Heights of algebraic numbers modulo multiplicative group actions ☆

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Abstract

Given a number field K and a subgroup $G \subset K^*$ of the multiplicative group of K, Silverman defined the G-height $\mathcal{H}(\theta; G)$ of an algebraic number θ as

$$\mathcal{H}(\theta; G) := \inf_{g \in G,\, n \in \mathbb{N}} \bigl\{ H\bigl(g^{1/n}\theta\bigr)\bigr\},$$

where H on the right is the usual absolute height. When $G = E_K$ is the units of K, such a height was introduced by Bergé and Martinet who found a formula for $\mathcal{H}(\theta; E_K)$ involving a curious product over the archimedean places of $K(\theta)$. We take the analogous product over all places of $K(\theta)$ and find that it corresponds to $\mathcal{H}(\theta; K^1)$, where K^1 is the kernel of the norm map from K^* to $\mathbb{Q}^*$. We also find that a natural modification of this same product leads to $\mathcal{H}(\theta; K^*)$. This is a height function on algebraic numbers which is unchanged under multiplication by K^*. For $G = K^1$, or $G = K^*$, we show that $\mathcal{H}(\theta; G) = 1$ if and only if $\theta^n \in G$ for some positive integer n. For these same G we also show that G-heights have the expected finiteness property: for any real number X and any integer N there are, up to multiplication by elements of G, only finitely many algebraic numbers θ such that $\mathcal{H}(\theta; G) < X$ and $[K(\theta) : K] < N$. For $G = E_K$, all of these statements were proved by Bergé and Martinet.

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1. Introduction

In 1987 Bergé and Martinet [BM1] defined a height function $\mathcal{H}(\theta; E_K)$ on algebraic numbers θ which was by construction invariant under the action of the group of units E_K of a fixed number field K. They defined

$$\mathcal{H}(\theta; E_K) := \inf_{g \in E_K,\, n \in \mathbb{N}} \{H(g^{1/n}\theta)\},$$

where H on the right is the usual absolute height [La2, p. 52]. The choice of root $g^{1/n}$ is immaterial since H is invariant under multiplication by roots of unity.

As Silverman suggested (cf. [BM2, p. 156]), E_K can be replaced by any multiplicative group G contained in the algebraic closure $\overline{\mathbb{Q}}$ of the rational field $\mathbb{Q}$. One can then define a G-height on non-zero algebraic numbers, constant on G-orbits, by

$$\mathcal{H}(\theta; G) := \inf_{g \in G,\, n \in \mathbb{N}} \{H(g^{1/n}\theta)\}. \tag{1.1}$$

One trivially has $\mathcal{H}(\theta; G) = 1$ when $\theta^n \in G$ for some $n \in \mathbb{N}$, but it is not clear to us whether this is the only kind of orbit with trivial G-height.

Despite the rather *ad hoc* appearance of the above definitions, Bergé and Martinet [BM1, BM2] found an explicit formula for $\mathcal{H}(\theta; E_K)$ which is reminiscent of a product expression for the classical height. Recall [La2, p. 51] that the absolute height H is given by

$$H(\theta) := \left(\prod_{2 \leqslant p \leqslant \infty} \prod_{\sigma : L \to \mathbb{C}_p} \max\left(\left|\sigma(\theta)\right|_p, 1\right) \right)^{1/[L:\mathbb{Q}]} \qquad (\theta \in L^*), \tag{1.2}$$

where the product over p runs over all places of $\mathbb{Q}$, L is any number field containing θ, and the product over σ runs over the $[L : \mathbb{Q}]$ embeddings of L into the completion $\mathbb{C}_p$ of the algebraic closure of the local field $\mathbb{Q}_p$. The field $\mathbb{C}_p$ is equipped with a unique absolute value $|\,|_p$ extending the usual one on $\mathbb{Q}_p$. (Usually H is defined as a product over places of L, with suitably normalized absolute values, but the above formulation is equivalent and notationally more convenient below.)

An alternative expression for $H(\theta)$, which treats the archimedean primes separately, is [La2, p. 53]

$$H(\theta)^{[L:\mathbb{Q}]} = \mathrm{Norm}_{L/\mathbb{Q}}\big(\mathfrak{d}(\theta)\big) \prod_{\sigma : L \to \mathbb{C}} \max\left(\left|\sigma(\theta)\right|_\infty, 1\right). \tag{1.3}$$

Here $\mathfrak{d}(\theta)^{-1}$ is the fractional ideal of L generated by θ and 1, and $|\,|_\infty$ is the usual absolute value on the complex field $\mathbb{C} = \mathbb{C}_\infty$. Equivalently, $\mathfrak{d} = \mathfrak{d}(\theta)$ is the denominator ideal of the principal fractional ideal $(\theta) = \mathfrak{n}\mathfrak{d}^{-1}$, with $\mathfrak{n}$ and $\mathfrak{d}$ relatively prime integral ideals of L.

To state Bergé and Martinet's formula giving $\mathcal{H}(\theta; E_K)$, for each archimedean embedding $\tau : K \to \mathbb{C}$, order the $[L : K]$ embeddings $\sigma_{\tau,i} : L \to \mathbb{C}$ of L extending τ so that

$$\left|\sigma_{\tau,1}(\theta)\right|_\infty \leqslant \left|\sigma_{\tau,2}(\theta)\right|_\infty \leqslant \cdots \leqslant \left|\sigma_{\tau,[L:K]}(\theta)\right|_\infty. \tag{1.4}$$

Then Bergé and Martinet's formula reads

$$\mathcal{H}(\theta;\, E_K)^{[L:\mathbb{Q}]} = \mathrm{Norm}_{L/\mathbb{Q}}\big(\mathfrak{d}(\theta)\big) \prod_{i=1}^{[L:K]} \max\Big(\prod_{\tau\,:\,K\to\mathbb{C}} \left|\sigma_{\tau,i}(\theta)\right|_\infty, 1\Big). \tag{1.5}$$

Comparing (1.5) with the expressions (1.2) and (1.3) for the classical height, it is natural to regard Bergé and Martinet's height function as an asymmetric version of one that would treat all places equally. To symmetrize (1.5) about the places, the non-archimedean contribution $\mathrm{Norm}_{L/\mathbb{Q}}(\mathfrak{d}(\theta))$ should be replaced by a product over all primes p, the product to be defined at each p in full analogy with the archimedean piece. Namely, given any place p of $\mathbb{Q}$ and $\theta \in L^*$, with $K \subset L$, for each embedding $\tau : K \to \mathbb{C}_p$ order the $[L:K]$ embeddings $\sigma_{\tau,i} : L \to \mathbb{C}_p$ of L extending τ so that

$$\left|\sigma_{\tau,1}(\theta)\right|_p \leqslant \left|\sigma_{\tau,2}(\theta)\right|_p \leqslant \cdots \leqslant \left|\sigma_{\tau,[L:K]}(\theta)\right|_p, \tag{1.6}$$

and define

$$f_K(\theta) := \Bigg(\prod_{2\leqslant p\leqslant\infty} \prod_{i=1}^{[L:K]} \max\Big(\prod_{\tau\,:\,K\to\mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p, 1\Big)\Bigg)^{1/[L:\mathbb{Q}]}. \tag{1.7}$$

One readily verifies that $f_K(\theta)$ is independent of the choice of field L containing $K(\theta)$ and that it shares elementary properties of the classical height function:

$$f_K(\theta) = f_K\big(\theta^{-1}\big), \qquad f_K\big(\theta^n\big) = \big(f_K(\theta)\big)^n \quad (n \in \mathbb{N}).$$

Having symmetrized about the places of $\mathbb{Q}$ to obtain a new function f_K on $\overline{\mathbb{Q}}^*$, one is lead to wonder what it might represent. When $K = \mathbb{Q}$, definition (1.7) reduces to the classical height, since then the ordering (1.6) plays no real role in (1.7). For other K it takes some effort to see what f_K measures. If we take $\theta \in K^*$, (1.2) and (1.7) yield

$$f_K(\theta)^{[K:\mathbb{Q}]} = H\big(\mathrm{Norm}_{K/\mathbb{Q}}(\theta)\big) \quad (\theta \in K^*). \tag{1.8}$$

One is thus lead to suspect that f_K is connected with the kernel K^1 of the norm map from K^* to $\mathbb{Q}^*$.

Theorem 1. *The function f_K defined in* (1.7) *above coincides with the K^1-height*:

$$f_K(\theta) = \mathcal{H}\big(\theta;\, K^1\big) := \inf_{g\in K^1,\, n\in\mathbb{N}} \big\{ H\big(g^{1/n}\theta\big)\big\} \quad (\theta \in \overline{\mathbb{Q}}^*).$$

Furthermore, $\mathcal{H}(\theta;\, K^1) = 1$ if and only if $\theta^n \in K^1$ for some positive integer n.

We also prove the finiteness property required of any respectable height function: for any real number X and any integer N, the conditions $\mathcal{H}(\theta;\, K^1) < X$ and $[K(\theta) : K] < N$ are satisfied by finitely many orbits θK^1 in $\overline{\mathbb{Q}}^*/K^1$.

Our proof of Theorem 1 suggests that a variation of f_K should also have a simple interpretation. Namely, in definition (1.7) we can shift the product over p into the maximum and let

$$g_K(\theta) := \left(\prod_{i=1}^{[L:K]} \max\left(\prod_{2 \leqslant p \leqslant \infty} \prod_{\tau : K \to \mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p,\ 1 \right) \right)^{1/[L:\mathbb{Q}]}. \tag{1.9}$$

The product formula

$$\prod_{2 \leqslant p \leqslant \infty} \prod_{\sigma : L \to \mathbb{C}_p} \left|\sigma(\theta)\right|_p = 1 \quad (\theta \in L^*) \tag{1.10}$$

shows that $g_K(\theta) = 1$ if $\theta \in K^*$. We shall prove

Theorem 2. *The function g_K defined in* (1.9) *above coincides with the K^*-height. Furthermore, $\mathcal{H}(\theta;\, K^*) = 1$ if and only if $\theta^n \in K^*$ for some positive integer n.*

Again, there are finitely many orbits θK^* in $\overline{\mathbb{Q}}^*/K^*$ satisfying $\mathcal{H}(\theta;\, K^*) < X$ and $[K(\theta) : K] < N$.

To prove Theorems 1 and 2 we establish a more general result in which K^* is replaced by the S-units of K, and K^1 by the S-units of norm 1. Here S is any set, finite or infinite, of places of $\mathbb{Q}$ containing ∞. When S consists of the single place ∞, we recover Bergé and Martinet's formula (1.5). When S consists of all places of $\mathbb{Q}$, we obtain Theorems 1 and 2. Our proof for finite S largely follows Bergé and Martinet's proof of the case $S = \{\infty\}$. The case of infinite sets S is obtained as a limit of the finite ones.

It would be interesting to estimate the number of orbits θK^1 having θ in a fixed extension L/K and K^1-height under a given large bound. When S is finite, the S-variant of this question can probably be solved using the geometry of numbers, as in [Sch] and [dM]. The corresponding asymptotic estimates should involve discriminants, S-unit regulators and class numbers. The case of infinite S seems quite different. For example, if $L = K = \mathbb{Q}[\sqrt{-1}]$ and $S = \{\text{all places of } \mathbb{Q}\}$, formula (1.8) shows that the problem is equivalent to estimating the cardinality

$$\mathrm{Card}\left\{\alpha \in \mathbb{Q}^* \;\middle|\; H(\alpha) < X,\ \alpha = x^2 + y^2 \text{ for some } x, y \in \mathbb{Q}\right\}$$

of the set of rational numbers which are sums of two rational squares and have height below a given large bound X.

This paper is organized as follows. In Section 2 we give a formula for G-heights when G is a suitable subgroup of K^*. In Section 3 we use this formula to show that $\mathcal{H}(\theta;\, G) = 1$ only when $\theta^n \in G$ for some $n \in \mathbb{N}$. In Section 4 we use a compactness argument to prove a finiteness property for these G-heights.

One of the outstanding problems in the theory of heights is that of finding lower bounds for $H(\theta)$ when θ is not a root of unity [GH]. Any lower bound found for the heights considered here would have applications to this problem since

$$H(\theta) \geqslant \mathcal{H}(\theta; E_K) \geqslant \mathcal{H}\big(\theta; K^1\big) \geqslant \mathcal{H}(\theta; K^*), \tag{1.11}$$

as follows from definition (1.1) and the inclusions $\{1\} \subset E_K \subset K^1 \subset K^*$. If θ has a small height, Smyth [Sm] proved that θ and θ^{-1} are algebraically conjugate units. If we let $K := \mathbb{Q}(\theta + \theta^{-1})$ and $L := \mathbb{Q}(\theta)$, then L/K is a quadratic extension. For this kind of K and θ, Theorem 2 shows that equality holds throughout (1.11).

2. f_K and g_K are G-heights

In this section we show $f_K(\theta) = \mathcal{H}(\theta; K^1)$ and $g_K(\theta) = \mathcal{H}(\theta; K^*)$, as claimed in Theorems 1 and 2. It proves convenient to take a set S of rational places containing $p = \infty$ and deal with S-versions of Theorems 1 and 2. We do not require S to be finite. On $\overline{\mathbb{Q}}^*$ define the functions $f_{K,S}$ and $g_{K,S}$ by

$$f_{K,S}(\theta)^{[L:\mathbb{Q}]} := \mathrm{Norm}_{L/\mathbb{Q}}\big(\mathfrak{d}_S(\theta)\big) \prod_{p\in S} \prod_{i=1}^{[L:K]} \max\Bigg(\prod_{\tau\,:\,K\to\mathbb{C}_p} \big|\sigma_{\tau,i}(\theta)\big|_p, 1\Bigg) \tag{2.1}$$

and

$$g_{K,S}(\theta)^{[L:\mathbb{Q}]} := \mathrm{Norm}_{L/\mathbb{Q}}\big(\mathfrak{d}_S(\theta)\big) \prod_{i=1}^{[L:K]} \max\Bigg(\prod_{p\in S} \prod_{\tau\,:\,K\to\mathbb{C}_p} \big|\sigma_{\tau,i}(\theta)\big|_p, 1\Bigg), \tag{2.2}$$

where L is a number field containing $K(\theta)$, the $\sigma_{\tau,i}$ are ordered just as before by (1.6), and $\mathfrak{d}_S = \mathfrak{d}_S(\theta)$ is the S-ideal denominator of θ. More precisely, if $\mathfrak{d} = \prod_{\mathfrak{p}} \mathfrak{p}^{e_\mathfrak{p}}$ is the factorization of $\mathfrak{d}$ in (1.3) into powers of prime ideals of L, then $\mathfrak{d}_S = \prod_{\mathfrak{p}\cap\mathbb{Z}\notin S} \mathfrak{p}^{e_\mathfrak{p}}$. Note that [La2, pp. 53–54]

$$\mathrm{Norm}_{L/\mathbb{Q}}(\mathfrak{d}_S) = \prod_{p\notin S} \prod_{\sigma\,:\,L\to\mathbb{C}_p} \max\big(\big|\sigma(\theta)\big|_p, 1\big). \tag{2.3}$$

When S consists of all places of $\mathbb{Q}$, we omit the subscript S and continue to write f_K and g_K instead of $f_{K,S}$ and $g_{K,S}$. Let K_S denote the S-units of K and K_S^1 the kernel of the norm map from K_S to $\mathbb{Q}^*$,

$$\begin{aligned} K_S &:= \big\{\alpha \in K \;\big|\; |\tau(\alpha)|_p = 1 \text{ for all } p \notin S \text{ and for all embeddings } \tau : K \to \mathbb{C}_p\big\},\\ K_S^1 &:= K_S \cap \ker(\mathrm{Norm}_{K/\mathbb{Q}}). \end{aligned} \tag{2.4}$$

Theorem 3. *The functions $f_{K,S}$ and $g_{K,S}$ defined in* (2.1) *and* (2.2) *above coincide with the K_S^1-and K_S-heights, respectively*:

$$f_{K,S}(\theta) = \mathcal{H}\big(\theta; K_S^1\big) := \inf_{g\in K_S^1,\, n\in\mathbb{N}} \big\{H\big(g^{1/n}\theta\big)\big\} \tag{2.5}$$

and

$$g_{K,S}(\theta) = \mathcal{H}(\theta; K_S) := \inf_{g\in K_S,\, n\in\mathbb{N}} \big\{H\big(g^{1/n}\theta\big)\big\}. \tag{2.6}$$

Proof. We first show that it suffices to consider finite sets S. Indeed, a straight-forward argument working directly from the definition (1.1) of G-heights shows

$$\mathcal{H}(\theta; K_S) = \inf_{S' \subset S} \{\mathcal{H}(\theta; K_{S'})\}, \qquad \mathcal{H}(\theta; K_S^1) = \inf_{S' \subset S} \{\mathcal{H}(\theta; K_{S'}^1)\},$$

where the infima are taken over finite subsets S' of S. To prove an analogous statement for $f_{K,S}(\theta)$ and $g_{K,S}(\theta)$, let

$$\widetilde{S} = \widetilde{S}(\theta) := \big(S \cap \{p \mid 2 \leqslant p < \infty,\ |\sigma(\theta)|_p > 1 \text{ for some } \sigma : L \to \mathbb{C}_p\}\big) \cup \{\infty\}.$$

Then $\widetilde{S}$ is finite and $f_{K,S}(\theta) = f_{K,\widetilde{S}}(\theta)$, $g_{K,S}(\theta) = g_{K,\widetilde{S}}(\theta)$, as one sees from definitions (2.1) and (2.2). These formulas can be rewritten using (2.3) as

$$f_{K,S}(\theta)^{[L:\mathbb{Q}]} = \Big(\prod_{p \notin S} \prod_{\sigma : L \to \mathbb{C}_p} \max(|\sigma(\theta)|_p, 1)\Big) \prod_{p \in S} \prod_{i=1}^{[L:K]} \max\Big(\prod_{\tau : K \to \mathbb{C}_p} |\sigma_{\tau,i}(\theta)|_p, 1\Big) \tag{2.7}$$

and

$$g_{K,S}(\theta)^{[L:\mathbb{Q}]} = \Big(\prod_{p \notin S} \prod_{\sigma : L \to \mathbb{C}_p} \max(|\sigma(\theta)|_p, 1)\Big) \prod_{i=1}^{[L:K]} \max\Big(\prod_{p \in S} \prod_{\tau : K \to \mathbb{C}_p} |\sigma_{\tau,i}(\theta)|_p, 1\Big).$$

Hence, if $S_1 \subset S_2$,

$$\frac{f_{K,S_1}(\theta)}{f_{K,S_2}(\theta)} = \frac{\prod_{p \in S_2 - S_1} \prod_{\sigma : L \to \mathbb{C}_p} \max(|\sigma(\theta)|_p, 1)}{\prod_{p \in S_2 - S_1} \prod_{i=1}^{[L:K]} \max(\prod_{\tau : K \to \mathbb{C}_p} |\sigma_{\tau,i}(\theta)|_p, 1)} \geqslant 1,$$

where the last inequality follows from

$$\max\Big(\prod_i a_i, 1\Big) \leqslant \prod_i \max(a_i, 1) \quad (a_i > 0). \tag{2.8}$$

Thus

$$f_{K,S}(\theta) = f_{K,\widetilde{S}}(\theta) = \inf_{S' \subset \widetilde{S}} \{f_{K,S'}(\theta)\} = \inf_{S' \subset S} \{f_{K,S'}(\theta)\}.$$

Since an analogous statement holds for $g_{K,S}$, in proving Theorem 3 we may assume that S is finite.

We now recall some formulas for the classical height. A useful S-version of (1.3) is (cf. [La2, p. 53]):

$$H(\gamma)^{[L:\mathbb{Q}]} = \mathrm{Norm}_{L/\mathbb{Q}}(\mathfrak{d}_S(\gamma)) \prod_{p \in S} \prod_{\sigma : L \to \mathbb{C}_p} \max(|\sigma(\gamma)|_p, 1) \quad (\gamma \in L^*). \tag{2.9}$$

Since $H(\gamma) = H(\gamma^{-1})$, a more symmetric form of (2.9) is

$$H(\gamma)^{2[L:\mathbb{Q}]} = N_S(\gamma) \prod_{p\in S} \prod_{\sigma\,:\, L\to\mathbb{C}_p} \max\left(\left|\sigma(\gamma)\right|_p, \left|\sigma(\gamma)\right|_p^{-1}\right) \quad (\gamma \in L^*), \tag{2.10}$$

where $N_S(\gamma) := \mathrm{Norm}_{L/\mathbb{Q}}(\mathfrak{d}_S(\gamma)\mathfrak{d}_S(\gamma^{-1}))$.

In calculating $H(g^{1/n}\theta)$ in (2.5), we will avoid passing to an extension field of L containing $g^{1/n}$ by using $H(g^{1/n}\theta) = (H(g\theta^n))^{1/n}$. Note also that $\mathfrak{d}_S(\theta^n) = (\mathfrak{d}_S(\theta))^n$ and $\mathfrak{d}_S(\theta) = \mathfrak{d}_S(g\theta)$ for $g \in K_S$. Hence,

$$\begin{aligned} H\left(g^{1/n}\theta\right)^{2[L:\mathbb{Q}]} &= H\left(g\theta^n\right)^{2[L:\mathbb{Q}]/n} \\ &= N_S(\theta)\left(\prod_{p\in S} \prod_{\sigma\,:\, L\to\mathbb{C}_p} \max\left(\left|\sigma(\theta)\right|_p^n \left|\sigma(g)\right|_p, \left|\sigma(\theta)\right|_p^{-n}\left|\sigma(g)\right|_p^{-1}\right)\right)^{1/n} \\ &= N_S(\theta) \prod_{p\in S} \prod_{\sigma\,:\, L\to\mathbb{C}_p} \max\left(\left|\sigma(\theta)\right|_p \left|\sigma(g)\right|_p^{1/n}, \left|\sigma(\theta)\right|_p^{-1}\left|\sigma(g)\right|_p^{-1/n}\right). \end{aligned} \tag{2.11}$$

We will also need a symmetric form of $f_{K,S}$. From (2.7) and the product formula (1.10),

$$\begin{aligned} &f_{K,S}(\theta)^{[L:\mathbb{Q}]} \\ &= \left(\prod_{p\notin S} \prod_{\sigma\,:\, L\to\mathbb{C}_p} \left|\sigma(\theta)\right|_p \max\left(1, \left|\sigma(\theta)\right|_p^{-1}\right)\right) \\ &\quad \cdot \prod_{p\in S} \prod_{i=1}^{[L:K]} \prod_{\tau\,:\, K\to\mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p \max\left(1, \prod_{\tau\,:\, K\to\mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p^{-1}\right) \\ &= \left(\prod_{p\notin S} \prod_{\sigma\,:\, L\to\mathbb{C}_p} \max\left(\left|\sigma\left(\theta^{-1}\right)\right|_p, 1\right)\right) \prod_{p\in S} \prod_{i=1}^{[L:K]} \max\left(\prod_{\tau\,:\, K\to\mathbb{C}_p} \left|\sigma_{\tau,[L:K]-i}\left(\theta^{-1}\right)\right|_p, 1\right) \\ &= f_{K,S}\left(\theta^{-1}\right)^{[L:\mathbb{Q}]}. \end{aligned} \tag{2.12}$$

In the next to last step we changed from $\sigma_{\tau,i}$ to $\sigma_{\tau,[L:K]-i}$ because the ordering determined by θ in (1.6) is the reverse of that determined by θ^{-1}. From (2.1) and (2.12) we obtain the symmetric form we sought:

$$f_{K,S}(\theta)^{2[L:\mathbb{Q}]} = N_S(\theta) \prod_{p\in S} \prod_{i=1}^{[L:K]} \max\left(\prod_{\tau\,:\, K\to\mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p, \prod_{\tau\,:\, K\to\mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p^{-1}\right). \tag{2.13}$$

Similar calculations yield

$$g_{K,S}(\theta)^{2[L:\mathbb{Q}]} = N_S(\theta) \prod_{i=1}^{[L:K]} \max\left(\prod_{p\in S} \prod_{\tau\,:\, K\to\mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p, \prod_{p\in S} \prod_{\tau\,:\, K\to\mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p^{-1}\right). \tag{2.14}$$

Comparing (2.5), (2.6) and (2.11) with (2.13) and (2.14), we see that the following lemma completes the proof of Theorem 3.

Lemma 1. *Let S be a finite set of places of $\mathbb{Q}$, including ∞, let L/K be an extension of number fields and let $\theta \in L^*$ determine an ordering of the embeddings $\sigma_{\tau,i} : L \to \mathbb{C}_p$ as in* (1.6)*. Then*

$$\inf_{g \in K_S^1,\, n \in \mathbb{N}} \left\{ \prod_{p \in S} \prod_{\sigma : L \to \mathbb{C}_p} \max\left(\left|\sigma(\theta)\right|_p \left|\sigma(g)\right|_p^{1/n},\, \left|\sigma(\theta)\right|_p^{-1} \left|\sigma(g)\right|_p^{-1/n} \right) \right\}$$
$$= \prod_{i=1}^{[L:K]} \prod_{p \in S} \max\left(\prod_{\tau : K \to \mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p,\, \prod_{\tau : K \to \mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p^{-1} \right), \tag{2.15}$$

and similarly

$$\inf_{g \in K_S,\, n \in \mathbb{N}} \left\{ \prod_{p \in S} \prod_{\sigma : L \to \mathbb{C}_p} \max\left(\left|\sigma(\theta)\right|_p \left|\sigma(g)\right|_p^{1/n},\, \left|\sigma(\theta)\right|_p^{-1} \left|\sigma(g)\right|_p^{-1/n} \right) \right\}$$
$$= \prod_{i=1}^{[L:K]} \max\left(\prod_{p \in S} \prod_{\tau : K \to \mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p,\, \prod_{p \in S} \prod_{\tau : K \to \mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p^{-1} \right). \tag{2.16}$$

Note that the only difference between the left-hand sides of (2.15) and (2.16) is that the infimum is taken over different g's. When S consists of the single place ∞, the above result is due to Bergé and Martinet [BM2, pp. 165–166]. In this case both K_S and K_S^1 coincide with the units of K. Lemma 1 explains the curious ordering involved in the products defining $f_{K,S}$ and $g_{K,S}$.

Proof. Following Bergé and Martinet, we shall prove the lemma by translating it to a convexity question in a finite-dimensional real vector space. We shall first prove (2.15). Let

$$V := \bigoplus_{p \in S} V^{(p)}, \qquad V^{(p)} := \bigoplus_{\sigma : L \to \mathbb{C}_p} \mathbb{R}_\sigma,$$

where $\mathbb{R}_\sigma$ is a copy of $\mathbb{R}$. We write elements $\mathbf{a} \in V$ as $\mathbf{a} = (a_{p,\sigma})$ and also as $\mathbf{a} = (a_{p,\sigma_{\tau,i}})$, where the $\sigma_{\tau,i}$ are the σ's in some order described below. Define the logarithm map

$$\mathcal{L} = \mathcal{L}_{L,S} : L^* \to V, \qquad \mathcal{L}(\theta)_{p,\sigma} := \log\left|\sigma(\theta)\right|_p,$$

and let $V_L := \mathbb{R}\mathcal{L}(L^*) \subset V$ be the closure of the image of L^* under $\mathcal{L}$. By the approximation theorem [La1, pp. 35–36], V_L consists of those $\mathbf{a} \in V$ such that for all $p \in S$, $a_{p,\sigma} = a_{p,\sigma'}$ whenever σ and σ' determine the same absolute value on L.

For $\mathbf{a} \in V_L$ let

$$\|\mathbf{a}\|_1 := \sum_{p,\sigma} |a_{p,\sigma}|, \qquad q(\mathbf{a}) := \inf_{\mathbf{w} \in \mathbb{Q}\mathcal{L}(K_S^1)} \left\{ \|\mathbf{a} + \mathbf{w}\|_1 \right\} = \inf_{\mathbf{w} \in \mathbb{R}\mathcal{L}(K_S^1)} \left\{ \|\mathbf{a} - \mathbf{w}\|_1 \right\}. \tag{2.17}$$

Thus, $q(\mathcal{L}(\theta))$ is the logarithm of the left-hand side of (2.15). Since q is a continuous function of $\mathbf{a}$, in proving that it is given by the logarithm of the right-hand side of (2.15), we may restrict $\mathbf{a}$ to a dense subset of V_L. In particular, we will assume that $a_{p,\sigma} \neq a_{p,\sigma'}$ for all pairs of embeddings $\sigma, \sigma' : L \to C_p$ inducing distinct absolute values on L.

Eq. (2.15) is implied by

$$q(\mathbf{a}) = \sum_{i=1}^{[L:K]} \sum_{p\in S} \Bigg| \sum_{\tau : K\to \mathbb{C}_p} a_{p,\sigma_{\tau,i}} \Bigg| \quad (\mathbf{a} \in V_L), \tag{2.18}$$

where for each p and τ the σ's restricting to τ on K have been re-christened $\sigma_{\tau,i}$ and ordered so that

$$a_{p,\sigma_{\tau,1}} < a_{p,\sigma_{\tau,2}} < \cdots < a_{p,\sigma_{\tau,[L:K]}}. \tag{2.19}$$

Note that if τ and τ' induce the same absolute value on K, then $a_{p,\sigma_{\tau,i}} = a_{p,\sigma_{\tau',i}}$ because we are assuming $\mathbf{a} \in V_L$.

The main point of the proof of (2.18) is to show that

$$F_{\mathbf{a}}(\mathbf{w}) := \|\mathbf{a} - \mathbf{w}\|_1 \tag{2.20}$$

is a convex function of $\mathbf{w} \in \mathbb{R}\mathcal{L}(K_S^1)$ which has a constant value $c_{\mathbf{a}}$ on a non-empty open set $Y \subset \mathbb{R}\mathcal{L}(K_S^1)$. It then follows that the minimum of this function is $c_{\mathbf{a}}$, i.e. $q(\mathbf{a}) = c_{\mathbf{a}}$. (Indeed, given any $\mathbf{y} \in Y$ and $\mathbf{w} \in \mathbb{R}\mathcal{L}(K_S^1)$, pick $\lambda < 1$ near enough to 1 that $\mathbf{y} + (1-\lambda)(\mathbf{w} - \mathbf{y})$ still belongs to the open set Y and $\lambda > 0$. Then

$$F_{\mathbf{a}}(\mathbf{y}) = F_{\mathbf{a}}\big(\mathbf{y} + (1-\lambda)(\mathbf{w}-\mathbf{y})\big) = F_{\mathbf{a}}\big(\lambda\mathbf{y} + (1-\lambda)\mathbf{w}\big) \leqslant \lambda F_{\mathbf{a}}(\mathbf{y}) + (1-\lambda)F_{\mathbf{a}}(\mathbf{w}),$$

whence the inequality claimed.)

The convexity of $F_{\mathbf{a}}(\mathbf{w})$, even for $\mathbf{w}$ ranging over the whole of V, is clear from the convexity of the ordinary absolute value function. The open set Y is defined by $\mathbf{y} = (y_{p,\sigma_{\tau,i}}) \in Y$ if and only if $\mathbf{y} \in \mathbb{R}\mathcal{L}(K_S^1)$ and

$$a_{p,\sigma_{\tau,k_p}} < y_{p,\sigma_{\tau,1}} < a_{p,\sigma_{\tau,k_p+1}} \quad (p \in S,\ \tau : K \to \mathbb{C}_p), \tag{2.21}$$

with the indices k_p determined as follows. If the expression $\sum_\tau a_{p,\sigma_{\tau,i}}$, which is monotone increasing in i, changes sign as i varies from 1 to $[L:K]$, then k_p is defined by

$$\sum_\tau a_{p,\sigma_{\tau,k_p}} < 0 < \sum_\tau a_{p,\sigma_{\tau,k_p+1}} \quad (p \in S).$$

As remarked at the beginning of the proof, we need only consider strict inequalities since we may restrict $\mathbf{a}$ to a dense open subset of V_L. If $\sum_\tau a_{p,\sigma_{\tau,i}} > 0$ for all i (i.e. $\sum_\tau a_{p,\sigma_{\tau,1}} > 0$), we let $k_p = 0$. Likewise, if for all i we have $\sum_\tau a_{p,\sigma_{\tau,i}} < 0$ (i.e. $\sum_\tau a_{p,\sigma_{\tau,[L:K]}} < 0$), we let $k_p = [L:K]$. In these two extreme cases we agree to omit any condition involving indices out of the range $1 \leqslant i \leqslant [L:K]$, for example in (2.21).

For any $\mathbf{w} \in \mathcal{L}(K^*)$, we naturally have $w_{p,\sigma_{\tau,i}} = w_{p,\sigma_{\tau,1}}$ for $1 \leqslant i \leqslant [L:K]$. Thus, (2.21) and the ordering (2.19) imply $a_{p,\sigma_{\tau,i}} < y_{p,\sigma_{\tau,i}}$ for $i \leqslant k_p$, while $a_{p,\sigma_{\tau,i}} > y_{p,\sigma_{\tau,i}}$ for $i > k_p$. Note also that any $\mathbf{w} = (w_{p,\sigma_{\tau,i}}) \in \mathbb{R}\mathcal{L}(K_S^1)$ satisfies

$$\sum_\tau w_{p,\sigma_{\tau,1}} = 0 \quad (p \in S). \tag{2.22}$$

This follows from the definition of K_S^1 and

$$\log\left|\mathrm{Norm}_{K/\mathbb{Q}}(\gamma)\right|_p = \sum_{\tau\,:\,K\to\mathbb{C}_p} \log\left|\sigma_{\tau,1}(\gamma)\right|_p \quad (\gamma\in K^*) \tag{2.23}$$

(see (2.4) and [La1, p. 39]).

We may now calculate $F_{\mathbf{a}}(\mathbf{y})$ in (2.20) for $\mathbf{y}\in Y$:

$$\begin{aligned} F_{\mathbf{a}}(\mathbf{y}) &= \sum_{p\in S}\sum_{\tau\,:\,K\to\mathbb{C}_p}\left(\sum_{i=1}^{k_p}(y_{p,\sigma_{\tau,1}} - a_{p,\sigma_{\tau,i}}) + \sum_{i=k_p+1}^{[L:K]}(a_{p,\sigma_{\tau,i}} - y_{p,\sigma_{\tau,1}})\right) \\ &= \sum_{i=1}^{[L:K]}\sum_p\left|\sum_\tau a_{p,\sigma_{\tau,i}}\right| + \sum_p\bigl(2k_p - [L:K]\bigr)\sum_\tau y_{p,\sigma_{\tau,1}} = \sum_{i=1}^{[L:K]}\sum_p\left|\sum_\tau a_{p,\sigma_{\tau,i}}\right|. \end{aligned} \tag{2.24}$$

Note that the result is indeed independent of $\mathbf{y}\in Y$.

To finish the proof of (2.15), we must still show that Y is not empty. To this end, we give an explicit description of $\mathbb{R}\mathcal{L}(K_S^1)$. The Dirichlet S-unit theorem [La1, p. 104] characterizes the larger space $\mathbb{R}\mathcal{L}(K_S)$. Indeed, a simple dimension count shows that $\mathbf{w}=(w_{p,\sigma_{\tau,i}})\in\mathbb{R}\mathcal{L}(K_S)$ if and only if the following three conditions hold

- $w_{p,\sigma_{\tau,i}} = w_{p,\sigma_{\tau,1}}$ for all $p\in S,\ \tau:K\to C_p,\ 1\leqslant i\leqslant[L:K],$ (2.25)
- $w_{p,\sigma_{\tau,1}} = w_{p,\sigma_{\tau',1}}$ whenever τ and τ' define the same absolute value on K, (2.26)
- $\sum_{p\in S}\sum_{\tau\,:\,K\to C_p} w_{p,\sigma_{\tau,1}} = 0.$ (2.27)

From (2.23) one checks that elements $\mathbf{w}\in\mathbb{R}\mathcal{L}(K_S^1)$ are characterized by (2.25), (2.26) and (2.22). Hence Y contains the explicitly given point $\mathbf{y}=(y_{p,\sigma_{\tau,i}})$ defined for $1\leqslant i\leqslant[L:K]$ by

$$y_{p,\sigma_{\tau,i}} := a_{p,\sigma_{\tau,k_p}} - \frac{s_{p,k_p}}{s_{p,k_p+1}-s_{p,k_p}}(a_{p,\sigma_{\tau,k_p+1}} - a_{p,\sigma_{\tau,k_p}}) \quad \bigl(1\leqslant k_p<[L:K]\bigr),$$

where

$$s_{p,j} := \sum_{\tau\,:\,K\to\mathbb{C}_p} a_{p,\sigma_{\tau,j}} \quad \bigl(1\leqslant j\leqslant[L:K]\bigr),$$

while in the two extreme cases

$$\begin{aligned} y_{p,\sigma_{\tau,i}} &:= a_{p,\sigma_{\tau,1}} - \frac{s_{p,1}}{[L:K]} \quad (k_p=0), \\ y_{p,\sigma_{\tau,i}} &:= a_{p,\sigma_{\tau,[L:K]}} - \frac{s_{p,[L:K]}}{[L:K]} \quad \bigl(k_p=[L:K]\bigr). \end{aligned}$$

This concludes the proof of (2.15).

The proof of (2.16) is very similar, the main difference being that (2.22) is replaced by (2.27). Instead of k_p we define k as the index at which $\sum_p\sum_\tau a_{p,\sigma_{\tau,i}}$ changes sign. Thus

$$\sum_{p\in S}\sum_{\tau:K\to\mathbb{C}_p} a_{p,\sigma_{\tau,k}} < 0 < \sum_{p\in S}\sum_{\tau:K\to\mathbb{C}_p} a_{p,\sigma_{\tau,k+1}}.$$

Again, $k=0$ or $k=[L:K]$ means that there is no sign change, i.e.

$$\sum_{p\in S}\sum_{\tau:K\to\mathbb{C}_p} a_{p,\sigma_{\tau,1}} > 0 \quad (k=0), \qquad \sum_{p\in S}\sum_{\tau:K\to\mathbb{C}_p} a_{p,\sigma_{\tau,[L:K]}} < 0 \quad \big(k=[L:K]\big).$$

We replace $Y\subset\mathbb{R}\mathcal{L}(K_S^1)$ by $X\subset\mathbb{R}\mathcal{L}(K_S)$ defined by $\mathbf{x}=(x_{p,\sigma_{\tau,i}})\in X$ if and only if

$$a_{p,\sigma_{\tau,k}} < x_{p,\sigma_{\tau,1}} < a_{p,\sigma_{\tau,k+1}} \quad (p\in S,\ \tau:K\to\mathbb{C}_p).$$

The set X contains the point $\mathbf{x}=(x_{p,\sigma_{\tau,i}})$ defined by

$$x_{p,\sigma_{\tau,i}} := a_{p,\sigma_{\tau,k}} - \frac{s_k}{s_{k+1}-s_k}(a_{p,\sigma_{\tau,k+1}} - a_{p,\sigma_{\tau,k}}) \quad \text{if } 1\leqslant k<[L:K],$$

where

$$s_j := \sum_{p\in S}\sum_{\tau:K\to\mathbb{C}_p} a_{p,\sigma_{\tau,j}} \quad \big(1\leqslant j\leqslant [L:K]\big),$$

while in the two extreme cases

$$\begin{aligned} x_{p,\sigma_{\tau,i}} &:= a_{p,\sigma_{\tau,1}} - \frac{s_1}{\sum_p\sum_\tau 1} \quad \text{if } k=0,\\ x_{p,\sigma_{\tau,i}} &:= a_{p,\sigma_{\tau,[L:K]}} - \frac{s_{[L:K]}}{\sum_p\sum_\tau 1} \quad \text{if } k=[L:K]. \end{aligned}$$

One then finds for $\mathbf{x}\in X$,

$$\begin{aligned} F_{\mathbf{a}}(\mathbf{x}) &= \sum_p\sum_\tau\left(\sum_{i=1}^{k}(x_{p,\sigma_{\tau,1}} - a_{p,\sigma_{\tau,i}}) + \sum_{i=k+1}^{[L:K]}(a_{p,\sigma_{\tau,i}} - x_{p,\sigma_{\tau,1}})\right)\\ &= \sum_{i=1}^{[L:K]}\left|\sum_p\sum_\tau a_{p,\sigma_{\tau,i}}\right| + \big(2k-[L:K]\big)\sum_p\sum_\tau x_{p,\sigma_{\tau,1}} = \sum_{i=1}^{[L:K]}\left|\sum_p\sum_\tau a_{p,\sigma_{\tau,i}}\right|, \end{aligned}$$

from which (2.16) follows. □

3. Trivial values of f_K and g_K

In this section we identify elements with $f_{K,S}(\theta)=1$ or $g_{K,S}(\theta)=1$ (see definitions (2.2) and (2.3)). We first characterize roots of elements of K by their absolute values.

Lemma 2. *Let L/K be an extension of number fields and let $\alpha \in L$. Then $\alpha^n \in K$ for some positive integer n if and only if for all rational places p $(2 \leqslant p \leqslant \infty)$, for all embeddings $\tau : K \to \mathbb{C}_p$, and for all pairs of embeddings $\sigma, \sigma' : L \to \mathbb{C}_p$ extending τ, we have*

$$\left|\sigma(\alpha)\right|_p = \left|\sigma'(\alpha)\right|_p. \tag{3.1}$$

Proof. One implication is clear: if $\alpha^n \in K$, then (3.1) holds after replacing α by α^n. Hence (3.1) itself holds. To prove the converse, take $\alpha \neq 0$ and let

$$\alpha' := \frac{\alpha^{[L:K]}}{\mathrm{Norm}_{L/K}(\alpha)}.$$

Then (3.1) implies that $|\sigma(\alpha')|_p = 1$ for all places p and all embeddings $\sigma : L \to \mathbb{C}_p$. Hence α' is a root of unity, whence the lemma. □

We now characterize trivial values of $g_{K,S}$.

Proposition. *Let θ be a non-zero algebraic number and let S be any set of places of $\mathbb{Q}$ with $\infty \in S$. Then $g_{K,S}(\theta) = 1$ if and only if $\theta^n \in K_S$ for some positive integer n.*

In particular, $g_K(\theta) = 1$ if and only if $\theta^n \in K^*$ for some positive integer n, as claimed in Theorem 2.

Proof. Let $L = K(\theta)$ and assume $g_{K,S}(\theta) = 1$. By the ordering defined in (1.6),

$$\left|\sigma_{\tau,i}(\theta)\right|_p \leqslant \left|\sigma_{\tau,i+1}(\theta)\right|_p \quad (p \in S), \tag{3.2}$$

for all embeddings $\tau : K \to \mathbb{C}_p$ and all indices i $(1 \leqslant i < [L:K])$. In particular, for $1 \leqslant i < [L:K]$,

$$\prod_{p \in S} \prod_{\tau : K \to \mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p \leqslant \prod_{p \in S} \prod_{\tau : K \to \mathbb{C}_p} \left|\sigma_{\tau,i+1}(\theta)\right|_p. \tag{3.3}$$

Note that (3.3) would be strict if (3.2) were strict for a single p and τ. From $g_{K,S}(\theta) = 1$, (2.2) and (2.3) we have

$$1 = \Bigl(\prod_{p \notin S} \prod_{\sigma : L \to \mathbb{C}_p} \max\bigl(\left|\sigma(\theta)\right|_p, 1\bigr)\Bigr) \prod_{i=1}^{[L:K]} \max\Bigl(\prod_{p \in S} \prod_{\tau : K \to \mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p, 1\Bigr),$$

whence

$$\prod_{p \in S} \prod_{\tau : K \to \mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p \leqslant 1 \quad \bigl(1 \leqslant i \leqslant [L:K]\bigr), \tag{3.4}$$

and

$$\left|\sigma(\theta)\right|_p \leqslant 1 \quad (p \notin S,\ \sigma : L \to \mathbb{C}_p). \tag{3.5}$$

The product formula (1.10) yields

$$\left(\prod_{p\notin S}\ \prod_{\sigma : L\to\mathbb{C}_p} \left|\sigma(\theta)\right|_p\right) \prod_{i=1}^{[L:K]} \left(\prod_{p\in S}\ \prod_{\tau : K\to\mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p\right) = 1.$$

From this, (3.4) and (3.5) we conclude

$$\prod_{p\in S}\ \prod_{\tau : K\to\mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p = 1 \quad \bigl(1 \leqslant i \leqslant [L:K]\bigr)$$

and

$$\left|\sigma(\theta)\right|_p = 1 \quad (p \notin S,\ \sigma : L \to \mathbb{C}_p). \tag{3.6}$$

It follows that we have equality in (3.3) for all i. As remarked above, this implies $|\sigma_{\tau,i}(\theta)|_p = |\sigma_{\tau,i+1}(\theta)|_p$ for all $p \in S, \tau$ and i. By (3.6), $|\sigma(\theta)|_p = |\sigma'(\theta)|_p$ trivially holds for $p \notin S$, $\sigma, \sigma' : L \to \mathbb{C}_p$. Lemma 2 now implies $\theta^n \in K^*$ for some positive integer n. From (3.6) we see that actually $\theta^n \in K_S$, as claimed.

To prove the converse, note that $\theta^n \in K^*$ implies $|\sigma_{\tau,i}(\theta)|_p = |\sigma_{\tau,j}(\theta)|_p$ for $1 \leqslant i, j \leqslant [L:K]$, and hence

$$\prod_{\tau : K\to\mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p = \left|\mathrm{Norm}_{L/\mathbb{Q}}(\theta)\right|_p^{1/[L:K]}.$$

If in addition, $\theta^n \in K_S$, then

$$\begin{aligned} g_{K,S}(\theta)^{[L:\mathbb{Q}]} &= \left(\prod_{p\notin S}\ \prod_{\sigma:L\to\mathbb{C}_p} \max\bigl(\left|\sigma(\theta)\right|_p, 1\bigr)\right) \prod_{i=1}^{[L:K]} \max\left(\prod_{p\in S}\ \prod_{\tau : K\to\mathbb{C}_p} \left|\sigma_{\tau,i}(\theta)\right|_p, 1\right) \\ &= \max\left(\prod_{p\in S} \left|\mathrm{Norm}_{L/\mathbb{Q}}(\theta)\right|_p, 1\right) = \max\left(\prod_{2\leqslant p\leqslant\infty} \left|\mathrm{Norm}_{L/\mathbb{Q}}(\theta)\right|_p, 1\right) = 1, \end{aligned} \tag{3.7}$$

where the last steps follow from the assumption that θ^n is an S-unit and from the product formula for $\mathbb{Q}$. □

We now deduce a characterization of trivial values of $f_{K,S}$.

Corollary. $f_{K,S}(\theta) = 1$ *if and only if* $\theta^n \in K_S^1$ *for some positive integer* n.

In particular, $f_K(\theta) = 1$ if and only if $\theta^n \in K^1$ for some positive integer n, as announced in Theorem 1.

Proof. Assume $f_{K,S}(\theta)=1$. The elementary inequality (2.8) shows that $f_{K,S}(\theta)\geqslant g_{K,S}(\theta)$. Hence $g_{K,S}(\theta)=1$ and, by the proposition, $\theta^n\in K_S$. A calculation analogous to (3.7), but now with the product over $p\in S$ outside the maximum, shows that for $\theta^n\in K_S$,

$$f_{K,S}(\theta)^{[L:\mathbb{Q}]}=\prod_{2\leqslant p\leqslant\infty}\max\bigl(\bigl|\mathrm{Norm}_{L/\mathbb{Q}}(\theta)\bigr|_p,1\bigr)=H\bigl(\mathrm{Norm}_{L/\mathbb{Q}}(\theta)\bigr). \tag{3.8}$$

This, together with the assumption $f_{K,S}(\theta)=1$, shows that $\mathrm{Norm}_{L/\mathbb{Q}}(\theta)=\pm1$. Hence $\theta^{2n}\in K_S^1$, as claimed.

The converse claim in the corollary follows from (3.8). □

We note that when $\theta\in K_S$, we may take $L=K$ in (3.8). This proves (cf. (1.8))

$$f_{K,S}(\theta)^{[K:\mathbb{Q}]}=H\bigl(\mathrm{Norm}_{K/\mathbb{Q}}(\theta)\bigr)\quad(\theta\in K_S).$$

4. Finiteness

In this section we prove a finiteness property of G-heights for $G=K_S$ or $G=K_S^1$. Namely, we now show that for any real number X and any integer N there are, up to multiplication by elements of G, only finitely many algebraic numbers θ such that $\mathcal{H}(\theta;G)<X$ and $[K(\theta):K]<N$. Once again, we do not assume that S is a finite set.

We first carry out the argument for $G=K_S^1$. Let $L=K(\theta)$. For the purpose of proving the finiteness property, we can take $[L:\mathbb{Q}]$ to be fixed. By Theorem 3 and (2.7)

$$\begin{aligned}&\mathcal{H}\bigl(\theta;K_S^1\bigr)^{[L:\mathbb{Q}]}\\&\quad=\Bigl(\prod_{p\notin S}\prod_{\sigma:L\to\mathbb{C}_p}\max\bigl(\bigl|\sigma(\theta)\bigr|_p,1\bigr)\Bigr)\prod_{p\in S}\prod_{i=1}^{[L:K]}\max\Bigl(\prod_{\tau:K\to\mathbb{C}_p}\bigl|\sigma_{\tau,i}(\theta)\bigr|_p,1\Bigr).\end{aligned}\tag{4.1}$$

We note that a non-trivial contribution to the right-hand side of (4.1) from any prime $p\neq\infty$ yields at least a factor of $p^{1/[L:\mathbb{Q}]!}$, since $[L:\mathbb{Q}]!$ is certainly a common multiple of all possible absolute ramification indices of primes of L. Let

$$T=T_{X,N}:=\bigl\{p\bigm| p<X^{[L:\mathbb{Q}][L:\mathbb{Q}]!}\bigr\}\cup\{\infty\},\qquad\widetilde{S}:=S\cap T.$$

The advantage of $\widetilde{S}$ over S is that the former is a finite set of places. Since we are assuming that $\mathcal{H}(\theta;K_S^1)<X$, we have $|\sigma(\theta)|_p\leqslant1$ and $|\sigma_{\tau,i}(\theta)|_p\leqslant1$ if $p\notin T$. Thus,

$$\begin{aligned}\mathcal{H}\bigl(\theta;K_S^1\bigr)^{[L:\mathbb{Q}]}&=\prod_{p\notin S,\,p\in T}\prod_{\sigma:L\to\mathbb{C}_p}\max\bigl(\bigl|\sigma(\theta)\bigr|_p,1\bigr)\prod_{p\in\widetilde{S}}\prod_{i=1}^{[L:K]}\max\Bigl(\prod_{\tau:K\to\mathbb{C}_p}\bigl|\sigma_{\tau,i}(\theta)\bigr|_p,1\Bigr)\\&=\prod_{p\notin\widetilde{S}}\prod_{\sigma:L\to\mathbb{C}_p}\max\bigl(\bigl|\sigma(\theta)\bigr|_p,1\bigr)\prod_{p\in\widetilde{S}}\prod_{i=1}^{[L:K]}\max\Bigl(\prod_{\tau:K\to\mathbb{C}_p}\bigl|\sigma_{\tau,i}(\theta)\bigr|_p,1\Bigr)\\&=\mathcal{H}\bigl(\theta;K_{\widetilde{S}}^1\bigr)^{[L:\mathbb{Q}]}.\end{aligned}\tag{4.2}$$

As $\widetilde{S}$ is finite, we can assume not only that $[L : K]$ is fixed, but also that the splitting pattern in K and L of all places above $p \in \widetilde{S}$ is fixed. Thus, using the notation of the proof of Lemma 1, the vector space $V_L = \mathbb{R}\mathcal{L}(L^*)$, its subspace $\mathbb{R}\mathcal{L}(K^1_{\widetilde{S}})$ and the full lattice $\mathcal{L}(K^1_{\widetilde{S}}) \subset \mathbb{R}\mathcal{L}(K^1_{\widetilde{S}})$ can all be fixed. Here $\mathcal{L} = \mathcal{L}_{L,\widetilde{S}} : L^* \to \bigoplus_{p\in\widetilde{S}} V^{(p)}$.

In the proof of Theorem 3 (see (2.13) and (2.24)) we established

$$\mathcal{H}\big(\theta;\, K^1_{\widetilde{S}}\big)^{2[L:\mathbb{Q}]} = N_{\widetilde{S}}(\theta)\, \mathrm{e}^{\|\mathcal{L}(\theta)+\mathbf{x}\|_1}$$

for some $\mathbf{x} \in \mathbb{R}\mathcal{L}(K^1_{\widetilde{S}})$, with $N_{\widetilde{S}}(\theta)$ as defined after (2.10). Note that $N_{\widetilde{S}}(\theta) = N_{\widetilde{S}}(\theta\varepsilon)$ for any $\varepsilon \in K_{\widetilde{S}}$. By compactness of the quotient space $\mathbb{R}\mathcal{L}(K^1_{\widetilde{S}})/\mathcal{L}(K^1_{\widetilde{S}})$, any $\mathbf{x} \in \mathbb{R}\mathcal{L}(K^1_{\widetilde{S}})$ can be written as $\mathbf{x} = \mathcal{L}(\varepsilon) + \mathbf{y}$, where $\varepsilon \in K^1_{\widetilde{S}}$, $\mathbf{y} \in \mathbb{R}\mathcal{L}(K^1_{\widetilde{S}})$ and $\|\mathbf{y}\|_1 < A$, where A is independent of θ and L ($\|\ \|_1$ is the 1-norm defined in (2.17)). Hence

$$\begin{aligned}\mathcal{H}\big(\theta;\, K^1_{\widetilde{S}}\big)^{2[L:\mathbb{Q}]} &= N_{\widetilde{S}}(\theta)\, \mathrm{e}^{\|\mathcal{L}(\theta)+\mathcal{L}(\varepsilon)+\mathbf{y}\|_1} = N_{\widetilde{S}}(\theta\varepsilon)\, \mathrm{e}^{\|\mathcal{L}(\theta\varepsilon)+\mathbf{y}\|_1} \geqslant N_{\widetilde{S}}(\theta\varepsilon)\, \mathrm{e}^{\|\mathcal{L}(\theta\varepsilon)\|_1 - \|\mathbf{y}\|_1} \\ &\geqslant N_{\widetilde{S}}(\theta\varepsilon)\mathrm{e}^{\|\mathcal{L}(\theta\varepsilon)\|_1}\mathrm{e}^{-A} = H(\theta\varepsilon)^{2[L:\mathbb{Q}]}\mathrm{e}^{-A}, \qquad (4.3)\end{aligned}$$

where in the last step we used (2.10) and $\mathbb{Q}(\theta\varepsilon) \subset K(\theta) = L$.

It follows from (4.2) and (4.3) that $H(\theta\varepsilon) < X \exp(A/(2[L : \mathbb{Q}]))$. By Northcott's finiteness theorem for the classical height [La2, p. 59], [HS, p. 177], $\theta\varepsilon$ can be only one of finitely many algebraic numbers. Since $\varepsilon \in K^1_{\widetilde{S}} \subset K^1_S$, we have proved our finiteness claim for K^1_S-heights.

The proof of the corresponding finiteness property for K_S-heights is nearly identical. One need only shift the product over $p \in S$ in (4.1) into the maximum, observe that $\mathcal{L}(K_{\widetilde{S}})$ is a full lattice inside $\mathbb{R}\mathcal{L}(K_{\widetilde{S}})$ and proceed exactly as above.

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