# Laws of large numbers for the number of weak records

Raúl Gouet[a], F. Javier López[b,*], Gerardo Sanz[b]

[a] *Dpto. de Ingeniería Matemática and Centro de Modelamiento Matemático, Universidad de Chile, Av. Blanco Encalada 2120, 837-0459 Santiago, Chile*

[b] *Dpto. de Métodos Estadísticos, Facultad de Ciencias, Universidad de Zaragoza, C/ Pedro Cerbuna, 12. 50009 Zaragoza, Spain*

**Abstract**

We obtain strong laws of large numbers for the number of weak records among the first $n$ observations from a sequence of nonnegative integer-valued independent identically distributed random variables.

## 1. Introduction and notation

The theory of records is a well-established topic; see for instance, the books by Arnold et al. (1998) and Nevzorov (2001). Weak records were introduced in Vervaat (1973) as a modification of records for discrete distributions. For random variables $X_1, X_2, \ldots$, let $M_n = \max\{X_1, \ldots, X_n\}$; an observation $X_n$ is a record if $X_n > M_{n-1}$ and a weak record if $X_n \geq M_{n-1}$. When the random variables $X_n$ are continuous the notions of record and weak record coincide almost surely (since ties occur with probability zero) but for discrete distributions they may exhibit quite different behaviours. Weak records have attracted much attention in recent years, starting from the work of Stepanov (1992). See Aliev (1998), Bairamov and Stepanov (2006), Dembińska and Stepanov (2006), Stepanov et al. (2003), Wesolowski and Ahsanullah (2001) and Wesolowski and López-Blázquez (2004), among others. In those papers, attention is placed mainly on weak record values rather than on their counting process.

We are interested here in the asymptotic behaviour of the number of weak records among the first $n$ observations in a discrete setting. That is, for a sequence $\{X_n, n \geq 1\}$ of nonnegative integer-valued independent identically distributed (iid) random variables, letting $I_n = \mathbf{1}_{\{X_n \geq M_{n-1}\}}$ ($M_0 = -1$ by convention), we study the almost sure limiting behaviour of $N_n^w = \sum_{k=1}^n I_k$. Unlike the continuous case, where the indicators $I_k$ of an observation being a record (or weak record) are independent with $P[I_k = 1] = 1/k$, the study of the number of weak records in the discrete case is a difficult task since the independence and distribution-freeness of the indicators $I_k$ are lost. Although we state our results for integer-valued random variables, it is clear that they apply equally to random variables taking values on any denumerable set of real numbers, without accumulation points.

Strong laws of large numbers for the number of (ordinary) records in discrete models were given in Gouet et al. (2001). Key (2005) obtained asymptotic results for the number of records and weak records for a limited

* Corresponding author.

*E-mail addresses:* rgouet@dim.uchile.cl (R. Gouet), javier.lopez@unizar.es (F. Javier López), gerardo.sanz@unizar.es (G. Sanz).

class of heavy-tailed random variables. Also, a central limit theorem for the number of weak records is contained in Gouet et al. (2007).

In this paper we obtain strong laws of large numbers for $N_n^w$, for a fairly complete range of discrete distributions: heavy-, moderate- and light-tailed (Theorem 2.1) and give examples including the most common distributions (Section 3). As in the case of ordinary records, the normalizing sequences depend on the tail of the distribution of $X_n$. We observe that the number of weak records and the number of ordinary records are asymptotically equivalent for heavy-tailed distributions whereas they differ significantly as the tail of the distribution becomes lighter (see Remark 3.1).

We use the following notation. For $k \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, let $p_k = P[X_1 = k] > 0$, $y_k = P[X_1 > k]$ and $r_k = P[X_1 = k | X_1 \geq k] = p_k / y_{k-1}$ (with $y_{-1} = 1$), the discrete hazard rate. Notice that $y_k = \prod_{i=0}^{k}(1 - r_i)$. Also, let $F(t) = P[X_1 \leq t]$ be the distribution function of $X_1$, $F^-(t) = P[X_1 < t]$ and $m(t) = \min\{j \in \mathbb{Z}_+ \mid y_j < 1/t\}$, $t > 0$.

For two sequences of real numbers $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$, we write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$. We use the superscripted arrow $\xrightarrow{a.s.}$ for almost sure convergence of sequences of random variables.

## 2. Main result

The next result relates $N_n^w$, the number of weak records among the first $n$ observations $X_1, \ldots, X_n$, with partial sums of minima of certain random variables.

**Proposition 2.1.** *Let $Z_n = 1 - F^-(X_n)$, $n \geq 1$, and $S_n = \sum_{k=2}^{n} \min\{Z_1, \ldots, Z_{k-1}\}$. Then, $N_n^w / S_n \xrightarrow{a.s.} 1$.*

**Proof.** Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ be the $\sigma$-algebra generated by $X_1, \ldots, X_n$, $n \geq 1$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Then, for $k \geq 1$, $E[I_k|\mathcal{F}_{k-1}] = P[X_k \geq M_{k-1}|\mathcal{F}_{k-1}] = 1 - F^-(M_{k-1}) = 1 - F^-(\max\{X_1, \ldots, X_{k-1}\}) = \min\{1 - F^-(X_1), \ldots, 1 - F^-(X_{k-1})\}$.

As the number of weak records tends to infinity a.s., the conditional Borel–Cantelli lemma (Corollary VII-2-6 of Neveu (1972)) yields $N_n^w / \sum_{k=1}^{n} E[I_k|\mathcal{F}_{k-1}] \xrightarrow{a.s.} 1$, that is $N_n^w / \sum_{k=2}^{n} \min\{1 - F^-(X_1), \ldots, 1 - F^-(X_{k-1})\} \xrightarrow{a.s.} 1$. □

By Proposition 2.1, the asymptotic behaviour of $N_n^w$ is equivalent to that of the sum of minima $S_n$. In the following lemma, we study some properties of the sequence $\{Z_n, n \geq 1\}$. In particular, we obtain an explicit expression for the $H$ function, defined in Proposition A.1, which will be useful in the proof of the main result.

**Lemma 2.1.** *Let $\{Z_n, n \geq 1\}$ be as in Proposition 2.1.*

(a) *The random variables $Z_n$ are iid and take values $y_{j-1}$ with probabilities $p_j$, $j \geq 0$. The distribution function of $Z_k$ is $G(z) = y_j$ for $y_j \leq z < y_{j-1}$ and its inverse $G^{\leftarrow}(t) := \inf\{x \geq 0 \mid G(x) \geq t\} = y_{j-1}$ for $y_j < t \leq y_{j-1}$ (that is, $G^{\leftarrow}(1/t) = y_{m(t)-1}$).*

(b) *Let $H(y) = \int_0^y G^{\leftarrow}(\mathrm{e}^{-u})\mathrm{e}^u \mathrm{d}u$, $y \geq 0$. For $t > 1$,*

$$H(\log t) = \sum_{k=0}^{m(t)} r_k/(1 - r_k) - \rho(t),$$

*where $\rho(t) = y_{m(t)-1}(y_{m(t)}^{-1} - t)$. Moreover, $0 < \rho(t) \leq r_{m(t)}/(1 - r_{m(t)})$.*

**Proof.** (a) It follows directly from the definition of the random variables $Z_k$.

(b) After a change of variable in the integral defining $H(y)$, we obtain

$$
\begin{aligned}
H(\log t) &= \int_{1/t}^{1} \frac{G^{\leftarrow}(x)}{x^2}\mathrm{d}x \\
&= y_{m(t)-1}\int_{1/t}^{y_{m(t)-1}} \frac{\mathrm{d}x}{x^2} + \sum_{k=0}^{m(t)-1} y_{k-1}\int_{y_k}^{y_{k-1}} \frac{\mathrm{d}x}{x^2} \\
&= y_{m(t)-1}(t - y_{m(t)-1}^{-1}) + \sum_{k=0}^{m(t)-1} y_{k-1}(y_k^{-1} - y_{k-1}^{-1}) \\
&= \sum_{k=0}^{m(t)} y_{k-1}\, r_k/y_k - y_{m(t)-1}(y_{m(t)}^{-1} - t).
\end{aligned}
$$

The inequalities for $\rho(t)$ follow from $y_{m(t)} < 1/t \le y_{m(t)-1}$. □

We state and prove the main result of the paper. Notice that only in the case $\lim_{k\to\infty} r_k = 1$ (light-tailed distributions) an extra assumption on $r_k$ is needed; more precisely, the ratio $(1 - r_{k-1})/(1 - r_k)$ should tend to 1 rapidly enough.

**Theorem 2.1.** *Let* $\{X_n, n \ge 1\}$*, be a sequence of iid random variables on the nonnegative integers with* $p_k = P[X_1 = k] > 0$, $k \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.

(a) *Let* $\limsup_{k\to\infty} r_k < 1$*, then*

$$\frac{N_n^w}{\sum_{k=0}^{m(n)} (r_k/(1 - r_k))} \xrightarrow{a.s.} 1, \tag{1}$$

*as* $n \to \infty$*. Moreover, if* $r_k \to r \in [0, 1)$*, then*

$$\frac{N_n^w}{\log n} \xrightarrow{a.s.} \frac{-r}{(1 - r)\log(1 - r)}, \tag{2}$$

*as* $n \to \infty$*, with* $-0/\log 1 = 1$.

(b) *Let* $r_k \to 1$*; if there exists* $\alpha > 1/2$ *such that*

$$\lim_{k\to\infty} k^\alpha (r_k - r_{k-1})/(1 - r_{k-1}) = 0, \tag{3}$$

*then,*

$$\frac{N_n^w}{\sum_{k=0}^{m(n)} (1/(1 - r_k))} \xrightarrow{a.s.} 1, \tag{4}$$

*as* $n \to \infty$.

**Proof.** Let $\{Z_n, n \ge 1\}$ be the sequence defined in Proposition 2.1 and $G$ the distribution function of $Z_1$. Note that, as $p_k > 0$ for all $k \in \mathbb{Z}_+$, we have $G(y) > 0$ for all $y > 0$; moreover, by Lemma 2.1, $H(\log n) > \sum_{k=0}^{m(n)-1} r_k$ which tends to infinity as $n \to \infty$, since $\sum_{k=0}^{\infty} r_k = \infty$ for any discrete distribution with $p_k > 0$ for all $k \in \mathbb{Z}_+$. Now, the idea of the proof is to check (14) and (15) of Proposition A.1 to obtain a strong law of large numbers for $S_n$ defined in Proposition 2.1

(a) As $\limsup_{k\to\infty} r_k < 1$, there exists $\delta > 0$ such that $1 - r_k > \delta$ for all $k \ge 0$. From the definition of $G^{\leftarrow}$ and $m(t)$, we obtain, for all $t > 0$,

$$1 \le tG^{\leftarrow}(1/t) < y_{m(t)-1}/y_{m(t)} = 1/(1 - r_{m(t)}) < 1/\delta. \tag{5}$$

On the other hand, for $y > 1$, (5) implies

$$0 < \frac{H(y + \log y) - H(y)}{H(y)} = \frac{\int_{e^y}^{ye^y} G^{\leftarrow}(1/t)\mathrm{d}t}{\int_1^{e^y} G^{\leftarrow}(1/t)\mathrm{d}t} < \frac{\log y}{\delta y},$$

and (14) follows.

For (15) it suffices to see that (5) implies

$$\frac{nG^{\leftarrow}(1/n)^2}{\left(\sum_{k=2}^{n} G^{\leftarrow}(1/k)\right)^2} < \frac{1/(\delta^2 n)}{\left(\sum_{k=2}^{n} 1/k\right)^2} \sim \frac{1}{\delta^2 n(\log n)^2}.$$

Hence, (15) is obtained from the convergence of the series $\sum_{n=2}^{\infty}(n(\log n)^2)^{-1}$. Therefore, from Propositions 2.1 and A.1, we have $N_n^w/H(\log n) \xrightarrow{a.s.} 1$ and (1) follows from Lemma 2.1 and $\rho(n) < 1/\delta$.

In the case $r_k \to 0$, from (5) we obtain $tG^{\leftarrow}(1/t) \to 1$ as $t \to 0$, so $H(\log n) \sim \log n$. When $r_k \to r \in (0, 1)$, from Lemma 2.1(b) we have

$$H(\log n) \sim \sum_{k=0}^{m(n)} \frac{r_k}{1-r_k} \sim \frac{r}{1-r} m(n).$$

On the other hand, as $r \in (0, 1)$, it is known (see Proposition 3.3 in Gouet et al. (2001)) that $m(n) \sim -\log n/\log(1-r)$ and (2) follows.

(b) Since condition (3) for $\alpha \geq 1$ implies the condition for $\alpha < 1$, in what follows we suppose $\alpha \in (1/2, 1)$.

Let $a_k = 1/(1-r_k)$. Then, (3) can be written as

$$k^{\alpha}\left(1 - \frac{a_{k-1}}{a_k}\right) \to 0, \tag{6}$$

so there exists $k_0 \in \mathbb{Z}^+$ such that, for any $k > k_0$,

$$a_{k-1} > \left(1 - \frac{1}{k^{\alpha}}\right) a_k$$

and thus,

$$a_l > \prod_{i=l+1}^{k} \left(1 - \frac{1}{i^{\alpha}}\right) a_k, \tag{7}$$

for $l = k_0, \ldots, k-1$. Now, from the elementary inequality $\log(1-x) \geq -2x$, for all $0 < x < 1/\sqrt{2}$, we obtain $\log(1 - 1/i^{\alpha}) \geq -2/i^{\alpha}$, for all $i \geq 2$ and, therefore,

$$\begin{aligned} \prod_{i=k+1}^{n} \left(1 - \frac{1}{i^{\alpha}}\right) &= \exp\left(\sum_{i=k+1}^{n} \log\left(1 - \frac{1}{i^{\alpha}}\right)\right) \geq \exp\left(-2\sum_{i=k+1}^{n} \frac{1}{i^{\alpha}}\right) \\ &\geq \exp\left(-2\int_k^n \frac{1}{x^{\alpha}} \mathrm{d}x\right) = \exp\left(-\frac{2(n^{1-\alpha} - k^{1-\alpha})}{1-\alpha}\right), \end{aligned} \tag{8}$$

for all $k \geq 1$.

We begin by checking (15). First, recall from Lemma 2.1(b), that $H(\log t) = \sum_{k=0}^{m(t)} r_k/(1-r_k) - \rho(t)$, with $0 < \rho(t) \leq r_{m(t)}/(1-r_{m(t)})$. From (6) we have $a_k/a_{k-1} \to 1$ and, by Lemma A.1, recalling that $r_k \to 1$,

$$0 < \frac{\rho(t)}{\sum_{k=0}^{m(t)} r_k/(1-r_k)} \leq \frac{r_{m(t)}/(1-r_{m(t)})}{\sum_{k=0}^{m(t)} r_k/(1-r_k)} \to 0.$$

Therefore,

$$H(\log n) \sim \sum_{k=0}^{m(n)} \frac{r_k}{1-r_k} \sim \sum_{k=0}^{m(n)} a_k. \tag{9}$$

The series in (15) can be written as

$$\sum_{n=2}^{\infty} \frac{nG^{\leftarrow}(1/n)^2}{\left(\sum\limits_{k=2}^{n} G^{\leftarrow}(1/k)\right)^2} = \sum_{l=1}^{\infty} \sum_{n:m(n)=l} \frac{nG^{\leftarrow}(1/n)^2}{\left(\sum\limits_{k=2}^{n} G^{\leftarrow}(1/k)\right)^2}.$$

As $H(\log n) = \int_1^n G^{\leftarrow}(1/x)\mathrm{d}x$ and $G^{\leftarrow}$ is nondecreasing, we have $\sum_{k=2}^{n} G^{\leftarrow}(1/k) \sim H(\log n)$. Thus, as $l \to \infty$,

$$\sum_{n:m(n)=l} \frac{nG^{\leftarrow}(1/n)^2}{\left(\sum\limits_{k=2}^{n} G^{\leftarrow}(1/k)\right)^2} \sim \sum_{n:m(n)=l} \frac{ny_{l-1}^2}{\left(\sum\limits_{k=2}^{l} a_k\right)^2} = \frac{y_{l-1}^2 h(l)}{\left(\sum\limits_{k=2}^{l} a_k\right)^2}, \tag{10}$$

where $h(l) = \sum_{n:m(n)=l} n$. From $r_k \to 1$, we obtain $h(l) \sim (y_l^{-2} - y_{l-1}^{-2})/2 \le r_l/y_l^2$, so the right-hand side of (10) is bounded above by $Ca_l^2/A_l^2$ for some $C > 0$ and all $l \ge 2$, with $A_l = \sum_{k=2}^{l} a_k$. From (7),

$$\frac{a_n}{A_n} < \frac{a_n}{\sum\limits_{k=k_0}^{n} a_k} < \frac{1}{\sum\limits_{k=k_0}^{n-1} \prod\limits_{i=k+1}^{n} \left(1 - \frac{1}{i^{\alpha}}\right)}. \tag{11}$$

Now, from (8), and letting $\beta = 2/(1-\alpha)$,

$$\sum_{k=k_0}^{n-1} \prod_{i=k+1}^{n} \left(1 - \frac{1}{i^{\alpha}}\right) \ge \mathrm{e}^{-\beta n^{1-\alpha}} \sum_{k=k_0}^{n-1} \mathrm{e}^{\beta k^{1-\alpha}} \ge \mathrm{e}^{-\beta n^{1-\alpha}} \int_{k_0-1}^{n-1} \mathrm{e}^{\beta x^{1-\alpha}} \mathrm{d}x.$$

Since $\int_0^y \mathrm{e}^{\beta x^{1-\alpha}} \mathrm{d}x \sim \mathrm{e}^{\beta y^{1-\alpha}} y^{\alpha}/2$ as $y \to \infty$, we have $\int_{k_0-1}^{n-1} \mathrm{e}^{\beta x^{1-\alpha}} \mathrm{d}x \ge (n-1)^{\alpha} \mathrm{e}^{\beta (n-1)^{1-\alpha}}/3$, for large enough $n$. Moreover, $n^{1-\alpha} - (n-1)^{1-\alpha} \le 1/(n-1)^{\alpha} \to 0$. Therefore, for large $n$,

$$\sum_{k=k_0}^{n-1} \prod_{i=k+1}^{n} \left(1 - \frac{1}{i^{\alpha}}\right) \ge \frac{(n-1)^{\alpha} \mathrm{e}^{\beta((n-1)^{1-\alpha} - n^{1-\alpha})}}{3} \ge \frac{n^{\alpha}}{4}$$

so, from (11),

$$a_n/A_n < 4n^{-\alpha} \tag{12}$$

and the convergence of the series in (15) is deduced from $\sum_{n=1}^{\infty} n^{-2\alpha} < \infty$, since $\alpha \in (1/2, 1)$.

To check (14), note that (9) implies $H(\log n + \log\log n)/H(\log n) \sim A_{m(n \log n)}/A_{m(n)}$, so (14) is equivalent to $(A_{m(n \log n)} - A_{m(n)})/A_{m(n)} \to 0$. Now, for large $n$,

$$\begin{aligned}
\frac{A_{m(n \log n)} - A_{m(n)}}{A_{m(n)}} &= \frac{1}{A_{m(n)}} \sum_{k=m(n)+1}^{m(n \log n)} a_k \\
&< \frac{1}{A_{m(n)}} \sum_{k=m(n)+1}^{m(n \log n)} \frac{a_{m(n)}}{\prod\limits_{i=m(n)+1}^{k} (1 - 1/i^{\alpha})} \\
&< \frac{a_{m(n)}(m(n \log n) - m(n))}{A_{m(n)} \prod\limits_{i=m(n)+1}^{m(n \log n)} (1 - 1/i^{\alpha})} \\
&< 4 \frac{m(n \log n) - m(n)}{m(n)^{\alpha}} \mathrm{e}^{\beta (m(n \log n)^{1-\alpha} - m(n)^{1-\alpha})} \\
&\le 4 \frac{m(n \log n) - m(n)}{m(n)^{\alpha}} \mathrm{e}^{\beta \frac{m(n \log n) - m(n)}{m(n)^{\alpha}}}
\end{aligned}$$

where the first inequality follows from (7), the third from (8) and (12) and the last one since $m(n)$ is increasing. Thus, (14) is proved if we show that

$$\frac{m(n\log n)-m(n)}{m(n)^{\alpha}}\to 0,$$

for $\alpha\in(1/2,1)$. By (7) and (8),

$$a_n<\frac{a_{k_0}}{\prod_{i=k_0+1}^{n}(1-1/i^{\alpha})}<a_{k_0}\mathrm{e}^{\beta n^{1-\alpha}}.$$

Since $y_{n-1}/y_n=a_n$, we obtain $y_{n-1}<a_{k_0}\mathrm{e}^{\beta n^{1-\alpha}}y_n$ and there exists some $C>0$ such that, for all $n>k_0$,

$$y_n>y_{k_0}a_{k_0}^{k_0-n}\exp\left(-\beta\sum_{k=k_0+1}^{n}k^{1-\alpha}\right)>\exp\left(-Cn^{2-\alpha}\right),$$

where the last inequality follows from $\alpha\in(1/2,1)$. Therefore, for large $n$,

$$\frac{1}{n}>y_{m(n)}>\exp\left(-Cm(n)^{2-\alpha}\right),$$

which implies $\log\log n<\log C+(2-\alpha)\log(m(n))$. Then, there exists $C'>0$ such that

$$\frac{\log\log n}{\log(m(n))}<C' \tag{13}$$

for all $n\geq 3$. On the other hand, it is known that $r_k\to 1$ implies $m(n\log n)-m(n)-1<\gamma\log\log n$, for some $\gamma>0$ and all large enough $n$, (see page 789 of Gouet et al. (2005)). Thus, by (13),

$$\frac{m(n\log n)-m(n)}{m(n)^{\alpha}}<\frac{\gamma\log\log n+1}{m(n)^{\alpha}}<\frac{\gamma C'\log(m(n))+1}{m(n)^{\alpha}}\to 0$$

and (14) is proved. Now (4) follows from Propositions 2.1 and A.1. □

## 3. Examples

**Example 3.1** (*Zeta Distribution*). An example of discrete distribution with $r_k\to 0$ is the Zeta distribution ($p_k=C(k+1)^{-a}, k\in\mathbb{Z}_+, a>1$). We obtain, from Theorem 2.1(a),

$$\frac{N_n^w}{\log n}\xrightarrow{a.s.}1.$$

**Example 3.2** (*Geometric and Negative Binomial Distributions*). The geometric distribution with parameter $p$ ($p_k=pq^k, k\in\mathbb{Z}_+, p\in(0,1), q=1-p$) has $r_k=p$ for all $k\geq 0$. For the negative binomial distribution ($p_k=(-1)^k\binom{-a}{k}p^aq^k$, for $k\in\mathbb{Z}_+, p\in(0,1), q=1-p$ and $a>1$) it is shown in Vervaat (1973) that $p-(a-1)q/k\leq r_k\leq p$ so $r_k\to p$. In both cases, we obtain

$$\frac{N_n^w}{\log n}\xrightarrow{a.s.}-p/(1-p)\log(1-p)$$

from Theorem 2.1(a).

**Example 3.3** (*Alternating Geometric*). For an example of nonconverging failure rates, we consider the distribution corresponding to the number of tails before the first head in a sequence of tosses with two alternating coins; that is, a coin with probability of heads $p_o\in(0,1)$ is used for odd tosses and another coin with probability of heads $p_e\in(0,1)$ is used for even tosses. We have $p_{2k}=q_o^kq_e^kp_o$ and $p_{2k+1}=q_o^{k+1}q_e^kp_e, k\geq 0$ (with $q_o=1-p_o, q_e=1-p_e$) so

$r_{2k} = p_o$ and $r_{2k+1} = p_e$, $k \geq 0$. It is easy to see that $m(n) \sim -2\log n/\log q_o q_e$ (Example 3 in Gouet et al. (2005)) and

$$\sum_{k=0}^{m(n)} \frac{r_k}{1-r_k} \sim \sum_{i=0}^{\lfloor m(n)/2 \rfloor} \frac{r_{2i}}{1-r_{2i}} + \sum_{i=0}^{\lfloor m(n)/2 \rfloor} \frac{r_{2i+1}}{1-r_{2i+1}} \sim -\left(\frac{p_o}{q_o} + \frac{p_e}{q_e}\right) \frac{\log n}{\log q_o q_e},$$

where $\lfloor . \rfloor$ denotes the largest integer less than or equal to its argument. Then, Theorem 2.1(a) yields

$$\frac{N_n^w}{\log n} \xrightarrow{a.s.} -\frac{p_o/q_o + p_e/q_e}{\log q_o q_e}.$$

**Example 3.4** (*Poisson Distribution*). Let $X$ have Poisson distribution with parameter $\lambda > 0$ (that is $p_k = \mathrm{e}^{-\lambda}\lambda^k/k!$, for $k \in \mathbb{Z}_+$). It can be found in Vervaat (1973) that

$$\frac{\lambda}{k+1} - \left(\frac{\lambda}{k+1}\right)^2 \leq 1 - r_k \leq \frac{\lambda}{k+1}.$$

Thus, $r_k \to 1$ and there exists $C > 0$ such that $(r_k - r_{k-1})/(1 - r_{k-1}) < C/k$ so condition (3) holds with $\alpha = 3/4$. Moreover, $a_k = 1/(1 - r_k) \sim k/\lambda$ and $m(n) \sim \log n/\log\log n$ so $\sum_{k=0}^{m(n)} a_k \sim \frac{1}{\lambda}\sum_{k=0}^{m(n)} k \sim \frac{1}{2\lambda}(\log n/\log\log n)^2$. Thus, by Theorem 2.1(b), we obtain

$$\frac{N_n^w}{(\log n/\log\log n)^2} \xrightarrow{a.s.} \frac{1}{2\lambda}.$$

**Remark 3.1.** It is interesting to compare $N_n^w$ with the counting process of ordinary records $N_n = \sum_{k=1}^n \mathbf{1}_{\{X_k > M_{k-1}\}}$, whose behaviour was analyzed in Gouet et al. (2001). For heavy-tailed distributions (those with $r_k \to 0$) such as the Zeta distribution, $N_n^w$ and $N_n$ are asymptotically equivalent, since $N_n/\log n \xrightarrow{a.s.} 1$. In the case of distributions with $r_k \to r \in (0, 1)$ (such as the geometric or negative binomial distributions), $N_n/\log n \xrightarrow{a.s.} -r/\log(1 - r)$ so the number of records also grows at a logarithmic speed, but with a smaller constant.

For light-tailed distributions (with $r_k \to 1$) the number of weak records grows at a higher speed than the number of records. In fact, under (3), Proposition 3.4(ii) of Gouet et al. (2001) implies $N_n/m(n) \xrightarrow{a.s.} 1$ while the normalizing sequence for $N_n^w$ is $\sum_{k=0}^{m(n)}(1/(1 - r_k))$, with $1/(1 - r_k) \to \infty$. In the the particular case of the Poisson distribution we have $N_n/(\log n/\log\log n) \xrightarrow{a.s.} 1$ whereas $N_n^w$ is normalized by $\frac{1}{2\lambda}(\log n/\log\log n)^2$.


## Acknowledgements

The first author thanks the Departamento de Métodos Estadísticos of Universidad de Zaragoza for their kind hospitality. The first author was supported by the Fondap in Applied Mathematics and Fondecyt grant 1060794. All the authors were supported by the project MTM2004-01175 of MEC and are members of the research group Modelos Estocásticos DGA.


## Appendix

**Proposition A.1.** *Let* $\{Y_n, n \geq 1\}$ *be a sequence of iid nonnegative random variables with common distribution function* $G$ *such that* $G(t) > 0$ *for all* $t > 0$ *and* $S_n = \sum_{i=1}^n \min\{Y_1, \dots, Y_i\}$. *Let* $G^{\leftarrow}(t) = \inf\{x \geq 0 \mid G(x) \geq t\}$, *for* $0 \leq t < 1$ *and* $H(y) = \int_0^y G^{\leftarrow}(\mathrm{e}^{-u})\mathrm{e}^u \mathrm{d}u$, *for* $y \geq 0$. *If* $\lim_{y\to\infty} H(y)$ *is finite, then* $S_n$ *grows a.s. to a finite limit. Otherwise, if* $\lim_{y\to\infty} H(y) = \infty$,

$$\frac{H(y + \log y)}{H(y)} \to 1 \quad \text{as } y \to \infty \quad \text{and} \tag{14}$$

$$\sum_{n=2}^{\infty} \frac{nG^{\leftarrow}(1/n)^2}{\left[\sum_{i=2}^{n} G^{\leftarrow}(1/i)\right]^2} < \infty, \tag{15}$$

*then*

$$\frac{S_n}{H(\log n)} \xrightarrow{a.s.} 1.$$

**Proof.** See Corollaire 4 of Deheuvels (1974).

**Lemma A.1.** *Let* $\{b_n, n \geq 1\}$ *be a sequence of positive terms such that* $b_n \to \infty$ *and* $b_n/b_{n-1} \to 1$. *Then* $b_n/\sum_{i=1}^n b_i \to 0$.

**Proof.** Let $\epsilon > 0$ and take $N \in \mathbb{N}$ such that $b_n - b_{n-1} < \epsilon b_n$, for all $n \geq N$. Then, for $n \geq N$,

$$b_n - b_0 = \sum_{i=1}^n (b_i - b_{i-1}) \leq b_N - b_0 + \epsilon \sum_{i=N+1}^n b_i \leq b_N - b_0 + \epsilon \sum_{i=1}^n b_i,$$

and the conclusion follows. $\square$

## References


Aliev, F.A., 1998. Characterization of distributions through weak records. J. Appl. Statist. Sci. 8, 13–16.

Arnold, B.C., Balakrishnan, N., Nagaraja, H.N., 1998. Records. Wiley, New York.

Bairamov, I., Stepanov, A.V., 2006. A note on large deviations for weak records. Statist. Probab. Lett. 76, 1449–1453.

Dembińska, A., Stepanov, A.V., 2006. Limit theorems for the ratio of weak records. Statist. Probab. Lett. 76, 1454–1464.

Deheuvels, P., 1974. Valeurs extrémales d'échantillons croissants d'une variable aléatoire réelle. Ann. Inst. H. Poincaré X 89–114.

Gouet, R., López, F.J., San Miguel, M., 2001. A martingale approach to strong convergence of the number of records. Adv. Appl. Probab. 33, 864–873.

Gouet, R., López, F.J., Sanz, G., 2005. Central limit theorems for the number of records in discrete models. Adv. Appl. Probab. 37, 781–800.

Gouet, R., López, F.J., Sanz, G., 2007. Asymptotic normality for the counting process of weak records and $\delta$-records in discrete models. Bernoulli 13, 754–781.

Key, E.S., 2005. On the number of records in an iid discrete sequence. J. Theoret. Probab. 18, 99–107.

Neveu, J., 1972. Martingales à Temps Discret. Masson, Paris.

Nevzorov, V.B., 2001. Records: Mathematical Theory. In: Translations of Mathematical Monographs, vol. 194. American Mathematical Society, Providence, RI.

Stepanov, A.V., 1992. Limit theorems for weak records. Theory Probab. Appl. 37, 570–574.

Stepanov, A.V., Balakrishnan, N., Hofmann, G., 2003. Exact distribution and Fisher information of weak record values. Statist. Probab. Lett. 64, 69–81.

Vervaat, W., 1973. Limit theorems for records from discrete distributions. Stochastic Processes Appl. 1, 317–334.

Wesolowski, J., Ahsanullah, M., 2001. Linearity of regression for non-adjacent weak records. Statist. Sinica 11, 39–52.

Wesolowski, J., López-Blázquez, F., 2004. Linearity of regression for the past weak and ordinary records. Statistics 38, 457–464.