_2(r)$ are two distinct solutions of (5)–(6) with at most one intersection in (a, b). Next we study some properties of these functions and then we will reach to a contradiction. We have

Lemma 2.2. *Suppose that $u_1(r)$ and $u_2(r)$ are two distinct solutions of* (5)–(6)*, with at most one intersection in (a, b), and assume that $u_1'(a) < u_2'(a)$. Then*

$$\frac{d}{dr}\left(\frac{u_1(r)}{u_2(r)}\right) > 0 \quad \textit{in } (a, b). \tag{8}$$

Proof. Setting $f(r) = r^\nu(u_1'(r)u_2(r) - u_1(r)u_2'(r))$, we have

$$\frac{d}{dr}\left(\frac{u_1(r)}{u_2(r)}\right) = \frac{1}{r^\nu u_2(r)^2} f(r),$$
$$\frac{df}{dr}(r) = r^\nu u_1 u_2 \left(u_2^{p-1} - u_1^{p-1}\right).$$

Under our assumption on u_1 and u_2 we have that either

(i) $f'(r) > 0$ in (a, σ) and $f'(r) < 0$ in (σ, b), for some $\sigma \in (a, b)$, or
(ii) $f'(r) > 0$ in (a, b).

Since $f(a) = f(b) = 0$, (ii) cannot take place. (i) implies that $f(r) > 0$ in (a, b), from where the conclusion follows. □

Next we introduce the change of variable $w(r) = r^\alpha u(r)$, where α is defined in (7) and $u(r)$ is a solution of (5)–(6), following [12] and [9]. Then $w(r)$ satisfies

$$r^\beta w'' + \frac{\beta}{2} r^{\beta-1} w' + w^p - \left(V(r) r^\beta + L r^{\beta-2}\right) w = 0,$$

where β and L are given in (7). For the two solutions under consideration we define, $w_j(r) = r^\alpha u_j(r)$ and

$$E(r; w_j) = \frac{1}{2} r^\beta w_j'(r)^2 + \frac{1}{p+1} w_j(r)^{p+1} - \frac{1}{2} G(r) w_j(r)^2,$$

where $G(r) = V(r) r^\beta + L r^{\beta-2}$ and $j = 1, 2$. We have the following

Lemma 2.3. *$E(r; w_j) > 0$ for $r \in [a, b]$ if $b < \infty$ and for $r \in [a, b)$ if $b = \infty$.*

Proof. We see that

$$\frac{d}{dr} E(r; w_j) = -\frac{1}{2} G'(r) w_j(r)^2,$$

with $G'(r) = r^{\beta-3} U(r)$. But we have that $E(a; w_j) > 0$ and $E(b; w_j) > 0$ for $j = 1, 2$, in case $b < \infty$, so that the conclusion follows from the hypothesis (V2). If $b = \infty$, $w_j(r)$ decays exponentially as $r \to \infty$ and we have $\lim_{r\to\infty} E(r; w_j) = 0$ for $j = 1, 2$. Thus we conclude from (V2′). □

Next we set

$$F(r) = E(r; w_2) - \left(\frac{w_2}{w_1}\right)^2 E(r; w_1), \tag{9}$$

as in the work by Kawano, Yanagida and Yotsutani [10]. Then we have

$$\frac{d}{dr} F(r) = -\left\{\frac{d}{dr}\left(\left(\frac{w_2}{w_1}\right)^2\right)\right\} E(r; w_1). \tag{10}$$

Noting $\frac{w_2}{w_1} = \frac{u_2}{u_1}$, it follows from Lemmas 2.2 and 2.3 that $\frac{d}{dr}(\frac{w_2}{w_1}) < 0$ and $\frac{d}{dr}F(r) > 0$ in (a, b). Thus we have

$$F(b) - F(a) > 0. \tag{11}$$

End of the proof of uniqueness. We claim that $F(a) = F(b) = 0$, contradicting (11). To prove the claim we compute $F(a)$ as

$$F(a) = \lim_{r \to a_+} F(r) = E(a; w_2) - \left(\frac{w_2'(a)}{w_1'(a)}\right)^2 E(a; w_1) = 0.$$

When $b < \infty$, we argue in a similar way to get $F(b) = 0$. When $b = \infty$, it follows from (8) that $\frac{d}{dr}(\frac{w_2}{w_1}) < 0$. Thus

$$0 < \frac{w_2(r)}{w_1(r)} \leqslant \frac{w_2(a)}{w_1(a)} \quad \text{for all } r \in (a, \infty).$$

As before, we have $E(r; w_j) \to 0$ as $r \to \infty$ and then $F(\infty) = 0$. Thus, the claim is proved and (5)–(6) has at most one positive solution. □

3. Non-degeneracy of the positive solution

In this section we prove the non-degeneracy of the unique solution by an indirect argument based on the analysis of the Morse index of a perturbed functional.

We first consider the case b is finite. Let $\varphi(r)$ be the unique positive solution of (5)–(6), then φ is a critical point of the functional

$$I(u) = \int_a^b \left(\frac{1}{2}\left(|u'|^2 + V(r)u^2\right) - \frac{1}{p+1}u_+^{p+1}\right) r^\nu \, dr,$$

where $I : H_0^1(a, b) \to \mathbf{R}$ is of class C^2. This functional has the mountain pass structure and the unique solution corresponds to a mountain pass solution. If we define the Morse index of φ as

$$i(I, \varphi) = \max\bigl\{\dim H \bigm| H \subset H_0^1(a, b) \text{ is a subspace such that } I''(\varphi)(h, h) < 0 \text{ for all } h \in H \setminus \{0\}\bigr\}.$$

Then $i(I, \varphi) \leqslant 1$, as follows from the work by Hofer [8]. Next, as in [9], we introduce a perturbed functional for small $\delta > 0$

$$J_\delta(u) = I(u) - \delta \int_a^b \left(\frac{1}{p+1}u_+^{p+1} - \frac{1}{2}\varphi(r)^{p-1}u^2\right) r^\nu \, dr.$$

By the maximum principle, we see that non-trivial critical points u of J_δ are positive solutions of the equation

$$u'' + \frac{\nu}{r}u' + (1+\delta)u^p - \big(V(r) + \delta\varphi(r)^{p-1}\big)u = 0, \tag{12}$$

$$u(a) = u(b) = 0, \tag{13}$$

and we see that $\varphi(r)$ is one of such positive solutions. Actually it is the only one, as we prove next.

Proposition 3.1. *Assume $b < \infty$, $\nu \geqslant 0$. For sufficiently small $\delta > 0$,* (12)–(13) *has at most one positive solution.*

Proof. We would like to apply the results of Section 2 to Eqs. (12)–(13), but we cannot guarantee that the potential $V_\delta(r) = V(r) + \delta\varphi(r)^{p-1}$ satisfies hypothesis (V2). Let S_δ denote the set of all positive solutions of (12)–(13) and let $w = r^\alpha u$ for $u \in S_\delta$. Then we define

$$G_\delta(r) = V_\delta(r)r^\beta + Lr^{\beta-2}$$

and the energy function

$$E_\delta(r; w) = \frac{1}{2}r^\beta w'(r)^2 + \frac{1}{p+1}w(r)^{p+1} - \frac{1}{2}G_\delta(r)w(r)^2.$$

We remark that the hypothesis (V2) was only used in the proof of Lemma 2.3 and for the proof of this proposition it suffices to show that for some $\delta_0 > 0$

$$E_\delta(r; w) > 0 \quad \text{for all } \delta \in (0, \delta_0],\ r \in [a, b] \text{ and } u \in S_\delta. \tag{14}$$

In fact, if there exist two distinct positive solutions $u_1, u_2 \in S_\delta$, as before we may assume that they have at most one intersection in (a, b). Define $F_\delta(r)$ by (9). Then $F_\delta(b) - F_\delta(a) > 0$ follows from (10) and (14). However we also have $F_\delta(a) = F_\delta(b) = 0$ and it is a contradiction.

To show (14), we first claim that there exist $C_0 > 0$, $\delta_0 > 0$ such that

$$\|u\|_{L^\infty(a,b)} \leqslant C_0, \quad \text{for all } \delta \in (0, \delta_0) \text{ and } u \in S_\delta.$$

We argue indirectly and assume that there exist $\delta_n > 0$, $u_n \in S_{\delta_n}$ and $r_n \in (a, b)$ such that $\delta_n \to 0$ and $\theta_n \equiv u_n(r_n) = \|u_n\|_{L^\infty(a,b)} \to \infty$ as $n \to \infty$. Then we use a standard rescaling argument setting

$$v_n(y) = \theta_n^{-1} u_n\big(\theta_n^{-(p-1)/2} y + r_n\big).$$

We find that, up to a subsequence, $v_n(y)$ converges in C^2_{loc} to a solution $v(y)$ of

$$v_{yy} + v^p = 0 \quad \text{in } I, \tag{15}$$

with $0 \leqslant v(y) \leqslant 1$ in I, $v(0) = 1$ and where I is an unbounded interval. We notice that it is crucial that $a > 0$ to get this autonomous equation. We have reached a contradiction, since every non-zero solution of (15) has a zero in a finite y, proving the claim.

Since each $u \in S_\delta$ solves (12)–(13), we can easily see that $\bigcup_{\delta \in [0,\delta_0]} S_\delta$ is a compact subset of $C^1([a,b])$. Since $S_0 = \{\varphi(r)\}$, we have

$$\sup_{u \in S_\delta} \|u - \varphi\|_{C^1([a,b])} \to 0 \quad \text{as } \delta \to 0.$$

Thus, since $E(r;\varphi) > 0$ in $[a,b]$—which is proved in Lemma 2.3—, (14) holds all $\delta \in (0,\delta_0]$, for a possibly smaller $\delta_0 > 0$. Thus the proof of Proposition 3.1 is completed. □

Proof of non-degeneracy for $b < \infty$ and $\nu \geqslant 0$. We argue indirectly assuming that the unique solution $\varphi(r)$ of (5)–(6) is degenerate, that is, there is a 2-dimensional subspace $H \subset H_0^1(a,b)$ such that

$$I''(\varphi)(h,h) \leqslant 0 \quad \text{for all } h \in H.$$

Since

$$J_\delta''(u)(h,h) = I''(u)(h,h) - \delta \int_a^b \bigl(pu_+^{p-1} - \varphi(r)^{p-1}\bigr)h^2 r^\nu\, dr,$$

we have

$$J_\delta''(\varphi)(h,h) = I''(\varphi)(h,h) - \delta(p-1) \int_a^b \varphi(r)^{p-1} h^2 r^\nu\, dr$$

and we see that

$$J_\delta''(\varphi)(h,h) < 0 \quad \text{for all } h \in H \setminus \{0\}$$

and then $i(J_\delta,\varphi) \geqslant 2$. On the other hand, since J_δ has the mountain pass structure and (12)–(13) has φ as its unique solution, we must have $i(J_\delta,\varphi) \leqslant 1$. This completes the proof. □

Next we consider the case $b = \infty$, $\nu > 0$. We introduce function spaces $H_{0,\nu}^1(a,\infty)$ for $\nu > 0$ by

$$H_{0,\nu}^1(a,\infty) = \left\{ u \in H_0^1(a,\infty) \;\middle|\; \|u\|_\nu^2 = \int_a^\infty \bigl(|u'|^2 + |u|^2\bigr) r^\nu\, dr < \infty \right\}.$$

We see that the unique solution $\varphi(r)$ of (5)–(6) is a critical point of the functional

$$I(u) = \int_a^\infty \left(\frac{1}{2}\bigl(|u'|^2 + V(r)u^2\bigr) - \frac{1}{p+1} u_+^{p+1} \right) r^\nu\, dr \in C^2\bigl(H_{0,\nu}^1(a,\infty), \mathbf{R}\bigr).$$

As in the case $b < \infty$, we introduce the perturbed functional for small $\delta > 0$

$$J_\delta(u) = I(u) - \delta \int_a^\infty \left(\frac{1}{p+1} u_+^{p+1} - \frac{1}{2} \varphi(r)^{p-1} u^2 \right) r^\nu \, dr \in C^2\big(H^1_{0,\nu}(a,\infty), \mathbf{R}\big).$$

Following the arguments for that case, we see that we only need to prove that $J_\delta(u)$ has a unique critical point to complete the proof of non-degeneracy. We devote the rest of the paper to prove this uniqueness statement, starting with the following estimate for φ, in addition to its exponential decay at infinity.

Lemma 3.2. *Let $\varphi(r)$ be the unique solution of* (5)–(6) *in* (a,∞). *Then there are positive constants $r_0 > 0$, C_1 and C_2 such that*

$$C_1 \varphi(r) \leqslant -\varphi'(r) \leqslant C_2 \varphi(r), \quad \textit{for } r \in [r_0, \infty). \tag{16}$$

Proof. Let $W(r) = V(r) - \varphi(r)^{p-1}$. Then there are constants $0 < W_0 < W_1$ such that $W(r) \in [W_0, W_1]$ for large r and we can write (5) as

$$\big(r^\nu \varphi'\big)' - W(r) r^\nu \varphi = 0.$$

Next we introduce the Prüfer transformation by

$$r^\nu \varphi(r) = R(r) \sin\theta(r), \qquad r^\nu \varphi'(r) = R(r)\cos\theta(r).$$

Since $(r^\nu\varphi')' = W(r) r^\nu \varphi > 0$ and $r^\nu \varphi'(r) \to 0$ as $r \to \infty$, we have $r^\nu\varphi'(r) < 0$ for large r. Thus we may assume

$$\theta(r) \in \left(\frac{\pi}{2}, \pi\right) \quad \text{for large } r. \tag{17}$$

Using the equation for φ we see that $\theta' = \cos^2\theta - W(r)\sin^2\theta + \frac{\nu}{r}\cos\theta\sin\theta$, and we can find $\delta > 0, \sigma > 0$ such that for large r

$$\cos^2\theta - W(r)\sin^2\theta + \frac{\nu}{r}\cos\theta\sin\theta \begin{cases} < -\sigma & \text{in } [\frac{\pi}{2}, \frac{\pi}{2} + \delta], \\ > \sigma & \text{in } [\pi - \delta, \pi]. \end{cases}$$

From this fact, we can easily see that if $\theta_0(r_0) \in (\frac{\pi}{2}, \frac{\pi}{2} + \delta)$ holds for some large r_0, $\theta(r)$ reaches $\frac{\pi}{2}$ in a finite r. This is a contradiction and we have $\theta(r) \geqslant \frac{\pi}{2} + \delta$ for large r. In a similar way we can show $\theta(r) < \pi - \delta$. Thus we have $\theta(r) \in [\frac{\pi}{2} + \delta, \pi - \delta]$ for large r, from where (16) follows. □

Proposition 3.3. *Assume $b = \infty$, $\nu > 0$. For sufficiently small $\delta > 0$,* (12)–(13) *has at most one positive solution.*

Proof. As in the case $b < \infty$, uniqueness follows after we establish

$$E_\delta(r; w) > 0 \quad \text{for all } \delta \in (0, \delta_0),\ r \in [a, \infty) \text{ and } u \in S_\delta. \tag{18}$$

We argue indirectly and assume the existence of a sequence $\delta_n \to 0$ and $u_n \in S_{\delta_n}$ such that $E_{\delta_n}(r; w_n) \leqslant 0$ for some $r \in [a, \infty)$. We remark that

$$\frac{d}{dr} E_{\delta_n}(r; w_n) = -\frac{1}{2} G'_{\delta_n}(r) w_n(r)^2,$$

and that $G'_{\delta_n}(r) = r^{\beta-3}U(r) + \delta_n(\varphi(r)^{p-1}r^{\beta})'$. Thus, by Lemma 3.2 and hypothesis (V2′), there exist $c_1 > 0$ and $r_1 > a$ such that

$$G'_{\delta}(r) \geqslant c_1 r^{\beta-3} > 0 \quad \text{in } [r_1, \infty) \text{ for small } \delta > 0, \tag{19}$$

which implies $\frac{d}{dr} E_{\delta_n}(r; w_n) < 0$ in $[r_1, \infty)$ and then $E_{\delta_n}(r; w_n) > 0$ in $[r_1, \infty)$. Therefore there exists $r_n \in [a, r_1]$ such that

$$E_{\delta_n}(r_n; w_n) = \inf_{r \in [a,\infty)} E_{\delta_n}(r; w_n) \leqslant 0 \tag{20}$$

and, since $E_{\delta_n}(a; w_n) > 0$, r_n satisfies $G'_{\delta_n}(r_n) = 0$. By the argument in the proof of Proposition 3.1, (u_n) is uniformly bounded in $L^\infty(a, \infty)$ and then there exists a solution u_0 of (5) such that, up to a subsequence, $u_n \to u_0$ in $C^2_{\mathrm{loc}}([a, \infty))$ and $r_n \to \bar{r} \in [a, r_1]$. We claim that $u_0 \not\equiv 0$. Assuming this for the moment we see that $u_0(r)$ is a bounded, positive and $u(a) = 0$. Moreover we have

$$E\big(\bar{r}, w_0(\bar{r})\big) \leqslant 0 \quad \text{and} \quad G'(\bar{r}) = 0 \quad \text{if } \bar{r} > a.$$

By the hypothesis (V2′) and $G'(r) = r^{\beta-3}U(r)$, we have

(a) $\bar{r} = a$, or
(b) $\bar{r} > a$ and $U(\bar{r}) = 0$, that implies, $G'(r) < 0$ in $(a, \bar{r})$.

But these cases cannot take place. In fact, if $\bar{r} = a$, then $E(\bar{r}; w) = E(a; w) > 0$. Otherwise, we have $\frac{d}{dr} E(r; w_0) = -\frac{1}{2} G'(r) w_0^2 > 0$ in $(a, \bar{r})$ and thus $E(\bar{r}; w_0) \geqslant E(a; w_0) > 0$. Thus (18) holds.

Finally we prove the claim by contradiction, assuming that $u_0 \equiv 0$. From Eqs. (12)–(13) we see that $\|u_n\|_{L^\infty} \geqslant (\inf V/2)^{1/(p-1)}$, then for every $\varepsilon_0 > 0$ small there exists $\tilde{r}_n > a$ such that

$$\tilde{r}_n = \inf\big\{r \in [a, \infty) \bigm| u_n(r) \geqslant \varepsilon_0\big\}.$$

Since $u_0(r) \equiv 0$, we have $\tilde{r}_n \to \infty$ as $n \to \infty$ and, making ε_0 smaller if necessary, by using the maximum principle we can prove that

$$\big|u_n(r_1)\big| + \big|u'_n(r_1)\big| \leqslant c_2 e^{-c_3(\tilde{r}_n - r_1)}, \tag{21}$$

for some c_2, c_3 independent of n, with r_1 given by (19). Now we see that for a constant c_4 independent of n it holds

$$\|u_n\|_{L^2(\tilde{r}_n, \tilde{r}_n+1)} \geqslant c_4. \tag{22}$$

Since $w_n(r) = r^\alpha u_n(r)$ satisfies

$$r^\beta w_n'' + \frac{\beta}{2} r^{\beta-1} w_n' + (1+\delta_n) w_n^p - G_{\delta_n}(r) w_n = 0,$$

$$w_n(a) = 0, \qquad r^{\beta/2} w_n(r), r^{\beta/2} w_n'(r) \to 0 \quad \text{as } r \to \infty,$$

multiplying by w_n' and integrating from r_1 to ∞ we find

$$\frac{1}{2} r_1^\beta \left| w_n'(r_1) \right|^2 + \frac{1+\delta_n}{p+1} w_n(r_1)^{p+1} - \frac{1}{2} G_{\delta_n}(r_1) w_n(r_1)^2 = \frac{1}{2} \int_{r_1}^{\infty} G_{\delta_n}'(r) w_n^2 \, dr. \tag{23}$$

By (21), the left-hand side of (23) can be estimated from above by $C' e^{-2c_3(\tilde{r}_n - r_1)}$. On the other hand, by (19), (22), we can find a constant C'' so that

$$\int_{r_1}^{\infty} G_{\delta_n}'(r) w_n^2 \, dr \geqslant \int_{\tilde{r}_n}^{\tilde{r}_n+1} c_1 r^{\beta-3} w_n^2 \, dr = \int_{\tilde{r}_n}^{\tilde{r}_n+1} c_1 r^{2\alpha+\beta-3} u_n^2 \, dr \geqslant C'' \tilde{r}_n^{2\alpha+\beta-3}.$$

For large n this leads to a contradiction, completing the proof of the claim. □

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