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New upper bounds on the spectral radius of unicyclic graphs ☆

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Abstract

Let $\mathscr{G} = (V(\mathscr{G}), E(\mathscr{G}))$ be a unicyclic simple undirected graph with largest vertex degree Δ. Let $\mathscr{C}_r$ be the unique cycle of $\mathscr{G}$. The graph $\mathscr{G} - E(\mathscr{C}_r)$ is a forest of r rooted trees $\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_r$ with root vertices $v_1, v_2, \ldots, v_r$, respectively. Let

$$k(\mathscr{G}) = \max_{1 \leqslant i \leqslant r} \{\max\{\text{dist}(v_i, u) : u \in V(\mathscr{T}_i)\}\} + 1,$$

where $\text{dist}(v, u)$ is the distance from v to u. Let $\mu_1(\mathscr{G})$ and $\lambda_1(\mathscr{G})$ be the spectral radius of the Laplacian matrix and adjacency matrix of $\mathscr{G}$, respectively. We prove that

$$\mu_1(\mathscr{G}) < \Delta + 2\sqrt{\Delta - 1} \cos \frac{\pi}{2k(\mathscr{G}) + 1},$$

whenever $\Delta > 2$ and

$$\lambda_1(\mathscr{G}) < 2\sqrt{\Delta - 1} \cos \frac{\pi}{2k(\mathscr{G}) + 1},$$

whenever $\Delta \geqslant 4$ or whenever $\Delta = 3$ and $k(\mathscr{G}) \geqslant 4$.

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1. Introduction

Let $\mathscr{G} = (V(\mathscr{G}), E(\mathscr{G}))$ be a simple undirected graph. Let $A(\mathscr{G})$ be the adjacency matrix of $\mathscr{G}$ and let $D(\mathscr{G})$ be the diagonal matrix of vertex degrees. The Laplacian matrix of $\mathscr{G}$ is the matrix $L(\mathscr{G}) = D(\mathscr{G}) - A(\mathscr{G})$. Both $A(\mathscr{G})$ and $L(\mathscr{G})$ are real symmetric matrices. Moreover, $L(\mathscr{G})$ is a positive semidefinite matrix and $(0, \mathbf{e})$ is an eigenpair of $L(\mathscr{G})$, where $\mathbf{e}$ is the all ones vector.

Let $\mu_1(\mathscr{G})$ and $\lambda_1(\mathscr{G})$ be the spectral radius of $L(\mathscr{G})$ and $A(\mathscr{G})$, respectively. It is known that if $\mathscr{H}$ is a subgraph of $\mathscr{G}$ then $\mu_1(\mathscr{H}) \leqslant \mu_1(\mathscr{G})$ and $\lambda_1(\mathscr{H}) \leqslant \lambda_1(\mathscr{G})$.

We recall that the distance $\operatorname{dist}(u, v), u, v \in V(\mathscr{G})$, is the length of the shortest path in $\mathscr{G}$ from u to v and that the degree $d(v)$, $v \in V(\mathscr{G})$, is the number of edges in $E(\mathscr{G})$ that are incident with v.

Let

$$\varDelta = \max\{d(v) : v \in V(\mathscr{G})\}.$$

A tree is a connected acyclic graph. In [6, 2003], Stevanović proves that for a tree $\mathscr{T}$ with largest vertex degree $\varDelta$,

$$\mu_1(\mathscr{T}) < \varDelta + 2\sqrt{\varDelta - 1}$$

and

$$\lambda_1(\mathscr{T}) < 2\sqrt{\varDelta - 1}.$$

In [2, 2007], Hu proves that if $\mathscr{G}$ is a unicyclic graph then

$$\mu_1(\mathscr{G}) \leqslant \varDelta + 2\sqrt{\varDelta - 1}$$

with equality if and only if $\mathscr{G}$ is the cycle $\mathscr{C}_n$ whenever n is even, and

$$\lambda_1(\mathscr{G}) \leqslant 2\sqrt{\varDelta - 1}$$

with equality if and only if $\mathscr{G}$ is the cycle $\mathscr{C}_n$.

From now on, let $\mathscr{G}$ be a unicyclic graph with largest vertex degree $\varDelta > 2$. Let $\mathscr{C}_r$ be the unique cycle of $\mathscr{G}$ and let $v_1, v_2, \ldots., v_r$ be the vertices of $\mathscr{C}_r$. Then, the graph $\mathscr{G} - E(\mathscr{C}_r)$ is a forest of r rooted trees $\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_r$ with root vertices $v_1, v_2, \ldots, v_r$, respectively.

Definition 1. For $i = 1, 2, \ldots, r$, we define

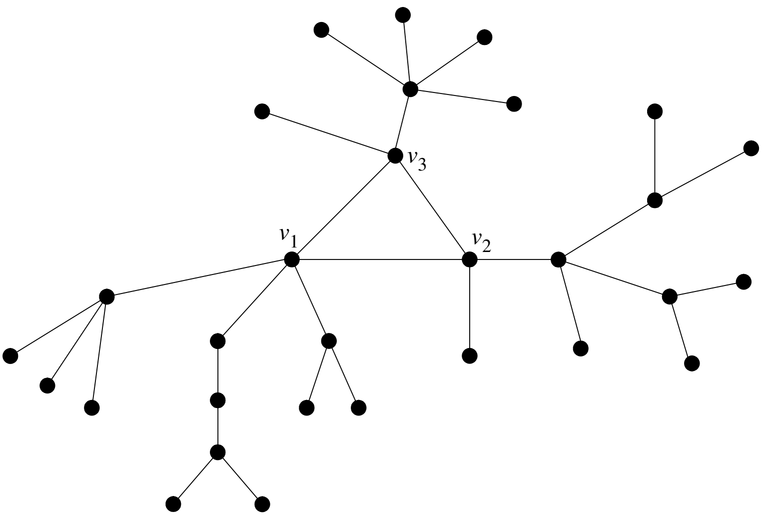
$$k_i = \max\{\operatorname{dist}(v_i, u) : u \in V(\mathscr{T}_i)\} + 1$$

and

$$k(\mathscr{G}) = \max\{k_i : 1 \leqslant i \leqslant r\}.$$

Let us illustrate this definition with the following example.

Example 1. Let $\mathscr{G}$ be the graph



For this graph, $\Delta = 5$,

$$k_1 = \max\{d(v_1, u) : u \in V(\mathscr{T}_1)\} + 1 = 4 + 1 = 5,$$
$$k_2 = \max\{d(v_2, u) : u \in V(\mathscr{T}_2)\} + 1 = 3 + 1 = 4,$$
$$k_3 = \max\{d(v_3, u) : u \in V(\mathscr{T}_3)\} + 1 = 2 + 1 = 3.$$

Then $k(\mathscr{G}) = \max\{5, 4, 3\} = 5$.

In this paper, we derive the new upper bounds

$$\mu_1(\mathscr{G}) < \Delta + 2\sqrt{\Delta - 1}\cos\frac{\pi}{2k(\mathscr{G}) + 1},$$

whenever $\Delta > 2$, and

$$\lambda_1(\mathscr{G}) < 2\sqrt{\Delta - 1}\cos\frac{\pi}{2k(\mathscr{G}) + 1},$$

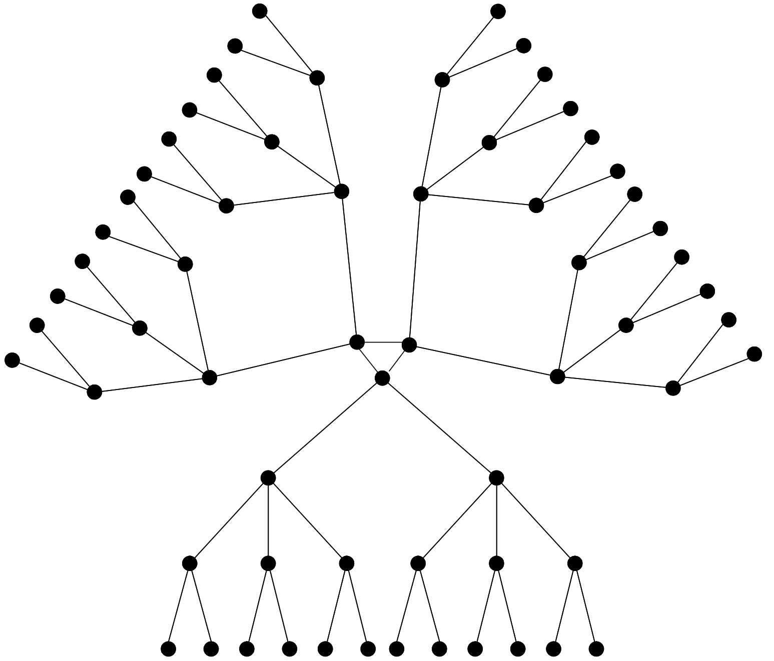
whenever $\Delta \geqslant 4$ or whenever $\Delta = 3$ and $k(\mathscr{G}) \geqslant 4$.

2. Finding the new upper bounds

We begin this section by recalling some results from [4, 2007] that will play an important role in this paper.

The level of a vertex in a rooted tree is one more than its distance from the root vertex. A tree $\mathscr{B}_k$ of k levels is a generalized Bethe tree [4] if vertices at the same level have equal degree. Let $\mathscr{B}_k$ be a generalized Bethe tree of k levels. Let $\mathscr{B}_k^{(r)}$ be the unicyclic graph obtained from the union of r copies of $\mathscr{B}_k$ and the cycle $\mathscr{C}_r$ connecting the r root vertices. We may consider $\mathscr{B}_k^{(r)}$ as a graph of $k > 1$ levels in which vertices at the same level have equal degree. We agree that the vertices of $\mathscr{C}_r$ are at the level 1. An example of a such graph is

Example 2



This graph has four levels of vertices in which the vertex degree sequence, from the pendant vertices to the vertices in $\mathscr{C}_3$, is (1, 3, 4, 4).

For the graph $\mathscr{B}_k^{(r)}$ with $k > 1$, let d_{k-j+1} be the degree of the vertices at the level $j\,(j = 1, 2, \ldots, k)$. Thus, d_k is the degree of the vertices at the level 1, and $d_1 = 1$ is the degree of the vertices at the level k (pendant vertices). Let

$$\mathbf{d} = (1, d_2, \ldots, d_k).$$

As usual, let $\rho(M)$ denotes the spectral radius of the matrix M.

In [4], we characterize completely the spectra of the adjacency matrix and Laplacian matrix of $\mathscr{B}_k^{(r)}$. In particular, we derive results concerning $\mu_1(\mathscr{B}_k^{(r)})$ and $\lambda_1(\mathscr{B}_k^{(r)})$, which we give in the following lemmas.

Lemma 1 [4, Theorem 3, part (c)]. *Let $r = 2s$ or $r = 2s + 1$. Let $L_{k,s}(\mathbf{d})$ be the $k \times k$ symmetric tridiagonal matrix*

$$L_{k,s}(\mathbf{d}) = \begin{bmatrix} 1 & \sqrt{d_2 - 1} & & & \\ \sqrt{d_2 - 1} & d_2 & \sqrt{d_3 - 1} & & \\ & \sqrt{d_3 - 1} & \ddots & \ddots & \\ & & \ddots & d_{k-1} & \sqrt{d_k - 2} \\ & & & \sqrt{d_k - 2} & d_k - 2\cos\frac{2\pi s}{r} \end{bmatrix}.$$

Then $\rho(L_{k,s}(\mathbf{d})) = \mu_1(\mathscr{B}_k^{(r)})$.

Lemma 2 [4, Theorem 7, part (d)]. *Let $A_{k,0}(\mathbf{d})$ be the $k \times k$ symmetric tridiagonal matrix*

$$A_{k,0}(\mathbf{d}) = \begin{bmatrix} 0 & \sqrt{d_2 - 1} & & & \\ \sqrt{d_2 - 1} & 0 & \sqrt{d_3 - 1} & & \\ & \sqrt{d_3 - 1} & \ddots & \ddots & \\ & & \ddots & 0 & \sqrt{d_k - 2} \\ & & & \sqrt{d_k - 2} & 2 \end{bmatrix}.$$

Then $\rho(A_{k,0}(\mathbf{d})) = \lambda_1(\mathscr{B}_k^{(r)})$.

Keep in mind that $\mathscr{G}$ is a unicyclic graph with largest vertex degree $\Delta > 2$, in which $\mathscr{C}_r$ is the unique cycle of $\mathscr{G}$ and that $k(\mathscr{G})$ is as in definition 1. For brevity, we write k instead of $k(\mathscr{G})$. Let $\mathscr{B}_k(\Delta)$ be the generalized Bethe tree with vertex degree sequence

$$(1, \Delta, \Delta, \ldots, \Delta, \Delta - 2),$$

from the pendant vertices to the root vertex. Then, each tree $\mathscr{T}_i$ is an induced subgraph of $\mathscr{B}_k(\Delta)$. Let $\mathscr{B}_k^{(r)}(\Delta)$ be the unicyclic graph obtained from r copies of $\mathscr{B}_k(\Delta)$ and the cycle $\mathscr{C}_r$ connecting the r root vertices. Therefore, $\mathscr{G}$ is an induced subgraph of $\mathscr{B}_k^{(r)}(\Delta)$. Consequently $\mu_1(\mathscr{G}) \leqslant \mu_1(\mathscr{B}_k^{(r)}(\Delta))$ and $\lambda_1(\mathscr{G}) \leqslant \lambda_1(\mathscr{B}_k^{(r)}(\Delta))$.

Observe that the vertex degree sequence for $\mathscr{B}_k^{(r)}(\Delta)$ is

$$\mathbf{d} = (1, \Delta, \Delta, \ldots, \Delta, \Delta).$$

We apply Lemma 1 to the graph $\mathscr{B}_k^{(r)}(\Delta)$ to get that the spectral radius of the $k \times k$ matrix

$$L_{k,s}(\Delta) = \begin{bmatrix} 1 & \sqrt{\Delta - 1} & & & \\ \sqrt{\Delta - 1} & \Delta & \sqrt{\Delta - 1} & & \\ & \sqrt{\Delta - 1} & \ddots & \ddots & \\ & & \ddots & \Delta & \sqrt{\Delta - 2} \\ & & & \sqrt{\Delta - 2} & \Delta - 2\cos\frac{2\pi s}{r} \end{bmatrix}$$

is equal to $\mu_1(\mathscr{B}_k^{(r)}(\Delta))$ and that the spectral radius of the $k \times k$ matrix

$$A_{k,0}(\Delta) = \begin{bmatrix} 0 & \sqrt{\Delta - 1} & & & \\ \sqrt{\Delta - 1} & 0 & \sqrt{\Delta - 1} & & \\ & \sqrt{\Delta - 1} & \ddots & \ddots & \\ & & \ddots & 0 & \sqrt{\Delta - 2} \\ & & & \sqrt{\Delta - 2} & 2 \end{bmatrix}$$

is equal to $\lambda_1(\mathscr{B}_k^{(r)}(\Delta))$.

Therefore in order to find upper bounds on $\mu_1(\mathscr{G})$ and $\lambda_1(\mathscr{G})$, we search for upper bounds on $\rho(L_{k,s}(\Delta))$ and $\rho(A_{k,0}(\Delta))$, respectively.

We recall the following result concerning the spectral radius of a special symmetric tridiagonal matrix.

Lemma 3 [3]. *The spectral radius of the $m \times m$ symmetric tridiagonal matrix*

$$D_m(b) = \begin{bmatrix} 0 & b & & & & \\ b & 0 & b & & & \\ & b & & \ddots & & \\ & & \ddots & \ddots & b & \\ & & & b & 0 & b \\ & & & & b & b \end{bmatrix}, \quad b > 0 \tag{1}$$

is

$$\rho(D_m(b)) = 2b \cos \frac{\pi}{2m+1}.$$

Theorem 4. *Let $\mathcal{G}$ be a unicyclic graph with largest vertex degree $\Delta > 2$. Let $k(\mathcal{G})$ as in definition 1. Then*

$$\mu_1(\mathcal{G}) < \Delta + 2\sqrt{\Delta - 1} \cos \frac{\pi}{2k(\mathcal{G}) + 1}. \tag{2}$$

Proof. We know that $\mu_1(\mathcal{G}) \leqslant \mu_1(\mathcal{B}_k^{(r)}(\Delta)) = \rho(L_{k,s}(\Delta))$. We have

$$L_{k,s}(\Delta) = \begin{bmatrix} 1 & \sqrt{\Delta-1} & & & \\ \sqrt{\Delta-1} & \Delta & \sqrt{\Delta-1} & & \\ & \sqrt{\Delta-1} & \ddots & \ddots & \\ & & \ddots & \Delta & \sqrt{\Delta-2} \\ & & & \sqrt{\Delta-2} & \Delta - 2\cos\frac{2\pi s}{r} \end{bmatrix}$$

$$\leqslant \begin{bmatrix} 1 & \sqrt{\Delta-1} & & & \\ \sqrt{\Delta-1} & \Delta & \sqrt{\Delta-1} & & \\ & \sqrt{\Delta-1} & \ddots & \ddots & \\ & & \ddots & \Delta & \sqrt{\Delta-2} \\ & & & \sqrt{\Delta-2} & \Delta + 2 \end{bmatrix} = E_k.$$

Since the spectral radius of an irreducible nonnegative matrix increases when any of its entries increases [8], we have

$$\rho(L_{k,s}(\Delta)) \leqslant \rho(E_k). \tag{3}$$

The matrix E_k has the LL^T-decomposition

$$E_k = LL^T,$$

where L is the lower bidiagonal matrix

$$L = \begin{bmatrix} 1 & & & & \\ \sqrt{\Delta-1} & 1 & & & \\ & \sqrt{\Delta-1} & \ddots & & \\ & & \ddots & 1 & \\ & & & \sqrt{\Delta-2} & 2 \end{bmatrix}.$$

Then E_k is a positive definite and the matrices $E_k = LL^T$ and $\widetilde{E_k} = L^T L$ have the same eigenvalues [9]. From this fact and (3), we obtain

$$\rho(L_{k,s}(\Delta)) \leqslant \rho(\widetilde{E_k}). \tag{4}$$

Computing the product $L^T L$, we obtain

$$\widetilde{E_k} = \begin{bmatrix} \Delta & \sqrt{\Delta-1} & & & \\ \sqrt{\Delta-1} & \Delta & \sqrt{\Delta-1} & & \\ & \sqrt{\Delta-1} & \ddots & \ddots & \\ & & \ddots & \Delta-1 & 2\sqrt{\Delta-2} \\ & & & 2\sqrt{\Delta-2} & 4 \end{bmatrix}.$$

We have

$$\begin{aligned} \widetilde{E_k} &= \operatorname{diag}\{\Delta, \Delta, \ldots, \Delta\} \\ &\quad + \begin{bmatrix} 0 & \sqrt{\Delta-1} & & & \\ \sqrt{\Delta-1} & 0 & \sqrt{\Delta-1} & & \\ & \sqrt{\Delta-1} & \ddots & \ddots & \\ & & \ddots & -1 & 2\sqrt{\Delta-2} \\ & & & 2\sqrt{\Delta-2} & 4-\Delta \end{bmatrix} \\ &= \operatorname{diag}\{\Delta, \Delta, \ldots, \Delta\} + F_k, \end{aligned}$$

where

$$F_k = \begin{bmatrix} 0 & \sqrt{\Delta-1} & & & \\ \sqrt{\Delta-1} & 0 & \sqrt{\Delta-1} & & \\ & \sqrt{\Delta-1} & \ddots & \ddots & \\ & & \ddots & -1 & 2\sqrt{\Delta-2} \\ & & & 2\sqrt{\Delta-2} & 4-\Delta \end{bmatrix}.$$

From (1),

$$D_k(\sqrt{\Delta-1}) = \begin{bmatrix} 0 & \sqrt{\Delta-1} & & & \\ \sqrt{\Delta-1} & 0 & \sqrt{\Delta-1} & & \\ & \sqrt{\Delta-1} & \ddots & \ddots & \\ & & \ddots & 0 & \sqrt{\Delta-1} \\ & & & \sqrt{\Delta-1} & \sqrt{\Delta-1} \end{bmatrix}.$$

We define

$$\begin{aligned} G_k &= D_k(\sqrt{\Delta-1}) - F_k \\ &= \begin{bmatrix} 0 & 0 & & & \\ 0 & \ddots & \ddots & & \\ & \ddots & 0 & 0 & \\ & & 0 & 1 & \sqrt{\Delta-1} - 2\sqrt{\Delta-2} \\ & & & \sqrt{\Delta-1} - 2\sqrt{\Delta-2} & \sqrt{\Delta-1} - 4 + \Delta \end{bmatrix}. \end{aligned}$$

Since $\varDelta \geqslant 3$, one can prove that the submatrix

$$\begin{bmatrix} 1 & \sqrt{\varDelta-1}-2\sqrt{\varDelta-2} \\ \sqrt{\varDelta-1}-2\sqrt{\varDelta-2} & -4+\varDelta+\sqrt{\varDelta-1} \end{bmatrix}$$

is a positive definite matrix. Hence G_k is positive semidefinite matrix. Since F_k is an irreducible nonnegative matrix, there exists unitary eigenvector $\mathbf{x} = \begin{bmatrix}\mathbf{x}_1^T & \mathbf{x}_2^T\end{bmatrix} > \mathbf{0}$ such that $\mathbf{x}^T F_k \mathbf{x} = \rho(F_k)$ [8]. Therefore $\mathbf{x}_2 \neq \mathbf{0}$ and

$$\begin{aligned} \mathbf{x}^T D_k(\sqrt{\varDelta-1})\mathbf{x} &= \mathbf{x}^T F_k \mathbf{x} + \mathbf{x}^T G_k \mathbf{x} \\ &= \rho(F_k) \\ &\quad + \mathbf{x}_2^T \begin{bmatrix} 1 & \sqrt{\varDelta-1}-2\sqrt{\varDelta-2} \\ \sqrt{\varDelta-1}-2\sqrt{\varDelta-2} & -4+\varDelta+\sqrt{\varDelta-1} \end{bmatrix} \mathbf{x}_2 \\ &> \rho(F_k). \end{aligned}$$

Then

$$\rho(F_k) < \mathbf{x}^T D_k(\sqrt{\varDelta-1})\mathbf{x} \leqslant \rho(D_k(\sqrt{\varDelta-1})).$$

From Lemma 3, the spectral radius of $D_{k-1}(\sqrt{\varDelta-1})$ is

$$\rho(D_k(\sqrt{\varDelta-1})) = 2\sqrt{\varDelta-1}\cos\frac{\pi}{2k+1}.$$

Therefore

$$\rho(F_k) < 2\sqrt{\varDelta-1}\cos\frac{\pi}{2k+1}.$$

Consequently

$$\begin{aligned} \lambda_1(\widetilde{E_k}) &= \varDelta + \lambda_1(F_k) \\ &< \varDelta + 2\sqrt{\varDelta-1}\cos\frac{\pi}{2k+1}. \end{aligned} \tag{5}$$

From (4) and (5), we get

$$\rho(L_{k,s}(\varDelta)) < \varDelta + 2\sqrt{\varDelta-1}\cos\frac{\pi}{2k+1}.$$

Since $\mu_1(\mathscr{G}) \leqslant \mu_1(\mathscr{B}_k^{(r)}) = \rho(L_{k,s}(\varDelta))$ and $k = k(\mathscr{G})$, the upper bound (2) is proved. □

At this point, it is convenient to recall some other known results on symmetric tridiagonal matrices [1,5,7].

Lemma 5. 1. *The characteristic polynomials*, $q_j(\lambda)$, *of the* $j \times j$ *leading principal submatrix of the* $k \times k$ *symmetric tridiagonal matrix*

$$Q_k = \begin{bmatrix} a_1 & b_1 & & & & \\ b_1 & a_2 & b_2 & & & \\ & b_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & a_{k-1} & b_{k-1} \\ & & & & b_{k-1} & a_k \end{bmatrix} \tag{6}$$

satisfy the three-term recursion formula

$$q_j(\lambda) = (\lambda - a_j)q_{j-1}(\lambda) - b_{j-1}^2 q_{j-2}(\lambda)$$

with $q_0(\lambda) = 1$ *and* $q_1(\lambda) = \lambda - a_1$.

2. *If all codiagonal entries* b_i $(i = 1, 2, \ldots, k-1)$ *in* (6) *are nonzero, the recursion polynomial* q_j $(j = 0, 1, \ldots, k)$ *has* j *real simple zeros. For* $1 \leqslant j \leqslant k-1$*, they strictly interlace those of* q_{j+1}.

Theorem 6. *Let* $\mathscr{G}$ *be a unicyclic graph with largest vertex degree* $\Delta > 2$. *Let* $k(\mathscr{G})$ *as in definition* 1. *Then*

$$\lambda_1(\mathscr{G}) < 2\sqrt{\Delta - 1}\cos\frac{\pi}{2k(\mathscr{G}) + 1}, \tag{7}$$

whenever $\Delta \geqslant 4$ *or whenever* $\Delta = 3$ *and* $k(\mathscr{G}) \geqslant 4$.

Proof. We know that $\lambda_1(\mathscr{G}) \leqslant \lambda_1(\mathscr{B}_k^{(r)}(\Delta)) = \rho(A_{k,0}(\Delta))$. Suppose $\Delta \geqslant 5$. Then

$$A_{k,0}(\Delta) = \begin{bmatrix} 0 & \sqrt{\Delta-1} & & & \\ \sqrt{\Delta-1} & 0 & \sqrt{\Delta-1} & & \\ & \sqrt{\Delta-1} & \ddots & \ddots & \\ & & \ddots & 0 & \sqrt{\Delta-2} \\ & & & \sqrt{\Delta-2} & 2 \end{bmatrix}$$

$$\leqslant \begin{bmatrix} 0 & \sqrt{\Delta-1} & & & \\ \sqrt{\Delta-1} & 0 & \sqrt{\Delta-1} & & \\ & \sqrt{\Delta-1} & \ddots & \ddots & \\ & & \ddots & 0 & \sqrt{\Delta-1} \\ & & & \sqrt{\Delta-1} & \sqrt{\Delta-1} \end{bmatrix} = D_k(\sqrt{\Delta-1})$$

with strict inequality in position $(k-1, k)$. Therefore

$$\rho(A_{k,0}(\Delta)) < \rho(D_k(\sqrt{\Delta-1})) = 2\sqrt{\Delta-1}\cos\frac{\pi}{2k+1}.$$

Thus the bound (7) is proved for $\Delta \geqslant 5$. It remains to study the cases $\Delta = 3$ and $\Delta = 4$. For $j = 1, 2, \ldots, k$, let $a_j(\lambda)$ and $d_j(\lambda)$ be the characteristic polynomials of the $j \times j$ leading principal submatrices of $A_{k,0}(\Delta)$ and $D_k(\sqrt{\Delta-1})$, respectively. Observe that $a_j(\lambda)$ and $d_j(\lambda)$ are identical polynomials $(j = 1, 2, \ldots, k-1)$, and that $a_k(\lambda)$ and $d_k(\lambda)$ are the characteristic polynomials of $A_{k,0}(\Delta)$ and $D_k(\sqrt{\Delta-1})$, respectively. From Lemma 5, part 1, we have

$$a_k(\lambda) = (\lambda - 2)a_{k-1}(\lambda) - (\Delta - 2)a_{k-2}(\lambda) \tag{8}$$

and

$$d_k(\lambda) = (\lambda - \sqrt{\Delta-1})a_{k-1}(\lambda) - (\Delta - 1)a_{k-2}(\lambda). \tag{9}$$

From (8) and (9), we obtain

$$a_k(\lambda) - d_k(\lambda) = (\sqrt{\Delta-1} - 2)a_{k-1}(\lambda) + a_{k-2}(\lambda). \tag{10}$$

From Lemma 5, part 2, we may write

$$a_k(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2)\cdots(\lambda - \alpha_{k-1})(\lambda - \alpha_k),$$

where

$$\alpha_k < \alpha_{k-1} < \cdots < \alpha_2 < \alpha_1$$

are the zeros of the polynomial $a_k(\lambda)$. Let $\Delta = \rho(D_k(\sqrt{\Delta - 1}))$. Then, Δ is the largest zero of $d_k(\lambda)$. Since, from Lemma 3,

$$\Delta = 2\sqrt{\Delta - 1}\cos\frac{\pi}{2k+1},$$

to show (7) it is sufficient to prove that $\Delta > \alpha_1$. Let β_1 be the largest zero of the identical polynomials $d_{k-1}(\lambda)$ and $a_{k-1}(\lambda)$. Since the zeros of these polynomials strictly interlace the zeros of the polynomials $a_k(\lambda)$ and $d_k(\lambda)$, we obtain that $\alpha_2 < \beta_1 < \alpha_1$ and $\beta_1 < \Delta$. Therefore $\alpha_2 < \Delta$, $a_{k-1}(\Delta) > 0$ and

$$\begin{aligned} a_k(\Delta) &= (\Delta - \alpha_1)(\Delta - \alpha_2)\cdots(\Delta - \alpha_{k-1})(\Delta - \alpha_k) \\ &= (\Delta - \alpha_1)P, \end{aligned}$$

where $P > 0$. Thus, to show that $\Delta > \alpha_1$ it is sufficient to prove that $a_k(\Delta) > 0$. From (9) and (10),

$$(\Delta - \sqrt{\Delta - 1})a_{k-1}(\Delta) - (\Delta - 1)a_{k-2}(\Delta) = 0$$

and

$$a_k(\Delta) = (\sqrt{\Delta - 1} - 2)a_{k-1}(\Delta) + a_{k-2}(\Delta).$$

Then

$$a_k(\Delta) = \left(\sqrt{\Delta - 1} - 2 + \frac{\Delta - \sqrt{\Delta - 1}}{\Delta - 1}\right)a_{k-1}(\Delta). \tag{11}$$

Let $\Delta = 4$. From (11)

$$\begin{aligned} a_k(\Delta) &= \left(\sqrt{3} - 2 + \frac{2\sqrt{3}\cos\frac{\pi}{2k+1} - \sqrt{3}}{3}\right)a_{k-1}(\Delta) \\ &= \left(\frac{2\sqrt{3}}{3} - 2 + \frac{2\sqrt{3}}{3}\cos\frac{\pi}{2k+1}\right)a_{k-1}(\Delta) \\ &\geqslant \left(\frac{2\sqrt{3}}{3} - 2 + \frac{2\sqrt{3}}{3}\cos\frac{\pi}{5}\right)a_{k-1}(\Delta) \\ &> 0.08a_{k-1}(\Delta) > 0. \end{aligned}$$

Let now $\Delta = 3$ and $k \geqslant 4$. From (11)

$$\begin{aligned} a_k(\Delta) &= \left(\sqrt{2} - 2 + \frac{2\sqrt{2}\cos\frac{\pi}{2k+1} - \sqrt{2}}{2}\right)a_{k-1}(\Delta) \\ &= \left(\frac{\sqrt{2}}{2} - 2 + \sqrt{2}\cos\frac{\pi}{2k+1}\right)a_{k-1}(\Delta) \end{aligned}$$

$$\geqslant \left(\frac{\sqrt{2}}{2} - 2 + \sqrt{2}\cos\frac{\pi}{9} \right) a_{k-1}(\Delta)$$
$$> 0.03 a_{k-1}(\Delta) > 0.$$

The proof is complete. □

Remark 1. The bound (7) does not hold for $\Delta = 3$ if $k(\mathscr{G}) = 2$ or $k(\mathscr{G}) = 3$. In fact, to four decimal places, the spectral radius of

$$A_{2,0}(3) = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

is 2.4142 and $2\sqrt{2}\cos\frac{\pi}{5} = 2.2882$, and the spectral radius of

$$A_{3,0}(3) = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

is 2.5616 and $2\sqrt{2}\cos\frac{\pi}{7} = 2.5483$.

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