A new family of expansive graphs[☆,☆☆]

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Abstract

An *affine graph* is a pair (G, σ) where G is a graph and σ is an automorphism assigning to each vertex of G one of its neighbors. On one hand, we obtain a structural decomposition of any affine graph (G, σ) in terms of the orbits of σ. On the other hand, we establish a relation between certain colorings of a graph G and the intersection graph of its cliques $K(G)$. By using the results we construct new examples of *expansive* graphs. The *expansive* graphs were introduced by Neumann-Lara in 1981 as a stronger notion of the *K-divergent* graphs. A graph G is *K-divergent* if the sequence $|V(K^n(G))|$ tends to infinity with n, where $K^{n+1}(G)$ is defined by $K^{n+1}(G) = K(K^n(G))$ for $n \geqslant 1$. In particular, our constructions show that for any $k \geqslant 2$, the complement of the Cartesian product C^k, where C is the cycle of length $2k + 1$, is K-divergent.

Keywords: Clique operator; Affine graphs; Expansivity

1. Introduction

Let G be a graph. The clique graph of G, denoted by $K(G)$, is the intersection graph of the cliques (maximal complete subgraphs) of G. A main question regarding the clique graph operator K is the study of the behavior of the iterated application of K. A graph G is *K-divergent* if the iterations $K^{i+1}(G) = K(K^i(G))$, for $i \geqslant 0$, generate a family of graphs whose sizes tend to infinity. On one hand, a graph can be proved to be K-divergent by explicitly computing all its iterated clique graphs. This approach was used in [4] to prove the K-divergence of the n-dimensional octahedron O_n and later, it was used to prove the K-divergence of locally C_6 graphs [1] and clockwork graphs [2]. On the other hand, it is known that clique divergence is preserved by some morphism (retractions [4], coverings [1]). Notions stronger than K-divergency are rank divergence [3] and expansivity [5].

In this work we provide new families of *expansive graphs*. This notion, introduced by Neumann-Lara in [5] was developed in the scope of coaffine graphs. A *coaffination* σ of a graph $G \neq \emptyset$ is an automorphism of G such that for

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every $u \in V(G)$, $u \neq \sigma(u)$ and $\sigma(u) \notin N(u)$. A *coaffine graph* is a pair (G, σ) where G is a graph and σ is a coaffine automorphism of G. For the purpose of this work it is more convenient to deal with the complement of coaffine graphs. An *affine graph* is a pair (G, σ), where G is a graph and σ is an automorphism such that the image of each vertex is one of its neighbors. Clearly, (G, σ) is an affine graph if and only if $(\overline{G}, \sigma)$ is a coaffine graph.

For (co)affine graphs (G, σ) and (H, ϕ), a morphism $f: (G, \sigma) \to (H, \phi)$ is *admissible* if $f \circ \sigma = \phi \circ f$. Moreover, if f is an isomorphism, then we say that (G, σ) and (H, ϕ) are *isomorphic*.

When G is a stable subgraph of H under $\phi(\phi(G) \subseteq G)$, we say that (G, ϕ_G) is an *admissible subgraph* of (H, ϕ), where ϕ_G is the restriction of ϕ to G.

A coaffine graph (G, σ) is *expansive* if there exist a sequence $n_1, n_2, \ldots$ of integers, with $n_i \to \infty$, and a sequence $(H_1, \phi_1), (H_2, \phi_2), \ldots$ of coaffine graphs such that:

- $H_i = H_{i,1} + \cdots + H_{i,r_i}$,[1] $r_i \to \infty$.
- For all i, (H_i, ϕ_i) is an admissible subgraph of $(K^{n_i}(G), \sigma_K^{n_i})$, where $\sigma_K^{n_i}$ is the canonical coaffination of $K^{n_i}(G)$ induced by σ [5].

As previously mentioned, the goal of this work is to provide new examples of expansive graphs. Our starting point is the following result.

Theorem 1 (*Neumann-Lara [5]*). *If (G, σ) and (H, ϕ) are affine graphs such that $(\overline{G} + \overline{H}, \sigma + \phi)$ is an admissible subgraph of $(K(\overline{G}), \sigma_K)$, then $(\overline{G}, \sigma)$ is expansive, where $\sigma_K(Q) = \sigma(Q)$ for all Q clique of $\overline{G}$.*

Since an edgeless graph is coaffinable we get the following.

Proposition 2. *Let (I, ϕ) be an edgeless graph with at least two vertices together with a coaffination, and let (G, σ) be an affine graph. If $(\overline{G} + I, \sigma + \phi)$ is an admissible subgraph of $(K(\overline{G}), \sigma_K)$, then $(\overline{G}, \sigma)$ is expansive.*

In order to use Proposition 2 we will proceed as follows. In Section 2 we give a complete characterization of affine graphs in terms of their affinations. In Section 3 we present sufficient conditions on (G, σ) to satisfy the *inclusion property* of Proposition 2: $(\overline{G} + I, \sigma + \phi)$ is an admissible subgraph of $(K(\overline{G}), \sigma_K)$. Finally, in Section 4, by using these results we construct new families of expansive graphs.

2. Affine graphs

In this section we give a characterization of affine graphs.

For an affine graph (G, σ), the orbit of a vertex $u \in V(G)$ by σ is the set

$$O(u) = \{v \in V(G): \exists i \in \mathbb{Z}, \sigma^i(u) = v\}.$$

It is clear that the set of orbits is a partition of the set of vertices of G. We define the orbit index of (G, σ), denoted by $\eta(G, \sigma)$, as the number of orbits induced by σ in G. We first study affine graphs with orbit index equals to one. Let $p \geqslant 2$ be an integer and let S be a set of elements in $\mathbb{Z}_p \backslash \{0\}$ such that $S = -S$. The *circulant graph* $D_p(S)$ of order p with connection set S, is the graph whose set of vertices is $V = \mathbb{Z}_p$ and such that ij is an edge of $D_p(S)$ if and only if $j - i \bmod p \in S$. In the rest of the paper we assume that all the arithmetic operations are carried out in $\mathbb{Z}_p$.

Let $D_p(S)$ be a circulant graph, if $1 \in S$ we say that the graph is *unitary*. Clearly, for any unitary circulant graph $D_p(S)$, the automorphism $\sigma_D(j) = j + 1$ is an affination. Hence, $(D_p(S), \sigma_D)$ is an affine graph.

Any affine graph (G, σ) isomorphic to an affine graph $(D_p(S), \sigma_D)$, where $D_p(S)$ is a unitary circulant graph, will be called a *transitively affine* graph. Let φ be an isomorphism between (G, σ) and $(D_p(S), \sigma_D)$. Since $\sigma = \varphi^{-1} \circ \sigma_D \circ \varphi$ and $\eta(D_p(S), \sigma_D) = 1$ we get that $\eta(G, \sigma) = 1$. Therefore, we have proved the forward implication of the following lemma.

[1] $G + H = (V(G) \cup V(H), E(G) \cup E(H) \cup \{\{i, j\}: i \in V(G), j \in V(H)\}$.

Lemma 3. *An affine graph (G, σ) is a transitively affine graph if and only if $\eta(G, \sigma) = 1$.*

Proof. Conversely, let us assume that $\eta(G, \sigma) = 1$ and $|G| = p$. Let u be any vertex of G and let $\phi: \mathbb{Z}_n \to V(G)$ be defined by $\phi(i) = \sigma^i(u)$. Since $\eta(G, \sigma) = 1$, the function ϕ is a bijection between $\mathbb{Z}_p$ and $V(G)$. Let $S = \{\phi^{-1}(w) - \phi^{-1}(v), \phi^{-1}(v) - \phi^{-1}(w): wv \in E(G)\} \subseteq \mathbb{Z}_p$. Then, $S = -S$ and, since $u\sigma(u)$ is an edge of G, we get that $1 \in S$. Hence, $D_p(S)$ is a unitary circulant graph. Moreover, ϕ is an isomorphism between $D_p(S)$ and G satisfying $\phi \circ \sigma_D = \sigma \circ \phi$. Therefore, (G, σ) and $(D_p(S), \sigma_D)$ are isomorphic. □

Let $U \subseteq V(G)$ be such that $\sigma(U) \subseteq U$. Then the pair $(G, \sigma)_U := (G[U], \sigma_U)$ is an affine graph which is an admissible subgraph of (G, σ), where $G[U]$ is the graph induced by U in G and σ_U is the restriction of σ to U.

Corollary 1. *Let O be an orbit of the affine graph (G, σ). Then, $(G, \sigma)_O$ is a transitively affine graph.*

Since each orbit of an affine graph is a transitively affine graph, a complete structural description of an affine graph can be obtained by describing how their orbits are connected.

Let A and B be two disjoint sets. We denote by $[A, B]$ the set of all subsets of $A \cup B$ with exactly one element of A and one element of B. When A and B are disjoint subsets of the vertex set of a graph G, we denote by $[A, B]_G$ the set of all edges between A and B in G.

Let $((V_1, E_1), \sigma_1)$ and $((V_2, E_2), \sigma_2)$ be two vertex-disjoint affine graphs and let $F \subseteq [V_1, V_2]$. We define the *affine-coupling* between $((V_1, E_1), \sigma_1)$ and $((V_2, E_2), \sigma_2)$ with *generator* F as the pair

$$(G, \sigma) := ((V_1, E_1), \sigma_1)\|_F((V_2, E_2), \sigma_2),$$

where G is the graph given by $V(G) = V_1 \cup V_2$ and

$$E(G) = E_1 \cup E_2 \cup \{\{\sigma_1^i(u_1), \sigma_2^i(u_2)\}: \{u_1, u_2\} \in F, u_1 \in V_1, u_2 \in V_2, i \in \mathbb{N}\}$$

and $\sigma(u) = \sigma_i(u)$ if $u \in V_i$ with $i = 1, 2$.

Lemma 4. *Let $((V_1, E_1), \sigma_1), ((V_2, E_2), \sigma_2)$ be two affine graphs and let $F \subseteq [V_1, V_2]$. Then, $(G, \sigma) := ((V_1, E_1), \sigma_1)\|_F((V_2, E_2), \sigma_2)$ is an affine graph.*

Proof. It is clear that $\sigma(u)$ is a neighbor of u in G, for each vertex u of G. Since, for $i = 1, 2$, σ_i is bijective and so is σ. It remains to prove that σ is a morphism. Since σ coincides with σ_i on V_i, $i = 1, 2$, we only need to consider the image under σ of an edge v_1v_2 in G with $v_i \in V_i$, $i = 1, 2$. By definition of the affine-coupling there are an integer j and an element $u_1u_2 \in F$ such that $\sigma_i^j(u_i) = v_i$, $i = 1, 2$. Therefore, $\sigma(v_1)\sigma(v_2) = \sigma_1^{j+1}(u_1)\sigma_2^{j+1}(u_2)$ and then $\sigma(v_1)\sigma(v_2)$ is an edge of G. □

Let (G, σ) be an affine graph with $\eta(G, \sigma) \geqslant 2$. We say that two orbits O_1, O_2 of σ are *adjacent* if $[O_1, O_2]_G \neq \emptyset$. We prove that the affine graph $(G, \sigma)_{O_1 \cup O_2}$ is the affine-coupling of $(G, \sigma)_{O_1}\|_F(G, \sigma)_{O_2}$, for some set F.

Lemma 5. *Let (G, σ) be an affine graph with $\eta(G, \sigma) \geqslant 2$. Let O_1, O_2 be two adjacent orbits of G. Then, there exists a set $F \subseteq [O_1, O_2]_G$ such that the affine graph $(G, \sigma)_{O_1 \cup O_2}$ is the affine-coupling $(G, \sigma)_{O_1}\|_F(G, \sigma)_{O_2}$.*

Proof. By definition, $(G, \sigma)_{O_1 \cup O_2} = (H, \sigma_{O_1 \cup O_2})$, where H is the graph induced by $O_1 \cup O_2$ in G. Let $u \in O_1$ be and let F be the set of all edges in $[O_1, O_2]_G$ incident with u. Let $(H', \sigma_{O_1 \cup O_2})$ be the affine-coupling $(G, \sigma)_{O_1}\|_F(G, \sigma)_{O_2}$. Then, we only have to prove that $H' = H$. As $V(H) = V(H')$, we prove that $E(H) = E(H')$. On one hand, since σ is an automorphism and $F \subseteq E(H)$, we get $E(H') \subseteq E(H)$. On the other hand, the edges in the graph induced by O_i are included in $E(H')$, for $i = 1, 2$. Moreover, every edge $v_1v_2 \in E(H)$ with $v_i \in O_i$, $i = 1, 2$, is associated with the edge $uu' \in F$, where $u' = \sigma^{-j}(v_2)$ and $\sigma^j(u) = v_1$, for some integer j. This shows that $v_1v_2 \in E(H')$. Therefore $E(H) \subseteq E(H')$. □

Since the set of orbits of an affine graph is a partition of the set of vertices of the graph, each edge belongs to an affine-coupling of two orbits. Hence a relevant information is given by the connections between the orbits. Let (G, σ)

be an affine graph, its *orbit-graph* is the graph H_σ where each orbit of G is a vertex of H_σ and two vertices are adjacent if and only if their corresponding orbits are adjacent too. We know that for each orbit u of G the affine graph $(G, \sigma)_u$ is a transitively affine graph. Moreover, from Lemma 5 we know that each edge e of H_σ is associated with a set of edges F_e. Hence, each affine graph G can be completely described by the triple $(H_\sigma, D_\sigma, F_\sigma)$, where $D_\sigma = \{(G, \sigma)_u : u \in V(H_\sigma)\}$ and $F_\sigma = \{F_e : e \in E(H_\sigma)\}$. Conversely, for a graph H, a family $D = \{(D_u, \sigma_u)\}_{u \in V(H)}$ of transitively affine graphs and a family $F = \{F_e\}_{e \in E(H)}$ of generators, we define the *affine construction* $((V, E), \sigma) := \Lambda(H, D, F)$ as follows: $V = \{(i, u) : i \in V(D_u), u \in V(H)\}$ and the function σ is the common extension of all the affine functions σ_u, $u \in V(H) : \sigma(i, u) := \sigma_u(i)$.The set E is such that for every edge $e = uv \in E(H)$ the affine graph $((V, E), \sigma)_{V(D_u) \cup V(D_v)}$ is the affine-coupling $(V(D_u), \sigma_u) \|_{F_e} (V(D_v), \sigma_v)$.

From Lemma 5 it is not difficult to see that $\Lambda(H, D, F)$ is an affine graph. Moreover, given an affine graph (G, σ) it follows that (G, σ) is isomorphic to $\Lambda(H_\sigma, D_\sigma, F_\sigma)$. Therefore, we have proved the following theorem which is a characterization of the affine graphs.

Theorem 6. *Let G be a graph. (G, σ) is an affine graph if and only if there exist a graph H, a family of transitively affine graphs $D = \{(D_u, \sigma_u)\}_{u \in V(H)}$ and a family of generators $F = \{F_e\}_{e \in E(H)}$ such that (G, σ) is isomorphic to $\Lambda(H, D, F)$.*

3. Locally bijective coloring

In this section we obtain sufficient conditions on affine graphs to satisfy the inclusion property of Proposition 2. These conditions will lead us to define a new class of graphs containing the complements of the n-dimensional octahedra and some cycles known to be K-divergent. Unfortunately, not every graph in this class is an affine graph.

A (proper) vertex coloring c of a graph $G = (V, E)$ is *locally bijective* if each color appears exactly once in the closed neighborhood of each vertex. Clearly, each monochromatic set defined by a locally bijective coloring is a dominating set and it is easy to see that all have the same size. Hence, if c uses p colors, then $|G| = rp$, where r is the size of each monochromatic set and the graph is $(p-1)$-regular. For some values of r and p we characterize the locally bijectively colorable graphs.

Lemma 7. *Let G be a graph. If G admits a locally bijective coloring with p colors then*:

(1) *$|G| = p$ if and only if $G = K_p$, the complete graph on p vertices.*
(2) *$p = 1$ if and only if G has no edges.*
(3) *$p = 2$ if and only if $G = \overline{O_r}$, the complement of the r-dimensional octahedron, with $r = |G|/2$.*
(4) *$p = 3$ if and only if G is the vertex-disjoint union of cycles of length $3k$, $k \in \mathbb{N}$.*

Proof. The sufficient conditions come from the fact that a $(p-1)$-regular graph which admits a locally bijective coloring uses p colors. The necessary conditions are proved as follows:

(1) The only graph with p vertices and $(p-1)$-regular is the complete graph on p vertices.
(2) If there is only one color, then the graph has no edges.
(3) If there are two colors, then each vertex has exactly one neighbor. Hence, the set of edges of G is a perfect matching of G. Therefore, $G = \overline{O_r}$.
(4) Let c be a locally bijective coloring with three colors. Then G is a 2-regular graph. Hence, it is the disjoint union of cycles. Since c induces a locally bijective coloring with three colors in each cycle, its size is a multiple of 3. □

Besides the fact that a graph admitting a locally bijective coloring is not necessarily an affine graph we can prove the following.

Theorem 8. *Let G be a K_3-free graph. If G admits a locally bijective coloring c with p colors, $2 \leqslant p \leqslant |G|/3$, then $\overline{G}_K + I_p$ is an induced subgraph of $K(\overline{G})$, where $\overline{G}_K$ is a subgraph of $K(\overline{G})$ isomorphic to $\overline{G}$.*

Proof. Let $\mathscr{M}$ be the set of monochromatic sets defined by c. By the choice of p the set $\mathscr{M}$ has at least two elements and each of its elements has size $|G|/p \geqslant 3$. Moreover, the size of the neighborhood of each vertex is $p-1 \geqslant 1$. Every monochromatic set is a maximal independent set, so it is a clique in $\overline{G}$. Hence $\mathscr{M} \subseteq V(K(\overline{G}))$. Since the elements of $\mathscr{M}$ are pairwise disjoint, the set $\mathscr{M}$ induces in $K(\overline{G})$ an independent set I_p, of size p.

For each vertex $v \in V(G)$ we define the set $M^v := M(v) \backslash \{v\} \cup N(v)$, where $M(v)$ is the (unique) element in $\mathscr{M}$ containing v. Let $\overline{G}_K$ be the intersection graph of the family $\{M^v : v \in V(G)\}$. We shall prove that $\overline{G}_K$ is an induced subgraph of $K(\overline{G})$ isomorphic to $\overline{G}$: We prove that M^v is a clique of $\overline{G}$ and that $M^v \cap M^u \neq \emptyset$ if and only if uv is not an edge of G.

To prove that M^v is a clique in $\overline{G}$ we show that it is a dominating independent set of G. Since G is K_3-free the set $N(v)$ is an independent set. Since for each $w \in N(v)$ the vertex v is the only neighbor of w with color $c(v)$, no neighbor of v is adjacent with other vertex in $M(v)$. Then M^v is an independent set of G. We now show that M^v is a dominating set of G. Since $N(v) \neq \emptyset$, the vertex v has a neighbor in M^v. Let $z \neq v$, $z \notin M^v$ be a vertex. Since c is a locally bijective coloring and $z \notin N(v)$, the vertex z has a neighbor in $M(v) \backslash \{v\} \subseteq M^v$. Hence M^v is a dominating set. Therefore $\overline{G}_K$ is an induced subgraph of $K(\overline{G})$.

We now prove that $\overline{G}_K$ is isomorphic to $\overline{G}$, by showing that $M^v \cap M^u \neq \emptyset$ if and only if u and v are not adjacent in G. Let us assume first that u and v are not adjacent. Since each monochromatic set has at least three elements, if $M(u) = M(v)$, then there is a vertex $w \in M(u) \backslash \{u, v\}$. Hence $w \in M^u \cap M^v$.

Let us assume that $M(u) \neq M(v)$. Then, $v \notin M(u)$ and there is a $w \in M(u) \cap N(v)$, $w \neq u$. Hence $w \in M^v \cap M^u$.

Conversely, let us suppose that u and v are adjacent in G. Since G is K_3-free $N(u) \cap N(v) = \emptyset$. Moreover, $N(v) \cap M(u) = \{u\}$, $N(u) \cap M(v) = \{v\}$ and $M(u) \cap M(v) = \emptyset$. Therefore, $M^v \cap M^u = \emptyset$. Then $\overline{G}_K$ is isomorphic to $\overline{G}$.

It is clear that for every $u \in V(G)$ and every $M \in \mathscr{M}$, $M^u \cap M \neq \emptyset$. Then $\overline{G}_K + I_p$ is an induced subgraph of $K(\overline{G})$. $\square$

In order to relate Theorem 8 with Proposition 2 we must provide an affination σ on G, such that $(\overline{G}_K + I_p, \phi)$ is an admissible subgraph of $(K(\overline{G}), \sigma_K)$, where ϕ is the restriction of σ_K to $V(\overline{G}_K) \cup V(I_p)$.

Let (G, σ) be an affine graph. A coloring c considered as a morphism from $V(G)$ onto K_p is an *admissible coloring* of (G, σ) if there exists a permutation π of the vertices of K_p such that $c \circ \sigma = \pi \circ c$. It is easy to see that the image of a monochromatic set under an admissible coloring is a monochromatic set.

From Theorem 8 and Proposition 2 we get the following corollary.

Corollary 9. *Let (G, σ) be an affine graph with G a K_3-free graph and $|G| \geqslant 2$. If (G, σ) admits an admissible locally bijective coloring c with at most $|G|/3$ colors, then $(\overline{G}, \sigma)$ is expansive.*

Proof. Let p be the number of colors used by c. As an affine graph has at least one edge, $2 \leqslant p \leqslant |G|/3$. By Theorem 8, the graph $\overline{G}_K + I_p$ is an induced subgraph of $K(\overline{G})$.

For each vertex v in G, let $M(v) = c^{-1}(c(v))$ and $M^v = M(v) \backslash \{v\} \cup N(v)$ be defined as in the proof of Theorem 8. Then $\overline{G}_K + I_p$ is the intersection graph of the set $\{M^v, M(v) : v \in V(G)\}$, where $\overline{G}_K$ and I_p are associated to the sets $\{M^v : v \in V(G)\}$ and $\{M(v) : v \in V(G)\}$, respectively.

We now prove that $\sigma_K(V(\overline{G}_K) \cup V(I_p)) \subseteq V(\overline{G}_K) \cup V(I_p)$. Since c is an admissible coloring, there is a permutation π of K_p such that for each vertex v in G we have $\sigma(c^{-1}(c(v))) = c^{-1}(\pi(c(v))) = c^{-1}(c(\sigma(v)))$. Therefore, $\sigma_K(M(v)) = \sigma(M(v)) = M(\sigma(v))$, for each vertex v in G. Hence, $\sigma_K(V(I_p)) \subseteq V(I_p)$. Moreover, $\sigma_K(M^v) = \sigma(M^v) = M^{\sigma(v)}$, since $\sigma(N(v)) = N(\sigma(v))$ for every vertex v in G. Hence, $\sigma_K(V(\overline{G}_K)) \subseteq V(\overline{G}_K)$. Finally from Proposition 2 we conclude that $(\overline{G}, \sigma)$ is expansive. $\square$

4. Applications

4.1. Cartesian product of cycles

We first show a direct application of Corollary 9. Let $k \geqslant 2$ and let C be a cycle of length $p = 2k+1$. Let C^k be the *Cartesian product* of C with itself k times given by $V(C^k) = (V(C))^k$ and two vertices $u = (u_1, u_2, \ldots, u_k)$ and $v = (v_1, v_2, \ldots, v_k)$ are adjacent if and only if for some $1 \leqslant i \leqslant k$, $u_i v_i \in E(C)$ and $u_j = v_j$ for all $j \neq i$. Clearly, $|C^k| \geqslant 2$.

The function $\sigma(u) = u + e_1$ is an affination for C^k, where e_i is the ith canonical vector of $\mathbb{Z}^k_{2k+1}$. Then, (C^k, σ) is an affine graph. Moreover, the closed neighborhood of $\mathbf{0} = (0, \dots, 0)$ in C^k, denoted by $N[\mathbf{0}]$, is the set $\{\mathbf{0}, \pm e_1, \dots, \pm e_k\}$ and for each vertex u in C^k its closed neighborhood is given by $N[u] = u + N[\mathbf{0}]$. Hence C^k is K_3-free.

Proposition 1. *For all $k \in \mathbb{N}$, $k \geqslant 2$ the graph $(\overline{C^k}, \sigma)$ is expansive.*

Proof. We shall prove that $c(v_1, \dots, v_k) = \sum_{i=1}^{k} i v_i$ is a coloring of (C^k, σ) satisfying the conditions of Corollary 9. Since we are working in $\mathbb{Z}_p$, the function c is a coloring with p colors. Moreover, since $k \geqslant 2$ and $p \geqslant 5$ we get $|C^k| = p^k \geqslant 3p$. Hence, $p \leqslant |C^k|/3$. It remains to show that c is an admissible locally bijective coloring.

Since $c(\pm e_i) = \pm i$ and $c(N[\mathbf{0}]) = \mathbb{Z}_p$ and, since c is linear, $c(N[v]) = c(v) + c(N[\mathbf{0}]) = c(v) + \mathbb{Z}_p = \mathbb{Z}_p$, so c is a locally bijective coloring of C^k. Moreover, the permutation $\pi(u) = u + 1$ defined on K_p satisfies $c \circ \sigma = \pi \circ c$. Hence, c is an admissible coloring. Therefore, by Corollary 9, $(\overline{C^k}, \sigma)$ is expansive. □

4.2. General constructions

In the sequel we shall discuss how to use Theorem 6 to construct expansive graphs. In view of Corollary 9, the idea is to choose a graph H, a family D of transitively affine graphs and a family of generators F such that the affine construction $(G, \sigma) := \Lambda(H, D, F)$ admits an admissible locally bijective coloring c and G is K_3-free.

Let $p \geqslant 5$ be an integer and let S_p be given by $\{1, p-1\}$ when p is odd and given by $\{1, p/2, p-1\}$ when p is even. Notice that $S_p = -S_p$ and $1 \in S_p$.

Let H be a graph. Let $D = D(p, S_p)$ be given by $D := \{(D_u, \sigma_u): u \in V(H)\}$, where $(D_u, \sigma_u) = (D_p(S_p), \sigma_D)$, for each $u \in V(H)$. Let $\mathscr{Z} = \{Z_{(u,v)}, Z_{(v,u)}: uv \in E(H)\}$ be a family of subsets of $\mathbb{Z}_p$ such that $Z_{(u,v)} = -Z_{(v,u)}$.

Let T be an orientation of H. We define the set of generators $F(\mathscr{Z}, T)$ by $F_e := \{(0, v)(j, u): j \in Z_{(v,u)}\}$, where the orientation of the edge e in T is (v, u).

Let $(G, \sigma) = \Lambda(H, D, F(\mathscr{Z}, T))$. Then $V(G) \cong \mathbb{Z}_p \times V(H)$ and $\sigma(i, u) = (i + 1, u)$, for each vertex (i, u) in G. From the definition of the affine construction we know that $(i, v)(j, u) \in E(G)$ if and only if $(0, v)(j - i, u) \in E(G)$ if and only if $(i - j, v)(0, u) \in E(G)$. Since $Z_{(u,v)} = -Z_{(v,u)}$ we deduce that (G, σ) only depends on H, D and $\mathscr{Z}$. We denote it by $\Lambda(H, D, F(\mathscr{Z}))$. Moreover, the set of neighbors of a vertex (i, u) in G is given by $\{(j, v): j - i \in Z_{(u,v)}, v \in N_H(u)\} \cup \{(j, u): j - i \in S_p\}$.

Let $c: V(G) \to \mathbb{Z}_p$ be given by $c(i, u) = i$, for every $(i, u) \in V(G)$. Note that if c is a coloring of G, then the permutation $\pi(i) = i + 1$ makes c admissible.

The function c assigns to the neighbors of vertex (i, u) colors in $(i + S_p) \cup \bigcup_{v \in N_H(u)} (i + Z_{(u,v)})$. If $\{Z_{(u,v)}: v \in N_H(u)\}$ is a partition of $\mathbb{Z}_p \backslash (S_p \cup \{0\})$, then c assigns to each neighbor of the vertex (i, u) a different color in $\mathbb{Z}_p \backslash \{i\}$. In this case, as $c(i, u) = i$, the function c is an admissible locally bijective coloring of G.

Let p, k be integers with $p \geqslant 5$ and $k \leqslant p - 4$. A family $\mathscr{Z} = \{Z_1, Z_2, \dots, Z_k\}$ of subsets of $\mathbb{Z}_p$ is a *local p-cover* if $\mathscr{Z}$ is a partition of $\mathbb{Z}_p \backslash (S_p \cup \{0\})$ and for each $i = 1, \dots, k$, the set Z_i is *sparse* that is, $\forall j \in \mathbb{Z}_p$, $j \in Z_i \Rightarrow j + 1 \notin Z_i$.

By instance, if p is an even integer, then $Z_1 = \{2, 4, \dots, p-2\} \backslash \{p/2\}$ and $Z_2 = \{3, 5, \dots, p-3\} \backslash \{p/2\}$ are sparse sets, and $\{Z_1, Z_2\}$ is a local p-cover.

A family $\mathscr{Z} = \{Z_{(u,v)}, Z_{(v,u)}: uv \in E(H)\}$ is a *p-cover* of H if for every $u \in V(H)$ the subfamily $\{Z_{(u,v)}: v \in N_H(u)\}$ is a *local p-cover* and for each $uv \in E(H)$, $Z_{(u,v)} = -Z_{(v,u)}$.

Theorem 10. *Let $p \geqslant 5$ be an integer and let H be a K_3-free graph with at least three vertices. Then the complement of $\Lambda(H, D, F(\mathscr{Z}))$ is expansive, where $\mathscr{Z}$ is any p-cover of H.*

Proof. Let $(G, \sigma) = \Lambda(H, D, \mathscr{Z})$. We have already proved that the function $c(i, v) := i$ is an admissible locally bijective coloring of (G, σ). Since H is K_3-free and each set in $\mathscr{Z}$ is sparse, the graph G is K_3-free too. Since $5 \leqslant p \leqslant |G|/|H|$ and $|H| \geqslant 3$, by Corollary 9 we conclude that $(\overline{G}, \sigma)$ is expansive. □

In the next lemmas we provide concrete p-covers for some classes of regular graphs.

Lemma 11. *Let p, k be integers with $p \geqslant 5$. Let H be a k-regular graph having a proper k-edge coloring c' with colors $\{1, 2, \ldots, k\}$. Let $\mathscr{Z} = \{Z_1, Z_2, \ldots, Z_k\}$ be any local p-cover.*

- *If $k \leqslant \lceil p/2 \rceil - 2$ and $Z_i = -Z_i$, for $i = 1, \ldots, k$, then $\{Z_{(u,v)} = Z_{c'(uv)}, Z_{(v,u)} = Z_{c'(uv)} : uv \in E(H)\}$ is a p-cover for H.*
- *If $H = (A \cup B, E)$ is bipartite and $p \geqslant k + 4$, then $\{Z_{(u,v)} = Z_{c'(uv)}, Z_{(v,u)} = -Z_{c'(uv)} : u \in A, v \in B, uv \in E\}$ is a p-cover for H.*

Proof. Since $\mathscr{Z}$ is a local p-cover, the conclusion comes from the fact that in both cases $Z_{(u,v)} = -Z_{(v,u)}$, for each $uv \in E(H)$. $\square$

Let H be a $2k$-regular graph. From a result of Petersen [6], the graph H can be splitted in k edge-disjoint 2-factors (vertex-disjoint union of cycles) $H_1, \ldots, H_k$. For $i = 1, \ldots, k$, let O_i be the orientation of H_i obtained by orienting cyclically each cycle of H_i.

Lemma 12. *Let p, k be integers with $p \geqslant 5$ and $k \leqslant \lceil p/2 \rceil - 2$. Let H be a $2k$-regular graph. Let $\mathscr{Z} = \{Z_1, Z_2, \ldots, Z_k\}$ be a partition of $\{2, 3, \ldots, \lceil p/2 \rceil - 1\}$, with Z_i a sparse set, for $i = 1, \ldots, k$. Let $i = 1, \ldots, k$ and let $uv \in E(H_i)$. Let $Z_{(u',v')} = Z_i$ and $Z_{(v',u')} = -Z_i$, where (u', v') is the orientation of uv in O_i. Then, $\{Z_{(u,v)}, Z_{(v,u)} : uv \in E(H)\}$ is a p-cover for H.*

Proof. Clearly, for each $uv \in E(H)$, the sets $Z_{(u,v)}, Z_{(v,u)}$ are sparse, and $Z_{(u,v)} = -Z_{(v,u)}$. We show that for each vertex u of H, the set $\{Z_{(u,v)} : v \in N_H(u)\}$ is a local p-cover. For each $i = 1, \ldots, k$, the vertex u has two neighbors w and w' in H_i such that (w, u) and (u, w') belong to O_i. Hence, $Z_{(u,w')} = Z_i$ and $Z_{(u,w)} = -Z_i$. Therefore,

$$\begin{aligned}
\bigcup_{v \in N_H(u)} Z_{(u,v)} &= \bigcup_{i=1}^{k} (Z_i \cup -Z_i) \\
&= \left\{2, \ldots, \left\lceil \frac{p}{2} \right\rceil - 1\right\} \cup \left\{p - 2, \ldots, p - \left(\left\lceil \frac{p}{2} \right\rceil - 1\right)\right\} \\
&= \{2, \ldots, p - 2\} \backslash \left\{\frac{p}{2}\right\}.
\end{aligned}$$

We conclude that $\{Z_{(u,v)}, Z_{(v,u)} : uv \in E(H)\}$ is a p-cover of H. $\square$

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