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Transversely isotropic nonlinear magneto-active elastomers

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Abstract Magneto-active elastomers are smart materials composed of a rubber-like matrix material containing a distribution of magneto active particles. The large elastic deformations possible in the rubber-like matrix allow the mechanical properties of magneto-active elastomers to be changed significantly by the application of external magnetic fields. In this paper, we provide a theoretical basis for the description of the nonlinear properties of a particular class of these materials, namely *transversely isotropic magneto-active elastomers*. The transversely isotropic character of these materials is produced by the application of a magnetic field during the curing process, when the magneto active particles are distributed within the rubber. As a result the particles are aligned in chains that generated a preferred direction in the material. Available experimental data suggest that this enhances the stiffness of the material in the presence of an external magnetic field by comparison with the situation in which no external field is applied during curing, which leads to an essentially random (isotropic) distribution of particles. Herein, we develop a general form of the constitutive law for such magnetoelastic solids. This is then used in the solution of two simple problems involving homogeneous deformations, namely simple shear of a slab and simple tension of a cylinder. Using these results and the experimental available data we develop a prototype constitutive equation, which is used in order to solve two boundary-value problems involving non-homogeneous deformations—the extension and inflation of a circular cylindrical tube and the extension and torsion of a solid circular cylinder.

1 Introduction

Magneto active elastomers, otherwise known as magneto-sensitive (MS) elastomers, are materials comprising a rubber like matrix within which microscopic magneto-active particles are distributed. The main characteristic of these materials is that their mechanical properties can be changed significantly by application of an external magnetic field. This characteristic makes them particularly suitable for applications that require a rapid change in the material properties, such as in vibration suppression, in variable stiffness devices for electronic control, and in tunable automotive suspensions, etc. see, for example, [1–3].

Experimental data on MS elastomers may be found in, for example, [4–10]. In particular the results in [4,5] and in [9,10] suggest that the application of a magnetic field during the curing process, in which the particles are added to the rubber-like base material, induces the particles to align in chain-like structures which endow the material with a preferred direction and enhance significantly the mechanical stiffness of the material as compared to the isotropic case in which the particles are randomly distributed. Indeed, the data illustrated in [2], which have been used in recent theoretical works concerning isotropic MS elastomers (see, for example, [11–15]), were actually obtained for transversely isotropic MS elastomers.

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An important early work on magnetoelastic interactions is the book by Brown [16], which has served as a basis for most of the subsequent research in the area. More recent account, within the general context of electromagnetic continuum mechanics, may be found in, for example, [17–20], and comparisons of the different formulations are provided in [21,22]. Interesting models of transversely isotropic MS elastomers, based on a micromechanical approach, have been developed by, for example, Borcea and Bruno [23] and by Yin et al. [24].

In many of the works that deal with magnetoelastic interactions the magnetization has been used as the independent magnetic variable (see, for example, [25]). In contrast, Dorfmann and Ogden [11–15] have recently developed the theory of MS elastomers using either the magnetic field or the magnetic induction as the independent magnetic variable instead the magnetization and a particular form of energy density function, which leads to relatively simple expressions for both the constitutive equations and the structure of the governing equations. While the theory of Dorfmann and Ogden [11–15] is quite general, it was applied explicitly only to the case of isotropic magnetoelastic solids. Their general formulation serves as a good starting point for the development of a theory of transversely isotropic MS elastomers. A parallel theory for transversely electro-active (or electro-sensitive) elastomers is presented in [26].

The set of all universal solutions for the electro elastic problem was found by Singh and Pipkin [27], while the similar set of solutions for the magneto elastic case, in analogy with the above reference, was found by Pucci and Saccomandi [28]. Universal relations for the particular case of the formulation of Dorfmann and Ogden [12] were found, for example, by Bustamante et al. [29].

In Sect. 2, we review the basic concepts of the nonlinear theory of magnetoelasticity as developed by Dorfmann and Ogden [11–15]. Then, in Sect. 3 we study the particular case of an MS elastomer with a preferred direction associated with the particle alignment in the reference configuration. We summarize the constitutive equations based on either the magnetic field or magnetic induction vector and then, using invariants (or quasi-invariants) that depend severally on the deformation, the preferred direction and the magnetic field, derive in detail the specific forms of constitutive laws for transversely isotropic magnetoelastic solid.

Two basic problems involving homogeneous deformations are studied in Sect. 4. The simple shear of a slab and the simple tension of a circular cylinder, are used subsequently in Sect. 5 in order to develop a specific prototype model, or first approximation, for the energy function. Here, we use the experimental data provided in [4,5,8], which were obtained essentially for the first two problems mentioned above, to provide a procedure for characterizing the magnetoelastic energy function.

In Sect. 6, two boundary-value problems involving non-homogeneous deformations and cylindrical symmetry are solved: the inflation and extension of a cylindrical tube, and the combined extension and torsion of a solid circular cylinder (see [15,12] for the counterparts of these problems for isotropic materials). The particular form of the constitutive equation developed in Sect. 5 is used in order to obtain closed-form solutions for these problems. Numerical results are then presented to illustrate the dependence of the mechanical characteristics on the magnitude of the mechanical field.

Finally, Sect. 7 contains some concluding remarks and outlines some possible directions for further research. Most of this paper is based on Chapter 5 of the Ph.D. thesis by Bustamante [30].

2 Basic equations

2.1 Kinematics

Consider a magnetoelastic material occupying the reference configuration $\mathcal{B}_0$ when undeformed and in the absence of a magnetic field. Let a material point in $\mathcal{B}_0$ be defined by its position vector $\mathbf{X}$ relative to an arbitrarily chosen origin. When the body is subjected to the deformation $\boldsymbol{\chi}$ the point $\mathbf{X}$ assumes a new position $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ in the resulting deformed configuration, which we denote by $\mathcal{B}$. We are considering quasi-static deformations with no time dependence. The deformation gradient tensor $\mathbf{F}$ relative to $\mathcal{B}_0$ and its determinant are [31],

$$\mathbf{F} = \mathrm{Grad}\,\boldsymbol{\chi}, \quad J = \det \mathbf{F} > 0, \tag{1}$$

respectively, where Grad is the gradient operator with respect to $\mathbf{X}$ and wherein the notation J is defined. The corresponding left and right Cauchy-Green deformation tensors, denoted here by $\mathbf{b}$ and $\mathbf{c}$, respectively, are defined by

$$\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \quad \mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}. \tag{2}$$

2.2 Magnetic field equations

Suppose that the deformed configuration $\mathcal{B}$ arises from the combined application of a magnetic field and boundary tractions. The magnetic field vector is denoted by $\mathbf{H}$ and the magnetic induction vector by $\mathbf{B}$. In the absence of material these are related by the standard equation (see, for example, [20] for a recent text on electromagnetic theory)

$$\mathbf{B} = \mu_0 \mathbf{H}, \tag{3}$$

where μ_0 is the magnetic permeability in vacuo. Inside the material, on the other hand, the two vectors are connected via

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \tag{4}$$

where $\mathbf{M}$ is known as the magnetization field.

In either case, the field equations

$$\operatorname{curl} \mathbf{H} = \mathbf{0}, \quad \operatorname{div} \mathbf{B} = 0, \tag{5}$$

are satisfied, where curl and div, respectively, are the curl and divergence operators with respect to $\mathbf{x}$. We are assuming here that there are no free currents.

Equation (5) is expressed in Eulerian form. Within a deformed material pull-back operations from $\mathcal{B}$ to $\mathcal{B}_0$ give the corresponding Lagrangian forms of the magnetic field vector, denoted $\mathbf{H}_l$, and magnetic induction vector, denoted by $\mathbf{B}_l$. The connections are (see, for example, [11,12,32])

$$\mathbf{H}_l = \mathbf{F}^{\mathrm{T}} \mathbf{H}, \quad \text{and} \quad \mathbf{B}_l = J \mathbf{F}^{-1} \mathbf{B}, \tag{6.1, 2}$$

where $^{\mathrm{T}}$ signifies the transpose of a second-order tensor. The counterparts of Eq. (5) for these vectors are (within the material)

$$\operatorname{Curl} \mathbf{H}_l = \mathbf{0}, \quad \operatorname{Div} \mathbf{B}_l = 0, \tag{7.1, 2}$$

where Curl and Div, respectively, are the curl and divergence operators with respect to $\mathbf{X}$.

2.3 Mechanical balance equations

Let ρ_0 and ρ be the mass densities of the material in the reference and deformed configurations, $\mathcal{B}_0$ and $\mathcal{B}$, respectively. Then, recalling that $J = \det \mathbf{F}$, the conservation of mass equation can be written simply as

$$J\rho = \rho_0. \tag{8}$$

The influence of the magnetic field on the mechanical stress in the deforming body may be incorporated through *magnetic body forces* or through a *magnetic stress tensor*. Here, we adopt the latter approach and denote the resulting *total (Cauchy) stress tensor* by $\boldsymbol{\tau}$, which has the advantage of being *symmetric*. In the absence of *mechanical* body forces, the equilibrium equation for a magnetoelastic solid has the (Eulerian) form

$$\operatorname{div} \boldsymbol{\tau} = \mathbf{0}. \tag{9}$$

For more details we refer to, for example, [12,13,25,32].

As in conventional nonlinear elasticity theory [31], we may define a 'nominal' stress tensor, here denoted $\mathbf{T}$ and referred to as the *total nominal stress tensor*, which is related to $\boldsymbol{\tau}$ by

$$\mathbf{T} = J \mathbf{F}^{-1} \boldsymbol{\tau}. \tag{10}$$

The equilibrium equation (9) may then be expressed simply in Lagrangian form as

$$\operatorname{Div} \mathbf{T} = \mathbf{0}. \tag{11}$$

2.4 Boundary conditions

At the interfaces between the considered material body and its exterior appropriate boundary conditions must be satisfied by the fields $\mathbf{H}$, $\mathbf{B}$ and $\boldsymbol{\tau}$. In particular, the vector fields $\mathbf{H}$ and $\mathbf{B}$ satisfy the standard jump conditions (see, for example, [20])

$$\mathbf{n} \times [\![\mathbf{H}]\!] = \mathbf{0}, \quad \mathbf{n} \cdot [\![\mathbf{B}]\!] = 0, \tag{12.1, 2}$$

where $[\![\cdot]\!]$ signifies the difference between the value of the enclosed quantity on the outside and the inside of the boundary $\partial\mathcal{B}$ of $\mathcal{B}$, and $\mathbf{n}$ is the outward unit normal to $\partial\mathcal{B}$. It is assumed here that there are no surface currents. The continuity condition involving the stress $\boldsymbol{\tau}$ may be written in the form

$$[\![\boldsymbol{\tau}]\!]\mathbf{n} = \mathbf{0}, \tag{13}$$

and we note that the traction $\boldsymbol{\tau}\mathbf{n}$ on the inner boundary must be matched to the combination of the traction associated with the Maxwell stress and any active (applied) mechanical traction [33] (or passive traction associated with a displacement boundary condition). Outside the material the Maxwell stress, denoted $\boldsymbol{\tau}^m$, has the standard form

$$\boldsymbol{\tau}^m = \mathbf{H} \otimes \mathbf{B} - \frac{1}{2}(\mathbf{H} \cdot \mathbf{B})\mathbf{I}, \tag{14}$$

where $\mathbf{I}$ is the identity tensor and $\mathbf{B} = \mu_0\mathbf{H}$.

Equations (12.1, 2) and (13) may also be expressed in terms of the Lagrangian quantities $\mathbf{H}_l$, $\mathbf{B}_l$ and $\mathbf{T}$ evaluated on the reference boundary (denoted $\partial\mathcal{B}_0$), for details of which we refer to [13], for example.

2.5 Constitutive equations

To solve boundary value problems we need to have available, in addition to mechanical and magnetic governing equations, constitutive equations for the total stress tensor $\boldsymbol{\tau}$ and one of the magnetic vectors. Here, we summarize briefly the formulation of Dorfmann and Ogden [11–13], initially adopting $\mathbf{F}$ and $\mathbf{B}_l$ to be independent variables for this purpose and defining an energy function Φ per unit mass: $\Phi = \Phi(\mathbf{F}, \mathbf{B}_l)$. For an unconstrained material, the total stress tensor is given by

$$\boldsymbol{\tau} = \rho\mathbf{F}\frac{\partial\Phi}{\partial\mathbf{F}} + \mu_0^{-1}[\mathbf{B} \otimes \mathbf{B} - \tfrac{1}{2}(\mathbf{B} \cdot \mathbf{B})\mathbf{I}], \tag{15}$$

where $\mathbf{B} = J^{-1}\mathbf{F}\mathbf{B}_l$. In vacuum, $\Phi \equiv 0$ and the stress $\boldsymbol{\tau}$ reduces to the Maxwell stress, here denoted $\boldsymbol{\tau}^m$ and given by

$$\boldsymbol{\tau}^m = \mu_0^{-1}[\mathbf{B} \otimes \mathbf{B} - \tfrac{1}{2}(\mathbf{B} \cdot \mathbf{B})\mathbf{I}], \tag{16}$$

with $\mathbf{B} = \mu_0\mathbf{H}$.

In terms of Φ the magnetization vector is

$$\mathbf{M} = -\rho J\mathbf{F}^{-\mathrm{T}}\frac{\partial\Phi}{\partial\mathbf{B}_l}. \tag{17}$$

Following Dorfmann and Ogden [12] we now supplement the energy function and define the new energy function Ω (per unit reference volume) as

$$\Omega(\mathbf{F}, \mathbf{B}_l) = \rho_0\Phi + \frac{1}{2}\mu_0^{-1}J^{-1}\mathbf{B}_l \cdot (\mathbf{c}\mathbf{B}_l), \tag{18}$$

which leads to the particularly simple forms

$$\mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}}, \quad \mathbf{H}_l = \frac{\partial\Omega}{\partial\mathbf{B}_l} \tag{19}$$

of the nominal stress tensor and the Lagrangian magnetic field, as in [12,13]. The corresponding total Cauchy stress tensor and Eulerian magnetic field are

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}}, \quad \mathbf{H} = \mathbf{F}^{-\mathrm{T}}\frac{\partial \Omega}{\partial \mathbf{B}_l}. \tag{20}$$

For an incompressible material, with the constraint

$$\det \mathbf{F} \equiv 1, \tag{21}$$

the total Cauchy and nominal stresses are given by

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{I}, \quad \mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{F}^{-1}, \tag{22}$$

where p is a Lagrange multiplier associated with the incompressibility constraint.

While $\mathbf{B}_l$ satisfies (7.2) and can therefore be expressed in terms of a magnetic vector potential, $\mathbf{H}_l$ satisfies (7.1) and is expressible in terms of a scalar potential, which has certain advantages, in particular in the formulation of variational principles [33] and the solution of boundary-value problems. It is therefore convenient to consider the alternative formulation of the constitutive laws based on use of $\mathbf{H}_l$ as the independent magnetic variable. We define another energy function, denoted Ω^*, as a function of $\mathbf{F}$ and $\mathbf{H}_l$ via the partial Legendre transformation [12,13]

$$\Omega^*(\mathbf{F}, \mathbf{H}_l) = \Omega(\mathbf{F}, \mathbf{B}_l) - \mathbf{H}_l \cdot \mathbf{B}_l. \tag{23}$$

This requires, in particular, that for every $\mathbf{F}$ there is a one-to-one relationship between $\mathbf{B}_l$ and $\mathbf{H}_l$ and, physically, that there is no magnetic hysteresis. Then, the counterparts of the Eqs. (19) and (20) are, respectively

$$\mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \mathbf{B}_l = -\frac{\partial \Omega^*}{\partial \mathbf{H}_l}, \tag{24}$$

and

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \mathbf{B} = -J^{-1}\mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}}, \tag{25.1, 2}$$

while for an incompressible material equation (22) is replaced by

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}} - p^*\mathbf{I}, \quad \mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}} - p^*\mathbf{F}^{-1}, \tag{26.1, 2}$$

where we have used p^* for the Lagrange multiplier in this case.

Note that neither of the above two formulations involves the magnetization.

3 Transversely isotropic MS elastomers

We now consider the situation in which the material has a preferred direction in the reference configuration denoted by the unit vector $\mathbf{a}_0$. For magnetoelastic elastomers this direction is generated by the alignment of magneto-active particles during curing in the presence of a magnetic field. In the present work we make use of the formulation based on the Lagrangian magnetic field $\mathbf{H}_l$ as the independent magnetic variable. A corresponding development with $\mathbf{B}_l$ as the independent magnetic variable is contained in the thesis by Bustamante [30]. Thus, we work with the energy function Ω^*, but now with the preferred direction $\mathbf{a}_0$ included explicitly as an argument: $\Omega^* = \Omega^*(\mathbf{F}, \mathbf{H}_l, \mathbf{a}_0)$. The relevant equations for the stresses and the magnetic induction remain in the general forms given in (25.1, 2) and (26.1, 2).

When a magnetic field is applied there are effectively two preferred directions in the reference configuration, namely $\mathbf{a}_0$ and $\mathbf{H}_l$, but the latter is not in general a unit vector. For a material with two preferred directions

Ω^* must be a function of ten invariants or quasi-invariants (see, for example, [34,35]). Here, we use the set of invariants defined by[1]

$$I_1 = \operatorname{tr}\mathbf{c}, \quad I_2 = \frac{1}{2}[(\operatorname{tr}\mathbf{c})^2 - \operatorname{tr}\mathbf{c}^2], \quad I_3 = \det\mathbf{c}, \tag{27.1–3}$$

$$I_4 = \mathbf{H}_l \cdot \mathbf{H}_l, \quad I_5 = \mathbf{H}_l \cdot (\mathbf{c}\mathbf{H}_l), \quad I_6 = \mathbf{H}_l \cdot (\mathbf{c}^2\mathbf{H}_l), \tag{28}$$

$$I_7 = \mathbf{a}_0 \cdot (\mathbf{c}\mathbf{a}_0), \quad I_8 = \mathbf{a}_0 \cdot (\mathbf{c}^2\mathbf{a}_0), \quad I_9 = \mathbf{a}_0 \cdot \mathbf{H}_l, \quad I_{10} = \mathbf{a}_0 \cdot (\mathbf{c}\mathbf{H}_l). \tag{29}$$

It should be noted that the signs of I_9 and I_{10} are changed if either $\mathbf{a}_0$ is changed to $-\mathbf{a}_0$ or $\mathbf{H}_l$ is changed to $-\mathbf{H}_l$ (but not both simultaneously). Thus, strictly, I_9 and I_{10} should be replaced by their squares in the above list or be multiplied by $\mathbf{a}_0 \cdot \mathbf{H}_l$ to give invariants that are independent of the senses of the preferred directions. However, we retain the forms as given above for simplicity since otherwise the expressions for the stress and magnetic induction would be rather longer than those that appear below. The difference is in any case properly accommodated in the properties of the energy function. It is also worth nothing that, according to Zheng [35], there should be an additional invariant, say I_{11}, defined by $I_{11} = \mathbf{a}_0 \cdot (\mathbf{c}^2\mathbf{H}_l)$. However, this invariant is not independent of the others, a result that is proved in Appendix B of [30].

In order to obtain explicit expressions for the total stress based on these invariants we need the derivatives

$$\frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}^{\mathrm{T}}, \quad \frac{\partial I_2}{\partial \mathbf{F}} = 2(I_1\mathbf{F}^{\mathrm{T}} - \mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{F}^{\mathrm{T}}), \quad \frac{\partial I_3}{\partial \mathbf{F}} = 2I_3\mathbf{F}^{-1}, \tag{30}$$

$$\frac{\partial I_5}{\partial \mathbf{F}} = 2\mathbf{H}_l \otimes \mathbf{F}\mathbf{H}_l, \quad \frac{\partial I_6}{\partial \mathbf{F}} = 2(\mathbf{H}_l \otimes \mathbf{F}\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{H}_l + \mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{H}_l \otimes \mathbf{F}\mathbf{H}_l), \tag{31}$$

$$\frac{\partial I_7}{\partial \mathbf{F}} = 2\mathbf{a}_0 \otimes \mathbf{F}\mathbf{a}_0, \quad \frac{\partial I_8}{\partial \mathbf{F}} = 2(\mathbf{a}_0 \otimes \mathbf{F}\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{a}_0 + \mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{a}_0 \otimes \mathbf{F}\mathbf{a}_0), \tag{32}$$

$$\frac{\partial I_{10}}{\partial \mathbf{F}} = \mathbf{a}_0 \otimes \mathbf{F}\mathbf{H}_l + \mathbf{H}_l \otimes \mathbf{F}\mathbf{a}_0. \tag{33}$$

Using these formulas it follows from (25.1) that for an unconstrained material the total Cauchy stress is given by

$$\begin{aligned}\boldsymbol{\tau} = J^{-1}[&2\Omega_1^*\mathbf{b} + 2\Omega_2^*(I_1\mathbf{b} - \mathbf{b}^2) + 2I_3\Omega_3^*\mathbf{I} + 2\Omega_5^*\mathbf{bH} \otimes \mathbf{bH} + 2\Omega_6^*(\mathbf{bH} \otimes \mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H} \otimes \mathbf{bH}) \\ &+ 2\Omega_7^*\mathbf{a} \otimes \mathbf{a} + 2\Omega_8^*(\mathbf{a} \otimes \mathbf{ba} + \mathbf{ba} \otimes \mathbf{a}) + \Omega_{10}^*(\mathbf{a} \otimes \mathbf{bH} + \mathbf{bH} \otimes \mathbf{a})],\end{aligned} \tag{34}$$

where Ω_i^* stands for the partial derivative of Ω^* with respect to I_i, $i = 1, 2, \ldots, 10$, and we have introduced the notation

$$\mathbf{a} = \mathbf{F}\mathbf{a}_0. \tag{35}$$

The corresponding result for an incompressible material is obtained from (26.1) in the form

$$\begin{aligned}\boldsymbol{\tau} = &2\Omega_1^*\mathbf{b} + 2\Omega_2^*(I_1\mathbf{b} - \mathbf{b}^2) - p^*\mathbf{I} + 2\Omega_5^*\mathbf{bH} \otimes \mathbf{bH} + 2\Omega_6^*(\mathbf{bH} \otimes \mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H} \otimes \mathbf{bH}) \\ &+ 2\Omega_7^*\mathbf{a} \otimes \mathbf{a} + 2\Omega_8^*(\mathbf{a} \otimes \mathbf{ba} + \mathbf{ba} \otimes \mathbf{a}) + \Omega_{10}^*(\mathbf{a} \otimes \mathbf{bH} + \mathbf{bH} \otimes \mathbf{a}),\end{aligned} \tag{36}$$

with $I_3 \equiv 1$.

To obtain an explicit expression for the magnetic induction $\mathbf{B}_l$ we need the derivatives

$$\frac{\partial I_4}{\partial \mathbf{H}_l} = 2\mathbf{H}_l, \quad \frac{\partial I_5}{\partial \mathbf{H}_l} = 2\mathbf{c}\mathbf{H}_l, \quad \frac{\partial I_6}{\partial \mathbf{H}_l} = 2\mathbf{c}^2\mathbf{H}_l, \quad \frac{\partial I_9}{\partial \mathbf{H}_l} = \mathbf{a}_0, \quad \frac{\partial I_{10}}{\partial \mathbf{K}_l} = \mathbf{c}\mathbf{a}_0. \tag{37}$$

Then, from (25.2) we obtain

$$\mathbf{B} = -J^{-1}(2\Omega_4^*\mathbf{bH} + 2\Omega_5^*\mathbf{b}^2\mathbf{H} + 2\Omega_6^*\mathbf{b}^3\mathbf{H} + \Omega_9^*\mathbf{a} + \Omega_{10}^*\mathbf{ba}) \tag{38}$$

for an unconstrained material, and

$$\mathbf{B} = -(2\mathbf{bH}\Omega_4^* + 2\mathbf{b}^2\mathbf{H}\Omega_5^* + 2\mathbf{b}^3\mathbf{H}\Omega_6^* + \mathbf{a}\Omega_9^* + \mathbf{ba}\Omega_{10}^*) \tag{39}$$

for an incompressible material.

[1] In the thesis by Bustamante [30] I_4, I_5, I_6, I_9, I_{10} are used to denote invariants involving $\mathbf{B}_l$, while invariants involving $\mathbf{H}_l$ are denoted with different symbols, namely K_4, K_5, K_6, K_9, K_{10}. Since in the present paper we only work with $\mathbf{H}_l$ as the independent magnetic variable, we have not considered a different notation here.

3.1 Some restrictions on the energy function

Some restriction on Ω^* may be obtained, as has been done in [26] for the electroelastic case, for example, by examining the situation in which there is no applied load or field. In the reference configuration we have $\mathbf{F} = \mathbf{I}$, $\mathbf{a} = \mathbf{a}_0$, and hence the invariants (27)–(29) reduce to

$$I_1 = I_2 = 3, \quad I_3 = 1, \quad I_4 = I_5 = I_6 = 0, \quad I_7 = I_8 = 1, \quad I_9 = I_{10} = 0. \tag{40}$$

Then, Eq. (34) reduces to

$$\bar{\boldsymbol{\tau}} = 2(\bar{\Omega}_1^* + 2\bar{\Omega}_2^* + \bar{\Omega}_{33}^*)\mathbf{I} + 2(\bar{\Omega}_7^* + 2\bar{\Omega}_8^*)\mathbf{a}_0 \otimes \mathbf{a}_0, \tag{41}$$

and (38) to

$$\bar{\mathbf{B}} = -(\bar{\Omega}_9^* + \bar{\Omega}_{10}^*)\mathbf{a}_0, \tag{42}$$

where the overbar signifies evaluation for the invariants given in (40).

If there is no residual stress and no residual magnetic field or magnetization then we must have $\bar{\boldsymbol{\tau}} = \mathbf{0}$ and $\bar{\mathbf{B}} = \mathbf{0}$ and it follows from (41) and (42) that

$$\bar{\Omega}_1^* + 2\bar{\Omega}_2^* + \bar{\Omega}_3^* = 0, \quad \bar{\Omega}_7^* + 2\bar{\Omega}_8^* = 0, \quad \bar{\Omega}_9^* + \bar{\Omega}_{10}^* = 0. \tag{43.1–3}$$

For an incompressible material (43.1) is replaced by

$$2\bar{\Omega}_1^* + 4\bar{\Omega}_2^* - p^* = 0, \tag{44}$$

but the restrictions (43.2, 3) remain in force.

Suppose now that in a deformed configuration under the action of an applied mechanical load in the absence of an applied magnetic field there is no induced magnetic induction (or magnetization). Then we would have the additional restriction

$$\mathbf{I}\breve{\Omega}_9^* + \mathbf{b}\breve{\Omega}_{10}^* = \mathbf{0}, \tag{45}$$

where $\breve{\Omega}_i^*$, $i = 9, 10$ indicates evaluation of Ω_i^* for $I_4 = I_5 = I_6 = I_9 = I_{10} = 0$, from which we deduce that

$$\breve{\Omega}_9^* = \breve{\Omega}_{10}^* = 0. \tag{46}$$

Of course, the restrictions (46) imply (43.3) in this case. For an isotropic material the invariants I_9 and I_{10} are absent, $\Omega_9^* = \Omega_{10}^* = 0$ and it follows automatically from (38) that $\mathbf{B}$, and hence the magnetization, vanishes when there is no applied magnetic field. For an anisotropic material it does not necessarily follow that there is no magnetization induced by mechanical loads. For the counterpart of this situation for electroelastic materials, in particular for piezoelectric materials, deformation causes a rearrangement of the distribution of charges leading to a non-zero polarization field (see, for example, [36]). Note that the restrictions (46) would not mean that there is no coupling between the magnetic and mechanical effects, but it does mean that such coupling requires the presence of an *applied* magnetic field.

A final special case worth mentioning corresponds to the maintenance of the undeformed configuration on the application of a magnetic field. In such a configuration the invariants reduce to $I_1 = I_2 = 3$, $I_3 = I_7 = I_8 = 1$ and $I_4 = I_5 = I_6 = \mathbf{H} \cdot \mathbf{H}$, $I_9 = I_{10} = \mathbf{H} \cdot \mathbf{a}_0$, and the stress required for an unconstrained material, for example, is

$$\begin{aligned}\boldsymbol{\tau} = {} & 2(\Omega_1^* + 2\Omega_2^* + \Omega_3^*)\mathbf{I} + 2(\Omega_5^* + \Omega_6^*)\mathbf{H} \otimes \mathbf{H} \\ & + 2(\Omega_7^* + 2\Omega_8^*)\mathbf{a}_0 \otimes \mathbf{a}_0 + \Omega_{10}^*(\mathbf{a}_0 \otimes \mathbf{H} + \mathbf{H} \otimes \mathbf{a}_0),\end{aligned} \tag{47}$$

while the magnetic induction becomes

$$\mathbf{B} = -2(\Omega_4^* + \Omega_5^* + \Omega_6^*)\mathbf{H} - (\Omega_9^* + \Omega_{10}^*)\mathbf{a}_0. \tag{48}$$

4 Boundary value problems: homogeneous deformations

In this section, we consider two simple boundary value problems for which the deformation is homogeneous, namely the simple shear of a slab and the uniaxial tension of a circular cylinder. The discussion of these two problems is motivated by the fact that there are few experimental results available for magneto-active elastomers suitable for a detailed characterization of the constitutive properties of such materials. Because of the complexity of the phenomena involved it is more difficult than in the purely nonlinear elastic case to obtain appropriate data sets, particularly because of the large number of invariants involved. As we will see in detail in Sect. 5, most of the available data have been obtained for rather simple problems. Thus, we focus on these two homogeneous deformation problems in order to develop a prototype specific form of Ω^*.

4.1 Simple shear

The simple shear problem for magnetoelastic materials has been treated several times in the literature; see, for example, [12,15] for the isotropic problem and [37], which also deals with isotropic materials but is based on a different formulation for the energy function. Here, we confine attention to incompressible materials.

Consider the slab described in the reference configuration by

$$-\infty \leq X_1 \leq \infty, \quad -L/2 \leq X_2 \leq L/2, \quad -\infty \leq X_3 \leq \infty. \tag{49}$$

Note that here and in the following sections we use semi-infinite geometries in order to avoid problems with the boundary conditions (12.1, 2), which for finite geometries in general lead to edge effects that discount the possibility of having homogeneous deformations. For an example of the influence of edge effects see [38].

We consider the (homogenous) simple shear deformation defined by

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3, \tag{50}$$

from which the component matrices of the deformation gradient and the left and right Cauchy deformation tensors, denoted F, b, c, respectively, are calculated as

$$\mathsf{F} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{b} = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{c} = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{51}$$

The two first invariants (27.1–3) are

$$I_1 = I_2 = 3 + \gamma^2, \tag{52}$$

and $I_3 = 1$.

We consider a uniform external magnetic field applied in the x_2 direction with component H^* outside the material, so that the corresponding component of the magnetic induction is $B^* = \mu_0 H^*$. Let H be the corresponding (uniform) component of the magnetic field within the slab. By the continuity condition (12.1) this is the only non-vanishing component with the material. However, $\mathbf{B}$ will in general have two components for this problem, and we write $\mathsf{B} = (B_1, B_2, 0)^{\mathrm{T}}$ as the vector of components. The continuity condition (12.2) gives $B^* = B_2$, which relates H to H^*. Parallel results for this problem based on use of the magnetic induction as the independent magnetic variable are given in [30], while for details of the experimental counterpart of this problem we refer to [2,8], in particular regarding the orientation field and the particle chains. Of course, the experiments can only be conducted on a slab of finite dimensions.

From the relation $\mathbf{H}_l = \mathbf{F}^{\mathrm{T}}\mathbf{H}$ we find that $\mathbf{H}_l = \mathbf{H}$ for this problem and the given field. Thus,

$$\mathsf{H}_l = (0, H, 0)^{\mathrm{T}}, \tag{53}$$

and it follows from (28) that

$$I_4 = H^2, \quad I_5 = H^2(1+\gamma^2), \quad I_6 = H^2(\gamma^4 + 3\gamma^2 + 1). \tag{54.1–3}$$

Options for the particle chain alignments include alignment in the x_1 or x_2 direction, which are the two cases considered here. For experimental results obtained for a slab under shear with these two alignments we refer to [8].

4.1.1 Particle alignment in the x_2 direction

In this case, the initial particle alignment is given by the field $\mathsf{a}_0 = (0, 1, 0)^{\mathrm{T}}$, and from (35) we obtain $\mathsf{a} = (\gamma, 1, 0)^{\mathrm{T}}$. The invariants (29) are then specialized to

$$I_7 = 1 + \gamma^2, \quad I_8 = \gamma^2 + (1 + \gamma^2)^2, \quad I_9 = H, \quad I_{10} = H(1 + \gamma^2). \tag{55}$$

The components of the total stress can now be obtained from (36) as

$$\begin{aligned} \tau_{11} &= -p^* + 2(1+\gamma^2)\Omega_1^* + 2(2+\gamma^2)\Omega_2^* + 2H^2\gamma^2[\Omega_5^* + (2+\gamma^2)\Omega_6^*] \\ &\quad +2\gamma^2[\Omega_7^* + 2(2+\gamma^2)\Omega_8^* + H\Omega_{10}^*], \end{aligned} \tag{56}$$

$$\tau_{22} = -p^* + 2\Omega_1^* + 4\Omega_2^* + 2H^2[\Omega_5^* + 2(1+\gamma^2)\Omega_6^*] + 2\Omega_7^* + 4(1+\gamma^2)\Omega_8^* + 2H\Omega_{10}^*, \tag{57}$$

$$\tau_{33} = -p^* + 2\Omega_1^* + 2(2+\gamma^2)\Omega_2^*, \tag{58}$$

$$\tau_{12} = 2\gamma\{\Omega_1^* + \Omega_2^* + H^2[\Omega_5^* + (3+2\gamma^2)\Omega_6^*] + \Omega_7^* + (3+2\gamma^2)\Omega_8^* + H\Omega_{10}^*\}, \tag{59}$$

with $\tau_{23} = \tau_{13} = 0$. From (39) the components of the magnetic induction are

$$B_1 = -\gamma\{2H[\Omega_4^* + (2+\gamma^2)\Omega_5^* + (3+4\gamma^2+\gamma^4)\Omega_6^*] + \Omega_9^* + (2+\gamma^2)\Omega_{10}^*\}, \tag{60}$$

$$B_2 = -\{2H[\Omega_4^* + (1+\gamma^2)\Omega_5^* + (1+3\gamma^2+\gamma^4)\Omega_6^*] + \Omega_9^* + (1+\gamma^2)\Omega_{10}^*\}, \tag{61}$$

with $B_3 = 0$, as anticipated.

Now, since the invariants (53), (54.1–3) and (55) depend collectively only on γ and H, we may introduce a reduced form of the energy function Ω^* that depends on these two variables. We denote this by $\omega^* = \omega^*(\gamma, H)$. Then it is straightforward to establish the simple formulas

$$\tau_{12} = \omega_\gamma^*, \quad B_2 = \omega_H^*, \tag{62}$$

where the subscripts γ and H signify partial derivatives.

4.1.2 Particle alignment in the x_1 direction

In this case, the initial preferred direction is given by $\mathsf{a}_0 = (1, 0, 0)^{\mathrm{T}}$ and hence by (35), $\mathsf{a} = (1, 0, 0)^{\mathrm{T}}$. The invariants we need to recalculate (29) are given by

$$I_7 = 1, \quad I_8 = 1 + \gamma^2, \quad I_9 = 0, \quad I_{10} = H\gamma. \tag{63.1–4}$$

The components of the total stress are now

$$\begin{aligned} \tau_{11} &= -p^* + 2(1+\gamma^2)\Omega_1^* + 2(2+\gamma^2)\Omega_2^* + 2H^2\gamma^2[\Omega_5^* + 2(2+\gamma^2)\Omega_6^*] + 2\Omega_7^* \\ &\quad +4(1+\gamma^2)\Omega_8^* + 2H\gamma\Omega_{10}^*, \end{aligned} \tag{64}$$

$$\tau_{22} = -p^* + 2\Omega_1^* + 4\Omega_2^* + 2H^2[\Omega_5^* + (1+\gamma^2)\Omega_6^*], \tag{65}$$

$$\tau_{33} = -p^* + 2\Omega_1^* + 2(2+\gamma^2)\Omega_2^*, \tag{66}$$

$$\tau_{12} = 2\gamma\{\Omega_1^* + \Omega_2^* + H^2[\Omega_5^* + (3+2\gamma^2)\Omega_6^*] + \Omega_8^*\} + H\Omega_{10}^*, \tag{67}$$

and again $\tau_{13} = \tau_{23} = 0$. The non-zero components of the magnetic induction are given by (39) as

$$B_1 = -\{2H\gamma[\Omega_4^* + (2+\gamma^2)\Omega_5^* + (3+4\gamma^2+\gamma^4)\Omega_6^*] + \Omega_9^* + (1+\gamma^2)\Omega_{10}^*\}, \tag{68}$$

$$B_2 = -\{2H[\Omega_4^* + (1+\gamma^2)\Omega_5^* + (1+3\gamma^2+\gamma^4)\Omega_6^*] + \gamma\Omega_{10}^*\}. \tag{69}$$

If we define ω^* as in the previous example then it can be shown that Eq. (62) holds again.

For each of the above two cases the difference $\tau_{22} - \tau_{22}^m$ is the applied mechanical traction required on the boundary $x = L/2$ in order to maintain the simple shear deformation along with the shear stress τ_{12}, where τ_{22}^m is the component of the Maxwell stress (14) exterior to the slab specialized for the present problem. The non-zero components of the Maxwell stress are

$$\tau_{11}^m = -\tau_{22}^m = \tau_{33}^m = -\frac{1}{2}\mu_0 H^{*2}. \tag{70}$$

4.2 Uniaxial tension of a bar

The uniform uniaxial tension of a circular cylindrical bar was used by Bellan and Bossis [4] and by Bossis et al. [5] to obtain some important experimental results, which we will use in Sect. 5 for the construction of some preliminary forms of the energy function. The (homogeneous) deformation is described by the equations

$$r = \lambda^{-1/2} R, \quad \theta = \Theta, \quad z = \lambda Z, \tag{71}$$

where (R, Θ, Z) are cylindrical polar coordinates in the reference configuration and (r, θ, z) the corresponding coordinates in the deformed configuration, λ being the stretch in the axial direction. For theoretical purposes the cylinder is considered to have 'infinite' length so as to avoid edge effects. The deformation gradient and the left and right Cauchy-Green tensors are diagonal with respect to the cylindrical axes, with components given by

$$\mathsf{F} = \begin{pmatrix} \lambda^{-1/2} & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \mathsf{b} = \mathsf{c} = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}, \tag{72}$$

and, from (27.1–3), the first and second invariants are

$$I_1 = 2\lambda^{-1} + \lambda^2, \quad I_2 = \lambda^{-2} + 2\lambda. \tag{73}$$

As in the previous example, we take the magnetic field as to be the independent magnetic variable, but here we consider only one example of the particle alignment, in the axial direction.

Let H_0 denote the uniform Lagrangian axial magnetic field. Then

$$\mathsf{H}_l = (0, 0, H_0)^{\mathrm{T}}, \quad \mathsf{H} = (0, 0, \lambda^{-1} H_0)^{\mathrm{T}}. \tag{74}$$

Let the initial alignment of the magneto-active particles be $\mathsf{a}_0 = (0, 0, 1)^{\mathrm{T}}$, so that, from (35), we obtain $\mathsf{a} = (0, 0, \lambda)^{\mathrm{T}}$. Then, the invariants given by (28)–(29) specialize to

$$I_4 = H_0^2, \quad I_5 = \lambda^2 H_0^2, \quad I_6 = \lambda^4 H_0^2, \quad I_7 = \lambda^2, \quad I_8 = \lambda^4, \quad I_9 = H_0, \quad I_{10} = \lambda^2 H_0. \tag{75}$$

From (36) the only non-zero components of the stress are

$$\tau_{rr} = \tau_{\theta\theta} = -p^* + 2\lambda^{-1}\Omega_1^* + 2\lambda^{-2}(1 + \lambda^3)\Omega_2^*, \tag{76}$$

$$\tau_{zz} = -p^* + 2\lambda^2\Omega_1^* + 4\lambda\Omega_2^* + 2H_0^2\lambda^2(\Omega_5^* + 2\lambda^2\Omega_6^*) + 2\lambda^2\Omega_7^* + 4\lambda^4\Omega_8^* + 2H_0\lambda^2\Omega_{10}^*, \tag{77}$$

and from (39) the only non-zero component of the magnetic induction is

$$B_z = -\lambda(2H_o\Omega_4^* + 2H_o\lambda^2\Omega_5^* + 2H_o\lambda^4\Omega_6^* + \Omega_9^* + \lambda^2\Omega_{10}^*). \tag{78}$$

As in Sect. 4.1, we define a reduced energy function ω^*, this time as a function of λ and H_0 such that $\omega^* = \omega^*(\lambda, H_0)$. Then, we have simply

$$\tau_{zz} - \tau_{rr} = \lambda\,\omega_\lambda^*, \quad B_z = -\lambda\,\omega_{H_0}^*, \tag{79}$$

where the subscripts λ and H_0 indicate partial derivatives.

By the continuity condition (12.1) the magnetic field exterior to the cylinder is axial and equal to $\lambda^{-1}H_0$. It follows that the non-zero components of the Maxwell stress (14) are, on use of (3), given by

$$\tau_{rr}^m = \tau_{\theta\theta}^m = -\tau_{zz}^m = -\frac{1}{2}\mu_0\lambda^{-2}H_0^2. \tag{80}$$

If we assume that there is no mechanical surface traction on the lateral surface of the cylinder then the traction continuity condition (13) requires $\tau_{rr} = \tau_{rr}^m$, and hence this condition implies

$$p^* = 2\lambda^{-1}\Omega_1^* + 2(\lambda^{-2} + \lambda)\Omega_2^* + \frac{1}{2}\mu_0\lambda^{-2}H_0^2, \tag{81}$$

and as a result we have from (77)

$$\tau_{zz} = 2(\lambda^2 - \lambda^{-1})\Omega_1^* + 2(\lambda - \lambda^{-2})\Omega_2^* + 2H_0^2\lambda^2(\Omega_5^* + 2\lambda^2\Omega_6^*) + 2\lambda^2\Omega_7^* \\ +4\lambda^4\Omega_8^* + 2H_0\lambda^2\Omega_{10}^* - \frac{1}{2}\mu_0\lambda^{-2}H_0^2, \tag{82}$$

which is equivalent to $\tau_{zz} = \lambda\,\omega_\lambda^* + \tau_{rr}^m$. For a long cylinder with an approximately uniform field the mechanical traction required on the ends of the cylinder to maintain the deformation is approximately $\tau_{zz} - \tau_{zz}^m$, which is given explicitly as $\lambda\,\omega_\lambda^* - \mu_0\lambda^{-2}H_0^2$.

5 Reduction of the energy function

From (36) and (39) we see that for the incompressible case it may be too difficult to determine all the constitutive functions Ω_i^* from experiments. But we do need specific functions for the solution of particular boundary value problems of practical interest, and so it is necessary to reduce the complexity of the general form of the constitutive equation. We do this systematically in the present section in several steps. Most of the experimental data used herein was taken from [4], in which the independent magnetic variable was the magnetic field, this is why we only consider the function Ω^*.

First, we consider the invariants. In the classical theory of incompressible isotropic nonlinear elasticity only the invariants I_1 and I_2 are required. A prototype model of rubber elasticity is the neo-Hookean model, which is linear in I_1 and independent of I_2, and there are many other successful models that are independent of I_2 (see, for example, the collection of papers on rubber elasticity in [39]). Against this background we assume, as a first approximation, that Ω^* does not depend on I_2.

Second, we recall the definitions of the invariants $I_4 = \mathbf{H}_l \cdot \mathbf{H}_l$, $I_5 = \mathbf{H}_l \cdot (\mathbf{c}\mathbf{H}_l)$ and $I_6 = \mathbf{H}_l \cdot (\mathbf{c}^2\mathbf{H}_l)$, the first of which accounts for the 'magnitude' of the (Lagrangian) field in the response of the material and will be retained in the energy function. The interaction of the magnetic field with the deformation is embodied in the two invariants K_5 and K_6. Of these two we include only K_5 in the energy function in order to reflect this interaction.

Next, we consider the invariants $I_7 = \mathbf{a}_0 \cdot (\mathbf{c}\mathbf{a}_0)$, $I_8 = \mathbf{a}_0 \cdot (\mathbf{c}^2\mathbf{a}_0)$. These are independent of the applied magnetic field and account for the 'transverse isotropy' associated with the particle alignment, as in the pure elasticity of transversely isotropic materials. The preferred direction $\mathbf{a}_0$ has an important influence on the materials (see [4]), and in the first instance we include this effect through I_7 alone.

Finally, there remain the 'invariants' $I_9 = \mathbf{a}_0 \cdot \mathbf{H}_l$, $I_{10} = \mathbf{a}_0 \cdot (\mathbf{c}\mathbf{H}_l)$. The first of these accounts for the interaction between the applied field and the particle alignment. This interaction has a key role in distinguishing the response of these materials from those without particle alignment. Indeed, some micro-mechanical models developed recently (see for example, [23,24]) incorporate the interaction of the magnetic field with the particle alignment as an important ingredient. Thus, we include I_9 in the energy function. We shall also include I_{10} since it provides a three-way coupling between the magnetic field, the particle alignment and the deformation. Note that if the magnitude of $\mathbf{H}_l$ is denoted by δ then I_4, I_5 and I_6 are all of order δ^2, while I_9 and I_{10} are of order δ.

Thus, from the above considerations only the invariants $I_1, I_4, I_5, I_7, I_9, I_{10}$ are retained in the energy function:

$$\Omega^* = \Omega^*(I_1, I_4, I_5, I_7, I_9, I_{10}). \tag{83}$$

The total Cauchy stress and the magnetic induction in Eulerian form now specialize to

$$\boldsymbol{\tau} = 2\mathbf{b}\Omega_1^* - p^*\mathbf{I} + 2\mathbf{bH} \otimes \mathbf{bH}\Omega_5^* + 2\mathbf{a} \otimes \mathbf{a}\Omega_7^* + (\mathbf{a} \otimes \mathbf{bH} + \mathbf{bH} \otimes \mathbf{a})\Omega_{10}^* \tag{84}$$

and

$$\mathbf{B} = -(2\mathbf{bH}\Omega_4^* + 2\mathbf{b}^2\mathbf{H}\Omega_5^* + \mathbf{a}\Omega_9^* + \mathbf{ba}\Omega_{10}^*). \tag{85}$$

Thus far, three invariants have been removed from the original list, leaving a function of six invariants. However, it remains a difficult task to characterize the form of Ω^*, as it is the analogous case of purely elastic deformations of a transversely isotropic material, and further simplifications are necessary.

The next step is to draw on the example of a transversely isotropic material (see, for example, [40,41]), for which the energy function is decomposed into the sum of two terms. Here, we separate Ω^* in the form

$\Omega^* = \hat{\Omega}^* + \tilde{\Omega}^*$, where $\hat{\Omega}^*$ corresponds to the contribution to the energy function from factors that do not depend on the orientation of the particles, while $\tilde{\Omega}^*$ is due to the particle alignment and its interaction with the magnetic field and the deformation. Thus, $\hat{\Omega}^*$ is the 'isotropic' part and $\tilde{\Omega}^*$ the 'anisotropic' part of the energy function. Explicitly, we then have

$$\Omega^*(I_1, I_4, I_5, I_7, I_9, I_{10}) = \hat{\Omega}^*(I_1, I_4, I_5) + \tilde{\Omega}^*(I_7, I_9, I_{10}). \tag{86}$$

The simplifications (83, 86) are substantial, but even so it may still be difficult to find an appropriate form for the energy function from the limited experimental data available. Thus, further assumptions are needed. It should be emphasized, however, that the simplifications to be introduced in what follows are not necessarily appropriate for representing the full range of magnetoelastic behaviour of a magneto-active elastomeric, and they must be considered only as a first attempt to develop a specific form of the constitutive equation that can be used in the solution of some boundary-value problems. Consider the form for $\hat{\Omega}^*$ given by

$$\hat{\Omega}^*(I_1, I_4, I_5) = f(I_1)g(I_4) + \nu(I_4) + \vartheta(I_5), \tag{87}$$

which is a slight generalization of the isotropic model used in [15]. We have assumed here that $\hat{\Omega}^*(I_1, I_4, I_5)$ depends primarily on I_1 and I_4 and is separable in these variables. In the absence of a magnetic field the term $f(I_1)$ described the mechanical response of the isotropic part of the energy, while the factor $g(I_4)$ reflects the change in this response due to the applied magnetic field. The function $\nu(I_4)$ represents the energy that the body accumulates due to the magnetic field alone when there is no deformation. It has been necessary to include the function $\vartheta(I_5)$ (which also contributes to the electroelastic energy in the undeformed configuration) in order to accommodate the Maxwell stresses in the traction boundary conditions, as will be seen in the examples in the following sections.

For the function $\tilde{\Omega}^*$ we suggest the form

$$\tilde{\Omega}^*(I_7, I_9, I_{10}) = h(I_7)\omega(I_9, I_{10}) + \eta(I_9), \tag{88}$$

which is motivated by a corresponding decomposition for transversely isotropic elastic materials (see, for example, [40,41]). The function $h(I_7)$ described the transversely isotropic mechanical response of the material when there is no applied magnetic field, and, similarly to $g(I_4)$, the factor $\omega(I_9, I_{10})$ determines how that response changes on application of a magnetic field. The term $\eta(I_9)$ represents the magnetic energy associated with the 'anisotropy' when there is no deformation, and reflects, in particular, the effect of the relative alignment of the magnetic field and the particles.

5.1 Application of some restrictions for Ω^*

We recall that the conditions that ensure that there is no residual stress and no residual magnetic induction in the reference configuration include (43.2, 3), which, for our specialization, become

$$\bar{\Omega}_7^* = 0, \quad \bar{\Omega}_9^* + \bar{\Omega}_{10}^* = 0, \tag{89}$$

evaluated for $I_1 = 3, I_4 = I_5 = 0, I_7 = 1, I_9 = I_{10} = 0$. These can be used to obtain some restrictions on the functions h, η and ω, namely

$$h'(1)\omega(0,0) = 0, \quad h(1)\omega_9(0,0) + \eta'(0) + h(1)\omega_{10}(0,0) = 0, \tag{90.1, 2}$$

where a prime signifies differentiation with respect to the argument of the function, and ω_9 and ω_{10} represent partial derivatives of ω with respect to I_9 and I_{10}, respectively.

The conditions (90.1, 2) admit several options for the individual functions. To examine the possibilities we replace in (84) the particular form of the energy given by (86) with (87) and (88), so that

$$\begin{aligned}\boldsymbol{\tau} = {} & 2f'(I_1)g(I_4)\mathbf{b} - p^*\mathbf{I} + 2\vartheta'(I_5)\mathbf{bH} \otimes \mathbf{bH} + 2h'(I_7)\omega(I_9, I_{10})\mathbf{a} \otimes \mathbf{a} \\ & + h(I_7)\omega_{10}(\mathbf{a} \otimes \mathbf{bH} + \mathbf{bH} \otimes \mathbf{a}).\end{aligned} \tag{91}$$

The option $\omega(0,0) = 0$ must be excluded since for $\mathbf{H} = \mathbf{0}$ there would be no transversely isotropic contribution to the (purely) mechanical stress, and a material with aligned particles could not be distinguished from an isotropic material. Thus, $\omega(0,0) \neq 0$ and from (90.1) we deduce that $h'(1) = 0$. Given this, we must then rule out the option $h(1) = 0$ for the same reason. Next, we assume that the conditions (46) hold, i.e. in the absence of an applied magnetic field there is no residual magnetic induction whatever the deformation. Since I_7 (> 0) is arbitrary this yields the conditions $\omega_9(0, 0) = \omega_{10}(0, 0) = \eta'(0) = 0$, and Eq. (90.2) then holds. In summary, therefore, we adopt the restrictions

$$h'(1) = 0, \quad \omega_9(0, 0) = \omega_{10}(0, 0) = \eta'(0) = 0, \tag{92}$$

and we insist that $\omega(0, 0) \neq 0$ and $h(1) \neq 0$.

5.2 Results for the uniaxial tension of a bar

Consider two cylindrical samples of an MS elastomer, one with a random distribution of particles and the other with a distinguished particle alignment, in this case in the axial direction. We suppose that the volume fraction of particles is the same in each cylinder. For each cylinder a uniaxial tension test is performed, and the axial stress is measured as a function of the axial stretch λ, without a magnetic field and with a uniform axial magnetic field. For a given magnitude of the magnetic field these experiments yield four separate profiles for the stress against the stretch. Of course, more data may be obtained by performing the experiment with different magnitudes of the external magnetic field, as has been done by Bellan and Bossis [4] (see, in particular, Figs. 2 and 4 therein).

We recall from Sect. 4.2 that, on accounting for the exterior Maxwell stress, the required mechanical axial load on the ends of the cylinder is given by $\lambda\,\omega^*_\lambda - \mu_0\lambda^{-2}H_0^2$, which we denote by t_z. With respect to the energy function (83) this gives explicitly

$$t_z = 2(\lambda^2 - \lambda^{-1})\Omega_1^* + 2H_0^2\lambda^2\Omega_5^* + 2\lambda^2\Omega_7^* + 2H_0\lambda^2\Omega_{10}^* - \mu_0\lambda^{-2}H_0^2, \tag{93}$$

the last term of which comes from the exterior Maxwell stress components. The relevant invariants are obtained from (73) and (75) as

$$I_1 = 2\lambda^{-1} + \lambda^2, \quad I_4 = H_0^2, \quad I_5 = \lambda^2 H_0^2, \quad I_7 = \lambda^2, \quad I_9 = H_0, \quad I_{10} = \lambda^2 H_0. \tag{94}$$

We emphasize that the problem considered here is idealized in that the magnetic field is assumed uniform both within the cylinder and its exterior and edge effects at the cylinder ends are neglected. We now write down the equations for the four experiments mentioned above.

- **Isotropic case,** $H_0 = 0$. For an isotropic cylinder with no external magnetic field, Eq. (93) with (87) yields

$$t_z = 2(\lambda^2 - \lambda^{-1}) f'(I_1) g(0). \tag{95}$$

  Consider the function f to be given by

$$f(I_1) = \frac{1}{k}\left[\frac{(I_1 - 1)^k}{2^k} - 1\right], \tag{96}$$

  where k is a constant such that $k \geq 1/2$ (see [42] for the derivation of this model in nonlinear elasticity).
- **Isotropic material with** $H_0 \neq 0$. For an isotropic material with $H_0 \neq 0$, Eq. (93) coupled with (87) and (88) yields

$$t_z = 2(\lambda^2 - \lambda^{-1}) f'(I_1) g(I_4) + 2H_0^2\lambda^2\vartheta'(I_5) - \mu_0\lambda^{-2}H_0^2. \tag{97}$$

  As a simple model for the function g we consider the linear approximation

$$g(I_4) = g_0 + g_1 I_4, \tag{98}$$

  where g_0 and g_1 are constants.
  Regarding the function ϑ, the results in [4] suggest that there is no effect of the Maxwell stress at the end of the cylinder when $\lambda = 1$. From (97), we see that the term contributes a compressive stress of magnitude

$-\mu_0\lambda^{-2}H_0^2$ for $H_0 \neq 0$, in particular when $\lambda = 1$. The invariant I_5 has been included in our formulation in order to rectify this discrepancy.[2] We assume that ϑ has the linear form

$$\vartheta(I_5) = \vartheta_0 I_5, \tag{99}$$

where ϑ_0 is a constant. The Maxwell stress can be nullified for $\lambda = 1$ if we set $\vartheta_0 = \mu_0/2$, and (97) then reduces to

$$t_z = 2(\lambda^2 - \lambda^{-1})f'(I_1)g(I_4) + \mu_0 H_0^2(\lambda^2 - \lambda^{-2}). \tag{100}$$

- **Transversely isotropic material with** $H_0 = 0$. For a transversely isotropic cylinder with no applied magnetic field we have

$$t_z = 2(\lambda^2 - \lambda^{-1})f'(I_1)g(0) + 2\lambda^2 h'(I_7)\omega(0,0). \tag{101}$$

A possible function h compatible with (92) is

$$h(I_7) = h_0 + h_1 \ln I_7 - \frac{h_1}{m}I_7^m, \quad m < 0. \tag{102}$$

We can check that for any $I_7 > 0$ the first and second derivatives of $h(I_7)$ exist. The second derivative is needed in the calculations of the moduli tensors.

- **Transversely isotropic material with** $H_o \neq 0$. In the previous cases the particular forms given for the functions f, g and h were designed in order to fit the data in Fig. 2 of [4]. The situation is more delicate with respect to the function ω. Remembering (99) we note that the required mechanical stress on the ends of the cylinder in the presence of an axial magnetic field is

$$\begin{aligned} t_z = {} & 2(\lambda^2 - \lambda^{-1})f'(I_1)g(I_4) + 2\lambda^2 h'(I_7)\omega(I_9, I_{10}) \\ & +2H_0\lambda^2 h(I_7)\omega_{10}(I_9, I_{10}) + \mu_0 H_0^2(\lambda^2 - \lambda^{-2}). \end{aligned} \tag{103}$$

Figure 4 of [4] shows the result for the difference between the axial loads with and without a magnetic field for different values of the external magnetic field. The difference between (103) and (101), written Δt_z, is

$$\begin{aligned} \Delta t_z = {} & 2(\lambda^2 - \lambda^{-1})f'(I_1)[g(I_4) - g(0)] + 2\lambda^2 h'(I_7)[\omega(I_9, I_{10}) - \omega(0,0)] \\ & +2H_0\lambda^2 h(I_7)\omega_{10}(I_9, I_{10}) + \mu_0 H_0^2(\lambda^2 - \lambda^{-2}). \end{aligned} \tag{104}$$

It has not been possible to find a simple form for the function ω such that (92.2, 3) hold and such that the behaviour of Δt_z fits the data accurately. We therefore adopt a simple bi-quadratic form for ω that satisfies (92.2, 3), which nevertheless works well for the data presented in Fig. 2 of [4]. We have

$$\omega(I_9, I_{10}) = \omega_0 + \omega_1 I_9^2 + \omega_2 I_{10}^2 + \omega_3 I_9 I_{10}, \tag{105}$$

where ω_i, with $i = 0, 1, 2, 3$, are constants. A problem with truncated Taylor expansions of this kind is that the choice of the optimal constants may not be unique and there may be other sets of values that might be equally suitable[3].

The resulting form for the energy function is

$$\begin{aligned} \Omega^* = {} & \frac{1}{k}\left[\frac{(I_1 - 3)^k}{2^k} - 1\right](g_0 + g_1 I_4) + \nu(I_4) + \frac{\mu_o}{2}I_5 + \left(h_0 + h_1 \ln I_7 - \frac{h_1}{m}I_7^m\right)(\omega_0 + \omega_1 I_9^2 \\ & +\omega_2 I_{10}^2 + \omega_3 I_9 I_{10}) + \eta(I_9) + \Omega_0^*, \end{aligned} \tag{106}$$

where Ω_0^* is a constant satisfying $\nu(0) + h_0\,\omega_0 + \Omega_0^* = 0$.

[2] Another way to handle this problem would be not to consider the Maxwell stress for the extremes of the cylinder; however, we would still have this stress around the lateral surface of the cylinder, and from the expression for p^* we would end up with a factor $-1/2\mu_0\lambda^{-2}H_0^2$ instead of $-\mu_0\lambda^{-2}H_0^2$ in (97).

[3] In fact, we tried higher order Taylor expansions to see whether we could find better approximations for Δt_z in relation with the data shown in Fig. 4 of [4]; however, those attempts proved to be a failure, truncated Taylor expansions of higher order did not do better. We could try an expression of the form

$$\omega = \omega_0 + \omega_1 I_9^{n_1} + \omega_2 I_{10}^{n_2} + \omega_3 I_9^{n_3} I_{10}^{n_4},$$

but for such function we could have problems if n_i, $i = 1, 2, 3, 4$, are not integers for negatives values of I_9 and I_{10}.

5.2.1 Fitting the different constants

Now, we need to find suitable values for the different constants that appear in (106) to fit the data presented in [4], but before that consider the following remarks:

- We only use the data of Fig. 2 of [4]. First, we start with the isotropic case with and without magnetic field, finding appropriate values for k, g_0 and g_1.
- Then, we proceed with the transversely isotropic material, with and without magnetic field, looking for h_0, h_1, m, ω_0, ω_1 and ω_2.
- There is a problem with the data presented in Fig. 2 of [4]. It is possible to see that the upper limit for the stretching λ is $\lambda = 1.1$, the question is: what was the reason that a higher stretching could not be reached? The magneto-active particles are usually made of iron or iron composites [4,5], therefore, they can be considered as almost rigid in comparison with the surrounding rubber-like matrix (see [23]). As a result, the stiffness of the MS elastomer increases, and we can expect a 'lower' level of deformation in comparison with a pure rubber-like material. Nevertheless, $\lambda = 1.1$ seems to be too low (some additional experimental results can be found in [9,10,43]). The new question is: what should we assume about the behaviour of the MS elastomers for values of λ greater than 1.1?
  The answer to the previous question is not simple, because we are using a truncated Taylor expansion (see 105), a good fitting for $1 \leq \lambda \leq 1.1$ may mean an unacceptable behaviour for $\lambda > 1.1$.
- Another issue is related with the behaviour of the transversely MS elastomer in compression. If we are considering a material composed of a rubber-like matrix with particles aligned in chain-like structures, in the case the particle alignment is in the axial direction for the cylinder, we can expect that the behaviour in tension and compression will be different, as was hinted in [9,10,43]. Unfortunately, we do not have data for the same samples as used by Bellan and Bossis [4] for compression tests, which is particularly problematic in the case of the transversely isotropic MS elastomers (see in particular the phenomenon described in Fig. 14 of [43]).

From Fig. 2 of [4] for the isotropic part of Ω^* we obtain the results shown in Table 1.

As for the values of the constants that appear in the anisotropic part of Ω^*, we study three different cases shown in Table 2.

The value of m in (102) is equal to -10 for all these three cases. In Table 2 the units of ω_0 are kPa, and the units of ω_1, ω_2 and ω_3 are kPa/(kA/m)^2.

Now, for the cases (b) and (c) we simply proposed some 'reasonable' values for t_z for higher values of λ. In Fig. 1 we see the behaviour of t_z for the transversely isotropic case with $H_0 = 123$[kA/m] calculated with (106) and (103) using the three sets of values presented in Table 2.

- The set (a) of Table 2 was obtained fitting the data presented in Fig. 2 of [4]. In Fig. 1 we see the behaviour of t_z for three different ranges of values for λ. It is clear that for this set the behaviour of t_z may not be physically reasonable for $\lambda > 1.5$, where t_z would become negative (with H_0 constant) in tension.
- The set (b) of Table 2 was obtained using the same data from [4], but extrapolating linearly the last values of Fig. 2 of that paper to the range $1.1 < \lambda \leq 1.5$. We see on the left side of Fig. 1 that its behaviour is

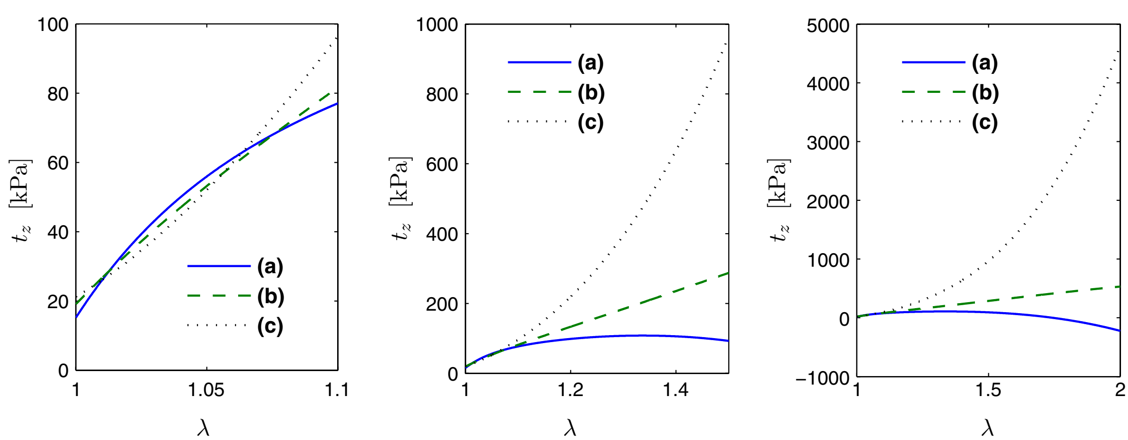

Fig. 1 Behaviour of t_z for three different sets of values for the constants for the transversely isotropic MS elastomer

Table 1 Values of the constants used in the isotropic part of the energy function

k	1	
g_0	100	kPa
g_1	-0.001	kPa/(kA/m)^2

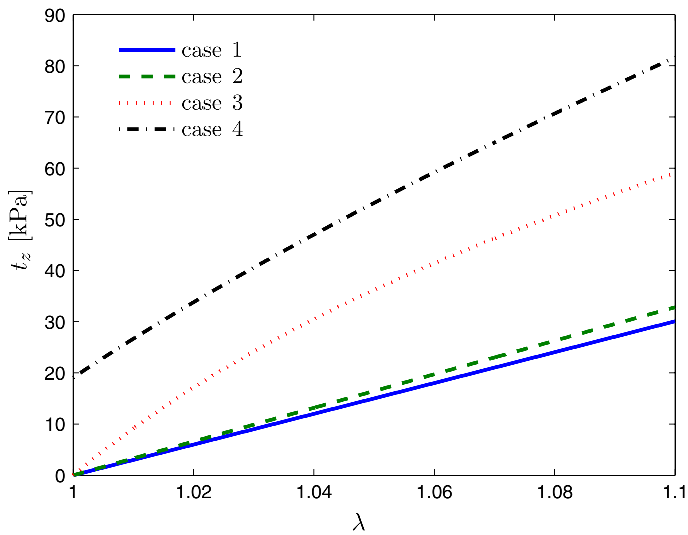

Fig. 2 Plots of t_z against λ for simple tension: case 1, isotropy, $H_0 = 0$; case 2, isotropy, $H_0 = 123$ [kA/m]; case 3, transverse isotropy $H_0 = 0$; case 4, transverse isotropy $H_0 = 123$ [kA/m]

reasonable for higher values of λ, but from the figure on the right, the fitting is not as good as using the set (a).

- Finally, the set of values (c) of Table 2 was obtained with an extrapolation into the range $1.1 < \lambda \leq 2$, assuming the typical behaviour of a rubber-like material for high values of the stretching. From the left side of Fig. 1 we see an even poorer fitting for $1 \leq \lambda \leq 1.1$ as using the set (b).

Since, we do not know the actual behaviour of this material for $\lambda > 1.1$, but, nevertheless, we want a moderately reasonable behaviour for some values of the stretching larger than 1.1, we choose the set (b) of values from Table 2 for the different constants that appear in (106) (including the values in Table 1). Additionally, we have used $\mu_0 = 1.2566 \times 10^{-3}$kN/kA^2. In Fig. 2 the stress t_z is plotted for the four different cases mentioned in this section using these values of the constants. Note, in particular, that the difference between the stresses is much larger for the transversely isotropic material than for the isotropic material.

5.3 Results for simple shear

There remain parts of the energy function that have not yet been used, in particular the functions ν and η. In order to determine these, two additional sources of experimental results are particularly useful. One is the paper by Jolly et al. [2], wherein Fig. 7 shows results for the 'shear modulus' as a function of the 'flux density' (magnetic induction) for a block, which is a problem very similar to that described in Sect. 4.1. Another important paper containing experimental results for the shear problem is that by Ginder et al. [8], especially Fig. 4, where the magnetization is plotted as a function of the magnetic field for a block under shear, for the two different alignments of the particles described in Sect. 4.1.

We now assess the data in these two papers on the basis of the particular form of the energy function Ω^*. We aim to calculate the 'shear modulus' for our problem and then to study its behaviour as a function of the magnetic field. Unfortunately, Fig. 7 of [2] was obtained assuming the magnetic induction **B** as the independent magnetic variable; moreover, the volume fraction of particles was different from that in [4], so we do not attempt to fit the data directly, but only to study quantitatively the behaviour of the model for this experiment.

Table 2 Proposed values of the constant for the transversely isotropic part of the energy function, for different limits for the stretching

(a)	$1 \leq \lambda \leq 1.1$	(b)	$1 \leq \lambda \leq 1.5$	(c)	$1 \leq \lambda \leq 2$
h_0	0.113578	h_0	0.010560	h_0	0.002881
h_1	0.007633	h_1	0.019229	h_1	0.012741
ω_0	2227.236	ω_0	884.0905	ω_0	1334.296
ω_1	0.040182	ω_1	−0.077869	ω_1	−0.063748
ω_2	−0.007653	ω_2	−0.005384	ω_2	0.198605
ω_3	0.019688	ω_3	0.061262	ω_3	−0.228273

- **Particle alignment in the x_2 direction.** Consider then the problem discussed in Sect. 4.1, with the particles aligned in the x_2 direction. Then, from (52), (54.1–3) and (55), we have

$$I_1 = 3 + \gamma, \quad I_4 = H_0^2, \quad I_5 = H_0^2(1+\gamma^2), \quad I_7 = 1+\gamma^2, \quad I_9 = H_0, \quad I_{10} = H_0(1+\gamma^2). \quad (107)$$

For the form of the constitutive equation we are using, the shear component of the total Cauchy stress and the x_2 component of the magnetic induction are given by (59) and (61) as

$$\tau_{12} = 2(\Omega_1^* + H_0^2\Omega_5^* + \Omega_7^* + H_0\Omega_{10}^*)\gamma, \quad (108)$$

$$B_2 = -[2H_0\Omega_4^* + 2H_0(1+\gamma^2)\Omega_5^* + \Omega_9^* + (1+\gamma^2)\Omega_{10}^*]. \quad (109)$$

In the linear theory of elasticity a shear deformation can be produced by application of a shear stress alone. This is not the case in nonlinear elasticity and certainly not here since, as can be seen from Sect. 4.1, normal components of stress are also necessary.

Let us write (108) in the form $\tau_{12} = G\gamma$, where G is a shear modulus given by

$$G(\gamma, H_0) = 2(\Omega_1^* + H_0^2\Omega_5^* + \Omega_7^* + H_0\Omega_{10}^*), \quad (110)$$

which depends on both the amount of shear γ and the magnetic field H_0. Substituting (106) into the above equation we obtain, after some algebraic manipulations,

$$G(\gamma, H_0) = \alpha_0(\gamma) + \alpha_1(\gamma)H_0^2, \quad (111)$$

where for brevity α_0 and α_1 are not given here.

Figure 7 of [2] shows the difference between the shear modulus for the cases either in the presence or absence of an external field. We form this difference

$$\Delta G \equiv G(\gamma, H_0) - G(\gamma, 0) = \alpha_1(\gamma)H_0^2. \quad (112)$$

Thus, independently of the value of $\alpha_1(\gamma)$, the 'shape' of the curve ΔG is a parabola in H_0. The results shown in Fig. 7 of [2] (see also Fig. 3 of [8]) suggest that the difference in the shear modulus increases until H_0 reaching a certain value (probably associated with magnetic saturation), and then tends to remain constant as H_0 increases further. This is not, of course, the behaviour of (112), so in the light of this data our model does not seem to work.

In order to use the data provided by Ginder et al. [8], let us determine the form of the B_2 component of the magnetic induction as a function of the external magnetic field. For this purpose, we use (106) in (109), which leads to

$$B_2 = -(\beta_0 + \beta_1 H_0), \quad (113)$$

where β_0 and β_1 are given by

$$\beta_0 = \eta'(K_9), \quad (114)$$

$$\beta_1 = \gamma^2 g_1 + 2\nu'(K_4) + (1+\gamma^2)\mu_0 + \left[h_0 + h_1\ln(1+\gamma^2) - \frac{h_1}{m}(1+\gamma^2)^m\right][2\omega_1 + \omega_3(1+\gamma^2)] + (1+\gamma^2)\left[h_0 + h_1\ln(1+\gamma^2) - \frac{h_1}{m}(1+\gamma^2)^m\right][2\omega_2(1+\gamma^2) + \omega_3]. \quad (115)$$

- **Particle alignment in the x_1 direction.** We also require results for the shear problem when the particles are aligned in the x_1 direction. The only invariants that we need to recalculate are I_7, I_9 and I_{10}, which, from (63.1, 3, 4) are $I_7 = 1$, $I_9 = 0$, $I_{10} = H_0\gamma$. From (67) and (69) we have

$$B_2 = -[2H_0\Omega_4^* + 2H_0(1+\gamma^2)\Omega_5^* + \gamma\Omega_{10}^*]. \tag{116}$$

Again using (106) and some manipulations, we obtain the expression

$$B_2 = -\left\{\gamma^2\left[g_0 + 2\omega_2\left(h_0 - \frac{h_1}{m}\right)\right] + (1+\gamma^2)\mu_0 + 2\nu'(I_4)\right\} H_0 \tag{117}$$

for the x_2 components of the magnetic induction.

5.3.1 A model for ν and η

Figure 4 of the paper by Ginder et al. [8] presents results for the magnetization **M** as a function of the magnetic field for the shear problem and for each of the two particle alignments considered. The material they used, however, had a different volume fraction of particles from that used in [4]. Nevertheless, since we do not expect at this stage to obtain definite expressions for the energy function, but rather a first approximation (as good as possible from the qualitative point of view), we will use this data, in particular to obtain the functions ν and η. In order to do so, consider the results for the case where the particle alignment is in the x_1 direction. Then, using (4) and (117) we have

$$\mu_0 M_2 = -[\zeta_0 + 2\nu'(I_4)]H_0, \tag{118}$$

where ζ_0 is defined by

$$\zeta_0 = \gamma_0^2\left[g_0 + 2\omega_2\left(h_0 - \frac{h_1}{m}\right)\right] + \mu_0(\gamma_0^2 + 2), \tag{119}$$

and γ_0 is a given (small) value of the shear.

Now, the data of Fig. 4 in [8] suggest that the magnetization is an odd function of H_0, and for values of $|H_0|$ greater than a certain threshold remains constant. This indicates that the magnetization has reached the saturation point for the magneto-active particles. We must also expect hysteresis, but here we assume that this effect is negligible. Then, an appropriate function that may be used in order to model the behaviour of M_2 is given by

$$\mu_0 M_2 = m_0 \tanh\left(\frac{H_0}{m_1}\right), \tag{120}$$

where the constant m_1 is related to the magnetic field necessary in order to reach the saturation point, while the constant m_0 corresponds to the value of the magnetization (times the magnetic constant μ_0) for that point of saturation.

From (118), (119), the particular value of I_4 for this problem given by (54.1) and (120), we obtain

$$\nu(I_4) = -m_0 m_1 \ln\left[\cosh\left(\frac{\sqrt{I_4}}{m_1}\right)\right] - \frac{1}{2}\zeta_0 I_4 + \nu_0, \tag{121}$$

where ν_0 is an arbitrary constant.

Next, in order to obtain η we consider the particles to be aligned in the x_2 direction and define η' as

$$\eta'(I_9) = \tilde{\eta}(I_9)I_9, \tag{122}$$

recalling that for this problem $I_9 = H_0$.

We also define ζ_1 as

$$\begin{aligned}\zeta_1 = (1+\gamma^2)\mu_0 + \left[h_0 + h_1\ln(1+\gamma^2) - \frac{h_1}{m}(1+\gamma^2)^m\right][2\omega_1 + \omega_3(1+\gamma^2)] \\ +(1+\gamma^2)\left[h_0 + h_1\ln(1+\gamma^2) - \frac{h_1}{m}(1+\gamma^2)^m\right][2\omega_2(1+\gamma^2) + \omega_3].\end{aligned} \tag{123}$$

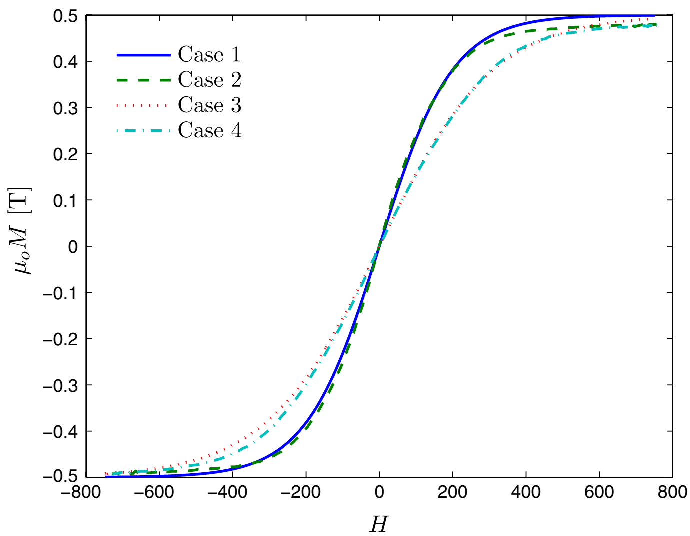

Fig. 3 Results for the shear experiment. Magnetization as a function of the magnetic field H [kA/m]. Case 1: parallel alignment (direction 2), theoretical model. Case 2: parallel alignment, experimental results. Case 3: perpendicular alignment (direction 1), theoretical model. Case 4: perpendicular alignment, experimental results

Then (113) becomes

$$B_2 = -[\tilde{\eta}(I_9) + \zeta_1 + 2\nu'(I_4)]H_0, \tag{124}$$

and the magnetization M_2 is then given from (4) by

$$\mu_0 M_2 = -[\tilde{\eta}(I_9) + \zeta_1 + \mu_0 + 2\nu'(I_9^2)]I_9. \tag{125}$$

As in the case of the alignment in the x_2 direction, the experimental data may be fitted by an hyperbolic tangent function, in this case

$$\mu_0 M_2 = m_0 \tanh\left(\frac{I_9}{m_2}\right), \tag{126}$$

where m_2 is a constant. Note that the experimental data suggest the same level of saturation for the magnetization for the two cases, which is to be expected since the volume fraction of particles is the same.

Then, using (126) and (121) in (125), we obtain

$$\tilde{\eta}(I_9) = \frac{m_0}{I_9}\left[\tanh\left(\frac{I_9}{m_2}\right) - \tanh\left(\frac{I_9}{m_1}\right)\right] - \zeta_1 - \mu_0 + \zeta_0. \tag{127}$$

As a result, from (122), we have

$$\eta(I_9) = m_0 \ln\left[\frac{\cosh^{m_2}\left(\frac{I_9}{m_2}\right)}{\cosh^{m_1}\left(\frac{I_9}{m_1}\right)}\right] + (\zeta_0 - \zeta_1 - \mu_0)\frac{I_9^2}{2} + \eta_0, \tag{128}$$

where η_0 is an arbitrary constant.

Figure 3 shows the results for the magnetization for our model and the data provided in Fig. 4 of the paper by Ginder et al. [8]. The values of the constants m_0, m_1 and m_2 are

$$m_0 = 0.4998[\mathrm{T}], \quad m_1 = 309.3395[\mathrm{kA/m}], \quad m_2 = 199.1828[\mathrm{kA/m}]. \tag{129}$$

It can be sen that the theory provides a good fit to the data.

5.4 Summary of results for the energy function

To summarize, we have constructed an energy function of the form

$$
\begin{aligned}
\Omega^* &= \left(\frac{I_1-3}{2}\right)(g_0+g_1 I_4) - \ln\left[\cosh\left(\frac{\sqrt{I_4}}{m_1}\right)\right] m_0 m_1 - \frac{1}{2}\zeta_0 I_4 + \frac{1}{2}\mu_0 I_5 \\
&\quad + \left(h_0 + h_1 \ln I_7 - \frac{h_1}{m} I_7^m\right)(\omega_0 + \omega_1 I_9^2 + \omega_2 I_{10}^2 + \omega_3 I_9 I_{10}) \\
&\quad + m_0 \ln\left[\frac{\cosh^{m_2}(I_9/m_2)}{\cosh^{m_1}(I_9/m_1)}\right] + (\zeta_0 - \zeta_1 - \mu_0)\frac{I_9^2}{2} + \Omega_0^*, \qquad (130)
\end{aligned}
$$

where the numerical values of the different constants that appear in the above expression are given in Tables 1, $2_{(b)}$ and in (129). The values of ζ_0 and ζ_1 may be obtained from the above expressions by evaluating for a given shear γ_0, which may be chosen to be small in the light of the small deformations used in the experiments. Finally, Ω_0^*, as indicated earlier, is given by the requirement that the energy is zero in the absence of both deformation and magnetic field.

We also record here the partial derivatives of Ω^* since they will be used in Sect. 6:

$$
\Omega_1^* = \frac{1}{2}(g_0 + g_1 I_4), \qquad (131)
$$

$$
\Omega_4^* = \frac{1}{2}(I_1-3)g_1 - \frac{1}{2}\tanh\left(\frac{\sqrt{I_4}}{m_1}\right)\frac{m_0}{\sqrt{I_4}} - \frac{1}{2}\zeta_0, \qquad (132)
$$

$$
\Omega_5^* = \frac{\mu_0}{2}, \qquad (133)
$$

$$
\Omega_7^* = \left(\frac{h_1}{I_7} - h_1 I_7^{m-1}\right)(\omega_0 + \omega_1 I_9^2 + \omega_2 I_{10}^2 + \omega_3 I_9 I_{10}), \qquad (134)
$$

$$
\begin{aligned}
\Omega_9^* &= \left(h_0 + h_1 \ln I_7 - \frac{h_1}{m} I_7^m\right)(2\omega_1 I_9 + \omega_3 I_{10}) + m_0\left[\tanh\left(\frac{I_9}{m_2}\right) - \tanh\left(\frac{I_9}{m_1}\right)\right] \\
&\quad + (\zeta_0 - \zeta_1 - \mu_0) I_9, \qquad (135)
\end{aligned}
$$

$$
\Omega_{10}^* = \left(h_0 + h_1 \ln I_7 - \frac{h_1}{m} I_7^m\right)(2\omega_2 I_{10} + \omega_3 I_9). \qquad (136)
$$

Note that (132) cannot be evaluated directly at $I_4 = 0$, but the limit as $I_4 \to 0$ exists. From (135) and (136) it is easy to check that the conditions (46) hold.

6 Boundary value problems: non-homogeneous deformations

In this section, the energy function (130) is used to obtain results for some controllable non-homogeneous deformations. Two problems with cylindrical symmetry are treated herein, namely the extension and inflation of a tube, and the extension and torsion of a cylinder.

Strictly, from the theoretical standpoint, these problems correspond to tubes and cylinders of ‘infinite’ lengths, which avoids difficulties with the magnetic boundary conditions [38]. In practice, specimens with length much larger than the diameter are used. For an isotropic magnetoelastic problem the effect of the boundary conditions (12.1, 2) on the magnetic field has been illustrated for a finite length tube in [38].

The general problem of finding controllable solutions for an isotropic magnetoelastic material was treated in [28]. In order to study the controllability here we consider the equations

$$
\operatorname{div}\boldsymbol{\tau} = \mathbf{0}, \quad \operatorname{curl}\mathbf{H} = \mathbf{0}, \quad \operatorname{div}\mathbf{B} = 0, \qquad (137)
$$

which, when specialized in cylindrical polar coordinates with no dependence on the angle θ, become (see, for example, [31])

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) = 0, \tag{138}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} = 0, \tag{139}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r}\tau_{rz} = 0, \tag{140}$$

$$\frac{\partial H_\theta}{\partial z} = 0, \quad \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = 0, \quad \frac{1}{r}\frac{\partial}{\partial r}(r H_\theta) = 0, \tag{141}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r B_r) + \frac{\partial B_z}{\partial z} = 0, \tag{142}$$

respectively.

6.1 Extension and inflation of a tube

Consider a circular cylindrical tube with reference geometry defined in cylindrical polar coordinates R, Θ, Z by $A_i \le R \le A_e$, $0 \le \Theta < 2\pi$ and $-\infty \le Z \le \infty$. The extension and inflation of the tube is described by the equations

$$r^2 = a_i^2 + \lambda_z^{-1}(R^2 - A_i^2), \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{143}$$

where a_i is the interior radius of the tube in the deformed configuration and λ_z is the (uniform) axial stretch. The corresponding exterior radius is denoted by a_e. The isotropic counterpart of this problem has been analysed in [15].

The deformation gradient, and the left and right Cauchy-Green deformation tensors have components

$$\mathsf{F} = \begin{pmatrix} (\lambda_z\lambda)^{-1} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_z \end{pmatrix}, \quad \mathsf{b} = \mathsf{c} = \begin{pmatrix} (\lambda_z\lambda)^{-2} & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda_z^2 \end{pmatrix}, \tag{144.1, 2}$$

where $\lambda = r/R$. The first and second invariants (27.1, 2) specialize to

$$I_1 = \operatorname{tr}\mathbf{c} = \lambda^2 + \lambda_z^2 + \lambda^{-2}\lambda_z^{-2}, \quad I_2 = \frac{1}{2}[(\operatorname{tr}\mathbf{c})^2 - \operatorname{tr}\mathbf{c}^2] = \lambda^{-2} + \lambda_z^{-2} + \lambda_z^2\lambda^2. \tag{145}$$

There are several options for the magnetic field and the particle alignment, we study one case.

6.1.1 Axial magnetic field and axial particle alignment

We consider a uniform magnetic field with Lagrangian value H_0. Then, on use of (6.1) and (144.1),

$$\mathsf{H}_l = (0, 0, H_0)^{\mathrm{T}}, \quad \mathsf{H} = (0, 0, \lambda_z^{-1} H_0)^{\mathrm{T}}, \tag{146}$$

and Eq. (141) is satisfied automatically.

The invariants I_4, I_5 and I_6 are obtained from (28) as

$$I_4 = H_0^2, \quad I_5 = \lambda_z^2 H_0^2, \quad I_6 = \lambda_z^4 H_0^2. \tag{147.1–3}$$

Consider also an initial uniform axial particle alignment and recall (35). Then, using (144.1), we obtain

$$\mathsf{a}_0 = (0, 0, 1)^{\mathrm{T}}, \quad \mathsf{a} = \mathsf{F}\mathsf{a}_0 = (0, 0, \lambda_z)^{\mathrm{T}}. \tag{148}$$

The remaining invariants (29) are

$$I_7 = \lambda_z^2, \quad I_8 = \lambda_z^4, \quad I_9 = H_0, \quad I_{10} = H_0\lambda_z^2. \tag{149}$$

The non-zero components of the total Cauchy stress (36) are

$$\tau_{rr} = -p^* + 2(\lambda\lambda_z)^{-2}\Omega_1^* + 2(\lambda^{-2} + \lambda_z^{-2})\Omega_2^*, \tag{150}$$

$$\tau_{\theta\theta} = -p^* + 2\lambda^2\Omega_1^* + 2[\lambda_z^{-2} + (\lambda\lambda_z)^2]\Omega_2^*, \tag{151}$$

$$\begin{aligned}\tau_{zz} = {} & -p^* + 2\lambda_z^2\Omega_1^* + 2[\lambda^{-2} + (\lambda\lambda_z)^2]\Omega_2^* + 2H_0^2\lambda_z^2\Omega_5^* + 4H_0^2\lambda_z^4\Omega_6^* + 2\lambda_z^2\Omega_7^* \\ & + 4\lambda_z^4\Omega_8^* + 2H_0\lambda_z^2\Omega_{10}^*,\end{aligned} \tag{152}$$

while the magnetic induction (39) has a single non-zero component

$$B_z = -(2H_0\lambda_z\Omega_4^* + 2H_0\lambda_z^3\Omega_5^* + 2H_0\lambda_z^5\Omega_6^* + \lambda_z\Omega_9^* + \lambda_z^3\Omega_{10}^*). \tag{153}$$

In what follows we shall require the components of the Maxwell stress. From (146), (14) and (3) the non-zero components of the Maxwell stress are[4]

$$\tau_{rr}^m = \tau_{\theta\theta}^m = -\tau_{zz}^m = -\frac{1}{2}\mu_0\lambda_z^{-2}H_0^2. \tag{154}$$

From (145), (147.1–3), (149) and the definition $\lambda = r/R$ we have that Ω_i^*, $i = 1, 2, \ldots, 10$ depend on r, λ_z and H_0 only, and so from (152) and (140) we conclude that $p^* = p^*(r)$. Therefore, integrating (138) in r we obtain

$$\tau_{rr}(r) = \int_{a_i}^{r} \frac{1}{\bar{r}}[\tau_{\theta\theta}(\bar{r}) - \tau_{rr}(\bar{r})]\,\mathrm{d}\bar{r} + c, \tag{155}$$

where $\bar{r}$ is a dummy variable of integration, and c is a constant to be determined from the boundary conditions for the stress.

We assume that the outer surface of the tube is free of mechanical loads, so that

$$\tau_{rr}(a_e) - \tau_{rr}^m(a_e) = 0, \tag{156}$$

and from (155) we obtain for c

$$c = \int_{a_i}^{a_e} \frac{1}{\bar{r}}[\tau_{rr}(\bar{r}) - \tau_{\theta\theta}(\bar{r})]\,\mathrm{d}\bar{r} + \tau_{rr}^m(a_e). \tag{157}$$

On the inner boundary the tube is subjected to a uniform pressure P, and hence

$$\tau_{rr}(a_i) = -P + \tau_{rr}^m(a_i), \tag{158}$$

and from (155), (157) and (158), considering that in this problem $\tau_{rr}^m(a_i) = \tau_{rr}^m(a_e)$, we obtain for the pressure

$$P = \int_{a_i}^{a_e} [\tau_{\theta\theta}(\bar{r}) - \tau_{rr}(\bar{r})]\frac{\mathrm{d}\bar{r}}{\bar{r}}. \tag{159}$$

Hence, from (150) and (151), we obtain

$$P = \int_{a_i}^{a_e} [(\bar{\lambda}^2 - \bar{\lambda}^{-2}\lambda_z^{-2})\Omega_1^* + (\bar{\lambda}^2\lambda_z^2 - \bar{\lambda}^{-2})\Omega_2^*]\frac{\mathrm{d}\bar{r}}{\bar{r}}, \tag{160}$$

[4] We assume that the body is completely surrounded by a free space, as a result we need to consider the presence of the Maxwell stresses for the extremes of the cylinder as external loads.

where $\bar{\lambda}$ is the value of λ for $r = \bar{r}$. This expression for P is the same as for an isotropic material except that Ω_1^* and Ω_2^* depend in general on the 'anisotropic' invariants I_7, I_8, K_9 and K_{10}. Moreover, it is unaffected by Maxwell stress, which has the same value on the inner and outer boundaries.

To extend the tube an axial mechanical load is required on the ends of the tube. We denote by $\mathcal{N}$ the resultant axial load. This is given by

$$\mathcal{N} = 2\pi \int_{a_i}^{a_e} t_z \, r \mathrm{d}r, \tag{161}$$

where $t_z = \tau_{zz} - \tau_{zz}^m$ is the axial component of the mechanical traction applied on the ends of the tube. Thus,

$$\mathcal{N} = 2\pi \int_{a_i}^{a_e} \tau_{zz} r \mathrm{d}r - \pi(a_e^2 - a_i^2)\tau_{zz}^m. \tag{162}$$

From (150) and (155), (157) we can obtain the explicit expression for $p^* = p^*(r)$, and so $\mathcal{N}$ would be completely determined as a function of Ω_i^*, λ_z and the external magnetic field, for brevity we do not show its full expression here.

6.1.2 Application to a particular energy function

For the energy function (130), we have $\Omega_1^* = \frac{1}{2}(g_0 + g_1 I_4)$, $\Omega_2^* = 0$. As a result, after integrating (160) and after some manipulations, we obtain

$$P = \frac{1}{2\lambda_z}(g_0 + g_1 H_0^2)\left[\lambda_z^{-1} \ln\left(\frac{\lambda_i}{\lambda_e}\right) - \frac{1}{2}\lambda_i^{-2}(\lambda_i^{-2} - \lambda_e^{-2})\right], \tag{163}$$

where $\lambda_i = a_i/A_i$ and $\lambda_e = a_e/A_e$, the values of λ on the inner and outer boundaries, with

$$\lambda_i^2\lambda_z - 1 = (\lambda_e^2\lambda_z - 1)\varsigma^2, \tag{164}$$

and $\varsigma = A_e/A_i$.

Remembering that the above expression for P is valid for both transversely isotropic and isotropic materials, Fig. 4 shows the function (163) for different values of the parameters λ_i, ς, λ_z and H_0.

For the energy function (130), Eq. (162) becomes

$$\begin{aligned}\mathcal{N} = 2\pi \int_{a_i}^{a_e} &[(2\lambda_z^2 - \lambda^2 - \lambda^{-2}\lambda_z^{-2})\Omega_1^* + 2H_0^2\lambda_z^2\Omega_5^* + 2\lambda_z^2\Omega_7^* + 2H_0\lambda_z^2\Omega_{10}^*] r \mathrm{d}r \\ &+\pi a_i^2 P - \pi(a_e^2 - a_i^2)\mu_0\lambda_z^{-2}H_0^2,\end{aligned} \tag{165}$$

where P is given by (163). Because of the presence of the terms Ω_7^* and Ω_{10}^* the difference between the results for transversely isotropic and isotropic materials is apparent, and we now consider two cases separately.

Isotropic case For the isotropic case the terms Ω_7^* and Ω_{10}^* are absent in (165) and the integration leads to the explicit result

$$\begin{aligned}\frac{\lambda_z^2\mathcal{N}}{\pi A_i^2} = (g_0 + g_1 H_0^2)&\left[\ln\left(\frac{\lambda_i}{\lambda_e}\right) + 2\frac{(\lambda_z^4 - 1)(\lambda_i^2 - \lambda_e^2)}{\lambda_e^2(\lambda_e^2\lambda_z - 1)} - \frac{\lambda_i^2 - \lambda_e^2}{\lambda_e^2}\right] \\ &+\mu_0\pi H_0^2(\lambda_z^4 - 1)(\lambda_e^2\varsigma^2 - \lambda_i^2).\end{aligned} \tag{166}$$

Figure 5 shows the normal force for different values of the parameters.

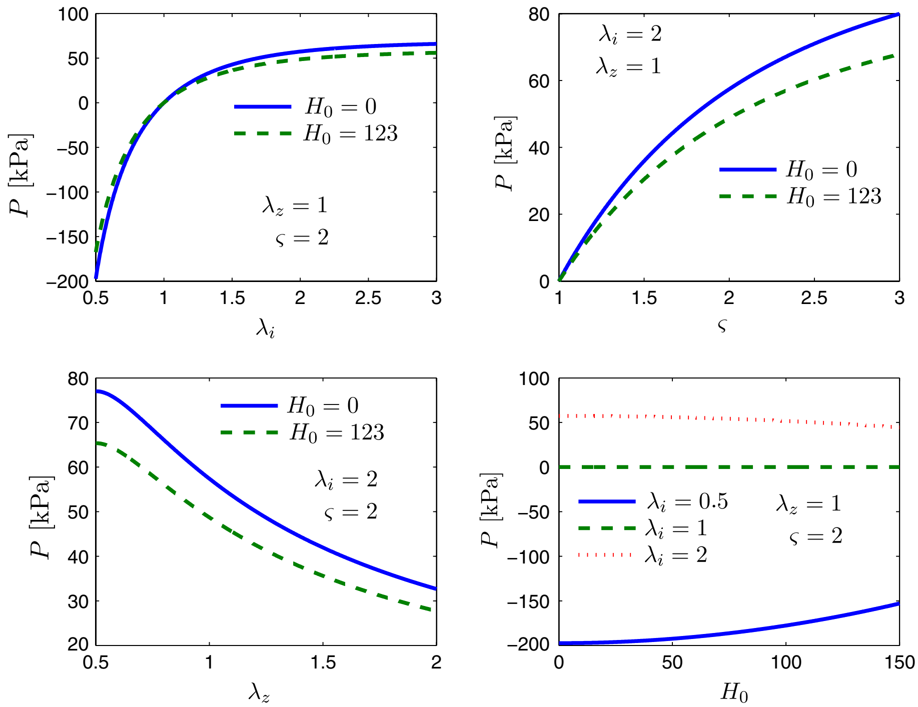

Fig. 4 Plot of the pressure P from (163) for selected values of λ_i, χ, λ_z and H_0 [kA/m]

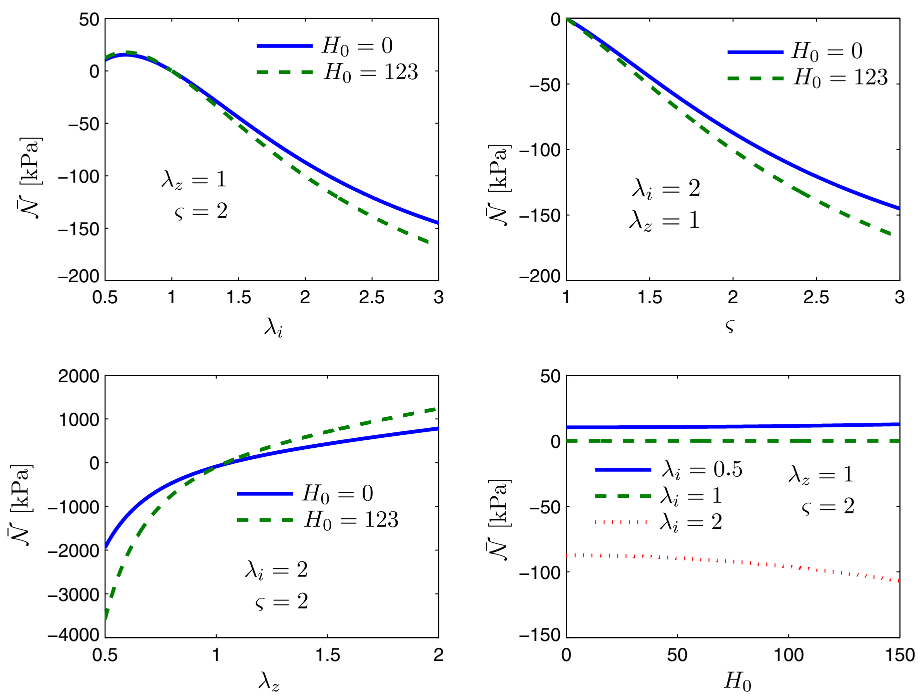

Fig. 5 Plot of the scaled axial load $\bar{\mathcal{N}} = \frac{\mathcal{N}}{\pi A_i^2}$ from (166) for selected values of λ_i, ς, λ_z and H_0 [kA/m]

Transversely isotropic case. For a transversely isotropic material, the terms Ω_7^* and Ω_{10}^* are given by (134) and (136), and the invariants I_7, I_9, I_{10} in (149) are independent of r. Equation (165) may then be written as

$$\frac{\mathcal{N}_{\text{tran}}}{\pi A_i^2} = \frac{\mathcal{N}_{\text{isot}}}{\pi A_i^2} + 2\lambda_z^2[\Omega_7^* + H_0\Omega_{10}^*](\varsigma^2 - 1), \tag{167}$$

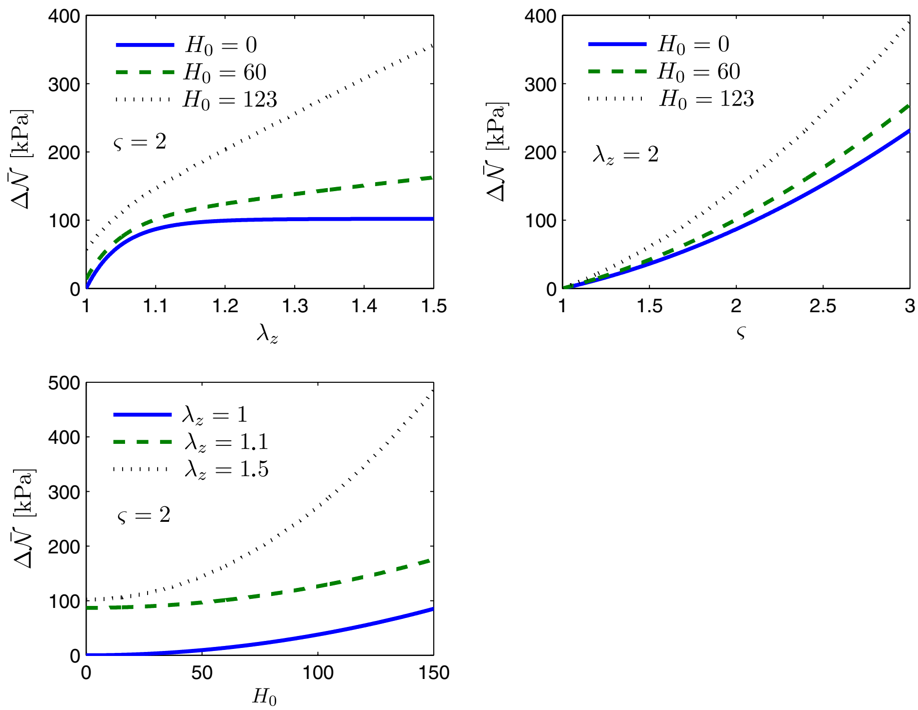

Fig. 6 Plot of the difference $\Delta\bar{\mathcal{N}}$ of the scaled axial loads against λ_z, ς and H_0 [kA/m]

where $\mathcal{N}_{\text{tran}}$ and $\mathcal{N}_{\text{isot}}$ correspond to the axial load for the transversely isotropic and for the isotropic cases, respectively. Consider the difference $\mathcal{N}_{\text{tran}} - \mathcal{N}_{\text{isot}}$, which we denote $\Delta\mathcal{N}$. Figure 6 illustrates the behaviour of $\Delta\mathcal{N}$, in the form $\Delta\bar{\mathcal{N}} = \frac{\Delta\mathcal{N}}{\pi A_i^2}$ for a range of parameter values.

Some remarks: From Figs. 4 and 5 we see that the magnetic field has only a moderate effect in the elastic response of the MS elastomer, for the internal pressure and the normal force needed for the inflation and extension of the tube in the 'isotropic' case. From the first three plots of Figs. 4 and 5 we see that for a certain range of the parameters the pressure P and the normal force $\mathcal{N}$ decrease for higher values of H_0. The 'isotropic' part of the behaviour of our MS elastomer is influenced in particular by the values of the constants shown in Table 1, where we see that g_1 is negative, being this the reason of the above phenomenon.

From Fig. 6 we see that for the normal force the difference between the 'isotropic' and 'transversely isotropic' parts of the response is rather marked, and that for higher values of the external field H_0 we have obtained larger values for $\Delta\mathcal{N}$, thus confirming that an alignment for the MS particles enhances significantly the response of the material to external magnetic fields.

In Figs. 4 and 5 we see, for example, that λ_z goes from 0.5 to 2. We have mentioned that the experimental data provided in [4] covers the range $[1, 1.1]$ for the stretch, therefore, we must take carefully the results presented in Figs. 4 and 5, especially for $0.5 \leq \lambda_z \leq 1$ and $1.1 < \lambda_z$.

For the same reason, considering the different problems we faced while fitting the energy function in the transversely isotropic case, in Fig. 6 we have only considered $\lambda_z \geq 1$ and for the first panel in Fig. 6 we have $1 \leq \lambda_z \leq 1.1$.

Other possibilities for the magnetic field and the particle alignment might be considered. For example, $\mathsf{H}_l = (0, 0, H_0)^{\mathrm{T}}$ and a radial uniform particle alignment field $\mathsf{a}_0 = (1, 0, 0)^{\mathrm{T}}$. But, in this case, from (36), it is not difficult to see that in general $\tau_{rz} \neq 0$, as a result, from the normal components of the stress and from (140) we would find that p^* would be a function of r and z. Thus, this solution is not controllable in general.

Another simple possibility may be to work with a radial uniform particle alignment as before, and a radial uniform magnetic field $\mathsf{H}_l = (H_0, 0, 0)^{\mathrm{T}}$. In such a case, we would have that $\tau_{r\theta} = \tau_{rz} = \tau_{\theta z} = 0$, then p^* may be obtained by simple integration from (138), but from (39) we would have that $B_r = B_r(r)$, thus, this implies that B_r would be singular at $r = 0$. Therefore, this solution is not admissible either.

Other possibilities may arise. We could try to work with the magnetic induction as the independent magnetic variable, but we do not study the problem of finding more controllable solutions for this case here.

6.2 Extension and torsion of a cylinder

The problem of extension and torsion for a cylinder has been studied previously in the context of isotropic magnetoelastic materials by Dorfmann and Ogden [12]. In reference cylindrical polar coordinates the cylinder is defined as $0 \le R \le A$, $0 \le \Theta \le 2\pi$, $-\infty \le Z \le \infty$, and the deformation is described by

$$r = \lambda_z^{-1/2} R, \quad \theta = \Theta + \lambda_z \tau Z, \quad z = \lambda_z Z, \tag{168}$$

where (r, θ, z) are cylindrical polar coordinates in the deformed configuration and τ is the angle of twist per unit deformed length of the cylinder. Referred to the two sets of coordinates, the deformation gradient has components

$$\mathsf{F} = \begin{pmatrix} \lambda_z^{-1/2} & 0 & 0 \\ 0 & \lambda_z^{-1/2} & \lambda_z\gamma \\ 0 & 0 & \lambda_z \end{pmatrix}, \tag{169}$$

where γ is defined by

$$\gamma = \tau r. \tag{170}$$

The associated left and right Cauchy-Green deformation tensors have components

$$\mathsf{b} = \begin{pmatrix} \lambda_z^{-1} & 0 & 0 \\ 0 & \lambda_z^{-1} + \lambda_z^2\gamma^2 & \lambda_z^2\gamma \\ 0 & \lambda_z^2\gamma & \lambda_z^2 \end{pmatrix}, \quad \mathsf{c} = \begin{pmatrix} \lambda_z^{-1} & 0 & 0 \\ 0 & \lambda_z^{-1} & \lambda_z^{1/2}\gamma \\ 0 & \lambda_z^{1/2}\gamma & \lambda_z^2(1+\gamma^2) \end{pmatrix}, \tag{171}$$

and the first and second invariants in (27.1–3) are calculated as

$$I_1 = 2\lambda_z^{-1} + \lambda_z^2(1+\gamma^2), \quad I_2 = 2\lambda_z + \lambda_z^{-2} + \lambda_z\gamma^2. \tag{172}$$

6.2.1 Axial magnetic field and axial particle alignment

As in the problem of extension and inflation of a tube, we consider a uniform (Lagrangian) axial magnetic field $\mathsf{H}_l = (0, 0, H_0)^{\mathrm{T}}$, and a uniform axial particle alignment $\mathsf{a}_0 = (0, 0, 1)^{\mathrm{T}}$. Then H is given by (146) and I_4 by (147.1). The invariants I_5 and I_6 in (28) become

$$I_5 = H_0^2\lambda_z^2(1+\gamma^2), \quad I_6 = H_0^2\lambda_z[\gamma^2 + (1+\gamma^2)^2\lambda_z^3], \tag{173}$$

while, from (169) and (35), we obtain $\mathsf{a} = (0, \gamma\lambda_z, \lambda_z)^{\mathrm{T}}$. The remaining invariants are obtained from (29) as

$$I_7 = (1+\gamma^2)\lambda_z^2, \quad I_8 = \gamma^2\lambda_z + (1+\gamma^2)^2\lambda_z^4, \quad I_9 = H_0, \quad I_{10} = H_0(1+\gamma^2)\lambda_z^2. \tag{174}$$

The components of the total Cauchy stress are given by (36) as

$$\tau_{rr} = -p^* + 2\lambda_z^{-1}\Omega_1^* + 2\lambda_z^{-2}[1 + (1+\gamma^2)\lambda_z^3]\Omega_2^*, \tag{175}$$

$$\begin{aligned}\tau_{\theta\theta} = &-p^* + 2(\lambda_z^{-1} + \gamma^2\lambda_z^2)\Omega_1^* + 2\gamma^2\lambda_z^2(H_0^2\Omega_5^* + \Omega_7^* + H_0\Omega_{10}^*) \\ &+2\lambda_z^{-2}\left[1 + (1+\gamma^2)\lambda_z^3\right]\left[\Omega_2^* + 2\gamma^2\lambda_z^2\left(H_0^2\Omega_6^* + \Omega_8^*\right)\right],\end{aligned} \tag{176}$$

$$\begin{aligned}\tau_{zz} = &-p^* + 2\lambda_z^2\Omega_1^* + 4\lambda_z\Omega_2 + 2\lambda_z^2\left(H_0^2\Omega_5^* + \Omega_7^* + H_0\Omega_{10}^*\right) \\ &+4(1+\gamma^2)\lambda_z^4\left(H_0^2\Omega_6^* + \Omega_8^*\right),\end{aligned} \tag{177}$$

$$\begin{aligned}\tau_{\theta z} = &\,2\gamma\lambda_z^2\Omega_1^* + 2\gamma\lambda_z\Omega_2^* + 2\gamma\lambda_z^2\left(H_0^2\Omega_5^* + \Omega_7^* + H_0\Omega_{10}^*\right) \\ &+2\gamma\lambda_z\left[1 + 2(1+\gamma^2)\lambda_z^3\right]\left(H_0^2\Omega_6^* + \Omega_8^*\right),\end{aligned} \tag{178}$$

with $\tau_{r\theta} = \tau_{rz} = 0$, and from (39) we obtain for the components of the magnetic induction

$$B_\theta = -\left\{2H_0\gamma\lambda_z\Omega_4^* + 2H_0\gamma\left[1+(1+\gamma^2)\lambda_z^3\right]\Omega_5^* + 2H_0\gamma\left[\lambda_z^{-1}+(1+2\gamma^2)\lambda_z^2 + (1+\gamma^2)^2\lambda_z^5\right]\Omega_6^* + \gamma\lambda_z\Omega_9^* + \gamma\left[1+(1+\gamma^2)\lambda_z^3\right]\Omega_{10}^*\right\}, \quad (179)$$

$$B_z = -\left\{2H_0\lambda_z\Omega_4^* + 2H_0(1+\gamma^2)\lambda_z^3\Omega_5^* + 2H_0\lambda_z^2\left[\gamma^2+(1+\gamma^2)^2\lambda_z^3\right]\Omega_6^* + \lambda_z\Omega_9^* + \lambda_z^3(1+\gamma^2)\Omega_{10}^*\right\}, \quad (180)$$

with $B_r = 0$.

Without giving all the details we now discuss briefly the controllability of the above solution. Since γ, and hence the invariants, depend only on r then Ω^* depends only on r. It follows from the above components of the stress that (139) is satisfied trivially, and from (140) we conclude that $p^* = p^*(r)$. As a result p^* may be obtained by integration from (138). Since B_θ and B_z also depend only on r, Eq. (142) is likewise satisfied trivially. Hence the considered solution is controllable.

6.2.2 Boundary conditions

Regarding the boundary conditions, we have essentially two quantities to calculate. One of these is the resultant mechanical traction $\mathcal{N}$ applied on the ends of the cylinder, and the other is the torque on the ends of the cylinder, denoted $\mathcal{M}$, required to effect the torsion. These are given by

$$\mathcal{N} = 2\pi\int_0^a t_z r\,\mathrm{d}r, \quad \mathcal{M} = 2\pi\int_0^a \tau_{\theta z} r^2\,\mathrm{d}r, \quad (181)$$

where t_z is the axial mechanical load per unit area applied at the ends of the cylinder.

Since $\mathbf{H}$ is the same as in the previous problem, the non-zero components of the Maxwell stress are given by (154). Then, on specializing (13), we obtain

$$t_z = \tau_{zz} - \tau_{zz}^m. \quad (182)$$

Hence

$$\mathcal{N} = 2\pi\int_o^a \tau_{zz} r\mathrm{d}r - \pi a^2\tau_{zz}^m, \quad (183)$$

which, on use of the radial equilibrium equation in the form

$$\tau_{rr} + \tau_{\theta\theta} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\tau_{rr}), \quad (184)$$

can be rewritten as

$$\mathcal{N} = \pi\int_0^a (2\tau_{zz} - \tau_{rr} - \tau_{\theta\theta})r\mathrm{d}r + \pi a^2\tau_{rr}(a) - \pi a^2\tau_{rr}^m. \quad (185)$$

If no mechanical tractions are applied on the surface $r = a$, then from (182) we obtain $\tau_{rr}(a) = \tau_{rr}^m$.

It is convenient to introduce a reduced energy function, here a function of λ_z, γ and H_0. We write this as $\omega^* = \omega^*(\lambda_z, \gamma, H_0)$. Then, it is easy to show that

$$\tau_{\theta z} = \omega_\gamma^*, \quad 2\tau_{zz} - \tau_{rr} - \tau_{\theta\theta} = 2\lambda_z\,\omega_{\lambda_z}^* - 3\gamma_\gamma^*, \quad B_z = -\lambda_z\,\omega_{H_0}^*, \quad (186)$$

where the subscripts λ_z, γ and H_0 represent partial derivatives.

Thus, we have the formulas

$$\mathcal{N} = \pi\int_o^a (2\lambda_z\,\omega_{\lambda_z}^* - 3\gamma\,\omega_\gamma^*)r\mathrm{d}r - \pi a^2\mu_0\lambda_z^{-2}H_0^2, \quad (187)$$

and

$$\mathcal{M} = 2\pi \int_0^a \omega_\gamma^* r^2 \mathrm{d}r, \tag{188}$$

which apply for a general form of the energy function within the considered class.

6.2.3 Boundary conditions for a particular energy function

For the energy function (130), the constitutive terms in the integrals (187) and (188) specialize to

$$\omega_\gamma^* = 2\gamma\lambda_z^2(\Omega_1^* + H_0^2\Omega_5^* + \Omega_7^* + H_0\Omega_{10}^*), \tag{189}$$

and, with the above,

$$2\lambda_z\,\omega_{\lambda_z}^* - 3\gamma\,\omega_\gamma^* = (2-\gamma^2)\frac{\omega_\gamma^*}{\gamma} - 4\lambda_z^{-1}\Omega_1^*. \tag{190}$$

First, we examine the isotropic case, for which $\Omega_7^* = \Omega_{10}^* = 0$.

Isotropic case. For the isotropic specialization we have

$$\mathcal{N} = 2\pi \int_0^a [\lambda_z^2(2-\gamma^2)(\Omega_1^* + H_0^2\Omega_5^*) - 2\lambda_z^{-1}\Omega_1^*]r\mathrm{d}r - \pi a^2\mu_0\lambda_z^{-2}H_0^2 \tag{191}$$

and

$$\mathcal{M} = 4\pi\lambda_z^2 \int_0^a \gamma(\Omega_1^* + H_0^2\Omega_5^*)r^2\mathrm{d}r. \tag{192}$$

Since we have taken Ω_1^* and Ω_5^* to be constant, we obtain finally

$$\mathcal{N} = \frac{1}{2}\pi a^2\lambda_z^2(\Omega_1^* + H_0^2\Omega_5^*)(4-\tau^2a^2) - 2\pi a^2\lambda_z^{-1}\Omega_1^* - \pi a^2\mu_0\lambda_z^{-2}H_0^2 \tag{193}$$

and

$$\mathcal{M} = \pi a^4\lambda_z^2\tau(\Omega_1^* + H_0^2\Omega_5^*). \tag{194}$$

Figure 7 illustrates the behaviour of $\bar{\mathcal{N}} = \frac{\mathcal{N}}{\pi a^2}$ and $\bar{\mathcal{M}} = \frac{\mathcal{M}}{\pi a^3}$ for selected values of λ_z, $\gamma_a = \tau a$ and H_0.

The special case in which $\lambda_z = 1$ is worth highlighting since we have $\Omega_5^* = \mu_0/2$ and hence

$$\mathcal{N} = -\frac{1}{2}\pi\tau^2a^2(2\Omega_1^* + \mu_0H_0^2) = -\frac{1}{2}\tau\mathcal{M}. \tag{195}$$

This generalizes a result which holds in nonlinear elasticity for the neo-Hookean material, for which $2\Omega_1^*$ is the shear modulus of the material in the reference configuration, and shows that the presence of the magnetic field enhances the compressive axial load generated by the purely mechanical torsion.

Transversely isotropic case. For the transversely isotropic material, based on the energy function (130), the differences in the values of $\mathcal{N}$ and $\mathcal{M}$ compared with the isotropic case are obtained from (187), (188), (193) and (194) as

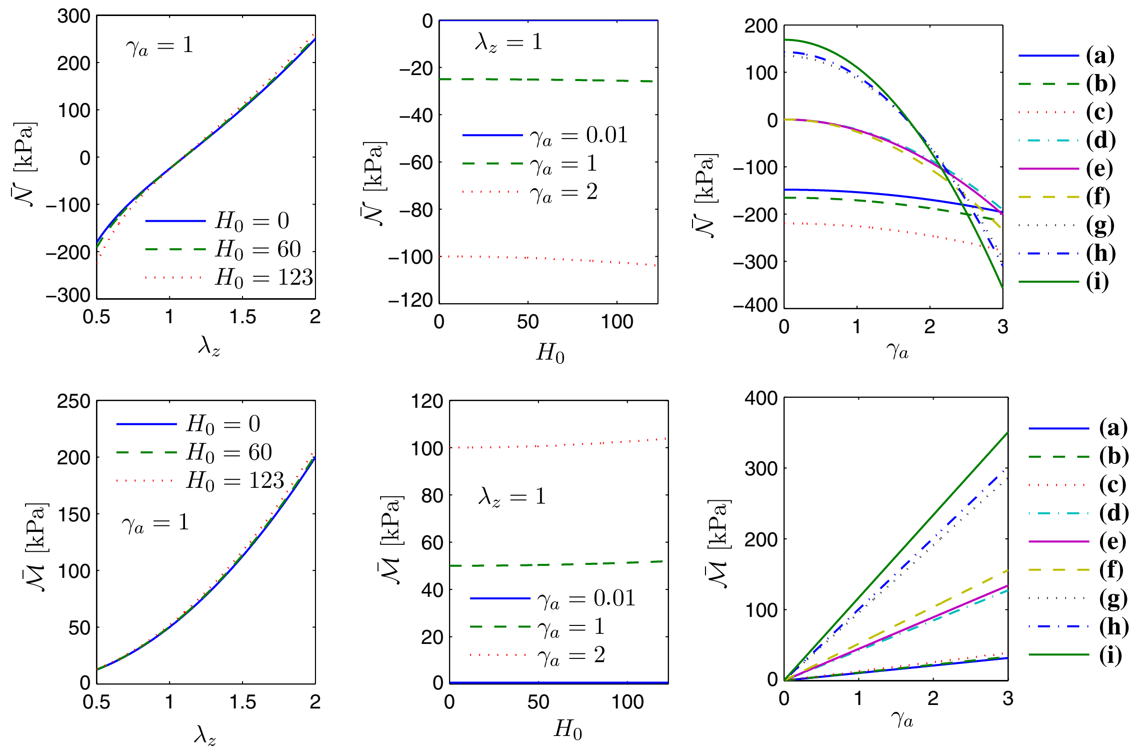

Fig. 7 Plot of the scaled axial load $\bar{\mathcal{N}}$ and the scaled torque $\bar{\mathcal{M}}$ from (193) and (194) for selected values of λ_z, γ_a and H_0 [kA/m]. For the two graphs on the right we have: *a* $\lambda_z = 0.5$, $H_0 = 0$; *b* $\lambda_z = 0.5$, $H_0 = 60$; *c* $\lambda_z = 0.5$, $H_0 = 123$; *d* $\lambda_z = 1$, $H_0 = 0$; *e* $\lambda_z = 1$, $H_0 = 60$; *f* $\lambda_z = 1$, $H_0 = 123$; *g* $\lambda_z = 1.5$, $H_0 = 0$; *h* $\lambda_z = 1.5$, $H_0 = 60$; *i* $\lambda_z = 1.5$, $H_0 = 123$

$$\mathcal{N}_{\text{diff}} = 2\pi \int_0^a (2 - \gamma^2)(\Omega_7^* + H_0 \Omega_{10}^*) r \, \mathrm{d}r, \tag{196}$$

$$\mathcal{M}_{\text{diff}} = 4\pi \int_0^a \gamma (\Omega^* + H_0 \Omega_{10}^*) r^2 \mathrm{d}r. \tag{197}$$

With Ω_7^* and Ω_{10}^* given by (134) and (136) these integrals can be evaluated in closed form. The resulting expressions are very lengthy and we do not include them here. Instead, we illustrate the results numerically, remembering the definition $\gamma_a = \tau a$. Figures 8 and 9 show the results for $\bar{\mathcal{N}}_{\text{diff}} = \frac{\mathcal{N}_{\text{diff}}}{\pi a^2}$ and $\bar{\mathcal{M}}_{\text{diff}} = \frac{\mathcal{M}_{\text{diff}}}{\pi a^3}$.

Like in the problem presented in the previous section, here from Figs. 7, 8, and 9 we see that an alignment for the particles enhances significantly the response of our MS elastomer in the presence of a magnetic field.

7 Conclusions

The general formulation presented in this paper can be used as a background against which the results of experiments on transversely isotropic magnetoelastic materials, in particular MS elastomers, can be assessed. These materials appear to be more important in terms of their capacity to respond to magnetic fields than isotropic MS elastomers (see, for example, [2,4]).

From Sect. 3, we see that these materials possess characteristics that are similar in terms of mechanics to those of fibre-reinforced materials with two families of fibres [40,41]. The number of invariants present in the energy function in its general form is too large to be able to fully characterize the material properties, and reduced forms for the energy function must be considered in order to make meaningful comparisons with currently available experimental data. Such reductions must at the same time produce physically meaningful forms of the energy function and lead to mathematical formulations that are well posed.

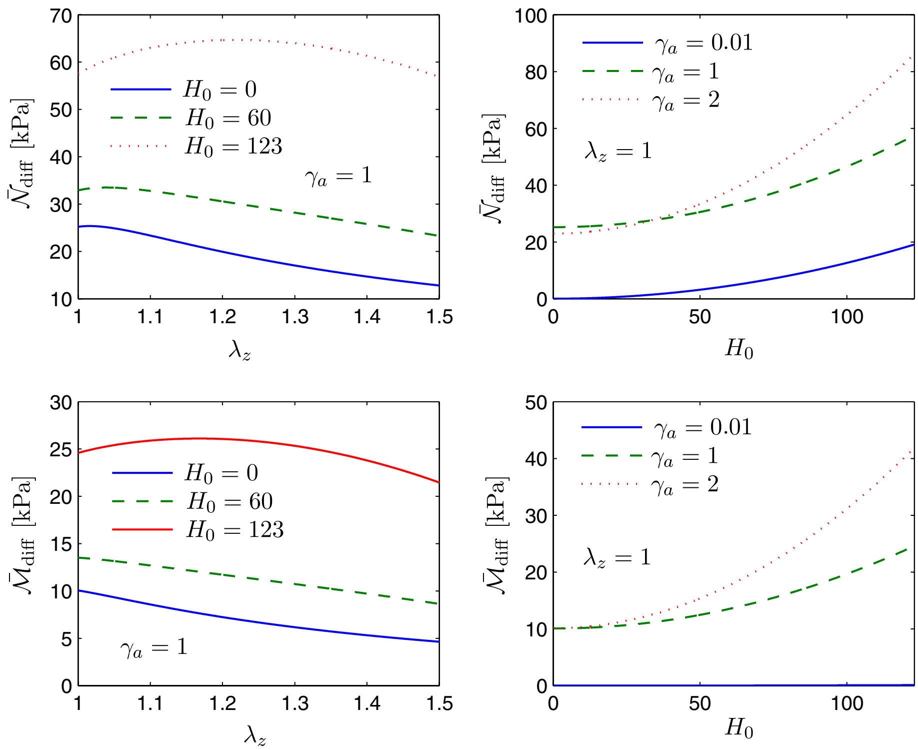

Fig. 8 Plots of the scaled difference in the axial load $\bar{\mathcal{N}}_{\mathrm{diff}}$ and the scaled difference in torque $\bar{\mathcal{M}}_{\mathrm{diff}}$ from (196) and (197) against λ_z and H_0 [kAm] for selected parameter values

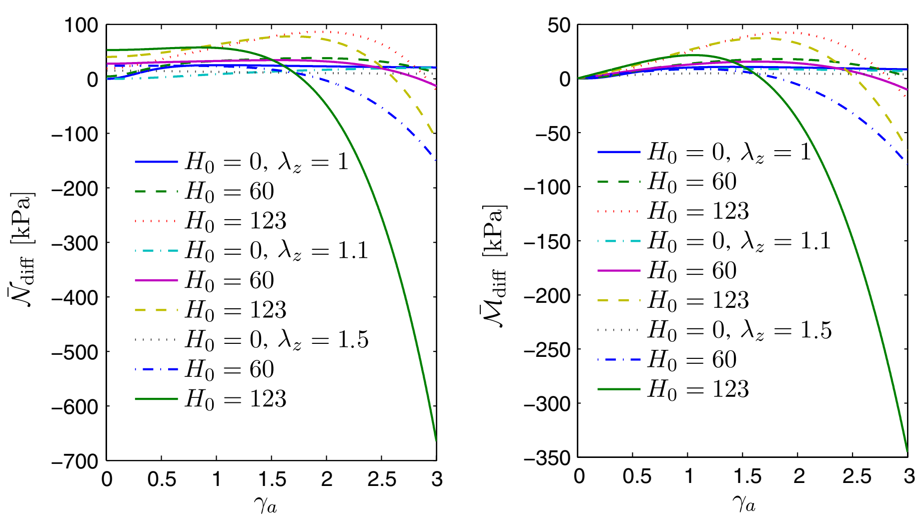

Fig. 9 Plot of the scaled difference in axial load $\bar{\mathcal{N}}_{\mathrm{diff}}$ and the scaled difference in torque $\bar{\mathcal{M}}_{\mathrm{diff}}$ from (196) and (197) against γ_a for selected parameter values. In this figure H_0 has units [kA/m]

There are a number of problems and simplifications that must be addressed in subsequent work. For example, the boundary-value problems presented in this paper assumed that the material is completely surrounded by free space (see [33]). This idealization is not realistic since application of mechanical traction to the surface of the material requires contact between the loading device and the material. Thus, formulations are needed that include the possible magnetic interaction between the material body and the material of the loading device. A preliminary formulation has recently been proposed by Bustamante [44].

Another problem is that in order to deal with the magnetic boundary conditions (12.1, 2) we have considered bodies of 'infinite' or 'semi-infinite' geometries, such as tubes and cylinders of finite radius but infinite length.

The boundary conditions (12.1, 2) make it difficult, if not impossible, in the nonlinear theory to obtain closed-form solutions of boundary-value problems with finite geometries. No exact solution of Eqs. (5) and (9) and the associated boundary conditions have yet been found for such problems. Thus, the solution requires a numerical approach, as exemplified in [38] using a finite difference method. For more complicated problems it would be natural to employ finite element methods, for which a suitable principle of virtual work or variational principle is required. Towards this objective certain magnetoelastic variational principles have been recently developed by Bustamante et al. [33].

The Mullins effect, which is a well-known (inelastic) stress softening in rubber-like materials containing particle fillers has also been detected recently in MS elastomers by Coquelle and Bossis [45]. An extension of the present theory to include this effect is a possible direction of future research. The question of the stability of bodies subjected to finite deformation and a magnetic field is also of interest. For example, Varga et al. [43] found that for a cube made of a transversely isotropic MS elastomer with a 'high' volume fraction of particles when under compression in the direction of the particle alignment there is a sort of 'collapse' or instability of the material (see Figs. 14 and 15 in [43]).

In a recent paper, Criscione [46] criticized the use of the invariants I_1 and I_2 for the characterization of isotropic hyperelastic materials. In accordance with his results, experimental errors are significantly magnified if an energy function W is defined in terms of such invariants, thus making impossible to find acceptable and accurate expressions for such a function for certain ranges of values for I_1 and I_2. Considering that for MS elastomers there is still little experimental data to start any discussion about which set of invariants may be better in order to characterize the behaviour of these materials, in our research we have used the classical set of invariants found, for example in [34].

The preliminary form of the energy function presented in (130) is just a first approximation. The development of improved models must wait until more experimental data becomes available.

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