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Translation numbers for a class of maps on the dynamical systems arising from quasicrystals in the real line

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Abstract. In this paper, we study translation sets for non-decreasing maps of the real line with a pattern-equivariant displacement with respect to a quasicrystal. First, we establish a correspondence between these maps and self maps of the continuous hull associated with the quasicrystal that are homotopic to the identity and preserve orientation. Then, by using first-return times and induced maps, we provide a partial description for the translation set of the latter maps in the case where they have fixed points and obtain the existence of a unique translation number in the case where they do not have fixed points. Finally, we investigate the existence of a semiconjugacy from a fixed-point-free map homotopic to the identity on the hull to the translation given by its translation number. We concentrate on semiconjugacies that are also homotopic to the identity and, under a boundedness condition, we prove a generalization of Poincaré's theorem, finding a sufficient condition for such a semiconjugacy to exist depending on the translation number of the given map.

1. *Introduction and results*

The aim of this paper is to provide a generalization of Poincaré's classification of the dynamics of homeomorphisms of the circle to a class of maps on the real line which are associated with quasicrystals.

On the one hand, Poincaré's classification of the dynamics of homeomorphisms of the circle is one of the oldest and most important results in the theory of dynamical systems (e.g. [**KH95**]). On the other hand, Delone sets and tilings have been recently and extensively studied in different contexts (e.g. [**LP03, Wan61**]) and in particular as models for quasicrystals.

1.1. *Delone sets and quasicrystals.* A discrete subset X of $\mathbb{R}^d$ is called a *Delone set* if it is uniformly discrete (there exists $r > 0$ such that every closed ball of radius r intersects X in at most one point) and relatively dense (there exists $R > 0$ such that every closed ball of radius R intersects X in at least one point). For $t \in \mathbb{R}^d$ and a discrete set $P \subset \mathbb{R}^d$, $P - t$ denotes the set $\{u - t \mid u \in P\}$, which is referred to as the *translate* of P by t. Let X be a Delone set in $\mathbb{R}^d$. We say that X is *aperiodic* if $X - t = X$ implies $t = 0$. A *patch* of X is a subset of the form $X \cap B$, where B is a closed ball in $\mathbb{R}^d$. The Delone set X has *finite local complexity* if for each $r > 0$ there is, up to translation, only a finite number of patches of diameter less than r.

Delone sets arise naturally as mathematical models for the description of solids. In this modelization, the solid is supposed to be infinitely extended and its atoms are represented by points. These atoms interact through a potential (for example, a Lennard–Jones potential). For a given specific energy, Delone sets are good candidates to describe the ground-state configuration: uniform discreteness corresponds to the existence of a minimum distance between atoms due to the repulsion forces between nuclei, and relative density corresponds to the fact that empty regions cannot be arbitrarily big because of the contraction forces. In perfect crystals, atoms are ordered in a repeating pattern extending in all three dimensions and can be modeled by lattices in $\mathbb{R}^3$. Quasicrystalline solids are those whose X-ray diffraction images have sharp spots indicating *long-range order* but without having a full lattice of periods. Typically, they exhibit symmetries that are impossible for a perfect crystal (see, e.g., [**SBGC84**]).

We restrict our attention to Delone subsets of the real line. A Delone set X is *repetitive* if for every patch P there is $M > 0$ such that every closed interval J of length $2M$ contains a closed subinterval I' such that $X \cap I'$ is a translated copy of P. Given a patch P of X, $t \in \mathbb{R}$ and $M > 0$, define

$$n(P, M, t) = \#\{P' \subset X \cap [t - M, t + M] \mid P' = P - u \text{ for some } u \in \mathbb{R}\}$$

to be the number of patches of X that are translated copies of P and are included in the interval $[t - M, t + M]$. We say that the Delone set X has *uniform pattern frequencies* if $n(P, M, t)/2M$ converges uniformly in t when $M \to +\infty$ and the limit is independent of t.

In our context, the term *quasicrystal* will be used for an aperiodic repetitive Delone set that has uniform pattern frequencies. It is plain that such sets can be seen as increasing sequences $X = (x_n)_{n\in\mathbb{Z}}$ with $\inf_{n\in\mathbb{Z}}(x_n - x_{n-1}) > 0$. In these terms, the Delone set X has finite local complexity if and only if the *prototile set* $\mathcal{L}(X) := \{x_{n+1} - x_n \mid n \in \mathbb{Z}\}$ is finite. Observe that each point of a Delone set is (possibly) decorated by a color. In this context, finite local complexity means finite prototile set and finite number of colors, and two patches will be considered translated if (without decorations) they are translated and they are decorated in the same way.

1.2. *Poincaré's classification for circle homeomorphisms.* Let us now recall Poincaré's classification. We deviate a little from the standard notation in rotation theory of the circle in order to simplify the comparison between Poincaré's classification and our work.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous non-decreasing map of the real line. The *displacement* of f is the continuous function ϕ defined by $t \to f(t) - t$. The *translation number of f at t* is defined by

$$\rho(f, t) = \lim_{n \to \infty} \frac{f^n(t) - t}{n}, \tag{1}$$

provided the limit exists. Let us suppose for the moment that ϕ is a periodic function and that f is one-to-one. In this case, the map f factors through an orientation-preserving homeomorphism of the circle $F : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. If $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ denotes the natural projection of the real line $\mathbb{R}$ onto the circle $\mathbb{R}/\mathbb{Z}$, then the former means that $\pi \circ f = F \circ \pi$.

The first result in this topic proved by Poincaré is that the translation number $\rho(f, t)$ exists for every $t \in \mathbb{R}$ and is independent of $t \in \mathbb{R}$. Therefore, there is a *translation number of f*. Moreover, if we take a map that, like f, factors over F, then its translation number differs from the translation number of f by an integer. This allows us to define the *rotation number of F* as the number $\rho(F) = \rho(f, t) \mod \mathbb{Z}$ for any $t \in \mathbb{R}$. Poincaré then proved that rotation numbers may be used to classify the dynamics of orientation-preserving homeomorphisms of the circle.

POINCARÉ'S THEOREM. *Let $F : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation-preserving homeomorphism. Then,*

(1) *the rotation number $\rho(F)$ is rational if and only if F has a periodic orbit;*

(2) *if the rotation number $\rho(F)$ is irrational, then F is semiconjugate to $R_{\rho(F)}$, which means that there exists a continuous, onto and orientation-preserving map $H : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that*

$$H \circ F = R_{\rho(F)} \circ H,$$

where $R_{\rho(F)}$ is the rotation $x \mapsto x + \rho(F) \mod \mathbb{Z}$ in $\mathbb{R}/\mathbb{Z}$;

(3) *if in (2) we assume that F is transitive (there is a dense orbit), then the map H defined in (2) is a homeomorphism.*

It is natural to attempt to generalize this result to more general systems. For instance, in the context of homeomorphisms of the torus $\mathbb{R}^k/\mathbb{Z}^k$, one finds that, in general, the rotation number depends on the orbit. This means that instead of a unique rotation number one obtains a *rotation set* (see [**GM99, MZ89**]). Even when the rotation set is reduced to a unique rotation number, there are examples with behaviors that are not compatible with Poincaré's classification (see, e.g., [**Her83, Man87**]). Under some extra (and natural) conditions, a Poincaré-like classification for torus homeomorphisms was recently obtained [**Jag08, JS06**].

1.3. *Translation numbers for maps on the quasicrystal systems homotopic to the identity.* In this paper we take a different direction: we drop the assumption of periodicity of ϕ and study the case when ϕ is pattern equivariant with respect to a quasicrystal. More precisely, given a quasicrystal X on the real line, a function $\phi : \mathbb{R} \to \mathbb{R}$ is called *strongly X-equivariant* or a *short-range potential* if there exists $S > 0$, called the *range* of ϕ, such that

$$(X - t) \cap [-S, S] = (X - u) \cap [-S, S]$$

implies that

$$\phi(t) = \phi(u).$$

We say that a continuous function $\phi : \mathbb{R} \to \mathbb{R}$ is *X-equivariant* if it is the uniform limit of a sequence of strongly X-equivariant continuous functions. In terms of the physical model, the value of a short-range potential at a given point t depends on the neighborhood of t up to a given radius. When $X = \mathbb{Z}$ (possibly decorated in an aperiodic way), continuous periodic functions are X-equivariant. Short-range potentials and X-equivariant functions have been studied in different contexts (see, e.g., [**GGP06, Hof95, Kel03**]).

In this paper we suppose that the displacement ϕ of f is X-equivariant, where X is a quasicrystal, and study the translation set of f. Our first result states the following.

Result 1. Let X be a quasicrystal and $f : \mathbb{R} \to \mathbb{R}$ a continuous non-decreasing map with X-equivariant displacement $\phi(t) = f(t) - t$. If ϕ is bounded away from zero then, for every $t \in \mathbb{R}$, the translation number $\rho(f, t)$ exists and does not depend on t. That is to say, f has a translation number, denoted by $\rho(f)$.

In the same way that a periodic lattice allows identification on the real line which yields a circle, a quasicrystal X allows us to define a metrizable topology on the real line, coarser than the standard one, that reflects several properties of X and for which the completion of the real line is a compact space Ω_X [**Rud89**], called *the continuous hull of X*, and its elements are Delone sets with the same prototile set as X. Thus, there is a natural action T of $\mathbb{R}$ on Ω_X by translation (see, e.g., [**Rob04**]), and this action is continuous. Moreover, in the situation we consider, it is minimal and uniquely ergodic (see, e.g., [**LMS02, Rob04**]).

There is a lamination structure on the continuous hull that is locally the product of an interval and a Cantor set. Moreover, T-orbits coincide with the leaves of the lamination [**BBG06**]. These results are reviewed in more detail in §2.

In the same way that periodic continuous functions with period 1 factor through a continuous function on the circle, we have that X-equivariant functions factor through continuous functions defined over the hull Ω_X (see, e.g., [**Kel03**]). More precisely, if ϕ is X-equivariant, then there is a unique continuous $\Phi : \Omega_X \to \mathbb{R}$ such that

$$\phi(t) = \Phi(X - t) \quad \text{for all } t \in \mathbb{R}. \tag{2}$$

We review this correspondence in §3 and then we use it to define a correspondence between self maps on Ω_X that are homotopic to the identity and continuous self maps of the real line with X-equivariant displacements. This correspondence reads as follows: for each continuous map f with X-equivariant displacement ϕ, there is a continuous map $F : \Omega_X \to \Omega_X$ defined by

$$F(Y) = Y - \Phi(Y) \quad \text{for all } Y \in \Omega_X, \tag{3}$$

where Φ is the function defined by equation (2). We refer to Φ as the *displacement* of F. We check that F is homotopic to the identity and that

$$X - f(t) = F(X - t) \quad \text{for all } Y \in \Omega_X. \tag{4}$$

In particular, F sends each leaf to itself. Since X is aperiodic, it is not difficult to check that every map that sends each leaf to itself can be written in the form (3), where the displacement Φ is not necessarily continuous. Hence, it is natural to ask whether every continuous orientation-preserving self map of Ω_X that sends each leaf to itself has a continuous displacement, and hence induces by equation (4) a map of the real line with X-equivariant displacement. We give an affirmative answer in §3 (cf. Theorem 3.4). It follows that every such map is homotopic to the identity.

In §4, we study translation sets for maps that are homotopic to the identity: let $F : \Omega_X \to \Omega_X$ be homotopic to the identity and $\Phi : \Omega_X \to \mathbb{R}$ its displacement. The following definitions are adapted from [**GM99**]. The *translation number of F at $Y \in \Omega_X$* is defined by

$$\rho(F, Y) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi(F^k(Y))$$

provided the limit exists. The (pointwise) *translation set* of F, denoted by $\rho_p(F)$, is defined as the set of all translation numbers of F. We observe that if $Y = X - t$ with $t \in \mathbb{R}$, then

$$\frac{1}{n} \sum_{k=0}^{n-1} \Phi(F^k(X - t)) = \frac{f^n(t) - t}{n}.$$

Hence, Result 1 is a consequence of the following more general result describing the translation set of F.

Result 2. Let X be a quasicrystal and Ω_X its continuous hull. Consider a continuous map $F : \Omega_X \to \Omega_X$ that is homotopic to the identity and denote its displacement by Φ. Then, exactly one of the following assertions holds.

(1) Φ changes sign: for every $Y \in \Omega_X$ the translation number of F at Y exists and equals 0 (cf. Proposition 4.1).

(2) Φ does not have zeros: F has a unique translation number, which is different from zero (cf. Theorem 4.6).

(3) Φ does not change sign but has zeros: there exists $\rho \in \mathbb{R}$ such that for almost every $Y \in \Omega_X$ the translation number of F at Y coincides with ρ (cf. Theorem 4.2).

We discuss some results in the literature that are related to Results 1 and 2: Kwapisz [**Kwa00**] proved results that correspond to Results 1 and 2 when the displacement of f is an almost-periodic function in the sense of Bohr. Clark [**Cla02**] studied rotation numbers for self maps of solenoids that are homotopic to the identity, so Result 2 may be seen as a generalization of some of the results of Clark. In particular, Result 2(3) already appears in [**Cla02**]. Finally, Shvetsov [**Shv03**] studied rotation numbers for continuous-time flows on the continuous hull of self-similar tilings. We observe that his results are true for the continuous hull arising from a (general) quasicrystal. However, since not every map considered in this paper is a time-t map for a flow, Results 1 and 2 are not consequences of the results in [**Shv03**].

1.4. *Classification of maps on Delone systems homotopic to the identity.* Finally, in §5 we develop the tools for obtaining a Poincaré-like classification for maps satisfying Result 2(2). Let $F : \Omega_X \to \Omega_X$ be a map satisfying Result 2(2). That is, F does not have fixed points, which means that it does not have periodic points either and it has a translation number $\rho(F)$. Our objective is to classify the dynamics of F in a Poincaré-like way. In particular, we ask whether there exists a semiconjugacy from F to $T_{\rho(F)}$, i.e. a continuous and surjective map $H : \Omega_X \to \Omega_X$ such that

$$H(F(Y)) = H(Y) - \rho(F) \quad \text{for every } Y \in \Omega_X.$$

Motivated by the fact that translation numbers are not necessarily preserved by semiconjugacies that are not homotopic to the identity, we require that semiconjugacies be homotopic to the identity. To simplify notation, we call such a semiconjugacy a *T-semiconjugacy*. This allows us to reduce the problem of finding a semiconjugacy to the problem of finding a continuous solution Ψ to the following cohomological equation:

$$\Psi(Y - \Phi(Y)) - \Psi(Y) = \rho(f) - \Phi(Y) \quad \text{for all } Y \in \Omega_X, \tag{5}$$

where Φ denotes the displacement of F and Ψ corresponds to the displacement of the desired T-semiconjugacy. Cohomological equations are very important objects in the theory of dynamical systems and appear in several contexts (see e.g. [**KH95**]).

A direct necessary condition for the existence of a continuous solution to equation (5) is the following: there exists $C > 0$ such that

$$|F^n(Y) - Y - n\rho(F)| < C \quad \text{for every } n \in \mathbb{N} \quad \text{and} \quad Y \in \Omega_X.$$

In the periodic case, this boundedness condition is always satisfied and it is key to prove Poincaré's theorem. In our setting, it is not known whether this condition is satisfied by all F's, so we say that a map that satisfies this condition is *ρ-bounded*. In the following, we suppose that F is ρ-bounded.

In our case, rational numbers are no longer available. We also point out that translations in Ω_X have no periodic points since X is aperiodic. To define an alternative set, we observe that if F is minimal and ρ-bounded then a well-known theorem of Gottschalk and Hedlund ensures the existence of a continuous solution to the cohomological equation, and hence also the existence of a semiconjugacy from F to $T_{\rho(F)}$. This implies that the translation $T_{\rho(F)}$ is minimal. The well-known fact that a rotation of the circle is minimal if and only if the angle that defines it is not rational motivates us to define

$$\mathcal{Q} = \{t \in \mathbb{R} \mid Y \in \Omega_X \mapsto Y - t \text{ is not minimal on } \Omega_X\}.$$

Finally, we introduce a replacement for periodic points when $\rho(F) \in \mathcal{Q}$. A cylinder in Ω_X is a set of the form

$$\{Z \in \Omega_X \mid Z \cap B = Y \cap B\},$$

where $Y \in \Omega_X$ and B is a closed ball around 0. It can be checked (see §2 for more details) that these cylinders are Cantor sets, and thus a *local vertical* is a clopen (both open and closed) subset of a cylinder in Ω_X. Given a local vertical V in Ω_X and two functions $\alpha, \beta : V \to \mathbb{R}$, the set

$$V[\alpha, \beta] = \{X - t \mid X \in V, \alpha(X) \le t \le \beta(X)\}$$

is called a *local strip*. If $\overline{V[\alpha, \alpha]} = \overline{V[\beta, \beta]} = V[\alpha, \beta]$, then $V[\alpha, \beta]$ is said to be *thin*.

The main result of this paper is the following.

Result 3. Suppose that F is a ρ-bounded map preserving the orientation in Ω_X with translation number $\rho(F)$. Then,

- if $\rho(F) \in \mathcal{Q}$, then F is not minimal and every minimal set is the finite disjoint union of thin local strips (cf. Theorem 5.10);
- if $\rho(F) \notin \mathcal{Q}$, then F is T-semiconjugate to $T_{\rho(f)}$ (cf. Theorem 5.12).

2. *Dynamical approach to the study of Delone sets*

We review here some basic facts about the approach of dynamical systems to the study of Delone sets on the real line. Most of these results are also valid for Delone sets in $\mathbb{R}^d$ with $d > 1$; see, e.g., [**KP00, LP03, Rob04**].

2.1. *Delone dynamical systems.*

Let $\mathcal{T} \subset \mathbb{R}$ be a finite set (of prototiles) and denote by $\Omega(\mathcal{T})$ the set of all Delone sets of $\mathbb{R}$ with prototile set included in $\mathcal{T}$. There is a natural action T of $\mathbb{R}$ on $\Omega(\mathcal{T})$, which is called a *translation action*, defined by

$$T_t(X) = X - t = \{x - t \mid x \in X\} \quad \text{for all } X \in \Omega(\mathcal{T}) \text{ and } t \in \mathbb{R}.$$

This action is continuous with respect to the topology induced by the distance defined below (see [**Rob04, Rud89**] for details). First, we introduce the following notation: for $Y \in \Omega(\mathcal{T})$ and $S > 0$, we define the *cylinder* of radius S around Y by

$$V_{Y,S} = \{Z \in \Omega(\mathcal{T}) \mid Z \cap [-S, S] = Y \cap [-S, S]\}.$$

Given $Y, Z \in \Omega(\mathcal{T})$, we set

$$d(Y, Z) = \min\left\{\frac{\sqrt{2}}{2}, \inf\{\epsilon > 0 \mid \exists t \in (-\epsilon, \epsilon) \text{ s.t. } Z - t \in V_{Y,1/\epsilon}\}\right\}.$$

Roughly speaking, two Delone sets are close to each other if, modulo a small translation, they coincide on a big ball around the origin. With this topology, $\Omega(\mathcal{T})$ is compact (see [**Rud89**]) and thus separable. A countable basis for $\Omega(\mathcal{T})$ can be formed by sets of the form

$$\{Z - t \mid Z \in V_{Y,S}, t \in (-\varepsilon, \varepsilon)\}. \tag{6}$$

Let Ω be a subset of $\Omega(\mathcal{T})$. The pair $(\Omega, \mathbb{R})$ is called a *Delone dynamical system* (in short, Delone system) if Ω is closed and invariant under the translation action. Let $(\Omega, \mathbb{R})$ be a Delone dynamical system. If the orbit $\mathcal{O}(X) = \{X - t \mid t \in \mathbb{R}\}$ of every $X \in \Omega$ is dense in Ω, then we say that $(\Omega, \mathbb{R})$ is *minimal*. If $X - t \neq X$ for every $t \in \mathbb{R} \setminus \{\mathbf{0}\}$ and every $X \in \Omega$, then we say that $(\Omega, \mathbb{R})$ is *aperiodic*. The main connection between Delone sets and Delone dynamical systems is summarized in the following theorem.

THEOREM 2.1. (See [**LP03, Sch00**]) *Let X be a Delone set with finite prototile set $\mathcal{T}$ and denote $\Omega_X = \overline{\mathcal{O}(X)}$. Then, $(\Omega_X, \mathbb{R})$ is a Delone dynamical system and $(\Omega_X, \mathbb{R})$ is minimal if and only if X is repetitive. In the repetitive case, one also has*

- $(\Omega_X, \mathbb{R})$ *is aperiodic if and only if X is aperiodic;*
- $(\Omega_X, \mathbb{R})$ *is uniquely ergodic (there is a unique T-invariant probability measure on Ω_X) if and only if X has uniform pattern frequencies.*

We say that $(\Omega_X, \mathbb{R})$ is a *quasicrystal system* whenever X is a quasicrystal. Thus, a quasicrystal system is an aperiodic, minimal, uniquely ergodic Delone system.

Let $(\Omega, \mathbb{R})$ be an aperiodic minimal Delone system. The *canonical transversal* Ω^0 of Ω is defined as the set of all Delone sets in Ω containing 0.

PROPOSITION 2.2. (See [**KP00**]) *The canonical transversal of an aperiodic minimal Delone system is a Cantor set.*

It follows that each cylinder $V_{Y,S}$, where $Y \in \Omega$ and $S > 0$, is a Cantor set (it is homeomorphic to a clopen subset of Ω^0). Moreover, $(\Omega, \mathbb{R})$ possesses a laminated structure, which means the following. There is a finite open cover $\{U_i\}_{i=1}^n$ of Ω. Each U_i can be written in the form

$$U_i = \{Z - t \mid Z \in V_{Y_i,S_i}, t \in I_i\},$$

where $Y_i \in \Omega$, $S_i > 0$ and I_i is an open interval. Moreover, each U_i is homeomorphic to $V_{Y_i,S_i} \times I_i$ through $h_i : I_i \times V_i \to U_i$ defined by $h_i(t, Z) = Z - t$, and the transition maps satisfy, where defined,

$$h_i^{-1} \circ h_j(t, Z) = (t - v_{i,j}, Z - v_{i,j}), \tag{7}$$

where $v_{i,j} \in \mathbb{R}$ depends only on i and j.

A *slice* is an interval $h_i(I_i \times \{Z\})$. The *leaves* of the lamination are the smallest connected subsets that contain all the slices they intersect. By equation (7), the leaves coincide with the orbits of the action T and are 1-manifolds isometric to $\mathbb{R}$ (for more details, see, e.g., [**BBG06, BG03, GGP06**]). Thus, we will prefer to use the term leaf to refer to a T-orbit.

Thus, the natural orientation of $\mathbb{R}$ induces an orientation on each leaf of Ω. The same is true for the natural order on $\mathbb{R}$: for $Y, Z \in \Omega$, we say that $Y \leq Z$ if there exists $t \geq 0$ such that $Y - t = Z$ (analogous definitions are given for $<, \geq, >$). It follows that a map $F : \Omega \to \Omega$ is *orientation preserving* if for every $Y, Z \in \Omega$

$$Y \leq Z \text{ implies } F(Y) \leq F(Z).$$

In particular, orientation-preserving maps send leaves to leaves.

Example 2.3. The standard examples of quasicrystals on the real line are constructed via substitution sequences (for more details, see [**DHS99, RS01**]), like the Fibonacci sequence. This sequence is constructed by iterating the substitution

$$\begin{cases} a \to ab, \\ b \to a. \end{cases}$$

Starting from the sequence $a.a$, one obtains a bi-infinite sequence $(w_n)_{n\in\mathbb{Z}}$, called the *Fibonacci sequence*, that is a periodic point (of period two) of the previous substitution. By taking its orbit closure under the shift map $\sigma((w_n)_{n\in\mathbb{Z}}) = (w_{n+1})_{n\in\mathbb{Z}}$, one obtains the so-called *Fibonacci shift*. Given $L, S > 0$, a *Fibonacci quasicrystal* is the Delone set $X_{\text{fib}} = \{x_n\}_{n\in\mathbb{Z}}$ such that $x_{n+1} - x_n$ is equal to L if $w_n = a$ and to S if $w_n = b$ (see Figure 1). If one lets Ω denote the continuous hull of X_{fib}, then it is easy to check

$L \; L$

$L\ S \; L\ S$

$L\ S\ L \; L\ S\ L$

$L\ S\ L\ L\ S \; L\ S\ L\ L\ S$

$L\ S\ L\ L\ S\ L\ S\ L \; L\ S\ L\ L\ S\ L\ S\ L$

FIGURE 1. Construction of the Fibonacci chain.

that the canonical transversal of Ω is conjugate with the Fibonacci shift. There are two important choices for L, S. First, if $L = S = 1$, then Ω is conjugate with the suspension of the Fibonacci shift. Second, if $L = \tau := (\sqrt{5}+1)/2$ and $S = 1$, then X_{fib} is self similar (see [**RS01**]) and there is a homeomorphism $\omega : \Omega \to \Omega$, the so-called *inflation–substitution* homeomorphism, which satisfies $\omega(X_{\text{fib}} - t) = X_{\text{fib}} - \tau t$ for every $t \in \mathbb{R}$.

2.2. *Return times and return maps.* We end this section by recalling some basic results that will be used throughout this paper. These are adaptations of standard tools used in the study of minimal Cantor systems (see, e.g., [**HPS92, Put89**]). Let X be an aperiodic repetitive Delone set and Ω_X its hull. A *local vertical* is a clopen subset of $V_{Y,S}$ for any $Y \in \Omega_X$ and $S > 0$. Let V be a local vertical. For every $Y \in \Omega$, the *first entry time* of Y to V is defined as

$$\mathbf{t}_V(Y) = \inf\{t > 0 \mid Y - t \in V\}.$$

When Y belongs to V, we refer to $\mathbf{t}_V(Y)$ as the *first-return time* of Y to V. The *first-return map to* V is the map $\sigma_V : V \to V$ defined by

$$\sigma_V(Y) = Y - \mathbf{t}_V(Y) \quad \text{for all } Y \in V.$$

For an illustration of first-return time and first-return map when $X = X_{\text{fib}}$ (defined in Example 2.3) see Figure 2.

LEMMA 2.4. *The first-return time $\mathbf{t}_V$ is continuous on Ω_X and, when restricted to V, it takes only finitely many values. Moreover, the map σ_V is a homeomorphism and the system (V, σ_V) is minimal.*

From this lemma, we immediately obtain the following.

COROLLARY 2.5. *For every $Y \in V$, the set of return times of Y to V, defined by*

$$\mathcal{R}(Y, V) = \{t \in \mathbb{R} \mid Y - t \in V\},$$

is a Delone set with finite local complexity.

Lemma 2.4 basically asserts that Ω_X is topologically conjugate to the special flow of the system (V, σ_V) under the first-return time $\mathbf{t}_V$, where V is any local vertical. This implies that each T-invariant measure induces a σ_V-invariant measure on V (see, e.g., [**CFS82**]). We provide some of the details. Let μ be a T-invariant probability measure on Ω_X.

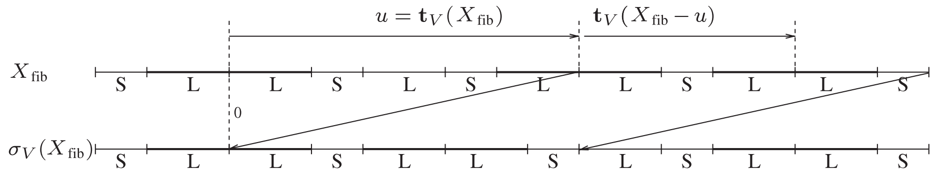

FIGURE 2. First-return time and first-return maps for Fibonacci quasicrystals. Here, $V = \{Y \in \Omega_{\text{Fib}} \mid Y \cap [-L, L] = \{-L, L\}\}$.

This measure induces a finite $\sigma_{\Omega_X^0}$-measure ν on the canonical transversal Ω_X^0. We call the measure ν the *transverse measure* on Ω_X^0 induced by μ. By Lemma 2.4, we write $\{h_1, \ldots, h_C\}$ for the image of $\mathbf{t}_{\Omega_X^0}$, where $C \in \mathbb{N}$, and define $E_i := \{Y \in \Omega_X^0 \mid \mathbf{t}_{\Omega_X^0}(Y) = h_i\}$ for each $i \in \{1, \ldots, C\}$. Given a clopen set V in Ω_X^0, its transverse measure is given by

$$\nu(V) = \sum_{i=1}^{C} \frac{1}{h_i}\mu(\{X - t \mid X \in V \cap E_i, 0 \le t < h_i\}).$$

One can check that

$$\sum_{i=i}^{C} \nu(E_i)h_i = 1. \tag{8}$$

The same process allows us to induce a σ_V-invariant measure for every clopen subset V of Ω_X^0.

If we assume that X is a quasicrystal, i.e. that the system $(\Omega_X, \mathbb{R})$ is uniquely ergodic, then there is a unique transverse measure on Ω_X^0, which we denote by ν, and, for each clopen subset V of Ω_X^0, the system (V, σ_V) is uniquely ergodic; the unique σ_V-invariant probability measure on V is $\nu/\nu(V)$.

The following lemma is an easy consequence of Birkhoff's ergodic theorem.

LEMMA 2.6. *Suppose that Ω_X is uniquely ergodic. For a clopen set V in Ω_X^0 and $Y \in V$, define*

$$\ell_V(t, Y) = \text{card}\{u \in [0, t) \mid Y - u \in V\}.$$

Then, for every clopen set V in Ω_X^0 and $Y \in V$, we have

$$\lim_{t \to +\infty} \frac{\ell_V(t, Y)}{t} = \nu(V).$$

3. *Maps with pattern-equivariant displacement*

3.1. *Pattern-equivariant functions.* Let X be an aperiodic repetitive Delone set and $\Omega_X = \overline{\mathcal{O}(X)}$ its associated aperiodic minimal Delone system. A function $\phi : \mathbb{R} \to \mathbb{R}$ is called *strongly X-equivariant (or a short-range potential)* if there exists $S > 0$ such that

$$(X - t) \cap [-S, S] = (X - u) \cap [-S, S]$$

implies that

$$\phi(t) = \phi(u).$$

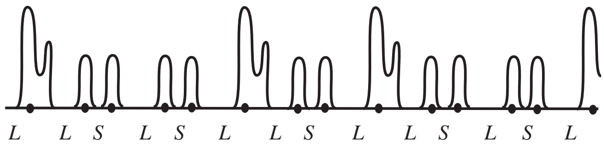

FIGURE 3. Example of a X_{fib}-equivariant function.

A continuous function ϕ is called *X-equivariant* if it is the uniform limit of a sequence of strongly X-equivariant continuous functions.

Example 3.1. Let X_{fib} be the Fibonacci quasicrystal defined in Example 2.3. A simple way to obtain X_{fib}-equivariant functions consists in choosing two real-valued smooth functions, $v_{L,L}$ and $v_{S,L}$, with compact support on the interval $(-I, I)$, where $0 < 2I < S\ (< L)$. A strongly X_{fib}-equivariant function ϕ_{fib} can be defined as follows (see Figure 3): if $\theta \in (x_n - I, x_n + I)$ for some $n \in \mathbb{Z}$, then we set

$$\phi_{\mathrm{fib}}(\theta) = \begin{cases} v_{L,L}(\theta - x_n) & \text{if } |x_n - x_{n-1}| = |x_n - x_{n+1}| = L, \\ v_{S,L}(\theta - x_n) & \text{if } |x_n - x_{n-1}| \neq |x_n - x_{n+1}|. \end{cases}$$

If not, we set $\phi_{\mathrm{fib}}(\theta) = 0$.

We denote by $C_X(\mathbb{R})$ the set of all X-equivariant functions. It is well known that X-equivariant functions form an algebra (see, e.g., [**Kel03**]). Moreover, there is an isomorphism between $C_X(\mathbb{R})$ and the set $C(\Omega_X)$ of real-valued continuous functions on Ω_X, as the following lemma states.

LEMMA 3.2. [**Kel03**] *The map $\mathcal{C} : C(\Omega_X) \to C_X(\mathbb{R})$ given by*

$$\mathcal{C}(\Phi)(t) = \Phi(X - t) \tag{9}$$

is an algebra isomorphism.

3.2. *Maps with pattern-equivariant displacement.* Let X be an aperiodic repetitive Delone set. For a continuous map $f : \mathbb{R} \to \mathbb{R}$, its *displacement* is defined by $\phi(t) = f(t) - t$. The map f has *X-equivariant displacement* if ϕ is an X-equivariant function.

In this section we shall concentrate on the set $\mathcal{F}_X^+(\mathbb{R})$ of non-decreasing continuous self maps of $\mathbb{R}$ with X-equivariant displacement. By using the correspondence defined in Lemma 3.2, we see that for each map f in $\mathcal{F}_X^+(\mathbb{R})$ there is a unique continuous map $F : \Omega_X \to \Omega_X$ that satisfies

$$F(X - t) = X - f(t) \quad \text{for every } t \in \mathbb{R}.$$

Indeed, F is defined by

$$F(Y) = Y - \Phi(Y) \quad \text{for all } Y \in \Omega_X, \tag{10}$$

where $\Phi = \mathcal{C}^{-1}(\phi)$ is the continuous function on Ω_X corresponding to ϕ in Lemma 3.2. The function Φ is called the *displacement* of F. We denote by $\mathcal{F}_0^+(\Omega_X)$ the set of all maps F of the form (10) with continuous displacement Φ. The following proposition states that every map in $\mathcal{F}_X^+(\mathbb{R})$ is induced by such a map.

PROPOSITION 3.3. *Let $F : \Omega_X \to \Omega_X$ be defined by $F(Y) = Y - \Phi(Y)$, where $\Phi : \Omega_X \to \mathbb{R}$. For each $Y \in \Omega_X$, let $f_Y : \mathbb{R} \to \mathbb{R}$ be defined by*

$$F(Y - t) = Y - f_Y(t) \quad \text{for every } t \in \mathbb{R}. \tag{11}$$

Then, for every $Y \in \Omega_X$, the map f_Y is well defined. Furthermore, if Φ is continuous, then for every $Y \in \Omega_X$ the map f_Y is continuous and has Y-equivariant displacement. We say that f_Y is the map induced by F on the leaf of Y. *Moreover, F preserves orientation if and only if f_Y is non-decreasing.*

Proof. Clearly, the map F sends each leaf to itself. Hence, for every $Y \in \Omega_X$, $F(Y - t)$ belongs to the leaf of Y. It follows from the aperiodicity of Y that $f_Y(t)$ is well defined for every $t \in \mathbb{R}$. Since Φ is continuous and $\Omega_X = \Omega_Y$, Lemma 3.2 implies that the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(t) = \Phi(Y - t)$ is continuous and Y-equivariant. Aperiodicity of Y implies that $f_Y(t) = t - \phi(t)$. The fact that f_Y is non-decreasing if F preserves orientation follows from the density of each leaf in Ω_X and the continuity of Φ. □

We remark that maps in $\mathcal{F}_0^+(\Omega_X)$ send each leaf to itself and are homotopic to the identity, the homotopy being defined by $F_\tau(Y) = Y - t\phi(Y)$ for $\tau \in [0, 1]$. It is natural to ask whether every map that is homotopic to the identity induces a map with pattern-equivariant displacement.

THEOREM 3.4. *Let $F : \Omega_X \to \Omega_X$ be a continuous map of the form*

$$Y \mapsto Y - \Phi(Y) \quad \text{for all } Y \in \Omega_X,$$

where $\Phi : \Omega_X \to \mathbb{R}$. Assume that F is orientation preserving. Then, the function Φ is continuous and, in particular, F is homotopic to the identity on Ω_X.

COROLLARY 3.5. *The set $\mathcal{F}_0^+(\Omega_X)$ coincides with the set of all self maps of Ω_X that are homotopic to the identity and preserve orientation.*

Proof of Theorem 3.4. We first prove that Φ is bounded. The sets $B_n = \{Y \in \Omega_X \mid |\Phi(Y)| \le n\}$ are closed for every $n \in \mathbb{N}$ by the continuity of F. As $\bigcup_{n \in \mathbb{N}} B_n = \Omega_X$, by the Baire category theorem there is $n \in \mathbb{N}$ such that B_n has non-empty interior, i.e. there are a non-empty open set $U \subset \Omega_X$ and a $C > 0$ such that $|\Phi(Y)| < C$ for every $Y \in U$. Fix $Y \in U$ and choose $S > 0$ to be big enough such that $V = V_{Y,S} \subset U$. By Corollary 2.5, $\mathcal{R}(Y, V)$ is a Delone set with finite local complexity, so we write $\mathcal{R}(Y, V)$ as an increasing sequence $(t_n)_{n \in \mathbb{Z}}$. As F preserves the leaf of Y and its orientation, one has

$$u + \Phi(Y - u) \le t + \Phi(Y - t)$$

for every $t, u \in \mathbb{R}$ such that $t \le u$. It follows that, for every $s \in \mathbb{R}$,

$$|\Phi(Y - u)| \le C + \sup_n (t_{n+1} - t_n) =: \tilde{C}. \tag{12}$$

Recall that the finite local complexity of $\mathcal{R}(Y, V)$ implies that $\tilde{C}$ is finite. Let $Z \in \Omega_X$. By minimality of $(\Omega, \mathbb{R})$, we can choose a sequence $(z_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$ for which $Y - z_n$ converges to Z. From equation (12), it follows that $(\Phi(Y - z_n))_{n \in \mathbb{N}}$ is bounded and, thus, by dropping to a subsequence if necessary, we assume that $\Phi(Y - z_n)$ converges to ϕ_z as

$n \to +\infty$ with $|\phi_z| \le \tilde{C}$. It follows that $F(Y - z_n) = Y - z_n - \Phi(Y - z_n)$ converges to $Z - \phi_z$. On the other hand, by the continuity of F we have that $F(Y - z_n)$ converges to $Z - \Phi(Z)$ as $n \to +\infty$ and then the aperiodicity of Z yields that $\phi_z = \Phi(Z)$, which implies that Φ is bounded.

Take an arbitrary $Y \in \Omega_X$ and let $(Y_n)_{n\in\mathbb{N}} \subseteq \Omega_X$ be a sequence that converges to Y. As Φ is bounded, it suffices to show that for every accumulation point ϕ_y of $(\Phi(Y_n))_{n\in\mathbb{N}}$ we have that $\Phi(Y) = \phi_y$. But, the latter follows from the same argument as above and this ends the proof. □

4. *Translation sets*

Let X be a quasicrystal and Ω_X its continuous hull. Consider a map F in $\mathcal{F}_0^+(\Omega_X)$ with displacement Φ. For each $n \in \mathbb{N}$, we define $\Phi^{(n)} = \sum_{i=0}^{n-1} \Phi \circ F^i$. For the definition of translation numbers, we follow [**GM99**]. The *translation number* of F at $Y \in \Omega_X$ is defined by

$$\rho(F, Y) := \lim_{n\to+\infty} \frac{\Phi^{(n)}(Y)}{n}, \tag{13}$$

provided the limit exists. The set of all translation numbers at all points where they exist is called the (pointwise) *translation set* of F and denoted by $\rho_p(F)$. If for every $Y \in \Omega_X$ the translation number of F at Y exists and is independent of Y, then we say that the *translation number* of F exists, or that F *has a translation number*, denoted by $\rho(F)$.

We remark that if f_X is the map induced by F on the leaf of X (see Proposition 3.3), then, for every $n \in \mathbb{N}$ and every $t \in \mathbb{R}$,

$$f_X^n(t) - t = \Phi^{(n)}(X - t).$$

This means that to study translation sets for maps with pattern-equivariant displacements it suffices to study translation sets for maps in $\mathcal{F}_0^+(\Omega_X)$.

Let $M_{\text{erg}}(\Omega_X, F)$ be the set of all ergodic F-invariant probability measures. We remark that for each measure $\mu_F \in M_{\text{erg}}(\Omega_X, F)$, the ergodic theorem states that the translation number of F exists at μ_F-almost every point and is equal to $\int_{\Omega_X} \Phi \, d\mu_F$. Thus, we call $\int_{\Omega_X} \Phi \, d\mu_F$ the *translation number* of μ_F and we define the *measure translation set* of F by

$$\rho_m(F) = \left\{ \int_{\Omega_X} \Phi \, d\mu_F \mid \mu_F \in M_{\text{erg}}(\Omega_X, F) \right\}.$$

It is clear that if F is uniquely ergodic, then F has a translation number. Moreover, it is well known (see for instance [**GM99**, Proposition 4.2]) that F has a translation number $\rho(F)$ if and only if $\rho_p(F) = \rho_m(F) = \{\rho(F)\}$.

The objective of this section is to describe the pointwise translation set of F. We distinguish the cases when F has fixed points or not.

4.1. *Translation sets for maps with fixed points.*

In this section we suppose that F has fixed points. It is plain that in this case 0 belongs to $\rho_p(F)$. This means that Φ has zeros, since these coincide with fixed points of F. Hence, there are two different cases to analyze:

(1) Φ changes sign;
(2) Φ does not change sign.

In case (1) we have the following easy result that describes the dynamics of every point (a similar result appears in [**Cla02**]).

PROPOSITION 4.1. *If Φ changes sign, then $\rho_p(F) = \{0\}$ and every point converges under iteration by F to a fixed point.*

Proof. Let Y and $Z \in \Omega_X$ be such that $\Phi(Y) < 0 < \Phi(Z)$. Because Φ is continuous, we can find neighborhoods U of Y and V of Z such that Φ is negative at every point of U and positive at every point of V. By the minimality of $(\Omega, \mathbb{R})$, there is $K > 0$ such that $\bigcup_{t\in[0,K]} T_t U = \bigcup_{t\in[0,K]} T_t V = \Omega_X$. It follows that for every $W \in \Omega_X$ there exist $-K < t_1 < t_2 < 0 < t_3 < t_4 < K$ such that $t \mapsto \Phi(W - t)$ changes sign between t_1 and t_2 and also between t_3 and t_4. Hence, applying the intermediate-value theorem, one deduces the existence of $-K < u' < 0 < t' < K$ such that $W - t'$ and $W - u'$ are fixed points of F. Let f_W be the map defined by Proposition 3.3. Then, t' and u' are fixed points of f_W and, since f_W is non-decreasing, the interval $[u', t']$ is f_W-invariant. It follows that the sequence $(f_W^n(0))_{n\in\mathbb{N}}$ is monotone and bounded. Since $\Phi^{(n)}(W) = f_W^n(0)$ for every $n \in \mathbb{N}$, it follows that $\rho(F, W) = 0$. Moreover, the sequence $(f_W^n(0))_{n\in\mathbb{N}}$ converges to $t_0 \in [u', t']$, which is necessarily a fixed point of f_W. By equation (11) and the continuity of the translation action, one sees that $W - t_0$ is a fixed point of F and that $F^n(W)$ converges to $W - t_0$. □

The case (2) is subtler. Because Φ does not change sign, either $\inf \Phi \geq 0$ or $\sup \Phi \leq 0$, and to fix ideas we will suppose that $\inf \Phi \geq 0$ (the other case may be treated in an analogous way).

We define

$$\Omega_X^+ = \left\{ Y \in \Omega_X \;\middle|\; \limsup_k \Phi^{(k)}(Y) = +\infty \right\}$$

and set $\Omega_X^{\mathrm{fp}} = \Omega_X \setminus \Omega_X^+$. We easily check that

- Ω_X^+ is F-invariant;
- the translation set of F restricted to Ω_X^{fp} is $\{0\}$ when Ω_X^{fp} is not empty;
- Ω_X^{fp} contains the fixed points of F. Moreover, Ω_X^{fp} is not empty if and only if F has fixed points.

Hence, we need only to compute the translation set of F restricted to Ω_X^+. We recall that (Ω_X, T) is uniquely ergodic and that μ is the unique T-invariant probability measure on Ω_X. We obtain the following result.

THEOREM 4.2. *Let F be a map in $\mathcal{F}_0^+(\Omega_X)$ and Φ its displacement. If* $\inf \Phi \geq 0$, *then there exists $\rho \geq 0$ such that for μ-almost every $Y \in \Omega_X^+$ the translation number $\rho(F, Y)$ exists and is equal to ρ.*

Theorem 4.2 implies that the set

$$A = \{Y \in \Omega_X \mid \rho(F, Y) = \rho\}$$

satisfies $\mu(A) = 1$. Let μ_F be an ergodic F-invariant probability measure. Since A is F-invariant, ergodicity implies that either $\mu_F(A) = 0$ or $\mu_F(A) = 1$. If $\mu_F(A) = 1$, then by the ergodic theorem there is a point $Y \in A$ for which the translation number at Y coincides with $\int_{\Omega_X} \phi \, d\mu_F$, and therefore $\int \phi \, d\mu_F = \rho$. Whether the case $\mu_F(A) = 0$

is possible or not is not known to the author. However, we believe that the case $\mu_F(A) = 0$ cannot occur, and thus we raise the following conjecture.

CONJECTURE 4.3. *Let F be a map satisfying the hypothesis of Theorem 4.2. Then, there exists $\rho \geq 0$ such that for every ergodic F-invariant measure μ_F supported on Ω_X^+ we have*

$$\int_{\Omega_X} \Phi \, d\mu_F = \rho.$$

To simplify the notation, $[Y, Z]$ will denote the set $\{W \in \Omega_X \mid Y \leq W \leq Z\}$ whenever $Y \leq Z$ (analogous definitions are given for $[Y, Z)$, $(Y, Z]$ and (Y, Z)).

Proof of Theorem 4.2. Let $(C_n)_{n \in \mathbb{N}}$ be a decreasing sequence of local verticals in Ω_X^0 with $\operatorname{diam} C_n \to 0$ as $n \to +\infty$. For the moment, we fix $n \in \mathbb{N}$, let σ_n denote the first-return map to C_n and define $k_n : C_n \to \mathbb{R} \cup \{+\infty\}$ by

$$k_n(Y) = \max\{k \geq 0 \mid F^k(Y) \leq \sigma_n(Y)\}, \quad Y \in C_n. \tag{14}$$

Let $Y \in \Omega_X^+$ and define the sequence $(Y_\ell)_{\ell \in \mathbb{N}}$ recursively by

$$Y_0 = \begin{cases} Y - \mathbf{t}_{C_n}(Y) & \text{if } Y \notin C_n, \\ Y & \text{if } Y \in C_n \end{cases}$$

and

$$Y_{\ell+1} = \sigma_n(Y_\ell), \quad \ell \in \mathbb{N}.$$

The idea is that $k_n(Y_\ell)$ is a good approximation to the number of points in the F-orbit of Y lying inside the slice $[Y_\ell, Y_{\ell+1})$. Hence, we have the following.

CLAIM 4.4. *For each $k \in \mathbb{N}$, set $t_k = \Phi^{(k)}(Y)$ and $\ell_k = \ell_{C_n}(t_k, Y)$ (see Lemma 2.6). Let $j_0 = \min\{j \in \mathbb{Z} \mid F^k(Y) > Y_0\}$. Then, for every $k \in \mathbb{N}$, we have*

$$j_0 + \sum_{\ell=0}^{\ell_k - 2} k_n(Y_\ell) \leq k \leq j_0 + \sum_{\ell=0}^{\ell_k - 1} k_n(Y_\ell) + \ell_k. \tag{15}$$

Proof. For each $\ell \in \mathbb{N} \setminus \{0\}$, we define j_ℓ to be the unique integer satisfying

$$F^{j_\ell}(Y) \in [Y_\ell, F(Y_\ell)). \tag{16}$$

Observe that by definition j_0 satisfies equation (16) and that the existence and uniqueness of j_ℓ follow from the facts that F is orientation preserving and Y belongs to Ω_X^+.

The difference $j_{\ell+1} - j_\ell$ is exactly the number of iterates of Y by F that lie inside the slice $[Y_\ell, Y_{\ell+1})$. We check that either

$$j_{\ell+1} = j_\ell + k_n(Y_\ell) \quad \text{or} \quad j_{\ell+1} = j_\ell + k_n(Y_\ell) + 1. \tag{17}$$

Indeed, applying $F^{k_n(Y_\ell)}$ to equation (16) yields, because F is orientation preserving, that $F^{j_\ell + k_n(Y_\ell)}(Y)$ belongs to $[F^{k_n(Y_\ell)}(Y_\ell), F^{k_n(Y_\ell)+1}(Y_\ell))$. By the definition of $k_n(Y_\ell)$, this slice clearly intersects $[Y_{\ell+1}, F(Y_{\ell+1}))$. Now, we distinguish two cases:

(1) $F^{j_{\ell+1}}(Y)$ belongs to the intersection of the two slices (see the left-hand part of Figure 4), in which case the equality $j_{\ell+1} = j_\ell + k_n(Y_\ell)$ follows from the fact that there is only one iterate of Y in either of the two slices;

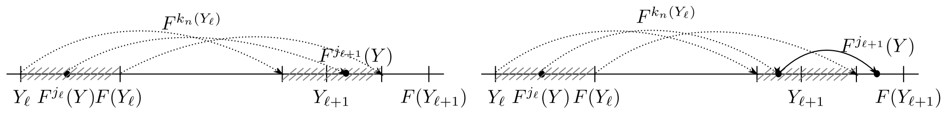

FIGURE 4. Proof of Theorem 4.2: on the left: $j_{\ell+1} = j_\ell + k_n(Y_\ell)$. On the right: $j_{\ell+1} = j_\ell + k_n(Y_\ell) + 1$.

(2) $F^{j_{\ell+1}}(Y) > F^{k_n(Y_\ell)+1}(Y_\ell)$, in which case one sees that $j_{\ell+1} = j_\ell + k_n(Y_\ell) + 1$ by using the uniqueness of j_ℓ (see the right-hand part of Figure 4).

Applying equation (17) recursively yields

$$j_\ell = j_0 + \sum_{\ell=0}^{\ell-1} k_n(Y_\ell) + e(\ell) \tag{18}$$

for every $\ell \in \mathbb{N}$, with an error $e(\ell)$ that satisfies $0 \le e(\ell) \le \ell$.

Because $F^k(Y) \in [Y_{\ell_k}, Y_{\ell_k+1})$ for every $k \in \mathbb{N}$, equation (16) and the preservation of orientation by F imply that

$$j_{\ell_k - 1} \le k \le j_{\ell_k} \quad \text{for every } k \in \mathbb{N}. \tag{19}$$

To conclude the proof of Claim 4.4, use equation (18) to replace j_{ℓ_k} and j_{ℓ_k-1} in equation (19), which yields equation (15). □

Now, divide equation (15) by $\Phi^{(k)}(Y)$ to obtain

$$\frac{j_0}{t_k} + \frac{\ell_k - 1}{t_k}\frac{1}{\ell_k - 1}\sum_{\ell=0}^{\ell_k-2} k_n(Y_\ell) \le \frac{k}{t_k} \le \frac{j_0}{t_k} + \frac{\ell_k}{t_k}\frac{1}{\ell_k}\sum_{\ell=0}^{\ell_k-1} k_n(Y_\ell) + \frac{\ell_k}{t_k}. \tag{20}$$

Then, let k go to infinity. Applying Lemma 2.6 and the ergodic theorem (which may be applied since k_n is measurable and positive) to the limit of equation (20) yields that, for μ-almost every $Y \in \Omega_X^+$,

$$\limsup_{k\to+\infty} \frac{k}{\Phi^{(k)}(Y)} - \nu(C_n) \le \int_{C_n} k_n \, d\nu \le \liminf_{k\to+\infty} \frac{k}{\Phi^{(k)}(Y)}. \tag{21}$$

Now, let n go to infinity in equation (21). Since $\operatorname{diam} C_n \to 0$, it follows that $\nu(C_n) \to 0$, and hence we obtain

$$\limsup_{n\to+\infty} \int_{C_n} k_n \, d\nu \le \liminf_{k\to+\infty} \frac{k}{\Phi^{(k)}(Y)} \le \limsup_{k\to+\infty} \frac{k}{\Phi^{(k)}(Y)} \le \liminf_{n\to+\infty} \int_{C_n} k_n \, d\nu \tag{22}$$

for μ-almost every Y in Ω_X^+.

Let $Y \in \Omega_X^+$ and suppose that equation (22) holds for Y. We distinguish two cases:

- the quantity $\liminf_{n\to+\infty} \int_{C_n} k_n \, d\nu$ is finite: in this case it follows from equation (22) that both sequences $(k/t_k)_{k\in\mathbb{N}}$ and $(\int_{C_n} k_n \, d\nu)_{n\in\mathbb{N}}$ have finite limits. The equality of both limits is clear and we need to check that they are not zero. Indeed, the first-return times to C_n go to infinity as $n \to +\infty$ because $\operatorname{diam} C_n$ goes to 0. Thus, we may suppose that $k_n \ge 1$ (it suffices to take n big enough), from which it follows that $\int_{C_n} k_n \, d\nu > 0$. Finally,

$$\rho(F, Y) = \lim_{n\to+\infty} \frac{1}{\int_{C_n} k_n \, d\nu};$$

- the quantity $\liminf_{n\to+\infty}\int_{C_n} k_n\,d\nu$ is infinite: in this case it follows from equation (22) that $\limsup_k t_k/k = 0$. Since $\Phi \geq 0$, $\liminf_k t_k/k \geq 0$ and therefore

$$\rho(F, Y) = 0. \qquad \square$$

4.2. *Translation numbers for maps without fixed points.* In this section we suppose that F is a map in $\mathcal{F}_0^+(\Omega_X)$ without fixed points. This means that Φ does not change sign and hence either $\inf\Phi > 0$ or $\sup\Phi < 0$. Again, we restrict to the case $\inf\Phi > 0$ (the other case may be treated in an analogous way). If in the proof of Theorem 4.2, the functions k_n were continuous, then by unique ergodicity, the ergodic theorem would ensure the uniform convergence to a unique translation number. But, the functions k_n are not necessarily continuous. Nevertheless, the following lemma will allow us to obtain uniform convergence to a unique translation number (see Theorem 4.6, below).

LEMMA 4.5. *Suppose that* $\inf\Phi > 0$. *Let C be a local vertical and let $\kappa : C \to \mathbb{N}$ be defined by*

$$\kappa(Y) = \max\{k \geq 0 \mid F^k(Y) \leq \sigma(Y)\},$$

where σ denotes the first-return map to C. Then, there exists $\widehat{\kappa} : C \to \mathbb{R}$ which is continuous and satisfies $\sup_{Y\in C} |\kappa(Y) - \widehat{\kappa}(Y)| \leq 1$.

Proof. Let $a = \inf\Phi > 0$. We show that κ is bounded. Indeed, $\Phi^{(k)}(Y) > ak$ for every $k \in \mathbb{N}$ and $Y \in C$. Since $\mathbf{t}_C$ is continuous (thus bounded) by Lemma 2.4, there exists $K > 0$ such that $\kappa(Y) < K$ for every $Y \in C$.

The family $\{\Phi^{(k)}\}_{k=1}^{K+1}$ is equicontinuous, which means that there is $\delta > 0$ such that for every $k \in \{1, \ldots, K+1\}$

$$d(Y, Z) < \delta \quad \text{implies that } |\Phi^{(k)}(Y) - \Phi^{(k)}(Z)| < a. \tag{23}$$

By Lemma 2.4, one can find a partition $\mathcal{C} = \{C_i\}_{i=1}^{L}$ of C by clopen sets with diameter smaller than δ such that $\mathbf{t}_C$ restricted to C_i is constant for each $i \in \{1, \ldots, L\}$. For each $i \in \{1, \ldots, L\}$, we define $Y_i \in C_i$ and k_i as follows: if there exist $Y^* \in C_i$ and $k^* \in \mathbb{N}$ such that

$$|\Phi^{(k^*)}(Y) - \mathbf{t}_C(Y^*)| < a, \tag{24}$$

then we let $Y_i = Y^*$, $k_i = k^*$ (take any pair if there is more than one). Suppose that we have this case and let $Z \in C_i$. If $\Phi^{(k)}(Z) \leq \mathbf{t}_C(Z)$, then $\Phi^{(k+1)}(Z) > \mathbf{t}_C(Z)$ follows from equation (23), and hence $\kappa(Z) = k$; else $\Phi^{(k-1)}(Z) \leq \mathbf{t}_C(Y)$ follows from equation (23), and hence $\kappa(Z) = k - 1$.

If there are no Y^* and k^* satisfying equation (24), then we let Y_i be any element of C_i and $k_i = \kappa(Y_i)$. Since $|\Phi^{(k)}(Z) - \mathbf{t}_C(Z)| > a$ for all $Z \in C_i$ and $k \in \mathbb{N}$, equation (23) implies that $\Phi^{(k_i)}(Z) \leq t_C(Z) < \Phi^{(k_i+1)}(Z)$ and thus $\kappa(Z) = k_i$.

Finally, define $\widehat{\kappa} : C \to \mathbb{N}$ by

$$\widehat{\kappa}(Y) = \sum_{i=1}^{L} \kappa(Y_i)\chi_{C_i}(Y) \quad \text{for every } Y \in C,$$

where, for a given set A, $\chi_A(Y)$ denotes the characteristic function of A. It is clear that $\widehat{\kappa}$ satisfies the assertions of the lemma, and the proof is finished. $\square$

THEOREM 4.6. *Let F be a map in $\mathcal{F}_0^+(\Omega_X)$ and Φ its displacement. Suppose that* $\inf \Phi > 0$. *Then, there exists $\rho > 0$ such that, for every $Y \in \Omega_X^+$, the translation number $\rho(F, Y)$ exists and is equal to ρ.*

Proof. The proof is a modification of the proof of Theorem 4.2. Let $(C_n)_{n\in\mathbb{N}}$ be a decreasing sequence of local verticals in Ω_X^0 with $\operatorname{diam} C_n \to 0$ as $n \to +\infty$, denote by σ_n the first-return map to C_n and define $k_n : C_n \to \mathbb{R} \cup \{+\infty\}$ by

$$k_n(Y) = \max\{k \geq 0 \mid F^k(Y) \leq \sigma_n(Y)\} \quad \text{for every } Y \in C_n.$$

Recall that $(Y_\ell)_{\ell\in\mathbb{N}}$ is defined recursively by

$$Y_0 = \begin{cases} Y - \mathbf{t}_{C_n}(Y) & \text{if } Y \notin C_n, \\ Y & \text{if } Y \in C_n \end{cases}$$

and

$$Y_{\ell+1} = \sigma_n(Y_\ell) \quad \text{for } \ell \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, applying Lemma 4.5 to k_n yields a continuous function $\widehat{k_n}$ that satisfies

$$|k_n(Y) - \widehat{k_n}(Y)| \leq 1 \quad \text{for all } Y \in \Omega_X.$$

Combining this inequality with Claim 4.4 and then dividing by t_k, we obtain

$$\frac{j_0}{t_k} + \frac{\ell_k - 1}{t_k}\left(-1 + \frac{1}{\ell_k - 1}\sum_{\ell=0}^{\ell_k-2} \widehat{k_n}(Y_\ell)\right) \leq \frac{k}{t_k} \leq \frac{j_0}{t_k} + \frac{\ell_k}{t_k}\left(\frac{1}{\ell_k}\sum_{\ell=0}^{\ell_k-1} \widehat{k_n}(Y_\ell) + 1\right) + \frac{\ell_k}{t_k} \tag{25}$$

for every $Y \in \Omega_X$ (recall that in this case $\Omega_X^+ = \Omega_X$). The system $(C_n, \nu/\nu(C_n), \sigma_n)$ is uniquely ergodic, so when we let k go to infinity and apply the ergodic theorem and Lemma 2.6 to equation (25), we obtain

$$\limsup_{k\to+\infty} \frac{k}{\Phi^{(k)}(Y)} - 2\nu(C_n) \leq \int_{C_n} \widehat{k_n}\, d\nu \leq \liminf_{k\to+\infty} \frac{k}{\Phi^{(k)}(Y)} + \nu(C_n) \tag{26}$$

for every $Y \in \Omega_X$. Because

$$\left| \int_{C_n} k_n(Y)\, d\nu - \int_{C_n} \widehat{k_n}(Y)\, d\nu \right| < \nu(C_n),$$

equation (26) implies that

$$\limsup_{k\to+\infty} \frac{k}{\Phi^{(k)}(Y)} - 3\nu(C_n) \leq \int_{C_n} k_n\, d\nu \leq \liminf_{k\to+\infty} \frac{k}{\Phi^{(k)}(Y)} + 2\nu(C_n)$$

for every $Y \in C_n$. Finally, we take the limit as n goes to infinity and proceed exactly as in the proof of Theorem 4.2 to conclude the proof. $\square$

5. *Towards Poincaré's theorem for maps on quasicrystal systems homotopic to the identity*

Let X be a quasicrystal and Ω_X its associated quasicrystal system. We focus on the set of homeomorphisms in $\mathcal{F}_0^+(\Omega_X)$ without fixed points. Denote this set by $\mathcal{H}^{++}(\Omega_X)$.

Let $F \in \mathcal{H}^{++}(\Omega_X)$. By Theorem 4.6, F has a translation number $\rho(F)$. In this section, we investigate the dynamics of the system $(\Omega,\ F)$ and provide a Poincaré-like classification theorem under a boundedness condition. Since in our setting rational numbers are no longer available, we need to define a proper replacement for them. Before doing so, we give some notation: for $K \subseteq \Omega_X$ and $G \in \mathcal{H}^{++}(\Omega_X)$ we say that K is *G-minimal* if K is closed, G-invariant and the system (K, G) is minimal.

5.1. *Replacing the rational numbers.*

It is well known that the rotation number of a monotone circle homeomorphism is rational if and only if there are periodic points. Since homeomorphisms in $\mathcal{H}^{++}(\Omega_X)$ do not have periodic points (regardless of their translation numbers), this characterization of rational numbers is not useful for our purposes. Another characterization of rational numbers is given by the fact that a rotation R_ρ of the circle $\mathbb{R}/\mathbb{Z}$ defined by $x \mapsto x + \rho \mod \mathbb{Z}$ is not minimal exactly when ρ is rational. This motivates the following definition:

$$\mathcal{Q} = \{t \in \mathbb{R} \mid (\Omega_X,\ T_t) \text{ is not minimal}\},$$

where the translation T_ρ is defined by $Y \in \Omega_X \mapsto Y - \rho$. We will say that $\rho \in \mathbb{R}$ is *T-rational* if $\rho \in \mathcal{Q}$, and $\rho \in \mathbb{R}$ is *T-irrational* otherwise. The following general lemma allows us to describe $\mathcal{Q}$ (for an outline of the proof, see [**Gla03**, 4.24.1]).

LEMMA 5.1. *Let M be a compact metric space and $\{T_t\}_{t\in\mathbb{R}}$ a minimal continuous action of $\mathbb{R}$ on M. Let $t \in \mathbb{R} \setminus \{0\}$ be such that (M, T_t) is not minimal. Then, there is an integer $k \neq 0$ such that k/t is a continuous eigenvalue of $(M, \{T_t\}_{t\in\mathbb{R}})$, i.e. there is a continuous function $g : M \to \mathbb{R}/\mathbb{Z}$ such that for every $u \in \mathbb{R}$ one has*

$$g(T_u x) = g(x) + u\frac{k}{t} \mod \mathbb{Z}. \tag{27}$$

Indeed, applying Lemma 5.1 to $(\Omega_X, \mathbb{R})$ gives the following description for $\mathcal{Q}$:

$$\mathcal{Q} = \left\{\frac{k}{\lambda} \,\middle|\, k \in \mathbb{Z},\ \lambda \neq 0 \text{ is a continuous eigenvalue of } (\Omega_X, \mathbb{R})\right\}. \tag{28}$$

In particular, the set $\mathcal{Q}$ is countable, and it can be either $\{0\}$ or dense in $\mathbb{R}$ (see [**Gla03**, 4.24.1] for details).

The next step is to describe minimal sets of (Ω, T_q) for $q \in \mathcal{Q}$, for which we need the following definition: a *local graph* is a set

$$V[\varphi] = \{X - \varphi(X) \mid X \in V\},$$

where V is a local vertical (i.e. a clopen subset of a cylinder $V_{Y,S}$) and the function $\varphi : V \to \mathbb{R}$ satisfies $\sup_{Y\in V} |\varphi(Y)| = (1/2) \inf_{Y\in V} \mathbf{t}_V(Y)$. The last condition ensures that the map $X \in V \mapsto X - \varphi(X)$ is a bijection from V to $V[\varphi]$. It is standard that $V[\varphi]$

is closed if and only if φ is continuous on V and, in this case, the previous map is a homeomorphism.

By rewriting the description of $\mathcal{Q}$ given in equation (28) in terms of local graphs, we obtain the following.

PROPOSITION 5.2. *Let I be a T_ρ-minimal set, where $\rho > 0$ is T-rational. Then, for every $\varepsilon > 0$ small enough, there are local verticals $V_1, \ldots, V_k$ of diameter smaller than ε and continuous functions $\varphi_i : V_i \to (-2\varepsilon, 2\varepsilon)$, $i \in \{1 \ldots, k\}$, such that I is the disjoint union of the local graphs $V_i[\varphi_i]$. In particular, I is a Cantor set.*

Proof. Take $k \in \mathbb{N}$ to be the smallest positive integer such that $\lambda := k/\rho$ is a continuous eigenvalue of $(\Omega_X, \mathbb{R})$ (this is well defined by equation (28)) and let g_λ be the associated continuous eigenfunction. Equation (27) implies that g_λ is T_ρ-invariant, so the minimality of (I, T_ρ) implies that g_λ is constant on I. Moreover, by equation (27), the set $K = \bigcup_{t\in[0,\rho)} T_t I$ is T-invariant and closed. Minimality of (Ω, T) implies that $K = \Omega$.

From equation (27) it follows that $J := g_\lambda^{-1}(g_\lambda(I)) = \bigcup_{\ell=0}^{k-1} T_{\ell\rho/k} I$. Since the sets $T_{\ell\rho/k} I$ for $\ell \in \{0, \ldots, k-1\}$ are T_ρ-minimal, they either coincide or are disjoint and thus I is clopen in J.

$I = g_\lambda^{-1}(g_\lambda(I))$ means that for every $Y \in I$ and $t \in \mathbb{R}$ one has that $g_\lambda(Y - t) \neq g(Y)$ if and only if $t \notin \rho/k\mathbb{Z}$. Hence, $Y - t$ does not belong to I if t is small enough, which implies that I is totally disconnected. By using the minimality of (I, T_ρ) and the aperiodicity of Ω_X, it is easy to check that I has no isolated points. Hence, I is a Cantor set.

Fix $\varepsilon > 0$ and suppose that $\tilde{g}_\lambda : \Omega_X \to \mathbb{R}$ is a lift of g_λ. By the continuity of $\tilde{g}_\lambda$ there exists $0 < \delta < \varepsilon$ such that $|\tilde{g}_\lambda(Z) - \tilde{g}_\lambda(Y)| < \varepsilon$ when $d(Y, Z) < \delta$. Thus, if we let $V = V_{Y,1/\delta}$, then $\sup_{Z\in V} |\tilde{g}_\lambda(Z) - \tilde{g}_\lambda(Y)| < \varepsilon$. Hence the function $\varphi : V \to \mathbb{R}$ given by

$$\varphi(Z) = \frac{\rho}{k}(\tilde{g}_\lambda(Y) - \tilde{g}_\lambda(Z))$$

defines a local graph $V[\varphi]$ when ε is sufficiently small. Moreover, equation (27) implies that for every $Z \in V$ one has that

$$g_\lambda(Z - \varphi(Z)) = \tilde{g}_\lambda(Y) \mod \mathbb{Z},$$

which means that $V[\varphi]$ is included in J. Since φ is continuous, it follows that $V[\varphi]$ is closed. Since $Y - t - \varphi(Y)$ does not belong to J for $Y \in V$ and t sufficiently small, one checks that $V[\varphi]$ is open in J. This construction yields a clopen cover of J by local continuous graphs. Since J is compact, there is a finite subcover $\{V_1[\varphi_1], \ldots, V_k[\varphi_k]\}$ of J. To finish, it is not difficult to check that if there exist i, j such that $C = V_i[\varphi_i] \cap V_j[\varphi_j]$ is non-empty, then C and $V_i[\varphi_i] \setminus C$ are local continuous graphs, so the subcover induces a partition of J the same form which in turn induces a partition of I of the same form. □

5.2. *T-semiconjugacies and ρ-bounded maps.* The next step consists in introducing the boundedness condition that will allow us to obtain the desired classification. We recall that given two homeomorphisms $F, G \in \mathcal{H}^{++}(\Omega_X)$, we say that F is *semiconjugate* to G if there is a continuous onto map $H : \Omega_X \to \Omega_X$ such that

$$H \circ F = G \circ H. \tag{29}$$

The map H is called a *semiconjugacy* from F to G. If H is also one-to-one, then F and G are said to be *conjugate* and the map H is called a *conjugacy*. In our context, it is natural to consider semiconjugacies that are homotopic to the identity. Thus, we say that $F \in \mathcal{H}^{++}(\Omega_X)$ is *T-semiconjugate* to $G \in \mathcal{H}^{++}(\Omega_X)$ if there exists a semiconjugacy H from F to G that is homotopic to the identity. In this case, we call H a *T-semiconjugacy* from F to G. It is left to the reader to see that translation numbers are preserved by T-semiconjugacies. The following example shows that this is no longer true if the semiconjugacy is not homotopic to the identity (a similar example appears in [**Shv03**]).

Example 5.3. Let $X = X_{\text{fib}}$ be the Fibonacci quasicrystal defined in Example 2.3. The inflation–substitution homeomorphism ω on Ω_X fixes X. Moreover,

$$\omega(Y - t) = \omega(Y) - \tau t \tag{30}$$

for every $Y \in \Omega_X$ and $t \in \mathbb{R}$, where τ is the golden ratio (see, e.g., [**RS01**]). Let F be a homeomorphism in $\mathcal{H}^{++}(\Omega_X)$ and Φ its displacement. We suppose that Φ is strictly positive and define Φ_G by

$$\Phi_G(Y) = \lambda\Phi(\omega^{-1}(Y)) \quad \text{for every } Y \in \Omega_X.$$

If we define $G : \Omega_X \to \Omega_X$ by $G(Y) = Y - \Phi_G(Y)$, then

$$\begin{aligned} G(\omega(Y - t)) = \omega(Y - t) - \tau\Phi(Y - t) &= Y - \tau(t + \Phi(Y - t)) \\ &= \omega(Y - t - \Phi(Y - t)) = \omega(F(Y - t)) \end{aligned}$$

for every $t \in \mathbb{R}$. Since F and ω are continuous maps, a density argument yields that F and G are conjugate by ω. It can be checked that G preserves orientation. Hence, G belongs to $\mathcal{H}^{++}(\Omega_X)$ and by Theorem 4.6 it has a translation number. We now compute $\rho(G)$. Iterating the semiconjugacy equation (29) gives

$$G^n(Y) = \omega \circ F^n \circ \omega^{-1}(Y) \quad \text{for all } Y \in \Omega_X \text{ and } n \in \mathbb{N}.$$

This equality applied to X reads

$$X - \Phi_G^{(n)}(X) = \omega(X) - \Phi^{(n)}(X) = X - \tau\Phi^{(n)}(X).$$

The aperiodicity of X then implies that

$$\rho(G) = \tau\rho(F).$$

It easily follows from Theorem 3.4 and equation (30) that ω is not homotopic to the identity.

Now, we concentrate on the question of determining when $F \in \mathcal{H}^{++}(\Omega_X)$ is T-semiconjugate to $T_{\rho(F)}$. Suppose that H is a T-semiconjugacy from F to $T_{\rho(F)}$ and let Φ and Ψ be the displacements of F and H, respectively. Then, the semiconjugacy equation (29) written in terms of Φ and Ψ yields a cohomological equation:

$$\Psi(F(Y)) - \Psi(Y) = \rho(F) - \Phi(Y) \quad \text{for all } Y \in \Omega_X. \tag{31}$$

This means that the problem of finding a T-semiconjugacy reduces to the problem of finding a continuous (and therefore bounded) solution Ψ to the cohomological equation (31). For every $n \in \mathbb{N}$ and $Y \in \Omega_X$, we let $\zeta(n, Y) = n\rho(F) - \Phi^{(n)}(Y)$. We say that F is *ρ-bounded* if there exists $C > 0$ such that

$$|\zeta(n, Y)| < C \quad \text{for all } Y \in \Omega_X \text{ and } n \in \mathbb{N}.$$

The following proposition is plain.

PROPOSITION 5.4. *Suppose that $F \in \mathcal{H}^{++}(\Omega_X)$ is T-semiconjugate to $T_{\rho(F)}$. Then, F is ρ-bounded.*

The next lemma provides a partial converse for the previous proposition. It is basically the Gottschalk–Hedlund theorem (see [**KH95**, §2.9] for an introduction to cohomological equations and a proof of this theorem) adapted to our context.

LEMMA 5.5. *Suppose that $F \in \mathcal{H}^{+}(\Omega_X)$ is ρ-bounded. Then, equation (31) has a bounded solution $\Psi : \Omega_X \to \mathbb{R}$. Moreover, its restriction to any F-minimal set is continuous, the map H defined by $H(Y) = Y - \Psi(Y)$ for $Y \in \Omega_X$ is orientation preserving and the following diagram commutes:*

$$\begin{tikzcd} K \arrow[r, "F|_K"] \arrow[d, "H"'] & K \arrow[d, "H"] \\ \Omega_X \arrow[r, "T_{\rho(F)}"] & \Omega_X \end{tikzcd} \tag{32}$$

Proof. We let

$$\Psi(Y) := \limsup_{n \in \mathbb{N}} \{\Phi^{(n)}(Y) - n\rho(F)\}.$$

Ψ is well defined and bounded because F is ρ-bounded, and it solves the cohomological equation (31) (for details see [**KH95**, Theorem 2.9.3]). Let K be an F-minimal set. The restriction of Ψ to K is continuous because it coincides with the continuous solution of equation (31) on K given by the Gottschalk–Hedlund theorem [**KH95**, Theorem 2.9.4]. To see that H is orientation preserving, it suffices to prove that the function h_Y defined by $t \in \mathbb{R} \mapsto t + \psi(Y - t)$ is non-decreasing for every $Y \in \Omega_X$. Indeed,

$$t + \Psi(Y - t) = \limsup_{n \in \mathbb{N}} \{t + \Phi^{(n)}(Y - t) - n\rho(F)\}$$

for every $t \in \mathbb{R}$. Besides, $t \mapsto t + \Phi^{(n)}(Y - t)$ is non-decreasing because F^n preserves orientation and $F^n(Y) = Y - \Phi^{(n)}(Y)$. As the lim sup of non-decreasing functions is non-decreasing, the proof is finished. $\square$

Lemma 5.5 means that when F is minimal, ρ-boundedness of F is also a sufficient condition for F to be T-semiconjugate to $T_{\rho(F)}$. Since minimality is preserved by semiconjugacies, it follows that $\rho(F)$ is T-irrational when F is ρ-bounded and minimal. This implies that, at least in the minimal case, T-rational numbers provide a good replacement for rational numbers.

5.3. *Two lemmas about F-minimal sets.*

LEMMA 5.6. *Let K be a non-empty closed F-invariant set. Then, K intersects every leaf of Ω_X and the set of return times*

$$\mathcal{R}(Y, K) = \{t \in \mathbb{R} \mid Y - t \in K\}$$

is relatively dense for every $Y \in \Omega_X$.

Proof. Let $Y \in \Omega_X$. First we prove that $\mathcal{R}(Y, K)$ is closed. Indeed, suppose that $(t_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}(Y, K)$ is a sequence converging to some $t \in \mathbb{R}$. By the continuity of T, $(Y - t_n)_{n\in\mathbb{N}}$ converges to $Y - t$ when n goes to infinity. But, by definition, the sequence $(Y - t_n)_{n\in\mathbb{N}}$ is included in K and, since K is closed, it follows that $Y - t$ belongs to K. This means that $t \in \mathcal{R}(Y, K)$.

Now, we prove that if $\mathcal{R}(Y, K)$ is not empty, then it is relatively dense. Indeed, suppose that $\mathcal{R}(Y, K)$ is not empty. There is $t \in \mathbb{R}$ such that $Y - t \in K$, and by F-invariance of K it follows that the F-orbit of $Y - t$ is included in K. Thus, the set of return times contains the set $\{t + \Phi^{(n)}(Y - t) \mid n \in \mathbb{Z}\}$. But, the latter set is relatively dense because $\{\Phi^{(n)}(Y) \mid n \in \mathbb{Z}\}$ is unbounded (from above and below), and its gaps are smaller than $\sup \Phi$, which is finite because Φ is continuous. This implies that $\mathcal{R}(Y, K)$ is relatively dense.

To conclude the proof of the lemma, it suffices to show that $\mathcal{R}(Y, K)$ is not empty for every $Y \in \Omega_X$. Fix $Y \in K$. By definition, $\mathcal{R}(Y, K)$ is not empty. Applying the previous argument to Y, we obtain that $\mathcal{R}(Y, K)$ is relatively dense. Since $(\Omega_X, \mathbb{R})$ is minimal, it follows that for every $Z \in \Omega_X$ there is a sequence $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that $(Y - t_n)_{n\in\mathbb{N}}$ converges to Z. The relative density of $\mathcal{R}(Y, K)$ implies that for each $n \in \mathbb{N}$ there is $u_n \in \mathcal{R}(Y, K)$ such that the sequence $(u_n - t_n)_{n\in\mathbb{N}}$ is bounded. By dropping to a subsequence, we may suppose that $(u_n - t_n)_{n\in\mathbb{N}}$ converges to some $u \in \mathbb{R}$. It follows that $(Y - u_n)_{n\in\mathbb{N}}$ converges to $Z - u$. But, by construction, the sequence $(Y - u_n)_{n\in\mathbb{N}}$ is included in K, which is closed. Hence, $Z - u$ belongs to K, which means that $u \in \mathcal{R}(Y, K)$, and the proof is done. □

LEMMA 5.7. *Let K be a non-empty F-minimal set and H the map defined in Lemma 5.5. Then, the set $H(K)$ is $T_{\rho(F)}$-minimal.*

Proof. By Lemma 5.5, the map $H|_K$ is continuous. Since K is compact, it follows that $H(K)$ is compact and thus closed. The $T_{\rho(F)}$-invariance of $H(K)$ follows directly from equation (32). Finally, let F be a closed $T_\rho(F)$-invariant set included in $H(K)$. Then, $H^{-1}(F) \cap K$ is a non-empty closed F-invariant set. From the minimality of the system (K, F), it follows that $H^{-1}(F) \supseteq K$, and thus $F \supseteq H(K)$, which concludes the proof. □

5.4. *The T-rational case.* In this section we consider the case when $\rho(F)$ is a T-rational number, and we give a description of F-minimal sets. Let Ψ, H be as defined in Lemma 5.5. The idea is to combine Lemma 5.7 and Proposition 5.2 to obtain some local F-invariant graphs. Since H may not be continuous outside a minimal set, these local graphs may not be closed. This motivates us to study the closures of local graphs, for which we need the following definitions: for a local vertical V in Ω_X, we let $R_V = (1/2)\inf_{Y\in V} \mathbf{t}_V(Y) > 0$. Given two functions $\alpha, \beta : V \to \mathbb{R}$ satisfying $-R_V < \alpha \leq \beta < R_V$, the set

$$V[\alpha, \beta] = \{Y - t \mid Y \in V, \alpha(Y) \leq t \leq \beta(Y)\}$$

is called a *local strip*. When $Y \in V$, then $Y[\alpha, \beta]$ is defined by

$$Y[\alpha, \beta] = \{Y - t \mid \alpha(Y) \leq t \leq \beta(Y)\}.$$

We check that, by the definition of R_V, the restriction of T to $V \times [\alpha, \beta]$ is a homeomorphism from $V \times [\alpha, \beta]$ to $V[\alpha, \beta]$. Also, $V[\alpha] = V[\alpha, \alpha]$. A set K is called a *strip* if it can be decomposed into a finite disjoint union of local strips. A local strip $V[\alpha, \beta]$ is *thin* if $\overline{V[\alpha]} = \overline{V[\beta]} = V[\alpha, \beta]$. A strip is *thin* if it is the finite union of disjoint thin local strips.

The following results follow directly from the fact that $V[\alpha, \beta]$ is homeomorphic to $V \times [\alpha, \beta]$ and standard results in the theory of semicontinuous functions.

LEMMA 5.8. *A local strip $V[\alpha, \beta]$ is closed (as a subset of Ω_X) if and only if α is lower semicontinuous and β is upper semicontinuous.*

PROPOSITION 5.9. *The closure of a local graph $V[\alpha]$ is included in a thin local strip of the form $V[\alpha^-, \alpha^+]$, where*

$$\alpha^+(x) = \inf\{\beta(x) \mid \beta \textit{ is upper semicontinuous and } \beta \geq \alpha \textit{ on } V\}$$

and

$$\alpha^-(x) = \sup\{\beta(x) \mid \beta \textit{ is lower semicontinuous and } \beta \leq \alpha \textit{ on } V\}.$$

Now, we can state the main result of this section.

THEOREM 5.10. *Let F be a ρ-bounded homeomorphism in $\mathcal{H}^{++}(\Omega_X)$ with $\rho(F) \in \mathcal{Q}$. Then, every F-minimal set is a thin strip.*

Before giving the proof, we need one more lemma.

LEMMA 5.11. *Given a local closed graph $V[\varphi] \subseteq \Omega_X$, there are bounded real-valued functions s, t defined on V such that*

$$V[s, t) \subseteq H^{-1}(V[\varphi]) \subseteq V[s, t].$$

Moreover, if $\|\varphi\|_\infty + \|\Psi\|_\infty < R_V$, then the set $V[s, t]$ is a local strip.

Proof. Since H preserves every leaf of Ω_X, it follows that, for every $Y \in V$, the set $\mathcal{A}$ containing the preimages of $Y - \varphi(Y)$ by H is included in the leaf of Y. This gives a correspondence between the elements of $\mathcal{A}$ and the solutions $t \in \mathbb{R}$ of the following equation:

$$t + \Psi(Y - t) = \varphi(Y). \tag{33}$$

Since H preserves orientation, the function $t \in \mathbb{R} \mapsto h_Y(t) = t + \Psi(Y - t)$ is non-decreasing. Moreover, it is unbounded from above and below because Ψ is bounded by Lemma 5.5. Hence, the set of solutions $t \in \mathbb{R}$ of equation (33) is a bounded interval. We denote by $s(Y)$ and $t(Y)$ its lower and upper extreme points, respectively. We observe that every solution t of equation (33) satisfies $|t| \leq \|\Psi\|_\infty + \|\varphi\|_\infty$. This implies that $t(Y)$ and $s(Y)$ are bounded and that $V[s, t]$ is a local strip when $\|\Psi\|_\infty + \|\varphi\|_\infty < R_V$. $\square$

Proof of Theorem 5.10. Let K be an F-minimal set and $I = H(K)$. By Lemma 5.7, we know that I is $T_{\rho(F)}$-minimal. Proposition 5.2 implies that I is the finite disjoint union of local closed graphs, say $V_1[\varphi_1], \ldots, V_n[\varphi_n]$. Without loss of generality, we may assume that $R_{V_i} > \|\Psi\|_\infty + \|\varphi_i\|_\infty$ (it suffices to take ε in Proposition 5.2 sufficiently small). For each $i \in \{1, \ldots, n\}$ and $Y \in V_i$, we define

$$K_i := H|_K^{-1}(V_i[\varphi_i]) = H^{-1}(V_i[\varphi_i]) \cap K.$$

The family $\{K_i \mid i \in \{1, \ldots, n\}\}$ defines a finite partition of K. For each $i \in \{1, \ldots, n\}$, K_i is closed in K, since $H|_K$ is continuous. It follows that K_i is closed. Applying Lemma 5.11, we obtain real-valued functions s_i, t_i on V_i such that

$$V_i[s_i, t_i) \cap K \subseteq K_i \subseteq V_i[s_i, t_i] \cap K \quad \text{for each } i \in \{1, \ldots, n\}. \tag{34}$$

For each $i \in \{1, \ldots, n\}$, we define

$$\alpha_i(Y) = \min\{t \in [s_i(Y), t_i(Y)] \mid Y - t \in K\}$$

and

$$\beta_i(Y) = \max\{t \in [s_i(Y), t_i(Y)] \mid Y - t \in K\}.$$

From equation (34), it easily follows that

$$K_i = V_i[\alpha_i, \beta_i] \cap K \quad \text{for each } i \in \{1, \ldots, n\}. \tag{35}$$

Suppose for now that the set $A = \bigcup_i V_i[\alpha_i]$ is F-invariant. By equation (35) and the definition of the K_i's, we see that A is included in K. Hence, the F-minimality of K implies that the closure of A coincides with K. But, A is the finite union of the local graphs $V_i[\alpha_i]$, whose closures are, by Proposition 5.9, included in thin local strips $V_i[\alpha_i^-, \alpha_i^+]$. Each of these strips is included in the corresponding K_i. Since the K_i's are disjoint, it follows that the local strips $V_i[\alpha_i^-, \alpha_i^+]$ are also disjoint, and thus K is the finite disjoint union of thin local strips, which means that K is a thin strip.

It remains to show that A is T-invariant. Indeed, let $i \in \{1, \ldots, n\}$ and $Y \in V_i$, so that $Y - \alpha_i(Y)$ belongs to A. Since I is $T_{\rho(F)}$-invariant, there are $j \in \{1, \ldots, n\}$ and $Z \in V_j$ such that

$$Y - \varphi_i(Y) - \rho(F) = Z - \varphi_j(Z). \tag{36}$$

j and Z are uniquely defined by equation (36). We will prove that

$$Z - \alpha_j(Z) = F(Y - \alpha_i(Y)), \tag{37}$$

from which it follows that A is F-invariant, since $Z - \alpha_j(Z)$ belongs to A. First, observe that equation (34) implies that

$$Y[s_i, t_i) \subseteq H^{-1}(Y - \varphi_i(Y)) \subseteq Y[s_i, t_i], \tag{38}$$

$$Z[s_j, t_j) \subseteq H^{-1}(Z - \varphi_j(Z)) \subseteq Z[s_j, t_j]. \tag{39}$$

Using equation (39) and the definitions of α_j and β_j, it is not difficult to check that

$$Z[\alpha_j, \beta_j] \subseteq H^{-1}(Z - \varphi_j(Z)).$$

Taking the preimage by F of this inclusion and using equation (32) in the resulting inclusion gives

$$F^{-1}(Z[\alpha_j, \beta_j]) \subseteq H^{-1}(Z - \varphi_j(Z) + \rho(F)). \tag{40}$$

Hence, replacing equations (36) and (38) in equation (40) yields

$$F^{-1}(Z[\alpha_j, \beta_j]) \subseteq Y[s_i, t_i]. \tag{41}$$

Since K is F-invariant, it follows from equations (41) and (34) that $Z - \alpha_j(Z)$ belongs to K_i. Now, we deduce from the definition of $\alpha_i(Y)$ that $F^{-1}(Z - \alpha_j(Z)) \geq Y - \alpha_i(Y)$. But, F preserves orientation, so this implies that

$$Z - \alpha_j(Z) \geq F(Y - \alpha_i(Y)). \tag{42}$$

The reverse inequality is obtained by a similar argument. Using equation (38) and the definitions of α_i and β_i, it is not difficult to check that

$$Y[\alpha_i, \beta_i] \subseteq H^{-1}(Y - \varphi_i(Y)). \tag{43}$$

Apply F to equation (43) and then use equations (36) and (32) in the resulting relation to obtain

$$F(Y[\alpha_i, \beta_i]) \subseteq H^{-1}(Z - \varphi_j(Z)).$$

Thus, applying equation (39) to the last inclusion yields

$$F(Y[\alpha_i, \beta_i]) \subseteq Z[s_j, t_j]. \tag{44}$$

Since K is F-invariant, it follows from equations (44) and (34) that $F(Y - \alpha_i(Y))$ belongs to K_j. The definition of α_j then implies that

$$Z - \alpha_j(Z) \leq F(Y - \alpha_i(Y)). \tag{45}$$

□

5.5. *The T-irrational case.* In this section we suppose that $\rho(F)$ is a T-irrational number and prove the existence of a T-semiconjugacy from F to $T_{\rho(F)}$. To this end, it suffices to prove that the map H defined in Lemma 5.5 is continuous on Ω_X and onto. Therefore, we have the following.

THEOREM 5.12. *Let F be a ρ-bounded homeomorphism in $\mathcal{H}^{++}(\Omega_X)$ and Φ its displacement. If $\rho(F)$ is T-irrational, then F is T-semiconjugate to $T_{\rho(F)}$.*

For the proof, we need the following definitions: let K be a non-empty closed F-invariant set. For each $Y \in \Omega_X$, we define

$$K^+(Y) = \sup\{t \leq 0 \mid t \in \mathcal{R}(Y, K)\}$$

and

$$K^-(Y) = \inf\{t \geq 0 \mid t \in \mathcal{R}(Y, K)\}.$$

It is easy to check that

$$-\|\Phi\|_\infty \leq K^+(Y) \leq 0 \leq K^-(Y) \leq \|\Phi\|_\infty. \tag{46}$$

Moreover, the following lemma states that the slice $Y[K^-, K^+]$ is included in a level set of H.

LEMMA 5.13. *Let K be a non-empty closed F-invariant set. Then, we have that $Y - K^-(Y) \in K$ and $Y - K^+(Y) \in K$ for every $Y \in \Omega_X$, and the maps $Y \mapsto K^+(Y)$ and $Y \mapsto K^-(Y)$ are, respectively, upper semicontinuous and lower semicontinuous. Moreover, the map H restricted to K is onto, and the equation*

$$H(Y - K^+(Y)) = H(Y) = H(Y - K^-(Y)) \tag{47}$$

holds for every $Y \in \Omega_X$.

Proof. We check the assertions for K^+, the assertions for K^- follow from similar arguments. Let $Y \in \Omega_X$. We show that $Y - K^+(Y)$ belongs to K. Indeed, Lemma 5.6 implies that the set $A_Y = (-\infty, 0] \cap \mathcal{R}(Y, K)$ is non-empty, closed and bounded from above. Hence, it attains its maximum and, since $K^+(Y) = \sup A_Y$, it follows that $K^+(Y) \in \mathcal{R}(Y, K)$, which (by definition of $\mathcal{R}(Y, K)$) means that $Y - K^+(Y) \in K$.

Now, we show that $Y \mapsto K^+(Y)$ is upper semicontinuous. Since $K^+(Y) < 0$ for every $Y \in \Omega_X$, it follows that $y := \limsup_{Z \to Y} K^+(Z)$ is finite. Consider a sequence $(Y_n)_{n\in\mathbb{N}}$ that converges to Y as $n \to +\infty$. Without loss of generality, we suppose that $K^+(Y_n)$ converges to y as $n \to +\infty$. By the continuity of T, we have that $Y_n - K^+(Y_n)$ converges to $Y - y$. Since this sequence is included in K and K is closed, we deduce that $y \in A_Y$. The definition of K^+ then implies that $y \leq K^+(Y)$. This proves that K^+ is upper semicontinuous.

We now show that H restricted to K is onto, i.e. that $H(K) = \Omega_X$. Indeed, by Lemma 5.7, we know that $H(K)$ is $T_{\rho(F)}$-minimal. But, $(\Omega_X, T_{\rho(F)})$ is minimal since $\rho(F)$ is T-irrational. Hence, $H(K) = \Omega_X$.

Finally, we suppose by contradiction that equation (47) does not hold. Then, there exists $Y \in \Omega_X$ for which $H(Y - K^+(Y)) \neq H(Y - K^+(Y))$. Since H preserves orientation, it is plain that $H(Y - K^+(Y)) \leq H(Y - K^-(Y))$, and thus there exists $Z \in \Omega_X$ such that

$$H(Y - K^+(Y)) < Z < H(Y - K^-(Y)).$$

The fact that each leaf is preserved by H implies that for every $Y' \in H^{-1}(Z) \cap K$, which is not empty since $H(K) = \Omega_X$, there is $t \in \mathbb{R}$ such that $Y' = Y - t$. Finally, the preservation of orientation by H implies that t belongs to $(K^+(Y), K^-(Y))$, which clearly contradicts either the definition of K^+ or the definition of K^-. □

Proof of Theorem 5.12. We need only to prove that H is continuous, since H satisfies equation (32) and is onto by Lemma 5.13. Take an arbitrary $Y \in \Omega_X$ and consider a sequence $(Y_n)_{n\in\mathbb{N}} \subseteq \Omega_X$ that converges to Y. Since Ψ is bounded, by dropping to a subsequence we have that the sequence $(H(Y_n))_{n\in\mathbb{N}}$ converges to some $Z \in \Omega_X$. By standard arguments, it suffices to prove that $Z = H(Y)$. To do this, we consider an F-minimal set K. By equation (46), the sequence $(K^+(Y_n))_{n\in\mathbb{N}}$ is bounded by $\|\Phi\|_\infty$. Thus, by dropping to a new subsequence, we have that $K^+(Y_n) \to x^+ \in [-\|\Phi\|_\infty, \|\Phi\|_\infty]$ as $k \to \infty$. The key step consists in applying equation (47) to each Y_n. This yields

$$H(Y_n) = H(Y_n - K^+(Y_n)) \quad \text{for every } k \in \mathbb{N}.$$

By taking the limit in this equation when $k \to +\infty$, we obtain

$$Z = H(Y - x^+).$$

From the upper semicontinuity of K^+, we know that $x^+ \leq K^+(Y)$, which means that $Y - x^+ \leq Y - K^+(Y)$. On the other hand, equation (47) applied to Y reads

$$H(Y) = H(Y - K^+(Y))$$

and, since H preserves orientation, it follows that $Z \leq H(Y)$. To see that $Z \geq H(Y)$, and thus that H is continuous, we apply almost the same argument to the sequence $(K^-(Y_n))_{n\in\mathbb{N}}$, the only difference being the fact that we use the lower semicontinuity of K^-. □

Lemma 5.13 also allows us to prove that (Ω_X, F) has a unique minimal set, as the following result shows.

PROPOSITION 5.14. *Suppose that F satisfies the hypotheses of Theorem 5.12. Then, the set K_H defined as the closure of*

$$A_H = \{Y \in \Omega_X \mid H^{-1}(H(Y)) = \{Y\}\}$$

is the unique minimal set of the system (Ω_X, F).

Proof. We will prove that A_H is F-invariant, non-empty and intersects every F-minimal set. The conclusion easily follows from these facts. Let K be an F-minimal set and take $Y \in \Omega_X$. If $Y \notin K$, then by definition of K^+ and K^- we have that $K^+(Y) < 0 < K^-(Y)$. Therefore, from equation (47), the set $H^{-1}(H(Y))$ is not a singleton, which means that Y does not belong to A_H. This means that A_H is included in K.

Let us check that A_H is F-invariant. From equation (32), for every $Y \in \Omega_X$ we have that

$$F^{-1} \circ H^{-1} \circ H(F(Y)) = H^{-1}(H(Y)).$$

Hence, if we assume that $Y \in A_H$, then the last equation implies that $F^{-1} \circ H^{-1} \circ H(F(Y)) = \{Y\}$. This implies that $F(Y) \in A_H$ and A_H is F-invariant.

Finally, we check that A_H is non-empty. Recall that the function $h_Y : \mathbb{R} \to \mathbb{R}$ induced by H on the leaf of Y is defined by $h_Y(t) = t + \Psi(Y - t)$. For each $t \in \mathbb{R}$, we let $a(t) = \min\{s \leq t \mid h_Y(s) = h_Y(t)\}$ and $b(t) = \max\{s \geq t \mid h_Y(s) = h_Y(t)\}$. Since H is continuous by Theorem 5.12, it is easy to check that $Y[a(t), b(t)] = H^{-1}(H(Y - t))$. If $Y[a(t), b(t)]$ is not a singleton, then we say that $Y[a(t), b(t)]$ is a *plateau* of H in the leaf of Y. By observing that $a(t') = a(t)$ and $b(t') = b(t)$ for every $t' \in [a(t), b(t)]$, we deduce that there is a countable number of plateaux of H in the leaf of Y (each plateau is indexed by a rational number). This implies that the image of H restricted to the union of all its plateaux in the leaf of Y is countable, and hence it does not cover the leaf. Since H is onto, it follows that A_H is not-empty. □

We end this paper by giving an example (which was hinted to the author by the anonymous referee).

Example 5.15. Let $X = X_{\text{fib}}$ be the Fibonacci quasicrystal defined in Example 2.3 with $L = S = 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be the lift of an orientation-preserving circle homeomorphism g. Since the displacement of f is X-equivariant, define $F : \Omega_X \to \Omega_X$ by $F(Y) = Y - \Phi(Y)$, where Φ is the continuous function on Ω_X corresponding to

$\phi(t) = f(t) - t$ (see Lemma 3.2). It is clear that for every $Y \in \Omega_X^0$ one has that $F(Y - t) = Y - f(t)$, and hence $\rho(F) = \rho(f)$. Moreover, F is ρ-bounded. On the other hand, it is known that the eigenvalues of Ω_X contain $(\sqrt{5}+1)/2$ and hence $\mathcal{Q}$ contains $\mathbb{Q}(\sqrt{5})$ (see e.g. [**Fog02**]).

Next, suppose that $\rho(f) = \sqrt{5}$, i.e. $\rho(f)$ is irrational but T-rational. We analyze two cases:

(1) g is transitive: in this case, by Poincaré's theorem f is conjugated to $t \mapsto t + \rho(f)$ by a map h whose displacement is also periodic (see e.g. [**KH95**, proof of Poincaré's theorem]). Hence, in this case h extends to a T-conjugacy between F and $T_{\sqrt{5}}$. Applying Theorem 5.2, one obtains that the minimal sets of F are local closed graphs (in this case H is one-to-one);

(2) g is a Denjoy example: in this case F is not transitive and therefore it is not T-conjugate with $T_{\sqrt{5}}$. Moreover, if E is the unique g-minimal Cantor subset of $\mathbb{R}/\mathbb{Z}$, then $K_H = \{Y - t \mid Y \in \Omega_X^0,\ t \in E\}$.

Open questions. Let F be a ρ-bounded homeomorphism in $\mathcal{H}^{++}(\Omega_X)$ and $\rho(F)$ its translation number.

- If $\rho(F)$ is T-irrational and F is transitive, is it true that F is T-conjugate with $T_{\rho(F)}$? If not, give a counterexample.
- If $\rho(F)$ is T-rational, give general conditions for the invariant sets to be closed graphs.
- Find more natural conditions for F to be ρ-bounded.

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REFERENCES

[BBG06] J. Bellissard, R. Benedetti and J.-M. Gambaudo. Spaces of tilings, finite telescopic approximations and gap-labeling. *Commun. Math. Phys.* **261**(1) (2006), 1–41.

[BG03] R. Benedetti and J.-M. Gambaudo. On the dynamics of $\mathbb{G}$-solenoids. Applications to Delone sets. *Ergod. Th. & Dynam. Sys.* **23**(3) (2003), 673–691.

[CFS82] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinaĭ. *Ergodic Theory (Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 245).* Springer, New York, 1982 (Translated from the Russian by A. B. Sosinskiĭ).

[Cla02] A. Clark. The dynamics of maps of solenoids homotopic to the identity. *Continuum Theory (Denton, TX, 1999) (Lecture Notes in Pure and Applied Mathematics, 230).* Dekker, New York, 2002, pp. 127–136.

[DHS99] F. Durand, B. Host and C. Skau. Substitutional dynamical systems, Bratteli diagrams and dimension groups. *Ergod. Th. & Dynam. Sys.* **19**(4) (1999), 953–993.

[Fog02] N. P. Fogg. *Substitutions in Dynamics, Arithmetics and Combinatorics (Lecture Notes in Mathematics, 1794)*. Eds. V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel. Springer, Berlin, 2002.

[GGP06] J.-M. Gambaudo, P. Guiraud and S. Petite. Minimal configurations for the Frenkel–Kontorova model on a quasicrystal. *Commun. Math. Phys.* **265**(1) (2006), 165–188.

[Gla03] E. Glasner. *Ergodic Theory via Joinings (Mathematical Surveys and Monographs, 101)*. American Mathematical Society, Providence, RI, 2003.

[GM99] W. Geller and M. Misiurewicz. Rotation and entropy. *Trans. Amer. Math. Soc.* **351**(7) (1999), 2927–2948.

[Her83] M.-R. Herman. Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnol ′d et de Moser sur le tore de dimension 2. *Comment. Math. Helv.* **58**(3) (1983), 453–502.

[Hof95] A. Hof. A remark on Schrödinger operators on aperiodic tilings. *J. Statist. Phys.* **81**(3–4) (1995), 851–855.

[HPS92] R. H. Herman, I. F. Putnam and C. F. Skau. Ordered Bratteli diagrams, dimension groups and topological dynamics. *Int. J. Math.* **3**(6) (1992), 827–864.

[Jag08] T. H. Jäger. Linearization of conservative toral homeomorphisms. *Invent. Math.* **176**(3) (2009), 601–616.

[JS06] T. H. Jäger and J. Stark. Towards a classification for quasiperiodically forced circle homeomorphisms. *J. London Math. Soc.* (2) **73**(3) (2006), 727–744.

[Kel03] J. Kellendonk. Pattern-equivariant functions and cohomology. *J. Phys. A: Math. Gen.* **36**(21) (2003), 5765–5772.

[KH95] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems (Encyclopedia of Mathematics and its Applications, 54)*. Cambridge University Press, Cambridge, 1995 (With a supplementary chapter by Katok and Leonardo Mendoza).

[KP00] J. Kellendonk and I. F. Putnam. Tilings, C^*-algebras, and K-theory. *Directions in Mathematical Quasicrystals (CRM Monograph Series, 13)*. American Mathematical Society, Providence, RI, 2000, pp. 177–206.

[Kwa00] J. Kwapisz. Poincaré rotation number for maps of the real line with almost periodic displacement. *Nonlinearity* **13**(5) (2000), 1841–1854.

[LMS02] J.-Y. Lee, R. V. Moody and B. Solomyak. Pure point dynamical and diffraction spectra. *Ann. Henri Poincaré* **3**(5) (2002), 1003–1018.

[LP03] J. C. Lagarias and P. A. B. Pleasants. Repetitive Delone sets and quasicrystals. *Ergod. Th. & Dynam. Sys.* **23**(3) (2003), 831–867.

[Man87] R. Mañé. *Ergodic Theory and Differentiable Dynamics (Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 8)*. Springer, Berlin, 1987 (Translated from the Portuguese by Silvio Levy).

[MZ89] M. Misiurewicz and K. Ziemian. Rotation sets for maps of tori. *J. London Math. Soc.* (2) **40**(3) (1989), 490–506.

[Put89] I. F. Putnam. The C^*-algebras associated with minimal homeomorphisms of the Cantor set. *Pacific J. Math.* **136**(2) (1989), 329–353.

[Rob04] E. Arthur Robinson Jr. Symbolic dynamics and tilings of $\mathbb{R}^d$. *Symbolic Dynamics and its Applications (Proceedings of Symposia in Applied Mathematics, 60)*. American Mathematical Society, Providence, RI, 2004, pp. 81–119.

[RS01] C. Radin and L. Sadun. Isomorphism of hierarchical structures. *Ergod. Th. & Dynam. Sys.* **21**(4) (2001), 1239–1248.

[Rud89] D. J. Rudolph. Markov tilings of $\mathbf{R}^n$ and representations of $\mathbf{R}^n$ actions. *Measure and Measurable Dynamics (Rochester, NY, 1987) (Contemporary Mathematics, 94)*. American Mathematical Society, Providence, RI, 1989, pp. 271–290.

[SBGC84] D. Shecthman, I. Blech, D. Gratias and J. W. Cahn. Metallic phase with long range orientational order and no translational symmetry. *Phys. Rev. Lett.* **53**(20) (1984), 1951–1954.

[Sch00] M. Schlottmann. Generalized model sets and dynamical systems. *Directions in Mathematical Quasicrystals (CRM Monograph Series, 13)*. American Mathematical Society, Providence, RI, 2000, pp. 143–159.

[Shv03] Y. Shvetsov. Rotation of flows on generalized solenoids. *PhD Thesis*, Montana State University, 2003.

[Wan61] H. Wang. Proving theorems by pattern recognition II. *Bell Syst. Tech. J.* **40** (1961), 1–41.