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NER Automata Dynamics on Random Graphs*

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Abstract: The average transient time, damage spreading and qualitative effects are determined for the NER automata parallel dynamics defined on random graphs. It was obtained that the NER automata converge with linear rate to fixed points, the average damage spreading presents a linear response without discontinuity at the origin for small damage limit and the hamming distance between the initial and steady configurations falls in the range [0.82,0.88]. These results can be interpreted as a generalization of ref. [8] to the case of random graphs where the global connectivity is present.

Keywords: NER Automata, Transient Time, Damage Spreading, Random Graphs.

Mathematics Subject Classification: 37M05, 68U10

1. Introduction

Enhancement, Segmentation, Description and Recognition techniques are methods used to obtain features of the image without any previous information, see refs. [5,13]. The enhancement sharpening techniques are applied to improve noisy or blurred images by locally increase or decrease the color levels differences of the image, see refs. [5,13]. The spatial sharpening techniques suggest the application of cellular automata methods for elementary image enhancement, see refs. [7,11,12].

The Nearest Extremum Rule Automaton (NER) was introduced as enhancement technique in ref. [1]. In this automaton, the local color differences of the image are increased by choosing the closest extremal local value, see refs. [3,6,7,8]. To determine the NER dynamical behavior the transient and stationary phase must be studied. The most common techniques used are Lyapunov functionals, see refs. [2,3,4]. It must be also performed a comparison numerical study with classical techniques to determine their typical dynamical behavior, see refs. [6,8].

2. NER Automata Definition and Previous Results

Let $G(V,E)$ be a finite, undirected and connected graph with V ($|V| = n$) the vertices set and $E \subseteq V \times V$ the edges set. To each vertex $i \in V$ are assigned values $x_i \in Q = \{0,1,...,q-1\}$ ($q \geq 3$ is the number of colors or gray scales). Let V_i be the vertex i neighborhood, defined by: $V_i = \{ j \in V : (i,j) \in E, j \neq i \}$, with $d_i = |V_i|$ the vertex i degree. The NER is a specific example of the general class of extremal rules automata ER, see refs. [3,6,7,8]. The ER parallel dynamics is defined by its local transition function:

$$x(t+1) = (x_1(t+1),...,x_n(t+1)),\ x_i(t+1) = f_{ER}(x_j(t) : j \in V_i) \tag{0.1}$$

$$f_{ER}(x_j(t) : j \in V_i) = \begin{cases} f_{ER}(x_j(t) : j \in V_i) \in \{ m_i(t), x_i(t), M_i(t) \} & \text{if } x_i \in (m_i(t), M_i(t)) \\ f_{ER}(x_j(t) : j \in V_i) = x_i(t) & \text{otherwise} \end{cases}$$

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where $m_i(t) = \min_{k \in V_i} x_k(t)$ and $M_i(t) = \max_{k \in V_i} x_k(t)$. Therefore, under any ER a vertex i can evolve only to its local extreme value. The ER was defined as a generalization of the FES Rules, see ref. [1], that were introduced in refs. [9,10] as earlier image processing techniques. The NER parallel dynamics is also defined by its local transition function:

$$x(t+1) = (x_1(t+1), ..., x_n(t+1)),\ x_i(t+1) = f_{NER}(x_j(t) : j \in V_i) \tag{0.2}$$

$$f_{NER}(x_j(t) : j \in V_i) = \begin{cases} m_i(t) \text{ if } x_i(t) - m_i(t) < M_i(t) - x_i(t) \text{ and } x_i(t) \in \left(m_i(t), M_i(t)\right) \\ M_i(t) \text{ if } M_i(t) - x_i(t) < x_i(t) - m_i(t) \text{ and } x_i(t) \in \left(m_i(t), M_i(t)\right) \\ x_i(t) \text{ otherwise} \end{cases}$$

The NER transition function evolves to the closer extreme value to $x_i(t)$.

In ref. [3], the ER sequential dynamics was characterized by the Lyapunov functional $L_{ER} : Q^n \to \mathbb{R}$ defined by:

$$L_{ER}(x) = -\tfrac{1}{2} \sum_{i=1}^{n} \sum_{j \in V_i} a^{|x_i - x_j|} \tag{0.3}$$

Proposition [3]**:** ER sequential dynamics

a) For $a > d \doteq \max_{i=1,...,n} |V_i|$, L_{ER} is a Lyapunov functional for the ER sequential dynamics.

b) An exponential bound for the maximal transient time τ can be obtained from L_{ER}:

$$\tau_{ER} \doteq \max_{t \geq 0} \left\{ t \,/\, x(t) \neq x(t-1), x(0) \in Q^n \right\} \sim O(n^{\log_2(n)}) \tag{0.4}$$

In ref. [3], polynomial sharp bounds were obtained for some particular ER.
For the ER parallel dynamics, a fixed point steady state was determined by direct proof, but a Lyapunov functional and the maximal transient time have not been yet determined.

Proposition [7]**:** The ER parallel iteration converges only to fixed points.

A $O(n^2)$ bound was obtained for the NER maximal parallel transient time and it was proved the convergence only to fixed points, see ref. [1].
In ref. [2] polynomial sharp bounds for the maximal transient time of the parallel dynamics of Max-Min automata (Mm) and NER were obtained by Lyapunov functionals. The max-min automata (Mm) is defined by: If $\{I_M, I_m\}$ is a partition of V then: $\forall i \in I_M : f_i(x_j : j \in V_i) = M_i$ and $\forall i \in I_m : f_i(x_j : j \in V_i) = m_i$.

Proposition [2]**:** $H(x) = \sum_{i \in I_m} x_i - \sum_{i \in I_M} x_i$ is a Lyapunov functional for the Mm parallel dynamics.

Corollary [2]**:** For the Mm: $\tau_{Mm}(G) \leq n(n-1)$ and this bound is attained.

Let S_t be the set of vertices with maximum jumps between steps t and $(t+1)$ defined by:

$$S_t = \left\{ i \in V : |x_i(t+1) - x_i(t)| \geq |x_k(t+1) - x_k(t)| \quad \forall k = 1, ..., n \right\} \text{ and } \underline{S}_t = \bigcup_{k=0}^{t} S_k \tag{0.5}$$

Proposition [2]: $H(x) = \sum_{i \in \underline{S}_t} \max \left\{ |x_i(t) - m_i(t)|, |M_i(t) - x_i(t)| \right\}$ is a Lyapunov functional for the NER parallel iteration.

Corollary [2]**:** For the NER: $\tau_{NER}(G) \leq n^2$ and this bound is attained.

In ref. [6] the parallel dynamics of four ER were studied numerically on the square lattice with von Neumann neighborhood: NER, PR (Potts Extremum Rule), MR (Mean Extremum Rule) and LN (Less Number of States Rule). It was determined that all these automata present logarithmic convergence rate to fixed points. It was also studied the damage spreading response ds to random and smoothed damage

of random images. Both kinds of studied damage present a linear response without discontinuity at the origin for small damage limit.
A numerical study on real two dimensional images for the ER studied in ref. [6] was developed in ref. [7]. All these automata present logarithmic convergence rate to fixed points, stability in front of random noise and simple parallel computing implementation. The qualitative effects of these ER over real images compared with a common use image processing software allow us to propose them as a first level enhancement method.

3. NER Dynamics on Random Graphs

The expected transient time, damage spreading and qualitative effects are determined by medium scale simulations. The methodology to perform the simulations was the following:

1) To compute the expected transient time:
1.1) The NER size was fixed to $n = 16384, 32768$.
1.2) Given n, for each $p \in \{0.01, 0.02, 0.03, 0.04, 0.05\}$, n random graphs $G(n, p)$ were generated.
1.3) For each $G(n, p)$, n random initial conditions $x(0) \in Q^n$ were generated.
1.4) Given: n, p and $G(n, p)$ the NER parallel dynamics was applied starting from $x(0)$, equation (2.2).
1.5) The expected transient time was computed:

$$\overline{\tau} = \left\langle \max_{t \geq 0} \left\{ t \,/\, x(t) \neq x(t-1), x(0) \in Q^n \right\} \right\rangle_{x(0)} \quad (3.1)$$

2) To compute the damage spreading response:
2.1) Repeat steps 1.1) to 1.3)
2.2) Given an initial condition $x(0)$ and $f \in \{0.01, 0.02, 0.03, 0.04, 0.05\}$, 1000 perturbed in a f fraction of sites random initial conditions $x_f(0)$ were generated.
2.3) Given: n, p and $G(n, p)$ the NER parallel dynamics was applied starting from $x(0), x_f(0)$, eq. (2.2).
2.4) The average normalized Hamming distance $\overline{H_1}$ between the steady configurations obtained from $x(0)$ and $x_f(0)$, denoted by $x^{steady}(x(0)), x^{steady}(x_f(0))$ respectively, was computed:

$$\overline{H_1}(p, f) = \left\langle d_H \left(x^{steady}(x(0)), x^{steady}(x_f(0)) \right) \right\rangle_{x(0)} \quad (3.2)$$

3) To compute the qualitative effects:
3.1) Repeat steps 1.1) to 1.4)
3.2) The average normalized Hamming distance $\overline{H_2}$ between $x(0)$ and its steady configuration $x^{steady}(x(0))$ was computed:

$$\overline{H_2}(p) = \left\langle d_H \left(x(0), x^{steady}(x(0)) \right) \right\rangle_{x(0)} \quad (3.3)$$

In what follows, the numerical results will be presented and discussed:

1) For $n = 16384, 32768$ the expected transient time is shown in Table 1:

Table 1. Expected transient time for $n = 16384, 32768$.

n / p	0.01	0.02	0.03	0.04	0.05
16384	3.12	2.83	2.45	2.21	2.00
32768	2.47	2.25	2.12	2.00	2.00

The NER automata converge with linear rate to fixed points. For $n = 16384$: $\overline{\tau}(p) = -28.6p + 3.38$ with correlation coefficient $R^2 = 0.99$. The global connectivity of random graphs allows fast information diffusion. For small neighborhood graphs the information diffusion is considerably slower, see refs. [6,8].

2) For each $p \in \{0.01, 0.02, 0.03, 0.04, 0.05\}$, the average damage spreading response $\overline{H_1}(p, f)$, equation (3.2), is a linear function of the perturbed sites fraction f. For instance, if $p = 0.01$: $\overline{H_1}(0.01, f) = 0.5f - 0.0001$ with correlation coefficient $R = 0.97$. We can affirm that NER is a stable cellular automaton.

3) The values obtained for $\overline{H_2}(p)$ are shown in Table 2:

Table 2. $\overline{H_2}(p)$ for $n = 16384, 32768$, $p \in \{0.01, 0.02, 0.03, 0.04, 0.05\}$

n / p	0.01	0.02	0.03	0.04	0.05
16384	0.82±0.03	0.85±0.03	0.87±0.04	0.88±0.04	0.88±0.04
32768	0.83±0.04	0.86±0.04	0.88±0.05	0.88±0.05	0.88±0.05

The $\overline{H_2}(p)$ quantity, equation (3.3), goes to a constant as n is increased. Since it measures how different are the initial and steady configuration, we can affirm that there is no noticeable finite size effects. This quantity is also a smooth increasing function of p: as the graph connectivity is increased the NER effect also increases. This can be explained since for greater connectivity the vertex neighborhood increases its size.

4. Conclusions

In this work, the NER parallel dynamics defined on random graphs was studied numerically by medium scale simulations. The following results were obtained: The NER automata converge with linear rate to fixed points because of the global connectivity presented in random graphs; the average damage spreading presents a linear response without discontinuity at the origin for small damage limit; the NER effect increases as the graph connectivity increases: The percentage of sites that change its value because of the NER dynamics is a smooth increasing function of the connectivity probability.

References

[1] W. Goddard and D.J. Kleitman, Convergence of a Transformation on a Weighted Graph, Congr. Numer. **82** (1991) 179-185.
[2] E. Goles, Lyapunov operators to study the convergence of extremal automata, Theoretical Computer Science **125** (1994) 329-337.
[3] E. Goles and G. Hernandez, Sequential Iteration for Extremal Automata, Proceedings of IV Workshop on Instabilities and Non Equilibrium Structures, Tirapegui et al eds., Kluwer, 1993.
[4] E. Goles and S. Martinez, Neural and Automata Networks, Maths. and Its Applications Series, Vol. 58, Kluwer, 1991.
[5] R.C. Gonzalez and R.E. Woods, Digital Image Processing (2nd Edition), Prentice Hall, 2002.
[6] G. Hernandez, H. J. Herrmann and E. Goles, Extremal Automata for Image Sharpening, International Journal of Modern Physics C **5** (1994) 923-932.
[7] G. Hernandez and H.J. Herrmann, Cellular Automata for Elementary Image Enhancement, GMIP: Graphical Models and Image Processing **58** (1996) 2-89.
[8] G. Hernandez and L. Salinas, Expected Transient Time and Damage Spreading for the NER Automaton on Geometrically Connected Graphs, Physica A **367 C** (2006) 173-180.
[9] C. Johnson, Convergence of a Nonlinear Sharpening Transformation for Digital Images, SIAM J. Alg. Disc. Meth. **6** No. 3 (1985).
[10] H.P. Kramer and J. B. Bruckner, Iterations of a Nonlinear Transformation for Enhancement of Digital Images, Pattern Recognition **7** (1975) 53-58.
[11] M. Mazzariol, B.A. Gennart and R.D. Hersch, Dynamic load balancing of parallel cellular automata, Proceedings SPIE Conference, Vol. **4118**, Parallel and Distributed Methods for Image Processing IV (2000) 21-29.
[12] P.L. Rosin, Training Cellular Automata for Image Processing, Proceedings SCIA Conference, Lecture Notes in Computer Science **3540** (2005) 195-204.
[13] J.C. Russ, The Image Processing Handbook, CRC Press - IEEE Press, 1999.