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Asymptotics for eigenvalues of the Laplacian in higher dimensional periodically perforated domains

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Abstract. This paper considers the periodic spectral problem associated with the Laplace operator written in $\mathbb{R}^N$ $(N = 3, 4, 5)$ periodically perforated by balls, and with homogeneous Dirichlet condition on the boundary of holes. We give an asymptotic expansion for all simple eigenvalues as the size of holes goes to zero. As an application of this result, we use Bloch waves to find the classical strange term in homogenization theory, as the size of holes goes to zero faster than the microstructure period.

Mathematics Subject Classification (2000). 35C20 · 35P15 · 47A55 · 35B27.

Keywords. Asymptotic analysis · Eigenvalues · Perturbation theory · Perforated domains · Homogenization · Bloch waves.

1. Introduction

In this paper we deal with the periodic spectral problem associated with the Laplace operator written in $\mathbb{R}^N$ $(N = 3, 4, 5)$ from which we remove a periodic network of spherical holes. It is well-known (see [22]) that each eigenvalue of this periodic spectral problem converges as the size of holes goes to zero, to its corresponding eigenvalue associated with the periodic Laplace operator without holes. The aim of our work is to study the rate of the above convergence under Dirichlet type boundary condition on the perforations. More precisely, we look for an asymptotic expansion with respect to the size of holes for all simple eigenvalues. The mathematical analysis of this expansion strongly depends on the dimension of space.

Concerning asymptotics on eigenelements of different spectral problems associated with the Laplace operator, plenty of references can be found in the literature. We briefly mention a few of them. The asymptotic expansion of simple eigenvalues of the Laplace operator in a bounded domain with an unique perforation is developed in the works of Ozawa, considering several boundary conditions on holes and some particular dimensions. For instance, the Dirichlet boundary condition is considered in [17, 19] for two and three dimensional domains, and in [18] for dimension four. Moreover, Neumann and Robin boundary conditions on holes are studied in [20, 21], respectively. We also mention Maz'ya, Nazarov and Plamenevskij [12], who obtained asymptotic expansions of the first eigenvalue and its corresponding eigenfunction of classical boundary value problems for the Laplace operator in two and three dimensional domains with small holes. In addition, a book by the same authors [13] presents a general and unified approach to the asymptotic analysis of elliptic boundary value problems in singularly perturbed domains. Asymptotics on the first eigenvalue of the periodic Laplace operator in periodically perforated domains are proved in Dupuy, Orive and Smaranda [9] in the setting of two and three dimensions with homogeneous Dirichet boundary condition on holes. As far as we know, there are no asymptotic expansions in dimension $N \geq 5$, nor for $N = 4$ in the periodic case.

The main novelty brought by our paper consists of finding asymptotic expansions in four and five dimensional spaces, for any simple eigenvalue of the periodic Laplace operator written in a periodically perforated domain. For this purpose, we work with iterated Green operators and we use appropriate L^p-norm estimates. The technique we employ here also allows us to recover the asymptotic expansion in the three dimensional domain, but with a better order of the error estimate than that given in [9].

As an interesting application of the above asymptotic analysis, we consider the Bloch wave homogenization of a Dirichlet type problem associated with the Laplace operator in a bounded periodically perforated domain. We assume that the size of holes tends to zero faster than the periodicity of the microstructure. Synthesizing the approach in [9], we use the asymptotic expansion of the first eigenvalue to completely characterize the critical size of the perforations for the situation in which the so-called strange term appears in the homogenized equation. This problem is classical in homogenization theory and it has been widely studied in the literature using different methods (see [1,3,11] and the references therein). The method employed here is a spectral one and is based on Bloch wave decomposition. We give a brief but far from exhaustive list of references in Bloch wave homogenization: Conca and Vanninathan [8], Ganesh and Vanninathan [10] studied the classical problem of elliptic operators in arbitrary domains; the problem of correctors can be found in Conca, Orive and Vanninathan [6,7]. In terms of using the Bloch wave decomposition to solve problems on periodically perforated domains, the reader is referred to papers by Conca, Gómez, Lobo and Pérez [4,5], Ortega, San Martín and Smaranda [15,16].

The paper is organized as follows. In the next section we state the problem and present our main result which is given in Theorem 2.1. Section 3 is devoted to properties of different Green functions and operators associated with our problem which are of particular interest to us. The following section deals with technical issues relevant to our main result, the proof of which appears in Sect. 5. Finally, Sect. 6 focuses on an application of our main result to the Bloch wave homogenization of a Dirichlet problem in periodically perforated domains.

2. Statement of main result

Let us first give some notation. For any positive real number a small enough, we denote by S^a the 2π-periodic network of balls of radius a in $\mathbb{R}^N$ ($N \geq 3$), that is,

$$S^a = \bigcup_{p\in\mathbb{Z}^N} \mathscr{B}(2\pi p, a).$$

Additionally, we consider the reference cell $Y = [-\pi, \pi)^N$ in $\mathbb{R}^N$ and the corresponding perforated reference cell $Y_a = Y \setminus \mathscr{B}(0, a)$ (see Fig. 1). Moreover, for any real number $p \geq 1$, we introduce the following space:

$$L^p_{\#}(Y) = \left\{\varphi \in L^p_{loc}\left(\mathbb{R}^N\right) \mid \varphi \text{ is } Y\text{-periodic}\right\}, \quad \|\varphi\|_{L^p_{\#}(Y)} = \left(\int_Y |\varphi(x)|^p \mathrm{d}x\right)^{1/p}.$$

We are going to use the symbol $O(\cdot)$ representing the classical "big-O" Landau notation, i.e., $f(x) = O(x) \iff |f(x)| \leq Cx$ for any $x > 0$ small enough and we denote $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

Let us now describe our problem. We are interested in studying the following periodic spectral problem written in $\mathbb{R}^N$ perforated by the network S^a: find $\lambda(a) \in \mathbb{R}$ and $\phi(a;\cdot) \not\equiv 0$ such that

$$\begin{cases} -\Delta\phi(a;\cdot) = \lambda(a)\phi(a;\cdot) & \text{in } \mathbb{R}^N \setminus \overline{S^a}, \\ \phi(a;\cdot) = 0 & \text{on } \partial S^a, \\ \phi(a;\cdot) \text{ is } Y\text{-periodic}. \end{cases} \tag{2.1}$$

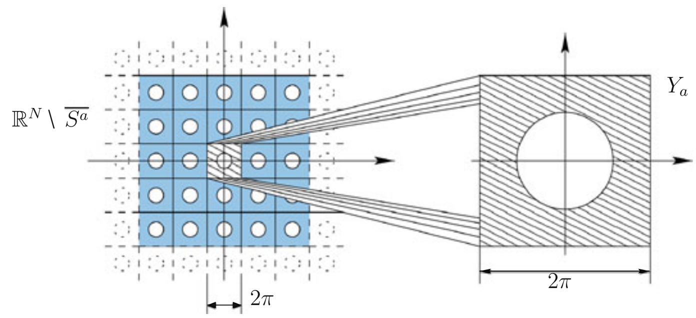

Fig. 1. Perforated domain and the reference cell

It is well-known that the above periodic spectral problem admits a countable sequence of strictly positive eigenvalues, each of them having finite multiplicity. As usual, we arrange them in increasing order repeating each eigenvalue according to its multiplicity:

$$0 < \lambda_1(a) < \lambda_2(a) \leq \cdots \leq \lambda_m(a) \leq \cdots \to \infty.$$

One important property is that, as a goes to zero, each eigenvalue converges to the corresponding eigenvalue of the periodic Laplace operator written in $\mathbb{R}^N$ without holes. More precisely,

$$\lambda_m(a) \to \lambda_m, \qquad \text{as } a \to 0, \tag{2.2}$$

where $\{\lambda_m\}$ are the eigenvalues of the following limit periodic spectral problem: find $\lambda \in \mathbb{R}$ and $\phi(\cdot) \not\equiv 0$ such that

$$\begin{cases} -\Delta\phi = \lambda\phi & \text{in } \mathbb{R}^N, \\ \phi \text{ is } Y\text{-periodic.} \end{cases} \tag{2.3}$$

For a proof of this property, we refer the reader to, for instance, Rauch and Taylor [22], Dupuy, Orive and Smaranda [9]. It is also well-known that the eigenvalues of (2.3) are positive and with finite multiplicity:

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots \to \infty.$$

Let us now state our main result that gives the rate of the convergence (2.2) in three, four and five dimensional domains:

Theorem 2.1. *For all a small enough and for any $m \in \mathbb{N}^*$ such that the mth eigenvalue λ_m of (2.3) is simple, the following asymptotic expansion of the mth eigenvalue of (2.1) holds:*

$$\lambda_m(a) = \lambda_m + (N-2)S_N\phi_m^2(0)\, a^{N-2} + \begin{cases} O\left(a^2\right) & \text{if } N=3, \\ O\left(a^3|\ln a|^{1/2}\right) & \text{if } N=4, \\ O\left(a^{7/2}\right) & \text{if } N=5, \end{cases} \tag{2.4}$$

where ϕ_m is the eigenvector of (2.3) corresponding to λ_m such that $\|\phi_m\|_{L^2_\#(Y)} = 1$ and S_N denotes the area of the unit sphere in $\mathbb{R}^N$.

3. Notation and preliminary results

3.1. Properties on Green functions in the reference cell

Let us consider the Green function $g(\cdot,\cdot)$ associated with the Laplace plus Identity operator in the reference cell Y and with periodicity boundary condition, i.e., for all $y \in Y$, $g(\cdot, y)$ satisfies

$$\begin{cases} (I-\Delta_x)\, g(x,y) = \delta(x-y) & \forall x \in Y, \\ g(\cdot, y) \text{ is } Y\text{-periodic}, \end{cases} \tag{3.1}$$

where δ denotes the Dirac mass. Moreover, for any $n \in \mathbb{N}^*$, we introduce the iterated Green functions $g^{(n)}(\cdot,\cdot)$, defined recursively as follows:

$$\begin{cases} g^{(1)}(x,y) = g(x,y) & \forall x, y \in Y, \\ g^{(n+1)}(x,y) = \displaystyle\int_Y g(x,z) g^{(n)}(z,y)\, \mathrm{d}z & \forall x, y \in Y. \end{cases} \tag{3.2}$$

For any real number $p > 1$, let us consider the linear operator $G \in \mathscr{L}(L^p_{\#}(Y))$ such that for any $f \in L^p_{\#}(Y)$, the function Gf is the unique solution of the following differential problem:

$$\begin{cases} (I-\Delta)\,(Gf) = f & \text{in } Y, \\ Gf \text{ is } Y\text{-periodic}. \end{cases} \tag{3.3}$$

Also, for any $n \in \mathbb{N}^*$, we consider the iterated linear operators $G^n \in \mathscr{L}(L^p_{\#}(Y))$ defined recursively as follows:

$$\begin{cases} G^1 = G, \\ G^{n+1} = G \circ G^n & \forall n \in \mathbb{N}^*. \end{cases} \tag{3.4}$$

Let us recall that for any p large enough, the operator G^n admits the following integral representation formula in terms of the Green function $g^{(n)}(\cdot,\cdot)$ defined in (3.2):

$$(G^n f)(x) = \int_Y g^{(n)}(x,y) f(y)\, \mathrm{d}y \quad \forall x \in Y, \quad \forall f \in L^p_{\#}(Y).$$

We remark that combining (3.3) with (3.4), for any $f \in L^p_{\#}(Y)$, the following identity holds:

$$(I-\Delta)^m (G^n f) = G^{n-m} f \quad \forall n \in \mathbb{N}^*, \quad \forall m \in \{0, \ldots, n-1\}. \tag{3.5}$$

Let us now recall some properties on the Green function $g(\cdot,\cdot)$ defined above. Its asymptotic behavior is well-known (see, for instance, [23]) and is given by:

$$g(x,y) = b(x,y) + \begin{cases} \dfrac{C_1}{|x-y|} + \dfrac{C_3}{|x-y|^3} + \cdots \dfrac{C_{N-2}}{|x-y|^{N-2}} & \text{if } N \text{ odd}, \\ C_0 \ln|x-y| + \dfrac{C_2}{|x-y|^2} + \cdots \dfrac{C_{N-2}}{|x-y|^{N-2}} & \text{if } N \text{ even}, \end{cases} \tag{3.6}$$

where $b(\cdot,\cdot)$ denotes a $\mathscr{C}^\infty$-class function and C_k, $k \in \{0, \ldots, N-2\}$, are real constants independent of x and y. In addition, the constant C_{N-2} is given by $C_{N-2} = \frac{1}{(N-2)S_N}$, for any $N \geq 3$.

As a direct consequence of this asymptotic behavior, the following pointwise estimate on the iterated Green functions holds:

Lemma 3.1. *For any $n \in \mathbb{N}^*$, there exists a positive constant C such that, for all $x, y \in Y$, $x \neq y$, we have*

$$|g^{(n)}(x,y)| \leq \begin{cases} \dfrac{C}{|x-y|^{N-2n}} & \text{if } n < \frac{N}{2}, \\ C(1 + |\ln|x-y||) & \text{if } n = \frac{N}{2}, \\ C & \text{if } n > \frac{N}{2}. \end{cases} \tag{3.7}$$

Proof. We proceed by induction on $n \in \mathbb{N}^*$. For $n = 1$, the estimate is a direct consequence of the asymptotic behavior (3.6).

Let us now assume that (3.7) is true for some $n \in \mathbb{N}^*$. We prove that this estimate remains true for $n+1$. To this end, using the definition (3.2) and the induction hypothesis, we get

$$\left|g^{(n+1)}(x,y)\right| \leq \begin{cases} C\displaystyle\int_Y \frac{1}{|x-z|^{N-2}}\cdot\frac{1}{|z-y|^{N-2n}}\,\mathrm{d}z & \text{if } n<\frac{N}{2},\\ C\displaystyle\int_Y \frac{1}{|x-z|^{N-2}}\cdot(1+|\ln|z-y||)\,\mathrm{d}z & \text{if } n=\frac{N}{2},\\ C\displaystyle\int_Y \frac{\mathrm{d}z}{|x-z|^{N-2}} & \text{if } n>\frac{N}{2}.\end{cases} \tag{3.8}$$

If $n<\frac{N}{2}$, using estimate (A.1) from Appendix with $\alpha+\beta = 2N-2(n+1)$, we obtain

$$\left|g^{(n+1)}(x,y)\right| \leq \begin{cases} \dfrac{C}{|x-y|^{N-2(n+1)}} & \text{if } n+1<\frac{N}{2},\\ C(1+|\ln|x-y||) & \text{if } n+1=\frac{N}{2},\\ C & \text{if } n+1\in(\frac{N}{2},\frac{N}{2}+1).\end{cases}$$

On the other hand, if $n \geq \frac{N}{2}$, the integrals in (3.8) are clearly bounded, then the last line of the above estimate can be extended to the case $n+1 \geq \frac{N}{2}+1$. ◻

Using the above Lemma, we give some properties on the L^p-norm of the iterated Green functions. To this end, we first introduce the following functional space: for any Lebesgue-measurable subset A of Y and for any $p \geq 1$, we denote

$$L^p_\#(A) = \left\{\varphi \in L^p_{loc}\left(\bigcup_{p\in\mathbb{Z}^N}(2\pi p + A)\right) \mid \varphi \text{ is } Y\text{-periodic}\right\},\quad \|\varphi\|_{L^p_\#(A)} = \left(\int_A |\varphi(x)|^p \mathrm{d}x\right)^{1/p}.$$

Proposition 3.2. *For all $y \in Y$, the following estimates hold:*

1. *If $\left(n \geq \frac{N}{2} \text{ and } p \geq 1\right)$ or $\left(n<\frac{N}{2} \text{ and } 1 \leq p < \frac{N}{N-2n}\right)$, there exists a constant $C>0$ such that*
$$\|g^{(n)}(\cdot,y)\|_{L^p_\#(Y)} \leq C. \tag{3.9}$$
2. *If $\left(n<\frac{N}{2} \text{ and } p \geq \frac{N}{N-2n}\right)$, there exists a constant $C>0$ such that, for all $r>0$ small enough,*
$$\|g^{(n)}(\cdot,y)\|_{L^p_\#(Y\setminus\mathscr{B}(y,r))} \leq \begin{cases} C|\ln r|^{\frac{1}{p}} & \text{if } p=\frac{N}{N-2n},\\ Cr^{\frac{N}{p}-(N-2n)} & \text{if } p>\frac{N}{N-2n}.\end{cases} \tag{3.10}$$
3. *There exists a positive constant C such that, for all $r>0$ sufficiently small,*
$$\|g^{(n)}(\cdot,y)\|_{L^p_\#(\mathscr{B}(y,r))} \leq \begin{cases} Cr^{\frac{N}{p}-(N-2n)} & \text{if } n<\frac{N}{2} \text{ and } p\in[1,\frac{N}{N-2n}),\\ Cr^{\frac{N}{p}}|\ln r| & \text{if } n=\frac{N}{2} \text{ and } p\geq 1,\\ Cr^{\frac{N}{p}} & \text{if } n>\frac{N}{2} \text{ and } p\geq 1.\end{cases} \tag{3.11}$$
4. *Moreover, if $n<\frac{N}{2}$, there exists a constant $C>0$ such that, for all $r>0$ sufficiently small and for all $p\geq 1$,*
$$\|g^{(n)}(\cdot,y)\|_{L^p_\#(\mathscr{B}(y,r)\setminus\mathscr{B}(y,\frac{r}{8}))} \leq Cr^{\frac{N}{p}-(N-2n)}. \tag{3.12}$$

Proof. Let A be a Lebesgue-measurable subset of Y. Taking the L^p-norm in the estimate (3.7) we get

$$\|g^{(n)}(\cdot,y)\|_{L^p_\#(A)} \le \begin{cases} C\left\|\dfrac{1}{|\cdot-y|^{N-2n}}\right\|_{L^p_\#(A)} & \text{if } n<\frac{N}{2},\\ C(|A|^{\frac{1}{p}}+\|\ln|\cdot-y|\|_{L^p_\#(A)}) & \text{if } n=\frac{N}{2},\\ C|A|^{\frac{1}{p}} & \text{if } n>\frac{N}{2}, \end{cases} \tag{3.13}$$

where $|A|$ denotes the measure of the set A.

The estimate (3.9) is a direct consequence of the above bound by taking $A=Y$. In fact, for $n<\frac{N}{2}$, the norm $\left\|\frac{1}{|\cdot-y|^{N-2n}}\right\|_{L^p_\#(Y)}$ is bounded for $p\in[1,\frac{N}{N-2n})$. Moreover, for $n=\frac{N}{2}$, the L^p-norm of the logarithm is bounded for any $p\ge 1$.

Let us now prove (3.10). To this end, in the estimate (3.13), we consider $n<\frac{N}{2}$, $p\ge\frac{N}{N-2n}$ and $A=Y\setminus\mathscr{B}(y,r)$. Then, we have

$$\|g^{(n)}(\cdot,y)\|_{L^p_\#(Y\setminus\mathscr{B}(y,r))} \le \begin{cases} C(1+|\ln r|)^{\frac{1}{p}} & \text{if } p=\frac{N}{N-2n},\\ C(1+r^{N-p(N-2n)})^{\frac{1}{p}} & \text{if } p>\frac{N}{N-2n}. \end{cases}$$

Thus, for all r small enough, we obtain that (3.10) is true.

In order to prove (3.11), in the estimate (3.13) we consider $A=\mathscr{B}(y,r)$. We obtain

$$\|g^{(n)}(\cdot,y)\|_{L^p_\#(\mathscr{B}(y,r))} \le \begin{cases} C\left\|\dfrac{1}{|\cdot-y|^{N-2n}}\right\|_{L^p_\#(\mathscr{B}(y,r))} & \text{if } n<\frac{N}{2},\\ C(r^{\frac{N}{p}}+\|\ln|\cdot-y|\|_{L^p_\#(\mathscr{B}(y,r))}) & \text{if } n=\frac{N}{2},\\ Cr^{\frac{N}{p}} & \text{if } n>\frac{N}{2}. \end{cases}$$

Then, easy computations give us the inequality (3.11).

Now, in the estimate (3.13), we consider $A=\mathscr{B}(y,r)\setminus\mathscr{B}(y,\frac{r}{8})$ and $n<\frac{N}{2}$. Then, we get

$$\|g^{(n)}(\cdot,y)\|_{L^p_\#(\mathscr{B}(y,r)\setminus\mathscr{B}(y,\frac{r}{8}))} \le C\left\|\frac{1}{|\cdot-y|^{N-2n}}\right\|_{L^p_\#(\mathscr{B}(y,r)\setminus\mathscr{B}(y,\frac{r}{8}))}.$$

Since the above L^p-norm is computed on $\mathscr{B}(y,r)\setminus\mathscr{B}(y,\frac{r}{8})$, we obtain that for any $p\ge 1$ the estimate (3.12) holds. □

Corollary 3.3. *For all $n\in\mathbb{N}^*$, there exists a positive constant C such that, for all a small enough, we have*

$$\|g^{(n)}(\cdot,0)\|_{L^2_\#(Y\setminus\mathscr{B}(0,\frac{a}{2}))} \le \begin{cases} C & \textit{if } n>\frac{N}{4},\\ C|\ln a|^{\frac{1}{2}} & \textit{if } n=\frac{N}{4},\\ Ca^{2n-\frac{N}{2}} & \textit{if } n<\frac{N}{4}. \end{cases} \tag{3.14}$$

3.2. Green operators in perforated domains

Let us first introduce the Green function $\widetilde{g}_a(\cdot,\cdot)$ corresponding to the perforated reference cell Y_a, with Y-periodicity conditions and Dirichlet boundary condition on $\partial\mathscr{B}(0,a)$, i.e., for all $y\in Y_a$, $\widetilde{g}_a(\cdot,y)$ satisfies

$$\begin{cases} (I-\Delta_x)\,\widetilde{g}_a(x,y)=\delta(x-y) & \forall x\in Y_a,\\ \widetilde{g}_a(x,y)=0 & \forall x\in\partial\mathscr{B}(0,a),\\ \widetilde{g}_a(\cdot,y) \text{ is } Y\text{-periodic}. \end{cases} \tag{3.15}$$

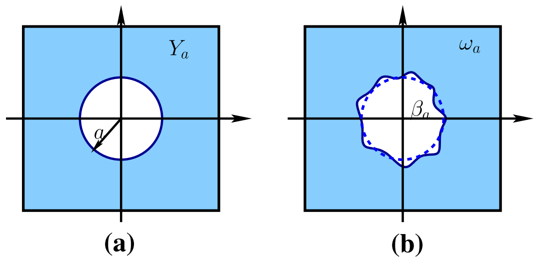

FIG. 2. Change of the perforated reference cell. (**a**) Original reference cell. (**b**) New reference cell

Additionally, for any $p > 1$ we consider the linear operator $\widetilde{G}_a \in \mathscr{L}(L^p_{\#}(Y_a))$ such that for any $f \in L^p_{\#}(Y_a)$, the function $\widetilde{G}_a f$ is the unique solution of the following differential problem:

$$\begin{cases} (I - \Delta)\,(\widetilde{G}_a f) = f & \text{in } Y_a, \\ \widetilde{G}_a f = 0 & \text{on } \partial\mathscr{B}(0,a), \\ \widetilde{G}_a f \ \text{is } Y\text{-periodic.} \end{cases} \tag{3.16}$$

The operator $\widetilde{G}_a$ admits the following integral representation formula in terms of the Green function $\widetilde{g}_a(\cdot,\cdot)$, for any p large enough:

$$(\widetilde{G}_a f)(x) = \int\limits_{Y_a} \widetilde{g}_a(x,y) f(y)\ \mathrm{d}y \quad \forall x \in Y_a,\ \forall f \in L^p_{\#}(Y_a).$$

An important aspect in the proof of our main result is the value of the Green function $g(\cdot,0)$ on the boundary of holes. It is clear that due to the asymptotic behavior (3.6), for any $N \geq 3$, we have $g(x,0) = K(a)\,(1 + O(a))$ for all $x \in \partial\mathscr{B}(0,a)$, where $K(a) = \frac{1}{(N-2)S_N a^{N-2}}$. This identity is not enough in our analysis and for this reason, we introduce a new perforated geometry delimited by the level set of the Green function $g(\cdot,0)$ corresponding to the value $K(a)$ (see Fig. 2). More precisely, we consider the sets ω_a and β_a defined as follows:

$$\begin{aligned} \omega_a &= \{x \in Y \mid g(x,0) < K(a)\}, \\ \beta_a &= Y \setminus \overline{\omega_a}. \end{aligned}$$

In this new reference perforated cell ω_a, let $g_a(x,y)$ be the Green function such that for all $y \in \omega_a$, $g_a(\cdot,y)$ satisfies

$$\begin{cases} (I - \Delta_x)\, g_a(x,y) = \delta(x-y) & \forall x \in \omega_a, \\ g_a(x,y) = 0 & \forall x \in \partial\beta_a, \\ g_a(\cdot,y) \ \text{is } Y\text{-periodic.} \end{cases} \tag{3.17}$$

Moreover, for any $n \in \mathbb{N}^*$, we introduce the corresponding iterated Green functions $g_a^{(n)}(\cdot,\cdot)$, defined recursively as follows:

$$\begin{cases} g_a^{(1)}(x,y) = g_a(x,y) & \forall x,y \in \omega_a, \\ g_a^{(n+1)}(x,y) = \displaystyle\int_{\omega_a} g_a^{(n)}(x,z) g_a(z,y)\ \mathrm{d}z & \forall x,y \in \omega_a. \end{cases} \tag{3.18}$$

In addition, for any real number $p > 1$ we consider the Green operator $G_a \in \mathscr{L}(L^p_\#(\omega_a))$ such that for any $f \in L^p_\#(\omega_a)$, the function $G_a f$ is the unique solution of the following differential problem:

$$\begin{cases} (I - \Delta)(G_a f) = f & \text{in } \omega_a, \\ G_a f = 0 & \text{on } \partial\beta_a, \\ G_a f \text{ is } Y\text{-periodic.} \end{cases} \tag{3.19}$$

Also, for any $n \in \mathbb{N}^*$, we consider the iterated Green operators $G_a^n \in \mathscr{L}(L^p_\#(\omega_a))$ defined recursively as follows:

$$\begin{cases} G_a^1 = G_a, \\ G_a^{n+1} = G_a \circ G_a^n \quad \forall n \in \mathbb{N}^*. \end{cases} \tag{3.20}$$

Let us recall that for any p large enough, the operator G_a^n admits the following integral representation formula in terms of the Green function $g_a^{(n)}(\cdot,\cdot)$ defined in (3.18):

$$(G_a^n f)(x) = \int\limits_{\omega_a} g_a^{(n)}(x,y) f(y)\, \mathrm{d}y \quad \forall x \in \omega_a, \quad \forall f \in L^p_\#(\omega_a).$$

We remark that combining (3.19) with (3.20), for any $f \in L^p_\#(\omega_a)$, the following identity holds:

$$(I - \Delta)^m (G_a^n f) = G_a^{n-m} f \quad \forall n \in \mathbb{N}^*, \quad \forall m \in \{0, \ldots, n-1\}. \tag{3.21}$$

In the sequel, let us denote by $\widetilde{\mu}_m(a)$ and $\mu_m(a)$ the mth eigenvalues of the operators $\widetilde{G}_a$ and G_a, respectively. The aim of this subsection is to establish a relation between these two eigenvalues. To this end, let us first prove the following geometrical result.

Lemma 3.4. *There exists a positive constant C such that, for all a small enough, we have*

$$\omega_{a+Ca^2} \subset Y_a \subset \omega_{a-Ca^2}.$$

Proof. The result is a direct consequence of the definition of the domain ω_a and of the asymptotic behavior around zero of the Green function $g(x,0)$. In fact, using (3.6) we have

$$g(x,0) = K(a) \left(\frac{a}{|x|} \right)^{N-2} (1 + O(|x|)). \tag{3.22}$$

Let α be a positive constant such that $|O(|x|)| \leq \alpha |x|$.

On one hand, if we consider $x \in Y_a$, i.e., $|x| \geq a$, then by using (3.22) we obtain that

$$g(x,0) \leq K(a)(1 + \alpha a) < K(a - \alpha a^2),$$

that is, $x \in \omega_{a-\alpha a^2}$.

Let us prove the inverse inclusion. We consider $x \in \mathscr{B}(0,a)$, then by using again (3.22) we get, for all a small enough,

$$g(x,0) \geq K(a)(1 - \alpha a) = \frac{K(a)}{1 + \frac{\alpha}{1-\alpha a} a}.$$

Since $\frac{\alpha}{1-\alpha a} \to \alpha$, as $a \to 0$, then $\frac{\alpha}{1-\alpha a} < 2\alpha$, for all a small enough. Therefore, we deduce that for all a small enough,

$$g(x,0) \geq \frac{K(a)}{1 + 2\alpha a} \geq K(a + 2\alpha a^2),$$

that is, $x \in \beta_{a+2\alpha a^2}$. □

As a direct consequence of Lemma 3.4 and the min-max principle, we deduce the following relations between the eigenvalues of the operators G_a and $\widetilde{G}_a$:

Proposition 3.5. *There exists a positive constant C such that, for all a small enough, the mth eigenvalues $\widetilde{\mu}_m(a)$ and $\mu_m(a)$ satisfy the inequalities*

$$\mu_m(a + Ca^2) \leq \widetilde{\mu}_m(a) \leq \mu_m(a - Ca^2).$$

4. Some technical results

4.1. Approximation of the Green operator G_a^n

In this subsection we shall approximate the iterated Green operator G_a^n defined in (3.20). To this end, for any $n \in \mathbb{N}^*$, let us define the function $h_a^{(n)}(\cdot,\cdot)$ in $\omega_a \times \omega_a$ by

$$h_a^{(n)}(x,y) = g^{(n)}(x,y) - K(a)^{-1} \sum_{k=1}^{n} g^{(k)}(x,0) g^{(n+1-k)}(y,0) \quad \forall x,y \in \omega_a \tag{4.1}$$

and the linear operator

$$(H_a^n f)(x) = (G^n \widetilde{f})(x) - K(a)^{-1} \sum_{k=1}^{n} g^{(k)}(x,0)(G^{n+1-k}\widetilde{f})(0) \quad \forall f \in L^q_\#(\omega_a),\ \forall q > 1, \tag{4.2}$$

where $\widetilde{f}$ represents the extension by zero of f outside of ω_a. It is clear that for any $q > \frac{N}{2}$, the following integral representation holds

$$(H_a^n f)(x) = \int_{\omega_a} h_a^{(n)}(x,y) f(y)\ \mathrm{d}y \quad \forall x \in \omega_a,\ \forall f \in L^q_\#(\omega_a).$$

We shall study the difference between G_a^n and the above operator H_a^n in the $\mathscr{L}(L^q_\#(\omega_a), L^2_\#(\omega_a))$-norm, for some $q > \frac{N}{2}$. For this purpose, for any $k \in \{1,\ldots,n\}$ let us define the following real numbers:

$$I_p^k(a) = \sup_{x \in \partial\beta_a} \left\| g^{(k)}(x,\cdot) - g^{(k)}(0,\cdot) \right\|_{L^p_\#(\omega_a)} \quad \forall p \in [1, \tfrac{N}{N-2k}), \tag{4.3}$$

and for any $p \geq 1$,

$$J_p^k(a) = \begin{cases} 0 & \text{if } k = 1, \\ K(a)^{-1} \displaystyle\sum_{j=1}^{k-1} \left[\sup_{x \in \partial\beta_a} \left| g^{(j+1)}(x,0) \right| \right] \left\| g^{(k-j)}(0,\cdot) \right\|_{L^p_\#(\omega_a)} & \text{if } k \in \{2,\ldots,n\}. \end{cases} \tag{4.4}$$

Using these definitions, we prove the following result:

Lemma 4.1. *Let us consider $n \in \mathbb{N} \setminus \{0,1\}$ and $p,q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, with $q > \frac{N}{2}$. For all $f \in L^q_\#(\omega_a)$ such that $\|f\|_{L^q_\#(\omega_a)} = 1$, the function $u \overset{\text{def}}{=} (G_a^n - H_a^n)f$ is Y-periodic and satisfies*

$$(I - \Delta)^n u = 0 \quad \text{in } \omega_a, \tag{4.5}$$

$$\sup_{\partial\beta_a} |(I-\Delta)^m u| \leq I_p^{n-m}(a) + J_p^{n-m}(a) \quad \forall m \in \{0,\ldots,n-1\}. \tag{4.6}$$

Proof. Using definition (4.2), we have that for any $x \in \omega_a$,

$$u(x) = (G_a^n f)(x) - (G^n \widetilde{f})(x) + K(a)^{-1} \sum_{k=1}^{n} g^{(k)}(x,0)(G^{n+1-k}\widetilde{f})(0).$$

It is easy to see that u is a Y-periodic function. Moreover, for all $m \in \{0, \dots, n-1\}$, applying the identities (3.21), (3.5) and (3.2), we obtain for any $x \in \omega_a$,

$$(I-\Delta)^m u(x) = (G_a^{n-m} f)(x) - (G^{n-m}\widetilde{f})(x) + K(a)^{-1} \sum_{k=m+1}^{n} g^{(k-m)}(x,0)(G^{n+1-k}\widetilde{f})(0). \tag{4.7}$$

In particular, for $m = n-1$, we have

$$(I-\Delta)^{n-1} u(x) = (G_a f)(x) - (G\widetilde{f})(x) + K(a)^{-1} g(x,0)(G\widetilde{f})(0).$$

Therefore, applying the operator $(I-\Delta)$ and using (3.19), (3.3) and (3.1), we get

$$(I-\Delta)^n u(x) = f(x) - f(x) = 0 \quad \forall x \in \omega_a.$$

Let us now deduce (4.6). To this end, we take an arbitrary $x \in \partial\beta_a$. Since $G_a^{n-m} f = 0$ on $\partial\beta_a$, for all $m \in \{0, \dots, n-1\}$, the relation (4.7) yields

$$(I-\Delta)^m u(x) = -(G^{n-m}\widetilde{f})(x) + K(a)^{-1} \sum_{k=m+1}^{n} g^{(k-m)}(x,0)(G^{n+1-k}\widetilde{f})(0).$$

Moreover, since $q > \frac{N}{2}$, the above identity is written using the integral representations of G^{n-m} and G^{n+1-k} as follows:

$$(I-\Delta)^m u(x) = \int_{\omega_a} \left(-g^{(n-m)}(x,y) + K(a)^{-1} \sum_{k=m+1}^{n} g^{(k-m)}(x,0) g^{(n+1-k)}(y,0) \right) f(y) \, dy.$$

Since $g(x,0) = K(a)$, we get

$$(I-\Delta)^m u(x) = \int_{\omega_a} \left(g^{(n-m)}(y,0) - g^{(n-m)}(x,y) + K(a)^{-1} \sum_{k=m+2}^{n} g^{(k-m)}(x,0) g^{(n+1-k)}(y,0) \right) f(y) \, dy,$$

for all $m \in \{0, \dots, n-2\}$, and

$$(I-\Delta)^{n-1} u(x) = \int_{\omega_a} (g(y,0) - g(x,y))\, f(y) \, dy.$$

Then, using the Hölder inequality and taking into account that $\|f\|_{L^q_\#(\omega_a)} = 1$, we get

$$|(I-\Delta)^m u(x)| \le \left\| g^{(n-m)}(x,\cdot) - g^{(n-m)}(0,\cdot) \right\|_{L^p_\#(\omega_a)}$$
$$+ K(a)^{-1} \sum_{j=1}^{n-m-1} \left| g^{(j+1)}(x,0) \right| \left\| g^{(n-m-j)}(0,\cdot) \right\|_{L^p_\#(\omega_a)},$$

for all $m \in \{0, \dots, n-2\}$, and

$$|(I-\Delta)^{n-1} u(x)| \le \left\| g(x,\cdot) - g(0,\cdot) \right\|_{L^p_\#(\omega_a)}.$$

From these inequalities, we easily get (4.6) by taking the supremum on $\partial\beta_a$. □

Let us now estimate the terms $I_p^k(a)$ and $J_p^k(a)$ defined in (4.3) and (4.4):

Lemma 4.2. *For all $p \in [1, \frac{N}{N-2})$, there exists a constant $C > 0$, independent of a, such that*

$$I_p^1(a) \le \begin{cases} Ca & \text{if } p \in \left[1, \frac{N}{N-1}\right), \\ Ca|\ln a|^{\frac{1}{p}} & \text{if } p = \frac{N}{N-1}, \\ Ca^{\frac{N}{p}-(N-2)} & \text{if } p \in \left(\frac{N}{N-1}, \frac{N}{N-2}\right), \end{cases} \tag{4.8}$$

for a small enough.

Moreover, for all $p \in [1, \frac{N}{N-2})$ and $k \in \{2, \dots, n\}$, there exists a positive constant C, independent of a, such that

$$I_p^k(a) \le Ca, \tag{4.9}$$

$$J_p^k(a) \le \begin{cases} Ca & \text{if } N = 3, \\ Ca^2|\ln a| & \text{if } N = 4, \\ Ca^2 & \text{if } N \ge 5, \end{cases} \tag{4.10}$$

for all a small enough.

Proof. Let us first prove the estimate (4.8). As a direct consequence of the asymptotic behavior (3.6), the following bound holds: for all $x, y \in Y$ such that $x \ne y$ and $y \ne 0$,

$$|g(x, y) - g(0, y)| \le C|x| + C\left|\frac{1}{|x-y|^{N-2}} - \frac{1}{|y|^{N-2}}\right|. \tag{4.11}$$

Due to this estimate, we obtain that

$$I_p^1(a) \le Ca + \sup_{x \in \partial\beta_a}\left(\int_{\omega_a}\left|\frac{1}{|x-y|^{N-2}} - \frac{1}{|y|^{N-2}}\right|^p \,\mathrm{d}y\right)^{\frac{1}{p}}. \tag{4.12}$$

Then, using Lemma A.2 from Appendix, we deduce the estimate (4.8).

In the sequel, we shall prove the estimate (4.9). Using the definition of the iterated Green function given in (3.2), for all $x, y \in Y$, $x \ne y$, $y \ne 0$ and for any $k \ge 2$ we get

$$|g^{(k)}(x, y) - g^{(k)}(0, y)| \le \int_Y |g(x, z) - g(0, z)||g^{(k-1)}(z, y)|\,\mathrm{d}z.$$

Then, due to the estimate (4.11) and Lemma 3.1, we deduce that

$$|g^{(k)}(x, y) - g^{(k)}(0, y)| \le C|x| + \begin{cases} C\displaystyle\int_Y \left|\frac{1}{|x-z|^{N-2}} - \frac{1}{|z|^{N-2}}\right| \frac{1}{|z-y|^{N-2(k-1)}}\,\mathrm{d}z & \text{if } k < \frac{N}{2} + 1, \\ C\displaystyle\int_Y \left|\frac{1}{|x-z|^{N-2}} - \frac{1}{|z|^{N-2}}\right| |\ln|z-y||\,\mathrm{d}z & \text{if } k = \frac{N}{2} + 1, \\ C\displaystyle\int_Y \left|\frac{1}{|x-z|^{N-2}} - \frac{1}{|z|^{N-2}}\right| \mathrm{d}z & \text{if } k > \frac{N}{2} + 1. \end{cases}$$

We now use Lemmas A.2 (with $p = 1$) and A.3 from Appendix and it follows that

$$|g^{(k)}(x, y) - g^{(k)}(0, y)| \le C|x| + \begin{cases} C|x|\displaystyle\int_0^1 \frac{1}{|tx-y|^{N-2k+1}}\mathrm{d}t & \text{if } k < \left[\frac{N}{2}\right] + 1, \\ C|x|\displaystyle\int_0^1 |\ln|tx-y||\mathrm{d}t & \text{if } k = \left[\frac{N}{2}\right] + 1, \\ C|x| & \text{if } k \in \left(\left[\frac{N}{2}\right] + 1, +\infty\right), \end{cases}$$

where $\left[\frac{N}{2}\right]$ denotes the integer part value of $\frac{N}{2}$. Taking the $L^p_\#(\omega_a)$-norm in the previous estimate and using Hölder and Fubini inequalities, we get

$$\|g^{(k)}(x, \cdot) - g^{(k)}(0, \cdot)\|_{L^p_\#(\omega_a)} \le C|x| + \begin{cases} C|x|\left\{\displaystyle\int_0^1\int_{\omega_a} \frac{1}{|tx-y|^{p(N-2k+1)}}\mathrm{d}y\mathrm{d}t\right\}^{\frac{1}{p}} & \text{if } k < \left[\frac{N}{2}\right] + 1, \\ C|x|\left\{\displaystyle\int_0^1\int_{\omega_a} |\ln|tx-y||^p\mathrm{d}y\mathrm{d}t\right\}^{\frac{1}{p}} & \text{if } k = \left[\frac{N}{2}\right] + 1, \\ C|x| & \text{if } k \in \left(\left[\frac{N}{2}\right] + 1, +\infty\right). \end{cases}$$

We observe that the integral of the logarithm is convergent for all p and the integral of the fraction is also convergent because $p < \frac{N}{N-2}$. Therefore,

$$\|g^{(k)}(x,\cdot) - g^{(k)}(0,\cdot)\|_{L^p_{\#}(\omega_a)} \leq C|x| \quad \forall x \in \partial\beta_a,$$

and this estimate finishes the proof of (4.9).

Let us now prove (4.10). To this end, we use Lemma 3.1 and we get that for all j,

$$\sup_{x\in\partial\beta_a} \left|g^{(j+1)}(x,0)\right| \leq \begin{cases} C & \text{if } j > \frac{N}{2} - 1, \\ C|\ln a| & \text{if } j = \frac{N}{2} - 1, \\ Ca^{-N+2(j+1)} & \text{if } j < \frac{N}{2} - 1. \end{cases}$$

On the other hand, since $p < \frac{N}{N-2}$, the estimate (3.9) gives us

$$\|g^{(k-j)}(\cdot,0)\|_{L^p_{\#}(\omega_a)} \leq C \quad \forall j \in \{1,\ldots,k-1\}.$$

Then, combining the above estimates and using the definition (4.4), we deduce (4.10). □

Let us now prove a technical result:

Lemma 4.3. *For all $n \in \mathbb{N}$, if u_a is a Y-periodic function satisfying $(I-\Delta)^{n+1}u_a = 0$ in ω_a, $|u_a| \leq M_a^0$ on $\partial\beta_a$ and $|(I-\Delta)^k u_a| \leq M_a^k$ on $\partial\beta_a$, $\forall k \in \{1,\ldots,n\}$, then*

$$|u_a(x)| \leq K(a)^{-1}\sum_{k=0}^{n} M_a^k g^{(k+1)}(x,0) \quad \forall x \in \omega_a.$$

Proof. We proceed by induction on $n \in \mathbb{N}$. For $n = 0$, it is easy to observe that the function $\Phi_a(x) = K(a)^{-1}M_a^0\, g(x,0)$ satisfies $(I-\Delta)\,\Phi_a = 0$ in ω_a and $\Phi_a = M_a^0$ on $\partial\beta_a$. Then, using the maximum principle we get that Φ_a is a Y-periodic upper bound of u_a.

Let us now suppose that the result is true for some $n \in \mathbb{N}$. We prove that it is still true for $n+1$. To this end, we consider a Y-periodic function w_a that satisfies

$$\begin{aligned} (I-\Delta)^{n+2}w_a &= 0 \text{ in } \omega_a, \\ |w_a| &\leq M_a^0 \text{ on } \partial\beta_a, \\ |(I-\Delta)^k w_a| &\leq M_a^k \text{ on } \partial\beta_a,\ \forall k \in \{1,\ldots,n+1\}. \end{aligned}$$

Is is clear that $v_a = (I-\Delta)w_a$ is a Y-periodic function satisfying

$$\begin{aligned} (I-\Delta)^{n+1}v_a &= 0 \text{ in } \omega_a, \\ |v_a| &\leq M_a^1 \text{ on } \partial\beta_a, \\ |(I-\Delta)^k v_a| &\leq M_a^{k+1} \text{ on } \partial\beta_a,\ \forall k \in \{1,\ldots,n\}. \end{aligned}$$

Since the induction hypothesis holds, we get that

$$|v_a(x)| \leq K(a)^{-1}\sum_{k=1}^{n+1} M_a^k g^{(k)}(x,0) \quad \forall x \in \omega_a,$$

that is,

$$|(I-\Delta)w_a| \leq K(a)^{-1}\sum_{k=1}^{n+1} M_a^k g^{(k)}(\cdot,0) \quad \text{in } \omega_a.$$

Let us now consider the function

$$\varphi_a(x) = K(a)^{-1}M_a^0\, g(x,0) + K(a)^{-1}\sum_{k=1}^{n+1} M_a^k g^{(k+1)}(x,0),$$

which is Y-periodic and satisfies

$$(I-\Delta)\varphi_a = K(a)^{-1}\sum_{k=1}^{n+1} M_a^k g^{(k)}(\cdot,0) \quad \text{in } \omega_a,$$
$$\varphi_a \geq M_a^0 \quad \text{on } \partial\beta_a.$$

Therefore, φ_a is a Y-periodic upper bound of w_a, that is,

$$|w_a(x)| \leq K(a)^{-1}M_a^0\, g(x,0) + K(a)^{-1}\sum_{k=1}^{n+1} M_a^k g^{(k+1)}(x,0).$$

□

As a direct consequence of the above Lemmas, we obtain the following result concerning the approximation of the iterated Green operator G_a^n by H_a^n:

Theorem 4.4. *For all $n \in \mathbb{N}\backslash\{0,1\}$ and $q > N$, there exists a positive constant C such that, for all a small enough, the following estimate holds:*

$$\|G_a^n - H_a^n\|_{\mathscr{L}(L^q_\#(\omega_a),L^2_\#(\omega_a))} \leq \begin{cases} Ca^2 & \text{if } N=3,\\ Ca^3|\ln a|^{\frac{1}{2}} & \text{if } N=4,\\ Ca^{\frac{N}{2}+1} & \text{if } N\geq 5.\end{cases}$$

Proof. Let us consider $q \in (N,\infty)$ and $f \in L^q_\#(\omega_a)$ such that $\|f\|_{L^q_\#(\omega_a)} = 1$.

We observe that due to Lemmas 4.1–4.3, for all $n \geq 2$, the function $(G_a^n - H_a^n)f$ satisfies the following estimate: for all $x \in \omega_a$,

$$\left|((G_a^n - H_a^n)f)(x)\right| \leq \sum_{k=0}^{n-1} Ca^{N-1}g^{(k+1)}(x,0).$$

Now, taking the $L^2_\#(\omega_a)$-norm and using estimate (3.14), we have

$$\|(G_a^n - H_a^n)f\|_{L^2_\#(\omega_a)} \leq Ca^{N-1}\|g(\cdot,0)\|_{L^2_\#(\omega_a)} \leq Ca^{N-1}\begin{cases} C & \text{if } N=3,\\ C|\ln a|^{\frac{1}{2}} & \text{if } N=4,\\ Ca^{2-\frac{N}{2}} & \text{if } N\geq 5.\end{cases}$$

Thus, we get the result. □

Remark 4.5. *Theorem 4.4 shows that, for all $n \geq 2$, the upper bound of $\|G_a^n - H_a^n\|_{\mathscr{L}(L^q_\#(\omega_a),L^2_\#(\omega_a))}$ is independent of the iteration number n. Therefore, for simplicity, we restrict our computations to $n=2$ in the remainder of the paper.*

4.2. Approximation of the operator H_a^2

In this subsection we shall introduce a new operator in $L^2_\#(Y)$ that we denote by $\widetilde{H}_a^2$ and we find an asymptotic expansion of its eigenvalues (see (4.16) and Theorem 4.7 below). Then, we prove that the eigenvalues of $\widetilde{H}_a^2$ are approximated eigenvalues of the operator H_a^2 (see Theorem 4.10 below).

Let us consider the following extension of the function $h_a^{(2)}(\cdot,\cdot)$: for all $x, y \in Y$,

$$\widetilde{h}_a^{(2)}(x,y) = g^{(2)}(x,y) - K(a)^{-1}\left[g^{(2)}(x,0)g(y,0)\psi_a(y) + \psi_a(x)g(x,0)g^{(2)}(y,0)\right], \tag{4.13}$$

where $\psi_a \in \mathscr{C}^\infty(\mathbb{R}^N)$ such that $\psi_a(x) = \begin{cases} 1 & \text{if } |x| \geq \frac{a}{2}\\ 0 & \text{if } |x| \leq \frac{a}{4}\end{cases}$.

Now, for any real number $q > 1$, we define the associated linear operator $\widetilde{H}_a^2 \in \mathscr{L}(L^q_\#(Y))$ by:

$$(\widetilde{H}_a^2 f)(x) = (G^2 f)(x) - K(a)^{-1}\left[g^{(2)}(x,0)G(\psi_a \cdot f)(0) + \psi_a(x)g(x,0)(G^2 f)(0)\right], \tag{4.14}$$

for any $f \in L^q_\#(Y)$. For any $q > \frac{N}{4}$, its integral representation is the following

$$(\widetilde{H}_a^2 f)(x) = \int_Y \widetilde{h}_a^{(2)}(x,y)f(y)\,\mathrm{d}y \quad \forall x \in Y,\ \forall f \in L^q_\#(Y).$$

In the sequel, we denote by $\mu_m = \frac{1}{1+\lambda_m}$ the mth eigenvalue of the Green operator G and we recall that its corresponding normalized eigenvector is denoted by ϕ_m. If μ_m is a simple eigenvalue, then the problem of finding a vector $\widetilde{\xi}_a$ orthogonal to ϕ_m such that

$$(G^2 - \mu_m^2)\widetilde{\xi}_a(x) = -2\mu_m^2\phi_m(0)\phi_m(x)G(\psi_a\cdot\phi_m)(0) + g^{(2)}(x,0)G(\psi_a\cdot\phi_m)(0) + \mu_m^2 g(x,0)\psi_a(x)\phi_m(0) \tag{4.15}$$

admits an unique solution in $L^2_\#(Y)$ because the right-hand side is orthogonal to ϕ_m.

We use this function $\widetilde{\xi}_a$ in order to construct an approximate eigenvalue of the operator $\widetilde{H}_a^2$. To this end, we consider the following quantities:

$$\nu(a) \stackrel{\text{def}}{=} \mu_m^2 - 2K(a)^{-1}\mu_m^2 G(\psi_a\cdot\phi_m)(0)\phi_m(0), \tag{4.16}$$

$$\widetilde{\phi}^*(a;x) \stackrel{\text{def}}{=} \phi_m(x) + K(a)^{-1}\widetilde{\xi}_a(x) \quad \forall x \in Y. \tag{4.17}$$

Let us also recall the following classical result of spectral perturbation theory (for a proof we refer the reader to Section III.1 from [14]):

Proposition 4.6. *Let X be a separable Hilbert real space with norm $\|\cdot\|$ and let us consider A a compact self-adjoint operator in X. Assume that $\lambda \in \mathbb{R}\setminus\{0\}$ and $v \in X$ satisfy the properties $\|(A-\lambda)v\| \le \delta$ (with $\delta > 0$) and $\|v\| \ge 1$, then there exists at least one eigenvalue $\lambda^\star$ of the operator A that satisfies $|\lambda^\star - \lambda| \le \delta$.*

We now prove that the quantity $\nu(a)$ is an approximate eigenvalue of the operator $\widetilde{H}_a^2$:

Theorem 4.7. *For all a small enough, there exists an eigenvalue of the operator $\widetilde{H}_a^2$, denoted by $\widetilde{\nu}(a)$, such that*

$$\left|\widetilde{\nu}(a) - \nu(a)\right| \le \begin{cases} Ca^2 & \text{if } N = 3,\\ Ca^4|\ln a|^{\frac{1}{2}} & \text{if } N = 4,\\ Ca^{\frac{3}{2}N-2} & \text{if } N \in \{5,6,7\},\end{cases} \tag{4.18}$$

where C is a positive constant independent of a.

Proof. First of all, using the definition (4.14) in the particular cases $f = \phi_m$ and $f = \widetilde{\xi}_a$, we have

$$(\widetilde{H}_a^2\phi_m)(x) = \mu_m^2\phi_m(x) - K(a)^{-1}\left[g^{(2)}(x,0)G(\psi_a\cdot\phi_m)(0) + \mu_m^2\psi_a(x)g(x,0)\phi_m(0)\right],$$

$$(\widetilde{H}_a^2\widetilde{\xi}_a)(x) = \mu_m^2\widetilde{\xi}_a(x) + (G^2-\mu_m^2)\widetilde{\xi}_a(x) - K(a)^{-1}\left[g^{(2)}(x,0)G(\psi_a\cdot\widetilde{\xi}_a)(0) + \psi_a(x)g(x,0)(G^2\widetilde{\xi}_a)(0)\right].$$

Then, combining these two identities with the definition (4.17), we get

$$\widetilde{H}_a^2\widetilde{\phi}^*(a;x) = \mu_m^2\widetilde{\phi}^*(a;x) - K(a)^{-1}\left[g^{(2)}(x,0)G(\psi_a\cdot\phi_m)(0) + \mu_m^2\psi_a(x)g(x,0)\phi_m(0)\right] + K(a)^{-1}(G^2-\mu_m^2)\widetilde{\xi}_a(x) - K(a)^{-2}\left[g^{(2)}(x,0)G(\psi_a\cdot\widetilde{\xi}_a)(0) + \psi_a(x)g(x,0)(G^2\widetilde{\xi}_a)(0)\right].$$

Using (4.15)–(4.17), we deduce the following identity:

$$(\widetilde{H}_a^2 - \nu(a))\widetilde{\phi}^*(a;x) = -K(a)^{-2}\Big[g^{(2)}(x,0)G(\psi_a \cdot \widetilde{\xi}_a)(0) + \psi_a(x)g(x,0)(G^2\widetilde{\xi}_a)(0)$$
$$-2\mu_m^2\phi_m(0)G(\psi_a \cdot \phi_m)(0)\widetilde{\xi}_a(x)\Big].$$

Let us now take the $L^2_\#(Y)$-norm in the above expression. Since the estimates (3.9)–(3.10) hold, we get that there exists a positive constant C, independent of a, such that, for all a small enough,

$$\| \left(\widetilde{H}_a^2 - \nu(a)\right) \widetilde{\phi}^*(a;\cdot)\|_{L^2_\#(Y)} \le \begin{cases} Ca^2 & \text{if } N = 3, \\ Ca^4|\ln a|^{\frac{1}{2}} & \text{if } N = 4, \\ a^{\frac{3}{2}N-2} & \text{if } N \in \{5,6,7\}. \end{cases}$$

Thus, applying the classical result on spectral perturbation theory given in Proposition 4.6, we conclude the result. □

In the remainder of this subsection, we shall compare the operators $\widetilde{H}_a^2$ and H_a^2. To this end, we first prove the following technical result:

Lemma 4.8.

(i) *If* $\big(N \in \{3,4\}$ *and* $p \ge 1\big)$ *or* $\Big(N \in \{5,6,7\}$ *and* $1 \le p < \frac{N}{N-4}\Big)$, *there exists a positive constant* C *such that, for all* a *small enough,*

$$\|\widetilde{h}_a^{(2)}(\cdot,\cdot)\|_{L^p_\#(\beta_a;L^2_\#(Y))} \le Ca^{\frac{N}{p}}. \tag{4.19}$$

(ii) *Moreover, if* $N \in \{3,\dots,7\}$ *and* $p \ge 1$, *there exists a positive constant* C *such that, for all* a *small enough,*

$$\|\widetilde{h}_a^{(2)}(\cdot,\cdot)\|_{L^p_\#(\omega_a;L^2_\#(Y))} \le C. \tag{4.20}$$

Proof. For all $x \in Y$, let us first compute the $L^2_\#(Y)$-norm of the function $\widetilde{h}_a^{(2)}(x,\cdot)$. To this end, we use the definition (4.13) and we obtain

$$\|\widetilde{h}_a^{(2)}(x,\cdot)\|_{L^2_\#(Y)} \le \|g^{(2)}(x,\cdot)\|_{L^2_\#(Y)} + K(a)^{-1}\Big[|g^{(2)}(x,0)|\,\|g(\cdot,0)\psi_a(\cdot)\|_{L^2_\#(Y)}$$
$$+ |\psi_a(x)g(x,0)|\,\|g^{(2)}(0,\cdot)\|_{L^2_\#(Y)}\Big].$$

We now estimate the $L^2_\#(Y)$-norm of $g^{(2)}(x,\cdot)$, respectively $g(\cdot,0)\psi_a(\cdot)$ by using (3.9) (with $n = 1,2$ and $p = 2$) and (3.10) (with $n = 1$ and $p = 2$). For any $N \in \{3,\dots,7\}$, we get

$$\|\widetilde{h}_a^{(2)}(x,\cdot)\|_{L^2_\#(Y)} \le C + Ca^{N-2}|\psi_a(x)g(x,0)|$$
$$+Ca^{N-2}|g^{(2)}(x,0)| \begin{cases} 1 & \text{if } N = 3, \\ |\ln a|^{1/2} & \text{if } N = 4, \\ a^{2-\frac{N}{2}} & \text{if } N \in \{5,6,7\}. \end{cases} \tag{4.21}$$

Taking the $L^p_\#(\beta_a)$-norm in the above estimate and using (3.12) (with $n = 1$) and (3.11) (with $n = 2$), we deduce

$$\|\widetilde{h}_a^{(2)}(\cdot,\cdot)\|_{L^p_\#(\beta_a;L^2(Y))} \le Ca^{\frac{N}{p}} + Ca^{\frac{N}{p}} \begin{cases} a & \text{if } N = 3 \text{ and } p \ge 1, \\ a^2|\ln a|^{3/2} & \text{if } N = 4 \text{ and } p \ge 1, \\ a^{4-\frac{N}{2}} & \text{if } N \in \{5,6,7\} \text{ and } p \in [1, \frac{N}{N-4}), \end{cases}$$

and thus we conclude (4.19).

If we now take the $L^p_\#(\omega_a)$-norm in the estimate (4.21) and we use (3.9)–(3.10), it follows that

$$\|\widetilde{h}^{(2)}_a(\cdot,\cdot)\|_{L^p_\#(\omega_a;L^2_\#(Y))} \le C + \begin{cases} Ca^{N-2} & \text{if } p < \frac{N}{N-2} \\ Ca^{N-2}|\ln a|^{\frac{1}{p}} & \text{if } p = \frac{N}{N-2} \\ Ca^{\frac{N}{p}} & \text{if } p > \frac{N}{N-2} \end{cases}$$
$$+ \begin{cases} Ca^{N-2} & \text{if } N = 3, \\ Ca^{N-2}|\ln a|^{1/2} & \text{if } N = 4, \\ Ca^{\frac{N}{2}} & \text{if } N \in \{5,6,7\} \text{ and } p < \frac{N}{N-4}, \\ Ca^{\frac{N}{2}}|\ln a|^{\frac{1}{p}} & \text{if } N \in \{5,6,7\} \text{ and } p = \frac{N}{N-4}, \\ Ca^{\frac{N}{p}+4-\frac{N}{2}} & \text{if } N \in \{5,6,7\} \text{ and } p > \frac{N}{N-4}. \end{cases}$$

The estimate (4.20) is a direct consequence of the above inequality. □

Let us now denote by $\widetilde{\phi}(a;\cdot)$ the normalized eigenvector in $L^2_\#(Y)$ of $\widetilde{H}^2_a$, corresponding to the eigenvalue $\widetilde{\nu}(a)$ defined in Theorem 4.7. Then, we split this eigenvector over ω_a and β_a as follows:

$$\widetilde{\phi}_{\omega_a}(a;\cdot) \overset{\text{def}}{=} \widetilde{\phi}(a;\cdot)\cdot\chi_{_{\omega_a}} \quad \text{and} \quad \widetilde{\phi}_{\beta_a}(a;\cdot) \overset{\text{def}}{=} \widetilde{\phi}(a;\cdot)\cdot\chi_{_{\beta_a}},$$

where $\chi_{_{\omega_a}}$ and $\chi_{_{\beta_a}}$ denote the characteristic functions on ω_a, respectively β_a.

Using the above notations, the identity $\widetilde{\nu}(a)\widetilde{\phi}(a;\cdot) = \widetilde{H}^2_a\widetilde{\phi}(a;\cdot)$ can be written as follows:

$$\widetilde{\nu}(a)\widetilde{\phi}_{\omega_a}(a;x) = \int_Y \widetilde{h}^{(2)}_a(x,y)\widetilde{\phi}(a;y)\mathrm{d}y \quad \forall x \in \omega_a, \tag{4.22}$$

$$\widetilde{\nu}(a)\widetilde{\phi}_{\beta_a}(a;x) = \int_Y \widetilde{h}^{(2)}_a(x,y)\widetilde{\phi}(a;y)\mathrm{d}y \quad \forall x \in \beta_a. \tag{4.23}$$

We prove an useful estimate on the vector $\widetilde{\phi}_{\omega_a}(a;\cdot)$ in $L^q_\#(\omega_a)$-norm:

Proposition 4.9. *For all $N \in \{3,\dots,7\}$ and $q \ge 1$, there exists a positive constant C such that, for all a small enough,*

$$\|\widetilde{\phi}_{\omega_a}(a;\cdot)\|_{L^q_\#(\omega_a)} \le C.$$

Proof. Using the identity (4.22) and the fact that $\|\widetilde{\phi}(a;\cdot)\|_{L^2_\#(Y)} = 1$, we obtain

$$\widetilde{\nu}(a)\|\widetilde{\phi}_{\omega_a}(a;\cdot)\|_{L^q_\#(\omega_a)} \le \|\widetilde{h}^{(2)}_a(\cdot,\cdot)\|_{L^q_\#(\omega_a;L^2_\#(Y))}\|\widetilde{\phi}(a;\cdot)\|_{L^2_\#(Y)} = \|\widetilde{h}^{(2)}_a(\cdot,\cdot)\|_{L^q_\#(\omega_a;L^2_\#(Y))}.$$

Since the estimate (4.20) from Lemma 4.8 is true, then for any $N \in \{3,\dots,7\}$ and $q \ge 1$,

$$\widetilde{\nu}(a)\|\widetilde{\phi}_{\omega_a}(a;\cdot)\|_{L^q_\#(\omega_a)} \le C.$$

Thus, the result is a direct consequence of the above estimate and the fact that $\widetilde{\nu}(a) \to 1$, as $a \to 0$. □

Let us now prove the following theorem concerning the comparison between the operators $\widetilde{H}^2_a$ and H^2_a:

Theorem 4.10. *For all $N \in \{3,\dots,7\}$, there exists a positive constant C such that, for all a small enough,*

$$\|(H^2_a - \widetilde{\nu}(a))\widetilde{\phi}_{\omega_a}(a;\cdot)\|_{L^2_\#(\omega_a)} \le Ca^N \tag{4.24}$$

and

$$\|\widetilde{\phi}_{\omega_a}(a;\cdot)\|_{L^2_\#(\omega_a)} \ge \frac{1}{2}. \tag{4.25}$$

Proof. Let us observe that

$$H_a^2\widetilde{\phi}_{\omega_a}(a;x) = \int\limits_{\omega_a} \widetilde{h}_a^{(2)}(x,y)\widetilde{\phi}(a;y)\mathrm{d}y,$$

because $\widetilde{h}_a^{(2)}(x,y) = h_a^{(2)}(x,y)$ and $\widetilde{\phi}(a;y) = \widetilde{\phi}_{\omega_a}(a;y)$ for all $x, y \in \omega_a$. We now use the identity (4.22), the fact that $\widetilde{\phi}(a;y) = \widetilde{\phi}_{\beta_a}(a;y)$ for all $y \in \beta_a$ and we get

$$(H_a^2 - \widetilde{\nu}(a))\widetilde{\phi}_{\omega_a}(a;x) = -\int\limits_{\beta_a} \widetilde{h}_a^{(2)}(x,y)\widetilde{\phi}_{\beta_a}(a;y)\mathrm{d}y \quad \forall x \in \omega_a.$$

Then, due to the symmetry of the function $\widetilde{h}_a^{(2)}(\cdot,\cdot)$, we deduce that

$$\big\|(H_a^2 - \widetilde{\nu}(a))\widetilde{\phi}_{\omega_a}(a;\cdot)\big\|_{L^2_\#(\omega_a)} \leq \|\widetilde{h}_a^{(2)}(\cdot,\cdot)\|_{L^2_\#(\beta_a;L^2_\#(\omega_a))}\|\widetilde{\phi}_{\beta_a}(a;\cdot)\|_{L^2_\#(\beta_a)}.$$

Since the estimate (4.19) from Lemma 4.8 holds, then

$$\big\|(H_a^2 - \widetilde{\nu}(a))\widetilde{\phi}_{\omega_a}(a;\cdot)\big\|_{L^2_\#(\omega_a)} \leq Ca^{\frac{N}{2}}\|\widetilde{\phi}_{\beta_a}(a;\cdot)\|_{L^2_\#(\beta_a)}, \tag{4.26}$$

for all $N \in \{3, \ldots, 7\}$ and a small enough.

On the other hand, using the identity (4.23) we get

$$\widetilde{\nu}(a)\big\|\widetilde{\phi}_{\beta_a}(a;\cdot)\big\|_{L^2_\#(\beta_a)} \leq \|\widetilde{h}_a^{(2)}(\cdot,\cdot)\|_{L^2_\#(\beta_a;L^2_\#(Y))}\|\widetilde{\phi}(a;\cdot)\|_{L^2_\#(Y)} = \|\widetilde{h}_a^{(2)}(\cdot,\cdot)\|_{L^2_\#(\beta_a;L^2_\#(Y))}.$$

Since the estimate (4.19) holds and $\widetilde{\nu}(a) \to 1$, as $a \to 0$, then

$$\big\|\widetilde{\phi}_{\beta_a}(a;\cdot)\big\|_{L^2_\#(\beta_a)} \leq Ca^{\frac{N}{2}}, \tag{4.27}$$

for all $N \in \{3, \ldots, 7\}$ and a sufficiently small.

Combining the estimates (4.26) and (4.27), we get (4.24).

Finally, the inequality (4.25) is a direct consequence of the estimate (4.27) and the fact that $\|\widetilde{\phi}_{\omega_a}(a;\cdot)\|^2_{L^2_\#(\omega_a)} + \|\widetilde{\phi}_{\beta_a}(a;\cdot)\|^2_{L^2_\#(\beta_a)} = 1.$ □

5. Proof of Theorem 2.1

In this section we shall prove the asymptotic expansion (2.4). Let us first observe that to prove (2.4) is equivalent to prove that

$$\Big|\widetilde{\mu}_m(a) - \big(\mu_m - \mu_m^2(N-2)S_N\phi_m^2(0)a^{N-2}\big)\Big| \leq \begin{cases} Ca^2 & \text{if } N=3,\\ Ca^3|\ln a|^{\frac{1}{2}} & \text{if } N=4,\\ Ca^{7/2} & \text{if } N=5,\end{cases}$$

where $\widetilde{\mu}_m(a)$, μ_m are the mth eigenvalues of the Green operators $\widetilde{G}_a$, respectively G defined in Sects. 3.2 and 3.1. Moreover, using Proposition 3.5, the above inequality is equivalent to

$$\Big|\mu_m(a) - \big(\mu_m - \mu_m^2(N-2)S_N\phi_m^2(0)a^{N-2}\big)\Big| \leq \begin{cases} Ca^2 & \text{if } N=3,\\ Ca^3|\ln a|^{\frac{1}{2}} & \text{if } N=4,\\ Ca^{7/2} & \text{if } N=5,\end{cases} \tag{5.1}$$

where $\mu_m(a)$ is the mth eigenvalue of the Green operator G_a defined in Sect. 3.2.

With these remarks it is enough to prove (5.1). To this end, let us first estimate the following norm:

$$\begin{aligned}\big\|(G_a^2 - \widetilde{\nu}(a))\widetilde{\phi}_{\omega_a}(a;\cdot)\big\|_{L^2_\#(\omega_a)} &\leq \|G_a^2 - H_a^2\|_{\mathscr{L}(L^q_\#(\omega_a),L^2_\#(\omega_a))}\|\widetilde{\phi}_{\omega_a}(a;\cdot)\|_{L^q_\#(\omega_a)}\\ &\quad + \|(H_a^2 - \widetilde{\nu}(a))\widetilde{\phi}_{\omega_a}(a;\cdot)\|_{L^2_\#(\omega_a)}.\end{aligned}$$

We use Theorems 4.4, 4.10, Proposition 4.9 and we deduce that, for all $q > N$ and $N \in \{3, \ldots, 7\}$,

$$\left\|(G_a^2 - \widetilde{\nu}(a))\widetilde{\phi}_{\omega_a}(a;\cdot)\right\|_{L^2_\#(\omega_a)} \leq \begin{cases} Ca^2 & \text{if } N = 3, \\ Ca^3|\ln a|^{\frac{1}{2}} & \text{if } N = 4, \\ Ca^{\frac{N}{2}+1} & \text{if } N \in \{5,6,7\}. \end{cases}$$

Then, since (4.25) holds, we use the classical result on spectral perturbation theory given in Proposition 4.6 and we obtain the existence of at least one eigenvalue of the operator G_a^2, denoted by $\mu(a)^2$, such that

$$|\mu(a)^2 - \widetilde{\nu}(a)| \leq \begin{cases} Ca^2 & \text{if } N = 3, \\ Ca^3|\ln a|^{\frac{1}{2}} & \text{if } N = 4, \\ Ca^{\frac{N}{2}+1} & \text{if } N \in \{5,6,7\}. \end{cases}$$

Combining the above estimate with (4.18) from Theorem 4.7, we get

$$|\mu(a)^2 - \nu(a)| \leq \begin{cases} Ca^2 & \text{if } N = 3, \\ Ca^3|\ln a|^{\frac{1}{2}} & \text{if } N = 4, \\ Ca^{\frac{N}{2}+1} & \text{if } N \in \{5,6,7\}. \end{cases} \tag{5.2}$$

Using the definition of $\nu(a)$ given in (4.16) and the fact that

$$|G(\psi_a \cdot \phi_m)(0) - \mu_m \phi_m(0)| \leq \|g(0,\cdot)\|_{L^1_\#(\mathscr{B}(0,a/2))} \|\phi_m(\cdot)\|_{L^\infty_\#(\mathscr{B}(0,a/2))} \leq Ca^2,$$

the estimate (5.2) becomes

$$\left|\mu(a)^2 - \mu_m^2\left(1 - 2(N-2)S_N\mu_m\phi_m^2(0)a^{N-2}\right)\right| \leq \begin{cases} Ca^2 & \text{if } N = 3, \\ Ca^3|\ln a|^{\frac{1}{2}} & \text{if } N = 4, \\ Ca^{\frac{N}{2}+1} & \text{if } N \in \{5,6,7\}, \end{cases}$$

that is,

$$\left|\mu(a) - \mu_m\left(1 - (N-2)S_N\mu_m\phi_m^2(0)a^{N-2}\right)\right| \leq \begin{cases} Ca^2 & \text{if } N = 3, \\ Ca^3|\ln a|^{\frac{1}{2}} & \text{if } N = 4, \\ Ca^{\frac{N}{2}+1} & \text{if } N \in \{5,6,7\}. \end{cases} \tag{5.3}$$

Passing to the limit in the previous expression, as a goes to zero, we get that $\lim_{a\to 0} \mu(a) = \mu_m$. Since μ_m is a simple eigenvalue of the operator G, we deduce that, for any a small enough, $\mu(a)$ is the mth eigenvalue of G_a, that is $\mu(a) = \mu_m(a)$. This identity implies that the estimate (5.1) is true, and therefore (2.4) is also true.

6. Application to homogenization

In this section we use our main result stated in Theorem 2.1 in order to prove a homogenization result. This result is based on Bloch waves associated with the Laplace operator which we define now. Let us consider a family of spectral problems parameterized by $\eta \in \mathbb{R}^N$: find $\lambda(a;\eta) \in \mathbb{R}$ and $\psi(a;\cdot;\eta)$ (not identically zero) such that

$$\begin{cases} -\Delta\psi(a;\cdot;\eta) = \lambda(a;\eta)\psi(a;\cdot;\eta) & \text{in } \mathbb{R}^N \setminus \overline{S^a}, \\ \psi(a;\cdot;\eta) = 0 & \text{on } \partial S^a, \\ \psi(a;\cdot;\eta) \text{ is } (\eta;Y)\text{-periodic}, \end{cases} \tag{6.1}$$

where the condition $\psi(a;\cdot;\eta)$ is $(\eta;Y)$-periodic means

$$\psi(a; y + 2\pi m; \eta) = e^{2\pi i m\cdot\eta}\psi(a;y;\eta) \quad \forall m \in \mathbb{Z}^N,\ \forall y \in \mathbb{R}^N.$$

Next, by Floquet theory, we define $\phi(a;y;\eta) = e^{-i\eta\cdot y}\psi(a;y;\eta)$ and the problem (6.1) can be rewritten in terms of ϕ as follows: find $\lambda(a;\eta) \in \mathbb{R}$ and $\phi(a;\cdot;\eta)$ (not identically zero) such that

$$\begin{cases} A(\eta)\phi(a;\cdot;\eta) = \lambda(a;\eta)\phi(a;\cdot;\eta) & \text{in } \mathbb{R}^N \setminus \overline{S^a}, \\ \phi(a;\cdot;\eta) = 0 & \text{on } \partial S^a, \\ \phi(a;\cdot;\eta) \text{ is } Y\text{-periodic}, \end{cases} \tag{6.2}$$

where the operator $A(\eta)$ is the so-called shifted operator, and is defined by

$$A(\eta) = -(\Delta + 2i\eta\cdot\nabla - |\eta|^2).$$

It is well-known that for each fixed $\eta \in Y' \stackrel{\text{def}}{=} \left[-\frac{1}{2}, \frac{1}{2}\right[^N$ (Y' is called the dual cell or Brilouin zone), the above spectral problem (6.2) admits a countable sequence of positive eigenvalues $\lambda_m(a;\eta)$. Their associated eigenvectors $\{\phi_m(a;\cdot;\eta)\}_{m\geq 1}$ (referred to as Bloch waves) enable us to describe the spectral resolution of the Laplace operator in the orthogonal basis $\{e^{i\eta\cdot y}\phi_m(a;y;\eta) \mid m \geq 1, \eta \in Y'\}$.

For any positive real number ε, let us consider a periodic network of balls of radius $r(\varepsilon)$ and centered in $2\pi\varepsilon\mathbb{Z}^N$:

$$T^\varepsilon = \bigcup_{p\in\mathbb{Z}^N} \mathscr{B}(2\pi p\varepsilon, r(\varepsilon)),$$

where $r : (0, +\infty) \longrightarrow (0, +\infty)$ is a continuous map satisfying the condition $r(\varepsilon) < \pi\varepsilon$.

We now introduce Bloch waves at the ε-scale:

$$\lambda_m^\varepsilon(r(\varepsilon);\xi) = \varepsilon^{-2}\lambda_m\left(\frac{r(\varepsilon)}{\varepsilon};\eta\right), \quad \phi_m^\varepsilon(r(\varepsilon);x;\xi) = \phi_m\left(\frac{r(\varepsilon)}{\varepsilon};y;\eta\right), \quad \psi_m^\varepsilon(r(\varepsilon);x;\xi) = \psi_m\left(\frac{r(\varepsilon)}{\varepsilon};y;\eta\right),$$

where the variables $(x;\xi)$ and $(y;\eta)$ are related by $y = \frac{x}{\varepsilon}$ and $\eta = \varepsilon\xi$. We note that $\phi_m^\varepsilon(r(\varepsilon);\cdot;\xi)$ is εY-periodic and $\psi_m^\varepsilon(r(\varepsilon);\cdot;\xi)$ is $(\varepsilon\xi;\varepsilon Y)$-periodic.

With these notations, we define the classical Bloch transform at the ε-scale of an arbitrary function $h \in L^2(\mathbb{R}^N\backslash\overline{T^\varepsilon})$ as follows (for more details, see for instance [2, p. 614]):

$$\left(B_m^\varepsilon h\right)(\xi) = \int_{\mathbb{R}^N\backslash\overline{T^\varepsilon}} h(x)e^{-i\xi\cdot x}\overline{\phi_m^\varepsilon}\,(r(\varepsilon);x;\xi)\,\mathrm{d}x \quad \forall m \in \mathbb{N}^*,\ \forall\xi \in \varepsilon^{-1}Y'. \tag{6.3}$$

Let us now state the homogenization problem that we are interested in. We consider $\Omega \subset \mathbb{R}^N$ (with $N \in \{3, 4, 5\}$) an open bounded set with a smooth enough boundary. We perforate this set by the network of balls T^ε obtaining the periodically perforated domain $\Omega^\varepsilon = \Omega \setminus \overline{T^\varepsilon}$.

For a given $f \in L^2(\Omega)$, we study the asymptotic behavior, as ε goes to zero, of the solution u^ε of the following homogeneous Dirichlet boundary-value problem:

$$\begin{cases} -\Delta u^\varepsilon = f & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases} \tag{6.4}$$

when the radius of balls tends to zero faster than the microstructure size, i.e.,

$$\lim_{\varepsilon\to 0} \frac{r(\varepsilon)}{\varepsilon} = 0. \tag{6.5}$$

Using the asymptotic expansion given in Theorem 2.1 and Bloch waves we prove the following homogenization result:

Theorem 6.1. *The extension by zero inside of balls of the solution u^ε of problem (6.4), denoted by $\widetilde{u^\varepsilon}$, weakly converges to u in $H_0^1(\Omega)$. Depending on the radius of balls $r(\varepsilon)$, we have the following characterizations of the limit u:*

(i) *If* $\lim_{\varepsilon\to 0} \varepsilon^{-N} r(\varepsilon)^{N-2} = \ell \geq 0$, *then* u *is the unique solution of the following homogenized problem:*

$$\begin{cases} \left((N-2)S_N\phi_1^2(0)\,\ell\right)\, u - \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.6}$$

where ϕ_1 *is the first normalized eigenvector of problem* (2.3) *in* $L^2_\#(Y)$.

(ii) *If* $\lim_{\varepsilon\to 0} \varepsilon^{-N} r(\varepsilon)^{N-2} = +\infty$, *then* u *is identically equal to zero.*

Proof. It is easy to prove that, up to a subsequence, $\widetilde{u^\varepsilon} \rightharpoonup u$ weakly in $H^1_0(\Omega)$, as $\varepsilon \to 0$.

In the sequel, our goal is to use the spectral method of Bloch waves in order to identify the limit u as solution of a partial differential equation, called the homogenized equation.

To this end, let us consider an arbitrary function $\varphi \in \mathscr{D}(\mathbb{R}^N)$, with supp $\varphi = K \subset \Omega$. Since u^ε satisfies (6.4), then $\varphi\widetilde{u^\varepsilon}$ is solution of the following problem:

$$\begin{cases} -\Delta(\varphi\widetilde{u^\varepsilon}) = F^\varepsilon & \text{in } \mathbb{R}^N \setminus \overline{T^\varepsilon}, \\ \varphi\widetilde{u^\varepsilon} = 0 & \text{on } \partial T^\varepsilon, \end{cases}$$

where the right-hand side is defined by $F^\varepsilon = f\varphi - 2\nabla\varphi\cdot\nabla\widetilde{u^\varepsilon} - \Delta\varphi\,\widetilde{u^\varepsilon}$.

We now apply the Bloch transform defined in (6.3) to the above problem and we obtain

$$\lambda^\varepsilon_m(r(\varepsilon);\xi)\; B^\varepsilon_m\left(\varphi\widetilde{u^\varepsilon}\right)(\xi) = B^\varepsilon_m\left(F^\varepsilon\right)(\xi) \quad \forall m \geq 1,\ \forall \xi \in \varepsilon^{-1}Y'.$$

It is well-known that all superior modes can be neglected in the homogenization process. Therefore, we pass to the limit as ε goes to zero, only in the first Bloch equation:

$$\varepsilon^{-2}\,\lambda_1\left(\frac{r(\varepsilon)}{\varepsilon};\varepsilon\xi\right)\; B^\varepsilon_1\left(\varphi\widetilde{u^\varepsilon}\right)(\xi) = B^\varepsilon_1\left(F^\varepsilon\right)(\xi) \quad \forall \xi \in \varepsilon^{-1}Y'. \tag{6.7}$$

For this purpose, we shall use some properties on the convergence of the first Bloch transform and on the analyticity of the first Bloch eigenvalue which are detailed, for instance, in Sect. 4 from [9].

First of all, since $\varphi\widetilde{u^\varepsilon} \rightharpoonup \varphi u$ weakly in $H^1(\mathbb{R}^N)$ and $F^\varepsilon \rightharpoonup F \stackrel{\text{def}}{=} f\varphi - 2\nabla\varphi\cdot\nabla u - \Delta\varphi\, u$ weakly in $L^2(\mathbb{R}^N)$, as $\varepsilon \to 0$, we have that

$$B^\varepsilon_1(\varphi\widetilde{u^\varepsilon}) \rightharpoonup \widehat{\varphi u} \quad \text{and} \quad B^\varepsilon_1(F^\varepsilon) \rightharpoonup \widehat{F}$$

weakly in $L^2_{loc}(\mathbb{R}^N)$, as $\varepsilon \to 0$. Here, we have denoted by $\widehat{\cdot}$ the classical Fourier transform.

Second, we write the following Taylor expansion of $\lambda_1\left(\frac{r(\varepsilon)}{\varepsilon};\eta\right)$ around $\eta = 0$:

$$\lambda_1\left(\frac{r(\varepsilon)}{\varepsilon};\eta\right) = \lambda_1\left(\frac{r(\varepsilon)}{\varepsilon}\right) + \delta_{kl}\eta_k\eta_l\left[1 + O\left(\lambda_1\left(\frac{r(\varepsilon)}{\varepsilon}\right)\right)\right] + O(\eta^3),$$

where we have used that $\lambda_1\left(\frac{r(\varepsilon)}{\varepsilon};0\right)$ is equal to the first eigenvalue of problem (2.1) for $a = \frac{r(\varepsilon)}{\varepsilon}$, i.e., $\lambda_1\left(\frac{r(\varepsilon)}{\varepsilon};0\right) = \lambda_1\left(\frac{r(\varepsilon)}{\varepsilon}\right)$.

With these computations, the equation (6.7) leads us to

$$\left[\varepsilon^{-2}\,\lambda_1\left(\frac{r(\varepsilon)}{\varepsilon}\right) + \delta_{kl}\xi_k\xi_l + O\left(\lambda_1\left(\frac{r(\varepsilon)}{\varepsilon}\right)\right) + O(\varepsilon\xi^3)\right] B^\varepsilon_1\left(\varphi\widetilde{u^\varepsilon}\right)(\xi) = B^\varepsilon_1\left(F^\varepsilon\right)(\xi), \tag{6.8}$$

and passing to the limit as $\varepsilon \to 0$, we get

$$\left[\lim_{\varepsilon\to 0}\left(\varepsilon^{-2}\,\lambda_1\left(\frac{r(\varepsilon)}{\varepsilon}\right)\right) + \delta_{kl}\xi_k\xi_l\right]\;\widehat{\varphi u}(\xi) = \widehat{F}(\xi), \tag{6.9}$$

for each $\xi \in Y'$.

Since the first eigenvalue $\lambda_1\left(\frac{r(\varepsilon)}{\varepsilon}\right)$ is simple, we use the asymptotic expansion given in Theorem 2.1 and we have

$$\mathscr{S} \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} \varepsilon^{-2}\ \lambda_1\left(\frac{r(\varepsilon)}{\varepsilon}\right)$$
$$= \lim_{\varepsilon\to 0}\left[(N-2)S_N\ \varepsilon^{-N}r(\varepsilon)^{N-2}\phi_1^2(0) + O\left(\varepsilon^{-N}r(\varepsilon)^{N-2}\begin{cases}\frac{r(\varepsilon)}{\varepsilon} & \text{if } N=3\\ \frac{r(\varepsilon)}{\varepsilon}\left|\ln\frac{r(\varepsilon)}{\varepsilon}\right|^{1/2} & \text{if } N=4\\ \left(\frac{r(\varepsilon)}{\varepsilon}\right)^{1/2} & \text{if } N=5\end{cases}\right)\right].$$

Let us prove (i). If we assume that $\lim_{\varepsilon\to 0}\varepsilon^{-N}r(\varepsilon)^{N-2}=\ell$, we get

$$\mathscr{S} = (N-2)S_N\ell\phi_1^2(0).$$

Therefore, the equation (6.9) becomes

$$(N-2)S_N\ell\phi_1^2(0)\widehat{\varphi u}(\xi) - \widehat{\Delta\,(\varphi u)}(\xi) = \widehat{F}(\xi).$$

Applying the inverse Fourier transform and taking into account that φ is arbitrary in $\mathscr{D}(\Omega)$, we get that u is the unique solution of problem (6.6).

Finally, if we assume $\lim_{\varepsilon\to 0}\varepsilon^{-N}r(\varepsilon)^{N-2} = +\infty$, the equation (6.9) is reduced to $\widehat{\varphi u}(\xi) = 0$, which implies that $u=0$ and thus (ii) is true. □

Appendix

We first recall that the reference cell is $Y=[-\pi,\pi)^N$. Let us prove some useful estimates:

Lemma A.1. *For any real numbers $\alpha,\beta<N$, there exists a positive constant C such that, for all $x,y\in Y$, $x\neq y$, the following bounds hold:*

$$\int_Y \frac{1}{|x-z|^\alpha}\cdot\frac{1}{|z-y|^\beta}\mathrm{d}z \le \begin{cases} C & \text{if } \alpha+\beta<N,\\ C(1+|\ln|x-y||) & \text{if } \alpha+\beta=N,\\ \dfrac{C}{|x-y|^{\alpha+\beta-N}} & \text{if } \alpha+\beta>N,\end{cases} \tag{A.1}$$

and

$$\int_Y \frac{1}{|x-z|^\alpha}\cdot|\ln|z-y||\mathrm{d}z \le C(1+|\ln|x-y||). \tag{A.2}$$

Proof. Let us estimate the first integral. To this end, we consider the change of variable $z=y+u|x-y|$ and we get

$$\int_Y \frac{1}{|x-z|^\alpha}\cdot\frac{1}{|z-y|^\beta}\mathrm{d}z \le \frac{1}{|x-y|^{\alpha+\beta-N}}\int_{\mathscr{B}\left(0,\frac{2\pi\sqrt{N}}{|x-y|}\right)}\frac{1}{\left|u-\frac{x-y}{|x-y|}\right|^\alpha|u|^\beta}\mathrm{d}u,$$

because $|u|\le\frac{2\pi\sqrt{N}}{|x-y|}$. Since $\alpha,\beta<N$, the above integral is convergent near $u=0$ and $u=\frac{x-y}{|x-y|}$. The estimate (A.1) is a consequence of the fact that this integral has an asymptotic behavior of type

$$\int_1^{\frac{2\pi\sqrt{N}}{|x-y|}}\frac{\mathrm{d}\rho}{\rho^{\alpha+\beta+1-N}}\quad\text{as } |u|\to\infty.$$

We now estimate the second integral. Using the above change of variable, we obtain

$$\int\limits_Y \frac{1}{|x-z|^\alpha} \cdot |\ln|z-y||\mathrm{d}z \le \frac{|\ln|x-y||}{|x-y|^{\alpha-N}} \int\limits_{\mathscr{B}\left(0,\frac{2\pi\sqrt{N}}{|x-y|}\right)} \frac{1}{\left|u-\frac{x-y}{|x-y|}\right|^\alpha}\mathrm{d}u + \frac{1}{|x-y|^{\alpha-N}} \int\limits_{\mathscr{B}\left(0,\frac{2\pi\sqrt{N}}{|x-y|}\right)} \frac{|\ln|u||}{\left|u-\frac{x-y}{|x-y|}\right|^\alpha}\mathrm{d}u.$$

Since $\alpha < N$, both integrals from the right-hand side of the above inequality are convergent near $u = \frac{x-y}{|x-y|}$. The estimate (A.2) is a consequence of the fact that these integrals have an asymptotic behavior of type

$$\int\limits_1^{\frac{2\pi\sqrt{N}}{|x-y|}} \frac{\mathrm{d}\rho}{\rho^{\alpha+1-N}}, \text{ respectively } \int\limits_1^{\frac{2\pi\sqrt{N}}{|x-y|}} \frac{\ln\rho}{\rho^{\alpha+1-N}}\mathrm{d}\rho \quad \text{as } |u| \to \infty.$$

□

Lemma A.2. *For any $p \in [1, \frac{N}{N-2})$, there exists a positive constant C such that, for all $x \in Y$, we have*

$$\int\limits_Y \left|\frac{1}{|x-y|^{N-2}} - \frac{1}{|y|^{N-2}}\right|^p \mathrm{d}y \le \begin{cases} C|x|^p & \text{if } p \in \left[1, \frac{N}{N-1}\right), \\ C|x|^p(1+|\ln|x||) & \text{if } p = \frac{N}{N-1}, \\ C|x|^{N-p(N-2)} & \text{if } p \in \left(\frac{N}{N-1}, \frac{N}{N-2}\right). \end{cases} \tag{A.3}$$

Proof. For all $x \in Y \setminus \{0\}$, we introduce the change of variable $y = |x|z$ and we obtain

$$\int\limits_Y \left|\frac{1}{|x-y|^{N-2}} - \frac{1}{|y|^{N-2}}\right|^p \mathrm{d}y \le |x|^{N-p(N-2)} \int\limits_{\mathscr{B}\left(0,\frac{\pi\sqrt{N}}{|x|}\right)} \left|\frac{1}{|\frac{x}{|x|}-z|^{N-2}} - \frac{1}{|z|^{N-2}}\right|^p \mathrm{d}z.$$

Since $p < \frac{N}{N-2}$ the above integral converges near $z = 0$ and $z = \frac{x}{|x|}$. Moreover, as $|z| \to \infty$, this integral is of type

$$\int\limits_1^{\frac{\pi\sqrt{N}}{|x|}} \frac{\mathrm{d}\rho}{\rho^{p(N-1)-N+1}},$$

then we obtain the result. □

Lemma A.3. *For any real number $k > 0$, there exists a positive constant C such that, for all $x, y \in Y$, $x \neq y$, we have*

$$\int\limits_Y \left|\frac{1}{|x-z|^{N-2}} - \frac{1}{|z|^{N-2}}\right| \frac{1}{|z-y|^{N-2k}}\mathrm{d}z \le \begin{cases} C|x|\displaystyle\int_0^1 \frac{1}{|tx-y|^{N-2k-1}}\mathrm{d}t & \text{if } k < \frac{N-1}{2}, \\ C|x|\displaystyle\int_0^1 (1+|\ln|tx-y||)\mathrm{d}t & \text{if } k = \frac{N-1}{2}, \\ C|x| & \text{if } k > \frac{N-1}{2}, \end{cases} \tag{A.4}$$

and

$$\int\limits_Y \left|\frac{1}{|x-z|^{N-2}} - \frac{1}{|z|^{N-2}}\right| |\ln|z-y||\,\mathrm{d}z \le C|x| \int\limits_0^1 (1+|\ln|tx-y||)\mathrm{d}t. \tag{A.5}$$

Proof. Let us consider the real function φ defined by

$$\varphi(t) = \frac{1}{|tx - z|^{N-2}},$$

which derivative is given by $\varphi'(t) = -\frac{(N-2)(tx-z)\cdot x}{|tx-z|^N}$. Then, the first integral can be bound as follows:

$$\begin{aligned}\int_Y |\varphi(1) - \varphi(0)| \frac{1}{|z-y|^{N-2k}} \, \mathrm{d}z &= \int_Y \left| \int_0^1 \frac{(N-2)(tx-z)\cdot x}{|tx-z|^N} \, \mathrm{d}t \right| \frac{1}{|z-y|^{N-2k}} \, \mathrm{d}z \\ &\leq C|x| \int_0^1 \int_Y \frac{1}{|tx-z|^{N-1} \, |z-y|^{N-2k}} \, \mathrm{d}z \, \mathrm{d}t.\end{aligned}$$

We now use the estimate (A.1) from Lemma A.1 and we obtain (A.4).

Similarly, using (A.2) we get the bound (A.5). □

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