MULTIPLE-END SOLUTIONS TO THE ALLEN-CAHN EQUATION IN $\mathbb{R}^2$

by

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Abstract. — We construct a new class of entire solutions for the Allen-Cahn equation $\Delta u + (1-u^2)u = 0$, in $\mathbb{R}^2(\sim \mathbb{C})$. Given $k \geq 1$, we find a family of solutions whose zero level sets are, away from a compact set, asymptotic to $2k$ straight lines (which we call the *ends*). These solutions have the property that there exist $\theta_0 < \theta_1 < \ldots < \theta_{2k} = \theta_0 + 2\pi$ such that $\lim_{r\to+\infty} u(re^{i\theta}) = (-1)^j$ uniformly in θ on compact subsets of (θ_j, θ_{j+1}), for $j = 0, \ldots, 2k-1$.

1. Introduction and statement of main results

1.1. Introduction. — In this paper, we are interested in the construction of a new class of entire solutions, in the entire space $\mathbb{R}^N$, for the semilinear elliptic equation

$$\Delta u + (1-u^2)\, u = 0\,, \tag{1.1}$$

known as the Allen-Cahn equation. This problem has its origin in the *gradient theory of phase transitions* [2], a model in which two distinct phases (represented by the values $u = \pm 1$) try to coexist in a domain Ω while minimizing their interaction which is proportional to the $(N-1)$-dimensional volume of the interface. Idealizing the phase as a regular function which takes values close to ± 1 in most of the domain, except in a narrow transition layer of width ε, one defines the Allen-Cahn energy,

$$J_\varepsilon(u) := \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2\, \mathrm{d}\mathbf{x} + \frac{1}{4\varepsilon}\int_\Omega (1-u^2)^2\, \mathrm{d}\mathbf{x},$$

whose critical points satisfy the Euler-Lagrange equation

$$\varepsilon^2 \Delta u + (1-u^2)\, u = 0 \quad \text{in} \quad \Omega. \tag{1.2}$$

Replacing u by $u(\cdot/\varepsilon)$ we obtain the equation

$$\Delta u + (1-u^2)\, u = 0 \quad \text{in} \quad \varepsilon^{-1}\,\Omega. \tag{1.3}$$

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Therefore, equation (1.1) appears as the limit problem in the blow up analysis of (1.2) as ε tends to 0. The relation between interfaces of least volume and critical points of J_ε was first established by Modica in [26]. Let us briefly recall the main results in this direction : If u_ε is a family of *local minimizers* of J_ε for which

$$\sup_{\varepsilon>0} J_\varepsilon(u_\varepsilon) < +\infty, \tag{1.4}$$

then, up to a subsequence, u_ε converges in L^1 to $\mathbf{1}_\Lambda - \mathbf{1}_{\Lambda^c}$, where $\partial\Lambda$ has minimal volume. Here $\mathbf{1}_\Lambda$ (resp. $\mathbf{1}_{\Lambda^c}$) is the characteristic function of the set Λ (resp. $\Lambda^c = \Omega - \Lambda$). Moreover, $J_\varepsilon(u_\varepsilon) \longrightarrow \frac{1}{\sqrt{2}}\,\mathcal{H}^{N-1}(\partial\Lambda)$.

For *critical points* of J_ε which satisfy (1.4), a related assertion is proven in [19]. In this case, the convergence of the interface holds with certain integer multiplicity to take into account the possibility of multiple transition layers converging to the same minimal hypersurface.

These results provide a link between solutions of equation (1.1) and the theory of minimal hypersurfaces which has been widely explored in the literature. For example, solutions concentrating along non-degenerate, minimal hypersurfaces of a compact manifold were found in [28] (see also [22]). As far as multiple transition layers are concerned, given a minimal hypersurface Γ (subject to some additional property on the sign of the potential of the Jacobi operator about Γ, which holds on manifolds with positive Ricci curvature) and given an integer $k \geq 1$, solutions of (1.2) with multiple transitions near Γ were built in [30] (see [13] for the 2-dimensional case, and [11] for the euclidean case), in such a way that $J_\varepsilon(u_\varepsilon) \longrightarrow \frac{k}{\sqrt{2}}\,\mathcal{H}^{N-1}(\Gamma)$.

This paper is concerned with the construction of a new and rather unexpected class of entire solutions of equation (1.1) satisfying the energy growth condition (1.5). Recall that, in dimension 1, solutions satisfying (1.5) are given by translations of the function H which is the unique solution of the problem

$$H'' + (1 - H^2)\,H = 0, \quad \text{with} \quad H(\pm\infty) = \pm 1 \quad \text{and} \quad H(0) = 0\,. \tag{1.5}$$

In fact, the function H is explicitly given by

$$H(y) = \tanh\left(\frac{y}{\sqrt{2}}\right).$$

Then, in any dimension and for all $\mathbf{a} \in \mathbb{R}^N$ with $|\mathbf{a}| = 1$ and for all $b \in \mathbb{R}$, the function $u(\mathbf{x}) = H(\mathbf{a}\cdot\mathbf{x} + b)$ solves (1.1). A celebrated conjecture due to De Giorgi states that, in dimension $N \leq 8$, these solutions are the only ones which are bounded and monotone in one direction. Let us recall that the monotonicity property is related to the fact that solutions u are local minimizers [14], [15].

In dimensions $N = 2, 3$, De Giorgi's conjecture has been proven in [16], [3] and (under some extra assumption) in the remaining dimensions in [29] (see also [14], [15]). When $N = 2$, the monotonicity assumption can even be replaced by a weaker stability assumption [18]. Finally, counterexamples in dimension $N \geq 9$ have recently been built in [12], using the existence of non trivial minimal graphs in higher dimensions.

In light of these results, it is natural to study the set of entire solutions of (1.1). The functions $u(\mathbf{x}) = H(\mathbf{a}\cdot\mathbf{x} + b)$ are obvious solutions. In dimension $N = 2$, nontrivial examples (whose nodal set is the union of two perpendicular lines) were built in [7] using the following strategy : A positive solution to (1.1) in the quadrant $\{(x,y) \, : \, x > |y|\}$ with zero boundary conditions is built by constructing appropriate super and subsolutions. This solution is then extended by odd reflections through the lines $x = y$ and $x = -y$ to yield u_2, a solution of (1.1) in all $\mathbb{R}^2$. The function u_2 is a solution of (1.1), whose 0-level set is the union of the two axis. It can easily be generalized to obtain solutions with dihedral symmetry by considering, for $k \geq 3$, the corresponding solution within the sector $\{(r\,\cos\theta, r\sin\theta) \, : \, r > 0\,,\, |\theta| < \frac{\pi}{2k}\}$ and extending it by $2k-1$ consecutive reflections to yield a solution u_k (we refer to [17] for the details, see also [6] where higher dimensional versions of this construction is given). The zero level set of u_k is constituted outside any ball by $2k$ infinite half lines with dihedral symmetry. To our knowledge, no other nontrivial examples of solutions are known in dimension $N = 2$ (up to the action of rigid motions).

1.2. Statement of the result. — We assume from now on that the dimension is equal to $N = 2$.

Definition 1. — *We say that u, solution of (1.1), has* $2k$-ends *if, away from a compact set, its nodal set is given by $2k$ connected curves which are asymptotic to $2k$ oriented half lines $\mathbf{a}_j \cdot \mathbf{x} + b_j = 0$, $j = 1, \ldots, 2k$ (for some choice of $\mathbf{a}_j \in \mathbb{R}^2$, $|\mathbf{a}_j| = 1$ and $b_j \in \mathbb{R}$) and if, along these curves, the solution is asymptotic to either $H(\mathbf{a}_j \cdot \mathbf{x} + b_j)$ or $-H(\mathbf{a}_j \cdot \mathbf{x} + b_j)$.*

Given any $k \geq 1$, we prove in this paper the existence of a wealth of $2k$-ended solutions of (1.1). In a forthcoming paper [8], we will complete this analysis and show that the solutions we construct in the present paper belong to some smooth $2k$-parameter family of $2k$-ended solutions of (1.1).

To state our result in precise way, we assume that we are given a solution $\mathbf{q} := (q_1, \ldots, q_k)$ of the *Toda system*

$$c_0\, q_j'' = e^{\sqrt{2}(q_{j-1}-q_j)} - e^{\sqrt{2}(q_j - q_{j+1})}\,, \tag{1.6}$$

for $j = 1, \ldots, k$, where $c_0 = \frac{\sqrt{2}}{24}$ and we agree that

$$q_0 \equiv -\infty \qquad \text{and} \qquad q_{k+1} \equiv +\infty\,.$$

The Toda system (1.7) is a classical example of integrable system which has been extensively studied. It models the dynamics of finitely many mass points on the line under the influence of an exponential potential. We recall in the next section some of the results which are concerned with the solvability of (1.7) and which will be needed for our purposes. We refer to [21] and [27] for the complete description of the theory. Of importance for us is the fact that solutions of (1.7) can be described (almost explicitly) in terms of $2k$ parameters. Moreover, if $\mathbf{q}$ is a solution of (1.7), then the long term behavior (i.e. long term scattering) of the q_j at $\pm\infty$ is well understood and it is known that, for all $j = 1, \ldots, k$, there exist $a_j^+, b_j^+ \in \mathbb{R}$ and $a_j^-, b_j^- \in \mathbb{R}$, all depending on the solution $\mathbf{q}$, such that

$$q_j(t) = a_j^\pm\, |t| + b_j^\pm + \mathcal{O}_{\mathcal{C}^\infty(\mathbb{R})}(e^{-\tau_0\,|t|})\,, \tag{1.7}$$

as t tends to $\pm\infty$, for some $\tau_0 > 0$. Moreover, $a^\pm_{j+1} > a^\pm_j$ for all $j = 1, \ldots, k-1$.

Given $\varepsilon > 0$, we define the vector valued function $\mathbf{q}_\varepsilon$, whose components are given by

$$q_{j,\varepsilon}(x) := q_j(\varepsilon\, x) - \sqrt{2}\left(j - \frac{k+1}{2}\right) \log \varepsilon \,. \tag{1.8}$$

It is easy to check that the $q_{j,\varepsilon}$ are again solutions of (1.7).

Observe that, according to the description of the asymptotics of the functions q_j, the graphs of the functions $q_{j,\varepsilon}$ are asymptotic to oriented half lines at infinity. In addition, for $\varepsilon > 0$ small enough, these graphs are disjoint and in fact their mutual distance is given by $-\sqrt{2}\, \log \varepsilon + \mathcal{O}(1)$ as ε tends to 0.

It will be convenient to agree that χ^+ (resp. χ^-) is a smooth cutoff function defined on $\mathbb{R}$ which is identically equal to 1 for $x > 1$ (resp. for $x < -1$) and identically equal to 0 for $x < -1$ (resp. for $x > 1$) and additionally $\chi^- + \chi^+ \equiv 1$. With these cutoff functions at hand, we define the 4 dimensional space

$$D := \mathrm{Span}\, \{x \longmapsto \chi^\pm(x),\; x \longmapsto x\, \chi^\pm(x)\} \,, \tag{1.9}$$

and, for all $\mu \in (0,1)$ and all $\tau \in \mathbb{R}$, we define the space $\mathcal{C}^{2,\mu}_\tau(\mathbb{R})$ of $\mathcal{C}^{2,\mu}$ functions h which satisfy

$$\|h\|_{\mathcal{C}^{2,\mu}_\tau(\mathbb{R})} := \|(\cosh x)^\tau\, h\|_{\mathcal{C}^{2,\mu}(\mathbb{R})} \;< \infty \,.$$

Keeping in mind the above notations, we have the :

Theorem 1.1. — *For all $\varepsilon > 0$ sufficiently small, there exists an entire solution u_ε of the Allen-Cahn equation (1.1) whose nodal set is the union of k disjoint curves $\Gamma_{1,\varepsilon}, \ldots, \Gamma_{k,\varepsilon}$ which are the graphs of the functions*

$$x \longmapsto q_{j,\varepsilon}(x) + h_{j,\varepsilon}(\varepsilon\, x) \,,$$

for some functions $h_{j,\varepsilon} \in \mathcal{C}^{2,\mu}_\tau(\mathbb{R}) \oplus D$ satisfying

$$\|h_{j,\varepsilon}\|_{\mathcal{C}^{2,\mu}_\tau(\mathbb{R}) \oplus D} \leq C\, \varepsilon^\alpha \,.$$

for some constants $C, \alpha, \tau, \mu > 0$ independent of $\varepsilon > 0$.

In other words, given a solution of the Toda system, we can find a one parameter family of $2k$-ended solutions of (1.1) which depend on a small parameter $\varepsilon > 0$. As ε tends to 0, the nodal sets of the solutions we construct become close to the graphs of the functions $q_{j,\varepsilon}$.

Going through the proof, one can be more precise about the description of the solution u_ε. If $\Gamma \subset \mathbb{R}^2$ is a curve in $\mathbb{R}^2$ which is the graph over the x-axis of some function, we denote by $\mathrm{dist}\,(\cdot, \Gamma)$ the signed distance to Γ which is positive in the upper half of $\mathbb{R}^2 \setminus \Gamma$ and is negative in the lower half of $\mathbb{R}^2 \setminus \Gamma$. Then, we have the :

Proposition 1.1. — *The solution of (1.1) provided by Theorem 1.1 satisfies*

$$\|e^{\varepsilon\, \hat\alpha\, |\mathbf{x}|}\, (u_\varepsilon - u^*_\varepsilon)\|_{L^\infty(\mathbb{R}^2)} \leq C\, \varepsilon^{\bar\alpha} \,,$$

for some constants $C, \bar{\alpha}, \hat{\alpha} > 0$ *independent of* ε*, where*

$$u^*_\varepsilon := \sum_{j=1}^{k} (-1)^{j+1} H\big(\mathrm{dist}(\cdot, \Gamma_{j,\varepsilon})\big) - \frac{1}{2}((-1)^k + 1)\,. \tag{1.10}$$

It is interesting to observe that, when $k \geq 3$, there are solutions of (1.7) whose graphs have no symmetry and our result yields the existence of entire solutions of (1.1) without any symmetry provided the number of ends is larger than or equal to 6.

1.3. Comments and open problems. — Our result raises some interesting questions :

(i) The classification of entire solutions of (1.1) remains an important and rather unexplored problem. In particular, the classification of entire solutions with finite Morse index is certainly an interesting problem (the Morse index of an entire solution u being defined as the supremum of the dimension of the space of smooth functions with compact support over which the quadratic form

$$\phi \longmapsto \int_{\mathbb{R}^N} (|\nabla \phi|^2 - (1 - 3\,u^2)\,\phi^2)\,\mathrm{d}\mathbf{x}$$

is negative definite). In dimension $N = 2$, we believe that these solutions are precisely the solutions with finitely many ends. In addition, there is strong evidence that the solutions with $2k$ ends we construct have Morse index equal to the Morse index of the Toda system.

(ii) Still in dimension $N = 2$, the understanding of the moduli space of all $2k$-ended solutions is far from being complete : the result in Theorem 1.1 (see also [7]) implies that this space is non empty and contains smooth families of solutions. Moreover, the result of [8] shows that this moduli space has formal dimension equal to $2k$ (the formal dimension is the dimension of the moduli space close to any non-degenerate solution). The main result of the present paper asserts that, there is a one to one correspondence between an open set of solutions of (1.7) and solutions of (1.1). In particular, this result provides a $2k$ dimensional family of solutions (even if it is not clear from our construction that this family is smooth) and this dimension count is in agreement with the result of [8]. Let us also mention that some *balancing conditions* on the directions of the ends is available (see [17]), it states that the sum of the unit vectors of the ends (oriented toward the ends) has to be 0.

(iii) It is tempting to conjecture that the solution u_k (whose nodal set has dihedral symmetry and whose construction is described in [17] and outlined before the statement of Definition 1) and the solutions given in Theorem 1.1 belong to the same connected component of the moduli space of $2k$-ended solutions.

(iv) When $k = 2$, it turns out that solutions of (1.7) are symmetric with respect to the reflections through two perpendicular lines. Equivalently, one can prove that, when $k=2$, the solutions of (1.1) which are provided by Theorem 1.1 also share this symmetry. In fact, we believe that any solution of (1.1) with 4 ends is symmetric with respect to reflections through two perpendicular lines.

These questions hint towards the classification of finite Morse index entire solutions of (1.1), a program on generalizing De Giorgi's conjecture.

1.4. Description of the proof. — Let us briefly describe the proof of Theorem 1.1. The method is based on an infinite dimensional version of the standard Lyapunov Schmidt reduction argument, as introduced in [28] or in [11] (see also [10], [13], [22] and [23], [24]).

Given a solution $\mathbf{q}$ of (1.7), we first build some infinite dimensional family of approximate solutions $u_{\varepsilon,\mathbf{h}}$, which depend on a small parameter $\varepsilon>0$ and a some (small) vector valued function $\mathbf{h}=(h_1,\ldots,h_k)$ whose components belong to $\mathcal{C}^{2,\mu}_a(\mathbb{R})\oplus D$, for some $a>0$, where D has been defined in (1.10). In essence, these approximate solutions are defined as in (1.11), the curves $\Gamma_{j,\varepsilon,\mathbf{h}}$ being the graphs of the functions $q_{j,\varepsilon}+h_j(\varepsilon\,\cdot)$.

For all ε small enough, we explain how these approximate solutions can be perturbed into genuine solutions of (1.1). To do so, we look for a solution of (1.1) of the form

$$u := u_{\varepsilon,\mathbf{h}} + \phi,$$

where the function ϕ is *small* in a sense to be made precise. Substituting this expression of u in (1.1), we reduce the problem to the solvability of the following nonlinear equation

$$(\Delta + 1 - 3\,u_{\varepsilon,\mathbf{h}}^2)\,\phi + S(u_{\varepsilon,\mathbf{h}}) - N(u_{\varepsilon,\mathbf{h}},\phi) = 0 \tag{1.11}$$

where we have defined

$$S(u) := \Delta u + (1-u^2)\,u\,,$$

and

$$N(u,\phi) := \phi^3 + 3\,u\,\phi^2\,.$$

One of the important task will be to analyze, as ε tends to 0, the mapping properties of the linear operator $\Delta+1-3\,u_{\varepsilon,\mathbf{h}}^2$ which appears on the left hand side of (1.12). It turns out that this analysis is quite delicate and involves some carefully designed weighted spaces. It also requires some Lyapunov-Schmidt type reduction argument.

To set up the analysis of the linearized operator $\Delta+1-3\,u_{\varepsilon,\mathbf{h}}^2$, we let ρ_j be a cutoff function such that $\rho_j\equiv 1$ in a tubular neighborhood of $\Gamma_{j,\varepsilon,\mathbf{h}}$ and identically equal to 0 outside some larger tubular neighborhood of $\Gamma_{j,\varepsilon,\mathbf{h}}$. We will show that for all f in a suitable weighted function space, there exists a function $\phi:\mathbb{R}^2\longmapsto\mathbb{R}$ and, for $j=1,\ldots,k$, a function $\kappa_j:\mathbb{R}^2\longmapsto\mathbb{R}$ which is defined in a tubular neighborhood of $\Gamma_{j,\varepsilon,\mathbf{h}}$ and only depend on the projection onto $\Gamma_{j,\varepsilon,\mathbf{h}}$, solutions of

$$(\Delta + 1 - 3\,u_{\varepsilon,\mathbf{h}}^2)\,\phi + \sum_{j=1}^k \kappa_j\,\rho_j\,H'(\mathrm{dist}\,(\cdot,\Gamma_{j,\varepsilon,\mathbf{h}})) = f\,, \tag{1.12}$$

and whose norms are uniformly controlled as ε tends to 0. Observe that we have introduced new unknown functions κ_j. These will be needed to overcome the fact that the solution of $(\Delta + 1 - 3\,u_{\varepsilon,\mathbf{h}}^2)\phi = f$ blows up as ε tends to 0 unless some orthogonality conditions are imposed on the function f.

In view of this result, instead of solving (1.12), we will look for ϕ and functions κ_j, for $j = 1, \ldots, k$, solutions of the following nonlinear problem

$$(\Delta + 1 - 3\,u_{\varepsilon,\mathbf{h}}^2)\,\phi + \sum_{j=1}^{k} \kappa_j\,\rho_j\,H'(\mathrm{dist}\,(\cdot, \Gamma_{j,\varepsilon,\mathbf{h}})) + S(\hat{u}_{\varepsilon,\mathbf{h}}) - N(u_{\varepsilon,\mathbf{h}}, \phi) = 0\,. \tag{1.13}$$

Now, a solution of (1.14) is a solution of (1.12) provided all functions κ_j are identically equal to 0. At this stage, it is worth remembering that our approximate solution $u_{\varepsilon,\mathbf{h}}$ depends on the vector valued functions $\mathbf{h}$ and we will see that it is possible to choose $\mathbf{h}$ appropriately so that the solution of (1.14) satisfies $\kappa_j = 0$, for $j = 1, \ldots, k$. This will complete the proof of the result.

2. The Toda system and its linearization

In this section, we gather some information about the theory which is necessary for solving (1.7) since this system is at the heart of our construction.

2.1. The Toda system. — We are interested in the understanding of the solutions of the Hamiltonian system

$$c_0\,q_j'' = e^{\sqrt{2}(q_{j-1}-q_j)} - e^{\sqrt{2}(q_j-q_{j+1})}, \tag{2.14}$$

where $c_0 = \frac{\sqrt{2}}{24}$ and we agree that $q_0 \equiv -\infty$ and $q_{k+1} \equiv +\infty$.

We introduce the functions

$$r_j := \sqrt{2}\,(q_{j+1} - q_j) + \log\left(\frac{c_0}{\sqrt{2}}\right), \tag{2.15}$$

for $j = 1, \ldots, k-1$. It is easy to check that, if $\mathbf{q}$ is a solution of (2.15), then $\mathbf{r} := (r_1, \ldots, r_{k-1})$ is a solution of the following nonlinear system

$$\mathbf{r}'' - \mathbf{M}\,e^{-\mathbf{r}} = 0 \tag{2.16}$$

where the $(k-1) \times (k-1)$ matrix $\mathbf{M}$ is given by

$$\mathbf{M} := \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \ldots & 0 & -1 & 2 \end{pmatrix},$$

and where $e^{-\mathbf{r}}$ is the vector whose entries are given by

$$e^{-\mathbf{r}} := \left(e^{-r_1}, \ldots, e^{-r_{k-1}}\right).$$

Conversely, given a solution $\mathbf{r}$ of (2.17) and $\bar p, \bar q \in \mathbb{R}$, the functions

$$q_j = \frac{1}{k}\left(\sum_{i=0}^{j-1} i\, r_i - \sum_{i=0}^{k-j} i\, r_{k-i}\right) + \bar p\, t + \bar q + \frac{1}{\sqrt{2}}\left(\frac{k-1}{2} - j\right)\log\left(\frac{c_0}{\sqrt{2}}\right), \tag{2.17}$$

for $j = 1, \ldots, k$ (we agree that $r_0 = r_k \equiv 0$), are solutions of (2.15).

The system (2.17) is an integrable system which has been extensively studied for example by J. Moser [27] and B. Kostant [21]. Some explicit formula of all solutions of (2.17) is available as well as a precise description of the asymptotic behavior of the solutions as t tends to $\pm\infty$. We briefly recall the main features of this theory.

The expression of the solutions of (2.17) can be found in section 7.7 of [21]. To describe it, we need to be given $\mathbf{w} := (w_1, \ldots, w_k) \in \mathbb{R}^k$ such that

$$\sum_{j=1}^{k} w_j = 0, \quad \text{and } w_{j+1} > w_j, \quad j = 1, \ldots, k-1 \tag{2.18}$$

and $\mathbf{g} := (g_1, \ldots, g_k) \in \mathbb{R}^k$ such that

$$\prod_{j=1}^{k} g_j = 1, \quad \text{and} \quad g_j > 0, \quad \text{for} \quad j = 1, \ldots, k\,. \tag{2.19}$$

Finally, for $j = 2, \ldots, k-1$, we define the function

$$\Phi_j(\mathbf{g}, \mathbf{w}; t) := \sum_{1 \le i_i < \cdots < i_j \le k} R_{i_1 \ldots i_j}(\mathbf{w})\, g_{i_1} \ldots g_{i_j}\, e^{-t\,(w_{i_1} + \ldots + w_{i_j})},$$

(see formula 7.7.10 in [21]) where $R_{i_1 \ldots i_j}$ are rational functions of the entries of the vector $\mathbf{w}$ whose precise form can be found in section 7.5 of [21]. We also agree that

$$\Phi_0 = \Phi_k \equiv 1\,.$$

It is proven in [21] that all solutions of (2.15) are of the form

$$r_j(t) = -\log \Phi_{j-1}(\mathbf{g}, \mathbf{w}; t) + 2\log \Phi_j(\mathbf{g}, \mathbf{w}; t) - \log \Phi_{j+1}(\mathbf{g}, \mathbf{w}; t) \tag{2.20}$$

for some choice of $\mathbf{g}$ and $\mathbf{w}$. Observe that we have a $2k$ family of solutions of (2.17) since $\mathbf{g}$ and $\mathbf{w}$ provide $2(k-1)$ independent parameters to which we have to add the parameters $\bar p$ and $\bar q$.

The next result is also borrowed from [21], [27]. It describes the asymptotics of the solutions of (2.17) (see Theorem 7.7.2 of [21]) :

Lemma 2.1. — *Let $\tau_0 > 0$ be defined by*

$$\tau_0 := \min_{j=1,\ldots,k-1} (w_{j+1} - w_j)\,. \tag{2.21}$$

Then, for $j = 1, \ldots, k-1$, the following expansion holds

$$r_j(t) = c_j\, t - d_j + e_j^+(\mathbf{c}) + \mathcal{O}_{\mathcal{C}^\infty}((\cosh t)^{-\tau_0}),$$

as t tends to $+\infty$ and

$$r_j(t) = -c_{k-j} t + d_{k-j} + e_j^-(\mathbf{c}) + \mathcal{O}_{\mathcal{C}^\infty}((\cosh t)^{-\tau_0}),$$

as t tends to $-\infty$, where, for $j = 1, \ldots, k-1$,

$$c_j := w_{j+1} - w_j, \quad d_j := \log g_{j+1} - \log g_j, \tag{2.22}$$

and where $e_j^{\pm}$ are smooth functions of $\mathbf{c} := (c_1, \ldots, c_{k-1})$.

Proof. — Thanks to (2.21), we can write as t tends to $+\infty$

$$\Phi_j(\mathbf{g}, \mathbf{w}; t) = R_{1\ldots j}(\mathbf{w})\, g_1 \ldots g_j\, e^{-(w_1+\ldots+w_j)t}\left(1 + \mathcal{O}_{\mathcal{C}^\infty}((\cosh t)^{-\tau_0})\right),$$

while we can write, as t tends to $-\infty$

$$\Phi_j(\mathbf{g}, \mathbf{w}; t) = R_{k-j\ldots k-1}(\mathbf{w})\, g_k \ldots g_{k-j+1}\, e^{-(w_k+\ldots+w_{k-j+1})t}\left(1 + \mathcal{O}_{\mathcal{C}^\infty}((\cosh t)^{-\tau_0})\right).$$

The expansions follow at once from elementary computations together with the definition of r_j. We leave the details to the reader. □

2.2. The linearized Toda system. — We assume that $\mathbf{q} = (q_1, \ldots, q_k)$ is a solution of (2.15) described in the previous section. The linearized system associated to linearization of (2.15) about the solution $\mathbf{q}$, reads as

$$c_0\, \mathbf{v}'' + \mathbf{N}\, \mathbf{v} = \mathbf{z}, \tag{2.23}$$

where the $k \times k$ matrix $\mathbf{N}$ has coefficients which are exponentially decaying at $\pm\infty$ (this follows from Lemma 2.1 which implies that the functions r_j tend to $+\infty$ as t tends to $\pm\infty$). We analyze the solvability of the above linear problem in the space $C^{\ell,\mu}_\tau(\mathbb{R}; \mathbb{R}^k)$ of $\mathcal{C}^{\ell,\mu}$ vector valued functions $\mathbf{v}$ which satisfy

$$\|\mathbf{v}\|_{\mathcal{C}^{\ell,\mu}_\tau(\mathbb{R};\mathbb{R}^k)} := \|(\cosh x)^\tau\, \mathbf{v}\|_{\mathcal{C}^{\ell,\mu}(\mathbb{R})} < \infty. \tag{2.24}$$

We take advantage of the fact that the solution $\mathbf{q}$, as described in (2.18), depends smoothly on the parameters $c_1, \ldots, c_{k-1}$ and $d_1, \ldots, d_{k-1}$ as well as the parameters $\bar{q}$ and $\bar{p}$. Differentiating with respect to any of these parameters yields $2k$ linearly independent solutions of the homogeneous problem $c_0\, \mathbf{v}'' + \mathbf{N}\, \mathbf{v} = 0$. We will write

$$\mathbf{v}_j^\sharp := \partial_{c_j}\, \mathbf{q} \qquad \text{and} \qquad \mathbf{v}_j^\flat := \partial_{d_j}\, \mathbf{q},$$

for $j = 1, \ldots, k-1$, and

$$\mathbf{v}_k^\sharp := \partial_{\bar{p}}\, \mathbf{q} \qquad \text{and} \qquad \mathbf{v}_k^\flat := \partial_{\bar{q}}\, \mathbf{q}.$$

It follows from the result of Lemma 2.1 that the vector valued functions $\mathbf{v}_j^\sharp$ are linearly growing at $\pm\infty$ while the vector valued functions $\mathbf{v}_j^\flat$ are bounded. More precisely, it follows from Lemma 2.1 that

Lemma 2.2. — *As t tends to $\pm\infty$, the vector valued functions $\mathbf{v}_j^\sharp$ and $\mathbf{v}_j^\flat$ can be decomposed as*

$$\mathbf{v}_j^\sharp = \mathbf{a}_{j,\pm}^\sharp\, t + \mathbf{b}_{j,\pm}^\sharp + \mathcal{O}_{\mathcal{C}^\infty}((\cosh t)^{-\tau_0}),$$

and

$$\mathbf{v}_j^\flat = \mathbf{b}_{j,\pm}^\flat + \mathcal{O}_{\mathcal{C}^\infty}((\cosh t)^{-\tau_0}),$$

where $\mathbf{a}_{j,\pm}^\sharp$ and $\mathbf{b}_{j,\pm}^\sharp, \mathbf{b}_{j,\pm}^\flat$ are fixed vectors in $\mathbb{R}^k$. Moreover, $\{\mathbf{a}_{j,\iota}^\sharp : j = 1, \ldots, k\}$ and $\{\mathbf{b}_{j,\iota}^\flat : j = 1, \ldots, k\}$ are basis of $\mathbb{R}^k$, for $\iota = \pm$.

We now define the *deficiency space*

$$\mathcal{D} := \text{Span}\left\{\chi^{\pm}\,\mathbf{v}_j^{\sharp}\,,\chi^{\pm}\,\mathbf{v}_j^{\flat}\,:\,j=1,\ldots,k\right\}, \tag{2.25}$$

where we recall that χ^+ (resp. χ^-) is a cutoff function identically equal to 1 for $t>1$ (resp. for $t<-1$) and identically equal to 0 for $t<-1$ (resp. for $t>1$) and $\chi^+ + \chi^- \equiv 1$. Observe that $\mathcal{D}$ is $4k$ dimensional and contains

$$\mathcal{K} := \text{Span}\left\{\mathbf{v}_j^{\sharp}\,,\,\mathbf{v}_j^{\flat}\,:\,j=1,\ldots,k\right\},$$

which is the $2k$ dimensional space of homogeneous solutions of $c_*\,\mathbf{v}'' + \mathbf{N}\,\mathbf{v} = 0$. Therefore, we can certainly decompose

$$\mathcal{D} = \mathcal{K}\oplus\mathcal{E}\,. \tag{2.26}$$

where $\mathcal{E}$ is a complement of $\mathcal{K}$ in $\mathcal{D}$. With this decomposition at hand, we have the following result which follows from standard arguments in ordinary differential equations.

Lemma 2.3. — *Assume that* $\tau>0$*. Then the mapping*

$$\begin{array}{rccc} T: & \mathcal{C}^{2,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)\oplus\mathcal{E} & \longrightarrow & \mathcal{C}^{0,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k) \\ & \mathbf{v} & \longmapsto & c_0\,\mathbf{v}'' + \mathbf{N}\,\mathbf{v} \end{array}$$

is an isomorphism.

Proof. — Standard arguments in ordinary differential equations imply that there exists a unique solution of (2.24) which satisfies $\mathbf{v}(0) = \mathbf{v}'(0) = 0$. We will denote $\mathbf{v} = S_0(\mathbf{z})$.

We now prove that $\mathbf{v}\in\mathcal{C}^{2,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)\oplus\mathcal{D}$. To do so, we observe that one can also find a (unique) solution $\bar{\mathbf{v}}$ of (2.24) which satisfies

$$|\bar{\mathbf{v}}(t)| \leq C\,e^{\tau t}\,\|\mathbf{z}\|_{\mathcal{C}^{0,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)},$$

in $(-\infty,0]$. Indeed, using the variation of parameters formula it is easy to show the existence of a unique solution decaying to 0 at $-\infty$ at some exponential rate. Integrating the equation twice over $(-\infty,t]$ shows that in fact $\bar{\mathbf{v}}\in\mathcal{C}^{2,\mu}_{\tau}((-\infty,0];\mathbb{R}^k)$. Then $\mathbf{v}-\bar{\mathbf{v}}$ is a linear combination of the functions $\mathbf{v}_j^{\sharp}$ and $\mathbf{v}_j^{\flat}$. This proves that, in $(-\infty,0]$, the vector valued function $\mathbf{v}$ can be decomposed into the sum of a linear combination of elements in $\mathcal{D}$ and a vector valued function which is bounded by a constant times $e^{\tau t}$. A similar decomposition can be derived on $[0,+\infty)$. Once this decomposition is proven, the estimates for the Hölder norm of $\mathbf{v}$ follow at once.

In other words, $S_0 : \mathcal{C}^{0,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)\longrightarrow\mathcal{C}^{2,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)\oplus\mathcal{D}$ is a right inverse for T. The decomposition $\mathcal{D}=\mathcal{K}\oplus\mathcal{E}$ induces the decomposition $S_0(\mathbf{z}) = \bar{S}_0(\mathbf{z}) + e(\mathbf{z}) + k(\mathbf{z})$, where $\bar{S}_0(\mathbf{z})\in\mathcal{C}^{2,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)$, $e(\mathbf{z})\in\mathcal{E}$ and $k(\mathbf{z})\in\mathcal{K}$. The operator $S := S_0 - k$ is also a right inverse of T and maps onto $\mathcal{C}^{2,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)\oplus\mathcal{E}$ as desired. This completes the proof of the Lemma. □

3. Linearized operator for a single interface

In this section we develop the relevant analysis which will allow us to find a right inverse for the operator which will appear in the linearization of (1.1) about an approximate solution.

3.1. Injectivity result. — We start by considering the linearized operator about H, namely

$$L_0 := \partial_y^2 + 1 - 3\,H^2 \,.$$

First, we recall that L_0 has a one dimensional kernel spanned by H' since $L_0\,H' = 0$ as can be checked by taking the derivative of $H'' + (1 - H^2)\,H = 0$. Since $H' > 0$ this implies that 0 is the bottom of the spectrum of $-L_0$. In fact more is known and we recall the following result from [1] :

Lemma 3.1. — [1] *The spectrum of the operator $-L_0$ is the union of the point spectrum, given by* 0 *(associated to the eigenfunction H') and $\frac{3}{2}$ (associated to the eigenfunction $H\,\sqrt{H'}$) and the continuous spectrum given by* $[2, +\infty)$.

In particular, for all $\xi \neq 0$, given $f \in L^2(\mathbb{R})$, the problem

$$(L_0 - \xi^2)\,\phi = f, \tag{3.27}$$

is uniquely solvable in $H^1(\mathbb{R})$.

Let us consider operator

$$L := \partial_x^2 + L_0 \,,$$

acting on functions defined in the plane. Obviously, we still have $L\,H' = 0$. Our first result shows any bounded solution of $L\,\phi = 0$ is colinear to H'. The proof of this fact follows the method first introduced in [28].

Lemma 3.2. — *Let ϕ be a bounded solution of*

$$L\,\phi = 0\,, \tag{3.28}$$

in $\mathbb{R}^2$. Then ϕ is colinear to H'.

Proof. — Let assume that ϕ is a bounded solution of $L\,\phi = 0$. We denote by $\hat\phi(\xi, y)$ the Fourier transform of $\phi(x, y)$ in the x variable. This distribution is defined by

$$\langle \hat\phi, f\rangle = \langle \phi(\cdot, y), \hat f\rangle \;= \int_{\mathbb{R}} \phi(x, y)\,\hat f(x)\,\mathrm{d}x\,,$$

where f is any smooth rapidly decreasing function and where $\hat f$ is its Fourier transform. Let us now consider a smooth rapidly decreasing function of the two variables $\psi(\xi, y)$. It follows from $L\,\phi = 0$ that

$$\int_{\mathbb{R}} \langle \hat\phi(\cdot, y), L_0\,\psi - \xi^2\,\psi\rangle\,\mathrm{d}y = 0. \tag{3.29}$$

Let $\varphi(y)$ and $\mu(\xi)$ be smooth and compactly supported functions such that 0 does not belong to the support of f. Then we can solve the family of equations (parameterized by $\xi \in \mathbb{R}$)

$$(L_0 - \xi^2)\,\psi(\xi, y) = f(\xi)\,\varphi(y)\,,$$

and obtain a smooth, rapidly decreasing function $\psi(\xi, y)$ such that $\psi(\xi, y) = 0$ whenever ξ is not the support of the function f. The fact that $y \longmapsto \psi(\xi, y)$ decays exponentially is standard and left to the reader. Using ψ in (3.30), we conclude that

$$\int_{\mathbb{R}} \langle \hat{\phi}(\cdot, y), f \rangle \, \varphi(y) \, \mathrm{d}y = 0 \, .$$

Since φ is arbitrary, we have proven that $\langle \hat{\phi}(\cdot, y), f \rangle = 0$ for all f whose support does not meet 0. This implies that the support of $\hat{\phi}(\cdot, y)$ is included in $\{0\}$.

It follows that $\hat{\phi}(\cdot, y)$ is a linear combination (with coefficients depending on y) of derivatives up to a finite order of Dirac masses at 0. Taking the inverse Fourier transform, we get that $\phi(x, y) = P_y(x)$, where for each $y \in \mathbb{R}$, P_y is a polynomial in x. Since ϕ is assumed to be bounded, we conclude that $P_y(x)$ is a constant polynomial and hence $\phi(x, y) = \phi(y)$ is a bounded function which satisfies $L_0 \, \phi = 0$. Therefore, ϕ is colinear to H'. □

3.2. ***A priori* estimates.** — Making use of the previous Lemma, we now obtain *a priori* estimates for solutions of the problem

$$L \, \phi = f \, , \tag{3.30}$$

in $\mathbb{R}^2$. The results of Lemma 3.2 shows that such an *a priori* estimate will not be possible without imposing any extra conditions on the solution ϕ. The classification of the bounded solutions of $L \, \phi = 0$ suggests to impose the following orthogonality condition on the function ϕ

$$\int_{\mathbb{R}} \phi(x, \cdot) \, H' \, \mathrm{d}y = 0, \tag{3.31}$$

for all $x \in \mathbb{R}$. With these restrictions imposed we have the following *a priori* estimates for this problem.

Lemma 3.3. — *There exists a constant $C > 0$ such that*

$$\|\phi\|_{L^\infty(\mathbb{R}^2)} \leq C \, \|L \, \phi\|_{L^\infty(\mathbb{R}^2)} \, ,$$

provided $\phi \in L^\infty(\mathbb{R}^2)$ satisfies (3.32).

Proof. — The proof of the Lemma is by contradiction (it is actually similar to the proof of Lemma 2.2 in [9]). If the result were not true, there would exist sequences of bounded functions ϕ_n and f_n satisfying

$$L \, \phi_n = f_n, \quad \text{in } \mathbb{R}^2, \tag{3.32}$$

$$\int_{\mathbb{R}} \phi_n \, H' \, \mathrm{d}y = 0, \quad \text{for all } x \in \mathbb{R} \tag{3.33}$$

with $\lim_{n \to \infty} \|f_n\|_{L^\infty} = 0$ while $\|\phi_n\|_{L^\infty} = 1$. For each $n \in \mathbb{N}$ we pick a point $(x_n, y_n) \in \mathbb{R}^2$ such that

$$|\phi_n(x_n, y_n)| \geq 1/2 \, . \tag{3.34}$$

We now consider the sequence of functions

$$\tilde{\phi}_n(x, y) = \phi_n(x + x_n, y + y_n) \, .$$

Using elliptic estimates together with Ascoli's theorem, we can assume (up to a subsequence) that the sequence $\tilde{\phi}_n$ converges, uniformly on compact sets, to a function $\tilde{\phi}$ which is defined in $\mathbb{R}^2$ and which is either a solution of

$$(\Delta - 2)\,\tilde{\phi} = 0,$$

if the sequence $(y_n)_n$ tends to $\pm\infty$ or a solution of

$$(\Delta + 1 - 3\,H^2)\,\tilde{\phi}(x, \cdot - y_\infty) = 0\,,$$

if $(y_n)_n$ converges to y_∞. Moreover, $\tilde{\phi}$ is bounded and $\tilde{\phi}$ is not identically equal to 0 since (3.35) guaranties that $\tilde{\phi}(0) \geq 1/2$. Finally, in the latter case, we can pass to the limit in (3.34) to get

$$\int_{\mathbb{R}} \tilde{\phi}(x, \cdot - y_\infty)\, H'\, \mathrm{d}y = 0\,,$$

for all $x \in \mathbb{R}$. The maximum principle implies that the former case does not occur and the result of Lemma 3.2 implies that the latter case does not occur either. Having found a contradiction in all cases, this completes the proof of the result. □

Using the maximum principle, we also get *a priori* estimates in weighted space.

Lemma 3.4. — *Assume that $\sigma \in [0, \sqrt{2})$ is fixed. There exists $C > 0$ such that*

$$\|(\cosh y)^\sigma\, \phi\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)} \leq C\, \left(\|\phi\|_{L^\infty(\mathbb{R}^2)} + \|(\cosh y)^\sigma\, L\,\phi\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} \right)\,. \tag{3.35}$$

Proof. — Since we have assumed that $\sigma^2 < 2$, we can choose $\nu > 0$ so that $\sigma^2 + 4\,\nu^2 \leq 2$. We consider the auxiliary function

$$W_\nu(x,y) := \left(e^{-\sigma y} + \nu\, e^{\sigma y}\right)\, \cosh(\nu\, x)\,.$$

We have

$$(\Delta - 2)\, W_\nu = -(2 - \sigma^2 - \nu^2)\, W_\nu\,.$$

The potential in L is given by $1 - 3\,H^2$, hence, for $|y|$ large enough, say $|y| \geq y_\sigma$, we can write

$$L\, W_\nu \leq -\left(\frac{2-\sigma^2}{2} - \nu^2\right)\, W_\nu\,.$$

Therefore, we get

$$L\, W_\nu \leq -\left(\frac{2-\sigma^2}{4}\right)\, e^{-\sigma\,|y|}\,,$$

in this range. We can now use the barrier W_ν and the maximum principle, to conclude that

$$\sup_{|y|\geq y_\sigma} |W_\nu^{-1}\,\phi| \leq C\, \left(\|\phi\|_{L^\infty(\mathbb{R}^2)} + \|(\cosh y)^\sigma\, f\|_{L^\infty(\mathbb{R}^2)} \right)\,.$$

Letting ν tend to 0 yields the desired estimate. □

For the time being, we have only considered the decay behavior of the solution in the y variable. The next result shows that some *a priori* weighted estimate with both decay in the x and y variables is also available. The key observation is that, according to Lemma 3.1, the least nonzero eigenvalue of $-L_0$ is $\frac{3}{2}$ and its continuous spectrum starts at 2, hence, if $\phi \in H^1(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}} \phi \, H' \, \mathrm{d}y = 0 \, ,$$

we have the inequality

$$\int_{\mathbb{R}} \left(|\partial_y \phi|^2 - (1 - 3\, H^2) \, \phi^2 \right) \, \mathrm{d}y \geq \frac{3}{2} \int_{\mathbb{R}} \phi^2 \, \mathrm{d}y \, . \tag{3.36}$$

Using this, we can prove the :

Lemma 3.5. — *Assume that $\sigma \in (0, \sqrt{2})$ is fixed. For all $a \in [0, \frac{1}{\sqrt{2}})$ such that*

$$\sigma^2 + a^2 < 2 \, ,$$

there exists a constant $C_a > 0$, which depends on a but remains bounded as a tends to 0, *such that*

$$\|(\cosh x)^a \, (\cosh y)^\sigma \, \phi\|_{L^\infty(\mathbb{R}^2)} \leq C_a \, \left(\|\phi\|_{L^\infty(\mathbb{R}^2)} + \|(\cosh x)^a \, (\cosh y)^\sigma \, L \, \phi\|_{L^\infty(\mathbb{R}^2)} \right) \, ,$$

provided $\phi \in L^\infty(\mathbb{R}^2)$ satisfies (3.32).

Before we proceed with the proof of the result, let us emphasize that the key property is that the constant C_a remains bounded as a tends to 0, we shall further comment on this at the end of this section. Also, the range in which the parameter a can be chosen is not optimal and it follows from the analysis of [8] that the optimal range is $[0, \sqrt{\frac{3}{2}})$ but we will not need this result in the present paper.

Proof. — We already have proven the appropriate decay in the y direction. We will now prove that, under the assumptions of the Lemma, the function ϕ has the appropriate decay in the x variable provided y remains in some compact set. Then, the result will follow from the use of suitable barrier functions as in the proof of the previous Lemma.

We consider the function

$$\psi(x) := \int_{\mathbb{R}} \phi^2(x, y) \, \mathrm{d}y \, ,$$

which, thanks to the result of Lemma 3.4, is well defined (notice that here we implicitly use the fact that $\sigma > 0$). We can compute

$$\psi''(x) = 2 \int_{\mathbb{R}} |\partial_x \phi|^2 \, \mathrm{d}y + 2 \int_{\mathbb{R}} \phi \, \partial_x^2 \phi \, \mathrm{d}y \, ,$$

where $'$ denote the derivative with respect to x. Using the fact that $L \, \phi = f$, we also have, using some integration by parts,

$$\int_{\mathbb{R}} \phi \, \partial_x^2 \phi \, \mathrm{d}y = \int_{\mathbb{R}} \left(|\partial_y \phi|^2 + (1 - 3\, H^3) \, \phi^2 + \phi \, f \right) \, \mathrm{d}y \, . \tag{3.37}$$

Collecting this together with (3.37), which holds since we have assumed that the orthogonality condition (3.32) was true for all $x \in \mathbb{R}$, we conclude easily that

$$\psi''(x) \geq 2 \int_{\mathbb{R}} |\partial_x \phi|^2 \, \mathrm{d}y + 3 \int_{\mathbb{R}} \phi^2 \, \mathrm{d}y + 2 \int_{\mathbb{R}} \phi \, f \, \mathrm{d}y \, .$$

Using Cauchy-Schwarz inequality to estimate the last term on the right hand side, we find that ψ satisfies the following differential inequality

$$\psi''(x) \geq 2\, \psi(x) - \int_{\mathbb{R}} f^2(x,y) \, \mathrm{d}y \, .$$

Therefore, we conclude that

$$-\psi''(x) + 2\, \psi(x) \leq C\, e^{-2a|x|} \, \|(\cosh x)^a \, (\cosh y)^\sigma \, f\|^2_{L^\infty(\mathbb{R}^2)} \, ,$$

for some constant $C > 0$. Observe that, thanks to the results of Lemma 3.2 and Lemma 3.4, we know that ψ is bounded and we have

$$|\psi(x)| \leq C \, \|(\cosh x)^a \, (\cosh y)^\sigma \, f\|^2_{L^\infty(\mathbb{R}^2)} \, .$$

Now, we can use the auxiliary function

$$\bar{\psi}_\nu(x) := M \, \|(\cosh x)^a \, (\cosh y)^\sigma \, f\|^2_{L^\infty(\mathbb{R}^2)} \, e^{-2a\,x} + \nu \, e^{2ax} \, ,$$

where the constant $M > 0$ is chosen sufficiently large and $\nu > 0$ is arbitrary small. If $a \in [0, \frac{1}{\sqrt{2}})$, this function can be used as a barrier and the maximum principle implies that $0 \leq \psi \leq \bar{\psi}_\nu$ for all $y \geq 0$ and letting ν tend to 0 we conclude that

$$\psi(x) \leq C \, \|(\cosh x)^a \, (\cosh y)^\sigma \, f\|^2_{L^\infty(\mathbb{R}^2)} \, e^{-2ax} \, ,$$

for all $x \geq 0$. A similar argument yields the corresponding estimate for $x \leq 0$. Hence we have obtained the bound

$$(\cosh x)^{2a} \int_{\mathbb{R}} \phi^2(x,y) \, \mathrm{d}y \leq C \, \|(\cosh x)^a \, (\cosh y)^\sigma \, f\|^2_{L^\infty(\mathbb{R}^2)} \, .$$

Local elliptic estimates then imply that, for all $y_0 > 0$, there exists a constant $C > 0$ (depending on the choice of y_0) such that

$$|\phi(x,y)| \leq C \, \|(\cosh x)^a \, (\cosh y)^\sigma \, f\|^2_{L^\infty(\mathbb{R}^2)} \, (\cosh x)^{-a} \, ,$$

uniformly in $x \in \mathbb{R}$ and $|y| \leq y_0$.

Having established such a decay in the x variable, the relevant estimate in the complementary region can be found using appropriately designed barriers. For instance, enlarging y_0 if this is necessary, in the quadrant $\{(x,y) \, : \, x > 0, y > y_0\}$ we may consider a barrier of the form

$$\tilde{\phi}_\nu(x,y) := M \, e^{-(ax+\sigma y)} \, \|(\cosh x)^a \, (\cosh y)^\sigma \, f\|^2_{L^\infty(\mathbb{R}^2)} + \nu \, e^{\frac{x}{2}+\frac{y}{2}} \, ,$$

with $\nu > 0$ arbitrarily small. Fixing M large enough (depending on y_0) and letting ν tend to 0 yields the desired estimate in the right upper quadrant of the plane. Similar argument also provide the relevant estimate in the other three quadrants, we leave the details to the reader. □

3.3. Surjectivity result. — As far as the existence of solutions of (3.31)-(3.32) is concerned, provided we assume that

$$\int_{\mathbb{R}} f(x,\cdot)\, H'\, \mathrm{d}y = 0\,, \tag{3.38}$$

for all $x \in \mathbb{R}$, we have the following result whose proof relies on the previous analysis :

Proposition 3.1. — *Assume that $\sigma \in (0, \sqrt{2})$ is fixed. For all $a \in [0, \frac{1}{\sqrt{2}})$ such that*

$$\sigma^2 + a^2 < 2\,,$$

there exists a constant $C_a > 0$, which depends on a but remains bounded as a tends to 0, *such that, for all f satisfying the orthogonality condition* (3.39) *and*

$$\|(\cosh x)^a\, (\cosh y)^\sigma\, f\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)} < +\infty\,,$$

there exists a unique function ϕ, solution of (3.31)-(3.32)*, which satisfies*

$$\|(\cosh x)^a\, (\cosh y)^\sigma\, \phi\|_{\mathcal{C}^{2,\mu}(\mathbb{R}^2)} \leq C_a\, \|(\cosh x)^a\, (\cosh y)^\sigma\, f\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^2)}\,.$$

Proof. — We first consider the equation on functions which are ζ-periodic in the x variable for some fixed $\zeta > 0$. Observe that 0 is in the spectrum of the operator $-L$ and the corresponding kernel is spanned by the function H'. The remaining part of the spectrum of $-L$ is positive and (according to Lemma 3.1) is larger than or equal to $\frac{3}{2}$, hence

$$\int_{\mathbb{R}^2_\zeta} (|\nabla \phi|^2 \;+ H\,(1 - 3H^3)\,\phi^2)\, \mathrm{d}\mathbf{x} \geq \frac{3}{2} \int_{\mathbb{R}^2_\zeta} \phi^2\, \mathrm{d}\mathbf{x}\,,$$

for any function ϕ satisfying

$$\int_{\mathbb{R}^2_\zeta} \phi\, H'\, \mathrm{d}\mathbf{x} = 0\,, \tag{3.39}$$

where $\mathbb{R}^2_\zeta := (\mathbb{R}/\zeta\mathbb{Z}) \times \mathbb{R}$.

As a consequence, given $f \in L^2(\mathbb{R}^2_\zeta)$ satisfying

$$\int_{\mathbb{R}^2_\zeta} f\, H'\, \mathrm{d}\mathbf{x} = 0\,,$$

there exists a unique solution $\phi \in H^1(\mathbb{R}^2_\zeta)$, also satisfying (3.40), of $L\,\phi = f$ and $\|\phi\|_{H^1(\mathbb{R}^2_\zeta)} \leq C\, \|f\|_{L^2(\mathbb{R}^2_\zeta)}$. Elliptic regularity theory then implies that

$$\|\phi\|_{L^\infty(\mathbb{R}^2_\zeta)} \leq C\, \left(\|f\|_{L^\infty(\mathbb{R}^2_\zeta)} + \|f\|_{L^2(\mathbb{R}^2_\zeta)} \right)\,.$$

Now, let us assume that, in addition the function f satisfies (3.39). Multiplying the equation $L\,\phi = f$ by functions of the form $\psi(x)\, H'(y)$ and integrating by parts, one checks that

$$\int_0^\zeta \left(\int_{\mathbb{R}} \phi\, H'\, \mathrm{d}y \right)\, \partial_x^2 \psi\, \mathrm{d}x = 0\,,$$

for any ζ-periodic function ψ. This implies that the function

$$x \longmapsto \int_{\mathbb{R}} \phi\, H'\, \mathrm{d}y$$

does not depend on x and, since its integral over $[0,\zeta]$ is 0, we conclude that ϕ satisfies (3.32).

We can now apply the result of Lemma 3.3 and 3.4 to get the estimate

$$\|(\cosh y)^\sigma\, \phi\|_{L^\infty(\mathbb{R}^2_\zeta)} < C\, \|(\cosh y)^\sigma\, f\|_{L^\infty(\mathbb{R}^2_\zeta)}\,,$$

where the constant $C > 0$ does not depend on ζ.

Now, given a function f satisfying the assumptions of the Proposition, we define f_ζ to be the restriction of f to $[0,\zeta]\times\mathbb{R}$ which is extended by periodicity in the x variable. Let ϕ_ζ be the corresponding solution of $L\,\phi_\zeta = f_\zeta$ obtained above. Elliptic estimates together with a simple compactness argument allows one to pass to the limit as ζ tends to ∞ to get the existence of ϕ, a bounded solution of (3.31)-(3.32). The estimate of ϕ follows from Lemma 3.5 together with classical elliptic estimates and the uniqueness of ϕ follows from Lemma 3.2. □

We end up this section with some comment on the orthogonality condition we impose on the function f. Given any (bounded) function f, with the appropriate decay as in the statement of Proposition 3.1, we want to solve the equation $L\,\phi = f$. We can certainly find a function $x \longmapsto c(x)$ such that $f - c\,H'$ satisfies (3.39). And then, we can apply the result of Proposition 3.1 to solve $L\,\phi = f - c\,H'$. Therefore, it just remains to solve the equation $L\,\psi = c\,H'$, but this is rather easy since it is enough to look for ψ of the form $\psi(x,y) = d(x)\,H'(y)$ in which case the equation reduces to the solvability of the equation $d'' = c$. Observe that, it is not possible to find a solution to this ordinary differential equation which decays exponentially at $\pm\infty$ unless the function c satisfies

$$\int_{\mathbb{R}} c(x)\,\mathrm{d}x = \int_{\mathbb{R}} x\,c(x)\,\mathrm{d}x = 0\,.$$

In fact this solution is explicitly given by

$$d(x) = x\int_{-\infty}^{x} c(z)\,\mathrm{d}z - \int_{-\infty}^{x} z\,c(z)\,\mathrm{d}z\,.$$

Now if c is bounded by a constant times $(\cosh x)^{-a}$, and satisfies the two conditions above, it is easy to check that d is also bounded by a constant (independent of $a\in(0,1)$) times $a^{-2}\,(\cosh x)^{-a}$. In particular, this solution blows up as a tends to 0. In the next section we will need to invert L on functions spaces corresponding to a tending to 0 and, in order to get a right inverse whose norm does not blow up, is will be necessary to impose the restriction (3.39) on the functions f.

4. The approximate solutions and the general set up

4.1. Description of the nodal curves of the approximate solutions. — We keep the notations introduced in the introduction and in section 2 to describe an infinite dimensional family of approximate solutions to our problem. We first choose the data which allow us to describe the curves which will be very close to the nodal sets of our solutions.

Remark 4.1. — *In order to simplify notations, if $\zeta \longmapsto \Xi(\pi_1, \ldots, \pi_m; \zeta)$ is a function or operator acting on ζ, which depends on parameters $\pi_1, \ldots, \pi_m$ (which might be integers, real numbers, functions, ...), we agree that we simply write Ξ instead of $\Xi(\pi_1, \ldots, \pi_m; \cdot)$ when no confusion is possible.*

Let us assume that we are given a solution $\mathbf{q} := (q_1, \ldots, q_k)$ of the Toda system (1.7), we define $\mathbf{q}_\varepsilon$ to be the vector valued function whose components are given by

$$q_{j,\varepsilon}(x) := q_j(\varepsilon\, x) - \sqrt{2}\left(j - \frac{k+1}{2}\right) \log \varepsilon \, .$$

We also assume that we are given $\mathbf{v} := (v_1, \ldots, v_k) \in \mathcal{E}$ (see section 2 for a precise definition of $\mathcal{E}$) such that

$$\|\mathbf{v}\|_{\mathcal{E}} \leq \delta_1\, \varepsilon^{\alpha_1}\, , \tag{4.40}$$

where the constants $\alpha_1 \in (0, 2)$ and $\delta_1 > 0$ will be fixed later, independent of $\varepsilon \in (0, 1/2]$.

Remark 4.2. — *In the following we have to estimate various quantities $\Xi(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$ which depend on ε, $\mathbf{v}$ or $\mathbf{h}$. In general, we will prove statements of the following form : there exists constants $C_0, \beta_0 > 0$ which does not depend on the choice of the parameters δ_1 and α_1 such that $\|\Xi(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)\| \leq C_0\, \epsilon^{\beta_0}$, provided ε is chosen small enough, say $\varepsilon \in (0, \varepsilon_0)$. And in general, ε_0 does depend on δ_1 and α_1. The idea behind this type of estimates is that there exists constants $C_0, \beta_0 > 0$ such that $\|\Xi(\varepsilon, \mathbf{0}, \mathbf{0}; \cdot)\| \leq \frac{C_0}{2}\, \epsilon^{\beta_0}$, while $\|\Xi(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)\| \leq \frac{C_0}{2}\, \epsilon^{\beta_0} + C_1\, \varepsilon^{\beta_1}$ provided (4.41) is satisfied. Here C_1 and β_1 do depend on δ_1 and α_1 but $\beta_1 > \beta_0$ and hence, for ε small enough, the term $C_1\, \varepsilon^{\beta_1}$ is certainly controled by $\frac{C_0}{2}\, \epsilon^{\beta_0}$ and this explains the general claim.*

With these data at hand, we define the planar curve $\bar{\Gamma}_j(\varepsilon, \mathbf{v})$ to be the image of

$$\gamma_j(x) := (x, q_{j,\varepsilon}(x) + v_j(\varepsilon\, x))\, .$$

Even though the definition of $\bar{\Gamma}_j$ also depends on the choice of $\mathbf{q}$, the solution of the Toda system, we shall not make this dependence explicit in the notation since we will assume from now on that $\mathbf{q}$ is fixed. Roughly speaking, the curves $\bar{\Gamma}_j$ will describe the nodal sets of our solution, or at least they will be close to them.

For each $j = 1, \ldots, k$, we introduce the Fermi coordinates (x_j, y_j) which are associated to the curve $\bar{\Gamma}_j$. More precisely, we consider the parameterization of a tubular neighborhood of $\bar{\Gamma}_j$ by $X_j = X_j(\varepsilon, \mathbf{v}; \cdot)$

$$X_j(x_j, y_j) := \gamma_j(x_j) + y_j\, n_j(x_j)\, , \tag{4.41}$$

where n_j is the normal vector about $\bar{\Gamma}_j$ (the curves are assumed to be positively oriented). Observe that the coordinate y_j is nothing but the signed distance to $\bar{\Gamma}_j$. In the sequel, we will make use of the convenient notation

$$X_j^*\, f(x_j, y_j) = (f \circ X_j)(x_j, y_j),$$

where f is a function defined in a neighborhood of $\bar{\Gamma}_j$.

4.2. An infinite dimensional family of approximate solutions. — Now that we have described the possible candidates for the nodal sets of our approximate solution, the basic idea is to consider the approximate solution which is close to the function $\pm H(\mathrm{dist}(\cdot, \bar{\Gamma}_j))$ (with alternative signs according to wether j is odd or even). A possible choice could be the function

$$\sum_{j=1}^{k} (-1)^{j+1} H(\mathrm{dist}(\cdot, \bar{\Gamma}_j)) - \frac{1}{2}((-1)^{k+1} + 1) . \tag{4.42}$$

We need to take care of two technical problems. The first one concerns the regularity of the distance function to the curves $\bar{\Gamma}_j$. This distance function is smooth in the neighborhood of $\bar{\Gamma}_j$ but is not smooth in the whole plane. More precisely, it is a simple exercise to check that, there exists $C_{\mathbf{q}} > 0$ (only depending on $\mathbf{q}$) such that the distance function to $\bar{\Gamma}_j$, is smooth in the set

$$V := \left\{ (x, y) \in \mathbb{R}^2 \, : \, |y| \leq C_{\mathbf{q}} \, \varepsilon^{-1} \, \sqrt{1 + |x|^2} \right\} . \tag{4.43}$$

This follows at once from the structure of $\mathbf{q}$ at infinity which implies that the curve $\bar{\Gamma}_j$ is exponentially close to half lines at infinity. Observe that the constant $C_{\mathbf{q}} > 0$ can be chosen independently of $\varepsilon \in (0, 1/2)$ and also observe that

$$\bar{\Gamma}_j \subset V , \tag{4.44}$$

for ε small enough.

To overcome the regularity issue, we take advantage of the fact that the function H is almost constant (equal to either $+1$ or -1) away from 0 and we make use of an appropriate cutoff function to connect the approximate solution (4.43) to the constant functions ± 1 away from the curves $\bar{\Gamma}_j$.

The second problem we have to face is more delicate to explain. As we will see shortly, it takes its origin in the orthogonality condition (3.39) we have to impose to produce a right inverse of L whose norm does not blow up as the weight parameter a tends to 0. This problem translates into the fact that, even though the nodal sets of the solutions we will construct are close to the curves $\bar{\Gamma}_j$ (say in Hausdorff topology), this topology is not refined enough to perform the construction. Hence, in some sense we need to improve the definition of the nodal sets of the approximate solutions by allowing more flexibility in the definition of the curves $\bar{\Gamma}_j$. This is the reason why we have already introduced the vector valued function $\mathbf{v}$ in the definition of $\bar{\Gamma}_j$. Unfortunately this is not quite enough and we need to introduce another vector valued function $\mathbf{h} := (h_1, \ldots, h_k) \in \mathcal{C}^{2,\mu}(\mathbb{R}; \mathbb{R}^k)$ satisfying

$$\|\mathbf{h}\|_{\mathcal{C}^{2,\mu}_\tau(\mathbb{R};\mathbb{R}^k)} := \|(\cosh x)^\tau \, \mathbf{h}\|_{\mathcal{C}^{2,\mu}(\mathbb{R};\mathbb{R}^k)} \leq \delta_1 \, \varepsilon^{\alpha_1} , \tag{4.45}$$

where $\tau > 0$ and the constants $\alpha_1 \in (0, 2)$ and $\delta_1 > 0$ will be fixed later on (independently of ε). It will be convenient to define the functions $H_j = H_j(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$ by the identity

$$X_j^* \, H_j(x_j, y_j) := H(y_j - h_j(\varepsilon \, x_j)) . \tag{4.46}$$

With these data and notations, we are now in a position to define a multiple-end approximate solution of (1.1). We start with the definition of $\bar u^0 = \bar u^0(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$ given by

$$\bar u^0 := \sum_{j=1}^k (-1)^{j+1} \, H_j - \frac{1}{2}((-1)^{k+1} + 1) \, .$$

We let $t \longmapsto \eta(t)$ be a smooth cutoff function such that $\eta(t) \equiv 1$ for $|t| \leq 1/2$ and $\eta(t) \equiv 0$ for $|t| \geq 1$ and we define for all $\varepsilon > 0$ small enough the function

$$\eta_\varepsilon (x,y) := \eta \left(\frac{\varepsilon \, y}{C_{\mathbf{q}} \, \sqrt{1+|x|^2}} \right) ,$$

where the constant $C_{\mathbf{q}}$ is the one introduced in the definition of V.

The cutoff function η_ε is now used to smooth this function and define the approximate solution $\bar u = \bar u(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$ in the following way

$$\bar u := \eta_\varepsilon \, \bar u^0 + (1-\eta_\varepsilon) \, \frac{\bar u^0}{|\bar u^0|} \, .$$

Let us emphasize that the approximate solution $\bar u$ depends on the choice of ε, $\mathbf{v} \in \mathcal{E}$ and $\mathbf{h} \in \mathcal{C}^{2,\mu}_\tau(\mathbb{R}; \mathbb{R}^k)$.

4.3. The set up of the nonlinear problem. — We now define an appropriate weighted norm for functions defined in $\mathbb{R}^2$. For all $\sigma, a > 0$, we need to build a weight function $W_{\sigma,a} = W_{\sigma,a}(\varepsilon, \mathbf{v}; \cdot)$ which is defined to be equal to

$$W_{\sigma,a} := \sum_{j=1}^k W_{\sigma,a,j}$$

where

$$X_j^* \, W_{\sigma,a,j}(x_j, y_j) = (\cosh x_j)^{-a} \, (\cosh y_j)^{-\sigma} \, ,$$

in V. In the lower part of $\mathbb{R}^2 \setminus V$, the weight function $W_{\sigma,a}$ is designed in such a way that

$$c \, e^{-(a \, |x_1| + \sigma \, |y_1|)} \leq W_{\sigma,a}(x,y) \leq C \, e^{-(a \, |x_1| + \sigma \, |y_1|)},$$

for all $(x_1, y_1) \in \mathbb{R}^2$ such that x_1 is coordinate in $\bar\Gamma_1$ of the point which realizes the (signed) distance y_1 from the point (x,y) to $\bar\Gamma_1$. Here $c < 1 < C$ are fixed constants.

This being understood, we have the :

Definition 2. — *Given $\sigma, a > 0$, we define $\mathcal{C}^{\ell,\mu}_{\sigma,a}(\mathbb{R}^2)$ to be the space of $\mathcal{C}^{\ell,\mu}$ functions for which the following norm is finite*

$$\|\phi\|_{\mathcal{C}^{\ell,\mu}_{\sigma,a}(\mathbb{R}^2)} := \sup_{\mathbf{x} \in \mathbb{R}^2} \left(W^{-1}_{\sigma,a}(\mathbf{x}) \, \|\phi\|_{\mathcal{C}^{\ell,\mu}(B_1(\mathbf{x}))} \right) . \tag{4.47}$$

In other words, σ is related to the rate of decay of the functions in the direction transverse to the curves $\bar\Gamma_j$ and a is related to the rate of decay of the functions along the curves $\bar\Gamma_j$. Observe that these definitions depend on ε even though this is not clear in the notations.

Granted the above notations and definitions, the equation we want to solve reads

$$\Delta(\bar{u}+\phi)+\bar{u}+\phi-(\bar{u}+\phi)^3=0\,, \tag{4.48}$$

where $\bar{u}=\bar{u}(\varepsilon,\mathbf{v},\mathbf{h};\cdot)$ for some $\phi\in\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R})$, some vector valued function $\mathbf{h}\in\mathcal{C}^{2,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)$ and some $\mathbf{v}\in\mathcal{E}$. We can then formally rewrite the equation (4.49) as

$$\mathbf{L}\,\phi=Q(\phi),$$

where the linear operator $\mathbf{L}=\mathbf{L}(\varepsilon,\mathbf{v},\mathbf{h};\cdot)$ is defined by

$$\mathbf{L}:=\Delta+1-3\,\bar{u}^2\,,$$

and where the nonlinear operator $Q=Q(\varepsilon,\mathbf{v},\mathbf{h};\cdot)$ is defined by

$$Q(\phi):=-(\Delta\bar{u}+(1-\bar{u}^2)\,\bar{u})+\phi^3+3\,\bar{u}\,\phi^2\,. \tag{4.49}$$

We now study the mapping properties of the linear operator $\mathbf{L}$ and the nonlinear operator Q when defined between appropriate weighted function spaces.

5. The linear theory for multiple interfaces

5.1. Laplacian in Fermi coordinates. — It will be useful to have the expression of the Laplacian in the above defined Fermi coordinates. Observe that in the coordinates (x_j,y_j) the Euclidean metric reads

$$X_j^*\,(dx^2+dy^2)=A_j\,\mathrm{d}x_j^2+\mathrm{d}y_j^2\,,$$

where the function A_j is explicitly given by

$$A_j:=1+\varepsilon^2\,B_j^2-2\,y_j\,\frac{\varepsilon^2\,C_j}{(1+\varepsilon^2\,B_j^2)^{1/2}}+y_j^2\,\frac{\varepsilon^4\,C_j^2}{(1+\varepsilon^2\,B_j^2)^2}\,,$$

where

$$B_j(x_j,y_j):=(q_j+v_j)'(\varepsilon x_j)\,,$$

and

$$C_j(x_j,y_j):=(q_j+v_j)''(\varepsilon x_j)\,.$$

In these coordinates, the expression of the Laplacian is given by

$$\Delta=\partial_{x_j}^2+\partial_{y_j}^2+\left(\frac{1}{A_j}-1\right)\partial_{x_j}^2+\frac{1}{2}\,\frac{\partial_{y_j}A_j}{A_j}\,\partial_{y_j}-\frac{1}{2}\,\frac{\partial_{x_j}A_j}{A_j^2}\,\partial_{x_j}$$

Observe that, there exists a constant $c>0$ such that

$$\varepsilon^2\,|y_j|\,(\cosh x_j)^{-\tau_0}\le C\,,$$

in V, uniformly as ε tends to 0. Using this, it is an easy exercise to check that the following estimates hold in V

$$A_j=1+\mathcal{O}_{\mathcal{C}^\infty(V)}(\varepsilon^2)+\mathcal{O}_{\mathcal{C}^\infty(V)}(\varepsilon^2\,|y_j|\,e^{-\tau_0\,\varepsilon\,|x_j|})$$

and hence

$$\left(1-\frac{1}{A_j}\right)=\mathcal{O}_{\mathcal{C}^\infty(V)}(\varepsilon^2)+\mathcal{O}_{\mathcal{C}^\infty(V)}(\varepsilon^2\,(1+y_j^2)^{1/2}\,(\cosh x_j)^{-\tau_0})\,,$$

$$\frac{\partial_{y_j} A_j}{A_j} = \mathcal{O}_{\mathcal{C}^\infty(V)}(\varepsilon^2\,(\cosh x_j)^{-\tau_0})$$

and

$$\frac{\partial_{x_j} A_j}{A_j^2} = \mathcal{O}_{\mathcal{C}^\infty(V)}(\varepsilon^3\,(1+y_j^2)^{1/2}\,(\cosh x_j)^{-\tau_0}).$$

We will also need the elementary fact which follows from the definition of the curves $\bar\Gamma_j$ and the Fermi coordinates together with elementary geometry. In V we have

$$\begin{aligned} y_i &= (i-j)\,\sqrt{2}\,\log\varepsilon + \mathcal{O}_{\mathcal{C}^\infty(V)}(1) + (1+\mathcal{O}_{\mathcal{C}^\infty(V)}(\varepsilon^2))\,y_j \\ &+ \varepsilon\,(a_j^\pm - a_i^\pm + \mathcal{O}_{\mathcal{C}^\infty(V)}(\delta_1\,\varepsilon^{\alpha_1}) + \mathcal{O}_{\mathcal{C}^\infty(V)}(\varepsilon^2))\,x_j\,, \end{aligned} \tag{5.50}$$

as ε tends to 0 (the superscript $^\pm$ is equal to $+$ (resp. $-$) when $x_j \geq 0$ (resp. $x_j \leq 0$). Recall that the parameters $a_j^\pm$ have been defined in (1.8). In other words, we evaluate the sign distance to $\bar\Gamma_i$ in therm of the Fermi coordinates associated to $\bar\Gamma_j$.

Observe that the term $\mathcal{O}_{\mathcal{C}^\infty(V)}(\delta_1\,\varepsilon^{\alpha_1})$ depends on δ_1 and α_1 and since we assume that $\alpha_1 \in (0,2)$, we can absorb the term $\mathcal{O}_{\mathcal{C}^\infty(V)}(\varepsilon^2)$ into it, keeping in mind that the estimate does depend on δ_1 and α_1.

5.2. Linear theory for multiple interfaces. — We now want to study the mapping properties of the operator

$$\mathbf{L} := \Delta + 1 - 3\,\bar u^2\,,$$

where the potential is built using the approximate solution $\bar u = \bar u(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$. The idea is to glue together parametrices which have been obtained in the previous section for the model operator $L = \Delta + 1 - 3\,H^2$, using a perturbation argument. We make use of the weighted function spaces $\mathcal{C}^{2,\mu}_{a,\sigma}(\mathbb{R}^2)$, $\mathcal{C}^{2,\mu}_a(\mathbb{R};\mathbb{R}^k)$ which have already been defined in (4.48) and (2.25), respectively.

Following (4.47), we introduce the functions $H'_j = H'_j(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$ by the identity

$$X_k^*\, H'_j(x_j, y_j) := H'(y_j - h_j(\epsilon\, x_j))\,.$$

We also define the cutoff functions $\rho_j = \rho_j(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$ by

$$X_j^*\, \rho_j(x_j, y_j) := \rho_\varepsilon(y_j - h_j(\epsilon\, x_j))\,,$$

where

$$\rho_\varepsilon(t) := \left(\frac{4t}{\sqrt{2}\,\log\frac{1}{\varepsilon}}\right)\,, \tag{5.51}$$

and where ρ is a cutoff function identically equal to 1 on $|t| < \frac{1}{2}$ and identically equal to 0 for $|t| > 1$ (Remember that the distance between two consecutive curves $\bar\Gamma_j$ and $\bar\Gamma_{j+1}$ can be estimated by $-\sqrt{2}\,\log\varepsilon + \mathcal{O}(1)$, so the supports of the cutoff functions ρ_j are disjoint for ε small).

We will consider the solvability of the linear problem

$$\mathbf{L}\,\phi + \sum_{j=1}^{k} \kappa_j\,\rho_j\,H'_j = f\,, \tag{5.52}$$

in $\mathbb{R}^2$, where the unknowns are the function ϕ and the functions κ_j which are defined in V in such a way that $X_j^* \kappa_j$ only depends on x_j. To keep notations short, we set

$$\mathbb{L}(\phi, \kappa) := \mathbf{L}\,\phi + \sum_{j=1}^{k} \kappa_j\, \rho_j\, H_j' \,,$$

where we have set $\kappa := (\kappa_1, \ldots, \kappa_k)$. Here, one has to keep in mind that $\mathbf{L}$, ρ_j and H_j' all depend on ε, $\mathbf{v}$ and $\mathbf{h}$ and hence so does $\mathbb{L}$. We will always assume that

$$\|\mathbf{h} + \mathbf{v}\|_{\mathcal{C}^{2,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)\oplus\mathcal{E}} \leq \delta_1\, \varepsilon^{\alpha_1} \,, \tag{5.53}$$

for some constants $\alpha_1 \in (0, 2)$ and $\delta_1 > 0$ which will be fixed later on. Building on the analysis of the previous section, we prove the :

Proposition 5.1. — *Assume that $\sigma \in (0, \sqrt{2})$ and $\tau > 0$ are fixed and assume that (5.54) is satisfied for some fixed α_1 and δ_1. Then, ther e exists $\varepsilon_0 > 0$ (depending on α_1 and δ_1) such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists a linear operator $\mathbb{G} = \mathbb{G}(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$*

$$\mathbb{G} : \mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2) \longrightarrow \mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2) \times \mathcal{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R}; \mathbb{R}^k) \,,$$

whose norm is bounded by a constant (independent of ε, δ_1 and α_1), such that, $(\phi; \kappa) := \mathbb{G}(f)$ is the unique solution of (5.53) which satisfies

$$\int_{\mathbb{R}} X_j^*(\rho_j\, H_j'\, \phi)\, \mathrm{d}y_j = 0 \;, \tag{5.54}$$

for all $x_j \in \mathbb{R}$.

The main idea in the proof of this proposition is to first handle the case where $\mathbf{h} = 0$. In this case we glue together parametrices of L which were obtained in Proposition 3.1 to get an approximate right inverse of $\mathbb{L}$ which is then perturbed into a genuine right inverse of $\mathbb{L}$. The general case, when $\mathbf{h} \neq 0$, can then be handled using a simple perturbation argument. We decompose the proof of this proposition in a sequence of intermediate results.

We start by considering the case where $\mathbf{h} = 0$ and $\mathbf{v} \in \mathcal{E}$ is fixed and prove the existence of $\mathbb{G}(\varepsilon, \mathbf{v}, \mathbf{0}; \cdot)$ in this case. This is the content of the following :

Lemma 5.1. — *Assume that $\mathbf{h} = 0$. Then, for all $\varepsilon > 0$ small enough, the existence of $\mathbb{G}(\varepsilon, \mathbf{v}, \mathbf{0})$ satisfying the statement of Proposition 5.1 holds.*

Proof. — We decompose the proof in three steps.

Step 1 - We make use of Proposition 3.1 to get the existence of ϕ_j solution of

$$\left(\partial^2_{x_j} + \partial^2_{y_j} + 1 - 3\, H^2\right)\, (X_j^*\, \phi_j) = \rho_\varepsilon\, (X_j^*\, f - \kappa_j^0\, H')$$

where H, H' and ρ_ε are functions of y_j and κ_j^0 are functions of x_j. The functions κ_j^0 are chosen so that the right hand side of this equation satisfies the orthogonality condition (3.39), hence

$$\kappa_j^0(x_j) \int_{\mathbb{R}} \rho_\varepsilon\, (H')^2\, \mathrm{d}y_j = \int_{\mathbb{R}} \rho_\varepsilon\, H'\, X_j^*\, f\, \mathrm{d}y_j \,.$$

Observe that $X_j^* \, \kappa_j^0$ only depends on x_j. It is easy to check that

$$\|\kappa^0\|_{\mathcal{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} + \left\| \sum_{j=1}^k \rho_j \, \phi_j \right\|_{\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \leq C \, \|f\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)}, \tag{5.55}$$

for some constant $C > 0$ independent of ε, δ_1 and α_1. The estimate for κ_j^0 follows at once from the definition while the estimate for $\sum_{j=1}^k \rho_j \, \phi_j$ follows directly from the result of Proposition 3.1. Observe that, by construction, we have

$$\int_{\mathbb{R}} H' \, X_j^* \phi_j \, \mathrm{d}y_j = 0 \, . \tag{5.56}$$

We define

$$f_0 := f - \mathbf{L} \left(\sum_{j=1}^k \rho_j \, \phi_j \right) - \sum_{j=1}^k \kappa_j^0 \, \rho_j \, H_j' \, .$$

Observe that there are two main reasons why f_0 is not identically equal to 0. The first being the effect of the cutoff function which implies that, away from the support of the functions ρ_j, we have $f_0 = f$. The second being that, close to the curves $\bar{\Gamma}_j$, even though $\rho_j = 1$, there is a small discrepancy between the Laplacian and the operator $\partial^2_{x_j} + \partial^2_{y_j}$.

We now give a more quantitative statement of these two facts. First we compute

$$f_0 = \left(1 - \sum_{j=1}^k \rho_j \right) \, f - \sum_{j=1}^k (\phi_j \, \Delta \rho_j + 2 \nabla \rho_j \nabla \phi_j) + \sum_{j=1}^k \rho_j \, (\partial^2_{x_j} + \partial^2_{y_j} - \Delta) \, \phi_j \, .$$

It is easy to check that we have

$$\|f_0\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \leq C \|f\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)}.$$

Moreover, in the region where $\rho_j \equiv 1$ we simply have $f_0 = (\partial^2_{x_j} + \partial^2_{y_j} - \Delta) \, \phi_j$ and still using the expression of the Laplacian in Fermi coordinates, once can check that the operator $\Delta - (\partial^2_{x_j} + \partial^2_{y_j})$ is a second order differential operator in ∂_{x_j} and ∂_{y_j} whose coefficients are bounded by a constant times $\varepsilon^2 \, \log \frac{1}{\varepsilon}$ in this region. Hence, we get

$$\|\chi_j \, f_0\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \leq C \, \varepsilon^2 \, \log \tfrac{1}{\varepsilon} \, \|f\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \, , \tag{5.57}$$

where the cutoff function $\chi_1, \ldots, \chi_k$ are defined by $X_j^* \, \chi_j(x_j, y_j) := \rho_\varepsilon(2 \, y_j)$.

Step 2 - We now solve

$$(\Delta - 2) \, \psi = f_0 \, . \tag{5.58}$$

The existence of ψ, bounded solution of this equation, is straightforward. We claim that

$$\|\psi\|_{\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \leq C \, \|f\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)}, \tag{5.59}$$

for some constant $C > 0$ independent of ε, α_1 and δ_1. Indeed, the maximum principle immediately implies that

$$\|\psi\|_{L^\infty(\mathbb{R}^2)} \leq C \, \|f\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \, .$$

Next, arguing as in the proof of Lemma 3.4, we define the auxiliary function $\bar W_{\sigma,\varepsilon\tau,\nu}$ by

$$X_j^*\,\bar W_{\sigma,\varepsilon\tau,\nu} = e^{-\sigma y_j}\left((\cosh x_j)^{-\varepsilon\tau} + \nu\,(\cosh x_j)^{\varepsilon\tau}\right),$$

and, using once more the expression of the Laplacian in Fermi coordinates, we check that

$$(\Delta - 2)\,\bar W_{\sigma,\varepsilon\tau,\nu} = -(2-\sigma^2 + \mathcal{O}(\varepsilon^2\,\log\tfrac{1}{\varepsilon}))\,\bar W_{\sigma,\varepsilon\tau,\nu}\,, \tag{5.60}$$

in the region $\bar V_j$ where $y_j \geq -\varepsilon\,\tau\,|x_j|$ and $y_{j+1} \leq \varepsilon\,\tau\,|x_{j+1}|$ (i.e. in a region which slightly encompasses the region between the curves $\bar\Gamma_j$ and $\bar\Gamma_{j+1}$). The maximum principle can then be used in $\bar V_j$ to prove that ψ is bounded by a constant (independent on ν) times $\bar W_{\sigma,\varepsilon\tau,\nu}$ times the norm of f in $\bar V_j$. Letting ν tend to 0 we obtain the estimate (5.60). A similar analysis can be carried out in the region of the plane which is above $\bar\Gamma_k$ or below $\bar\Gamma_1$.

We define the cutoff functions $\hat\chi_1,\ldots,\hat\chi_k$ by $X_j^*\,\hat\chi_j(x_j,y_j) := \rho_\varepsilon(4\,y_j)$. Observe that we also have the following estimate

$$\|\hat\chi_j\,\psi\|_{\mathcal{C}^{2,\mu}_{0,\varepsilon\tau}(\mathbb{R}^2)} \leq C\,(\varepsilon^2\,\log\tfrac{1}{\varepsilon} + \varepsilon^{\frac{\sqrt{2}\sigma}{16}})\,\|f\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)}, \tag{5.61}$$

which again follows from the maximum principle, using the barrier function, $\bar W_{0,\varepsilon\tau,\nu}$ together with (5.58) to evaluate the right hand side in (5.59) and (5.60) to evaluate ψ the boundary of the set $\{X_j(x_j,y_j)\,:\,|y_j| \leq \frac{\sqrt{2}}{16}\,\log\frac{1}{\varepsilon}\}$.

Step 3 - We set

$$\bar\phi := \psi + \sum_{j=1}^k \rho_j\,\phi_j - \sum_{j=0}^k \lambda_j\,\rho_j\,H_j'\,,$$

and

$$\bar\kappa_j := \kappa_j^0 + \Delta\,\lambda_j\,,$$

where the functions $\lambda_1,\ldots,\lambda_k$ are defined by the identity

$$X_j^*\,\lambda_j(x_j,y_j)\,\int_{\mathbb{R}} \rho_\varepsilon^2\,(H')^2\,\mathrm{d}y_j = \int_{\mathbb{R}} \rho_\varepsilon\,H'\,X_j^*\left(\psi + \sum_{j=1}^k \rho_j\,\phi_j\right)\,\mathrm{d}y_j\,.$$

Observe that $X_j^*\,\lambda_j$ only depends on x_j. We consider the operator

$$\bar{\mathbb{G}}(f) := (\bar\phi, \bar\kappa)\,.$$

It follows from (5.56), (5.60) that

$$\bar{\mathbb{G}} : \mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2) \longrightarrow \mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2) \times \mathcal{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)\,,$$

is well defined and has norm bounded by a constant independent of ε, δ_1 and α_1.

We compute

$$\begin{aligned}\mathbf{L}\,\bar\phi + \sum_{j=1}^k \bar\kappa_j\,\rho_j\,H_j' \quad &= \quad f + 3\,(1-\bar u^2)\,\psi - 2\sum_{j=1}^k \nabla(\lambda_j\,\rho_j)\,\nabla H_j' \\ &- \quad \sum_{j=1}^k \lambda_j\,\rho_j\,\mathbf{L}\,H_j' - \sum_{j=1}^k (\lambda_j\,\Delta\rho_j + 2\nabla\lambda_j\,\nabla\rho_j)\,H_j'\,.\end{aligned}$$

Using (5.57), we can estimate

$$\|\lambda_j\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R})} \leq C\, \varepsilon^\alpha \, \|f\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \, ,$$

and using (5.62) together with the fact that $\sigma < \sqrt{2}$, we check that

$$\|(1-\bar u^2)\, \psi\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \leq C\, \varepsilon^\alpha \, \|f\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \, ,$$

for some $\alpha > 0$ (independent of ε and f). Then, it is easy to check that

$$\|\mathbb{L} \circ \bar{\mathbb{G}}(f) - f\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \leq C\, \varepsilon^\alpha \, \|f\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)},$$

for all ε small enough. When $\mathbf{h} = 0$, the existence of $\mathbb{G}(\varepsilon, \mathbf{v}, \mathbf{0}; \cdot)$ follows at once from a standard perturbation argument. This completes the proof of the Lemma. □

We now assume that $\mathbf{h} \neq 0$ and, using the previous Lemma together with a perturbation argument, we prove the :

Lemma 5.2. — *For all $\varepsilon > 0$ small enough, the existence of $\mathbb{G}(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$ satisfying the statement of Proposition 5.1 holds.*

Proof. — Again, the proof of this result relies on some perturbation argument. To distinguish the operators when $\mathbf{h} = 0$ and $\mathbf{h} \neq 0$, we adorn them with the subscript $\mathbf{h}$ writing for example $\mathbb{L}_{\mathbf{h}}$, $\mathbb{G}_{\mathbf{h}}$, $H_{j,\mathbf{h}}, \ldots$ instead of $\mathbb{L}(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$, $\mathbb{G}(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$, $H_j(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$, ...

We set $(\phi, \kappa) := \mathbb{G}_{\mathbf{0}}(f)$ and define the operator $\bar{\mathbb{G}}_{\mathbf{h}}$ by $\bar{\mathbb{G}}_{\mathbf{h}}(f) := (\bar\phi, \bar\kappa)$ where

$$\bar\phi := \phi - \sum_{j=0}^{k} \lambda_j\, \rho_{j,\mathbf{h}}\, H'_{j,\mathbf{h}} \qquad \text{and} \qquad \bar\kappa_j := \kappa_j + \Delta \lambda_j \, ,$$

and where the functions $\lambda_1, \ldots, \lambda_k$ are defined by the identity

$$X_j^*\, \lambda_j(x_j, y_j) \int_{\mathbb{R}} \rho_\varepsilon^2\, (H')^2 \, \mathrm{d}y_j = \int_{\mathbb{R}} X_j^*\, (\rho_{j,\mathbf{h}}\, H'_{j,\mathbf{h}}\, \phi)\, \mathrm{d}y_j \, .$$

Observe that $X_j^*\, \lambda_j$ only depends on x_j and, by construction, we have

$$\int_{\mathbb{R}} X_j^*\, (\rho_{j,\mathbf{0}}\, H'_{j,\mathbf{0}}\, \phi)\, \mathrm{d}y_j = 0 \, ,$$

hence we can also write

$$X_j^*\, \lambda_j(x_j, y_j) \int_{\mathbb{R}} \rho_\varepsilon^2\, (H')^2 \, \mathrm{d}y_j = \int_{\mathbb{R}} X_j^*\, \left((\rho_{j,\mathbf{h}}\, H'_{j,\mathbf{h}} - \rho_{j,\mathbf{0}}\, H'_{j,\mathbf{0}})\, \phi \right)\, \mathrm{d}y_j \, .$$

Since we already know that $\|\phi\|_{\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\,\tau}(\mathbb{R}^2)} \leq C\, \|f\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\,\tau}(\mathbb{R}^2)}$, we get

$$\|\lambda_j\|_{\mathcal{C}^{2,\mu}_{\varepsilon\,\tau}(\mathbb{R};\mathbb{R}^k)} \leq C\, \|\mathbf{h}\|_{\mathcal{C}^{0,\mu}_{\varepsilon\,\tau}(\mathbb{R};\mathbb{R}^k)}\, \|f\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\,\tau}(\mathbb{R}^2)} \, . \tag{5.62}$$

In particular, this implies that

$$\bar{\mathbb{G}}_{\mathbf{h}} : \mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2) \longrightarrow \mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2) \times \mathcal{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)$$

is well defined and has norm bounded by a constant independent of ε, α_1 and δ_1.

We claim that

$$\|\mathbb{L}_{\mathbf{h}} \circ \bar{\mathbb{G}}_{\mathbf{h}}(f) - f\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \leq C\, \|\mathbf{h}\|_{\mathcal{C}^{0,\mu}_{\varepsilon\,\tau}(\mathbb{R};\mathbb{R}^k)}\, \|f\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \, .$$

Assuming we have already proved the claim, the existence of $\mathbb{G}(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$ follows again from a standard perturbation argument. Therefore, it remains to prove the claim. To this aim, we compute

$$\begin{aligned} \mathbb{L}_{\mathbf{h}}(\bar\phi, \bar\kappa) - f &= 3\,(\bar u_{\mathbf{0}}^2 - \bar u_{\mathbf{h}}^2)\,\phi + \sum_{j=1}^{k} \kappa_j\,(\rho_{j,\mathbf{h}}\,H'_{j,\mathbf{h}} - \rho_{j,\mathbf{0}}\,H'_{j,\mathbf{0}}) \\ &- 2\sum_{j=1}^{k} \nabla(\lambda_j\,\rho_{j,\mathbf{h}})\,\nabla H'_{j,\mathbf{h}} - \sum_{j=1}^{k} \lambda_j\,\rho_{j,\mathbf{h}}\,\left(\Delta + 1 - 3\,\bar u_{h}^2\right)\,H'_{j,\mathbf{h}} \\ &- \sum_{j=1}^{k} (\lambda_j\,\Delta\rho_{j,\mathbf{h}} + 2\nabla\lambda_j\,\nabla\rho_{j,\mathbf{h}})\,H'_{j,\mathbf{h}}\,. \end{aligned}$$

Using the result of the previous proposition to evaluate the norm of f and κ in terms of the norm of f and using (5.63), it is straightforward to check that

$$\left\|\mathbb{L}_{\mathbf{h}}(\bar\phi, \bar\kappa) - f\right\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \le C\,\|\mathbf{h}\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)}\,\|f\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)}\,,$$

for some constant $C > 0$ which does not depend on ε. This completes the proof of the claim. □

Finally, it remains to prove the uniqueness of $\mathbb{G}$. This is the content of :

Lemma 5.3. — *For all $\varepsilon > 0$ small enough, the operator $\mathbb{G}$ described in the statement of Proposition 5.1 is unique.*

Proof. — The proof is decomposed into two steps.

Step 1 - We first prove an *a priori* estimate for the solutions of the homogeneous problem $\mathbb{L}(\phi, \kappa) = 0$ satisfying (5.55). More precisely, we claim that there exists a constant $C > 0$ and $\alpha > 0$ (independent of ε, ϕ and κ) such that

$$\|\kappa\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} \le C\,\varepsilon^{\alpha}\,\|\phi\|_{\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)}\,,$$

for any such solution.

To simplify notations, we identify $X_j^*\phi$ with ϕ and $X_j^*\bar u$ with $\bar u$. We start by multiplying $\mathbb{L}(\phi, \kappa) = 0$ by $\rho_j\,H'_j$ and integrate over y_j to get with little work

$$\begin{aligned} -X_j^*\,\kappa_j \int_{\mathbb{R}} \rho_\varepsilon^2\,(H')^2\,\mathrm{d}y_j &= \int_{\mathbb{R}} \rho_j\,H'_j\,\partial^2_{x_j}\phi\,\mathrm{d}y_j \\ &+ \int_{\mathbb{R}} \rho_j\,H'_j\,(\partial^2_{y_j} + 1 - 3H_j^2)\,\phi\,\mathrm{d}y_j \\ &+ 3\int_{\mathbb{R}} \rho_j\,H'_j\,(H_j^2 - \bar u^2)\,\phi\,\mathrm{d}y_j \\ &+ \int_{\mathbb{R}} \rho_j\,H'_j\,(\Delta - \partial^2_{x_j} - \partial^2_{y_j})\,\phi\,\mathrm{d}y_j\,. \end{aligned}$$

We evaluate each consecutive term. Observe that thanks to (5.55) we can write

$$\int_{\mathbb{R}} \rho_j\,H'_j\,\partial^2_{x_j}\phi\,\mathrm{d}y_j = -\int_{\mathbb{R}} \phi\,\partial^2_{x_j}(\rho_j\,H'_j)\,\mathrm{d}y_j - 2\int_{\mathbb{R}} \partial_{x_j}\phi\,\partial_{x_j}(\rho_j\,H'_j)\,\mathrm{d}y_j\,.$$

Since

$$\partial_{x_j} (\rho_j \, H_j') = -\varepsilon \, h_j' \, \left(\rho_j' \, H_j' + \rho_j \, H_j''\right) \,,$$

and

$$\partial_{x_j}^2 (\rho_j \, H_j') = \varepsilon^2 \, (h_j')^2 \, \left(\rho_j'' \, H_j' + 2 \, \rho_j' \, H_j'' + \rho_j \, H_j'''\right) - \varepsilon^2 \, h_j'' \, \left(\rho_j' \, H_j' + \rho_j \, H_j''\right) \,,$$

it is easy to check that

$$\| \int_{\mathbb{R}} \rho_j \, H_j' \, \partial_{x_j}^2 \phi \, \mathrm{d}y_j \|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R})} \leq C \, \varepsilon^\alpha \, \|\phi\|_{\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \,,$$

for some $\alpha > 0$ which does not depend on ε, ϕ and κ.

Using an integration by parts and the fact that $(\partial_{y_j}^2 + 1 - 3H_j^3) \, H_j' = 0$, we see that the second term can also be written as

$$\int_{\mathbb{R}} \rho_j \, H_j' \, (\partial_{y_j}^2 + 1 - 3H_j^3) \, \phi \, \mathrm{d}y_j = \int_{\mathbb{R}} (\rho_j'' \, H_j' + 2 \, \rho_j' \, H_j'') \, \phi \, \mathrm{d}y_j$$

from which it follows at once that (reducing α if this is necessary)

$$\left\| \int_{\mathbb{R}} \rho_j \, H_j' \, (\partial_{y_j}^2 + 1 - 3H_j^3) \, \phi \, \mathrm{d}y_j \right\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R})} \leq C \, \varepsilon^\alpha \, \|\phi\|_{\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \,.$$

Using the fact that the approximate solution $\bar{u}$ is close to H_j near $\bar{\Gamma}_j$, we check that (reducing α if this is necessary)

$$\left\| \int_{\mathbb{R}} \rho_j \, H_j' \, (H_j^2 - \bar{u}^2) \, \phi \, \mathrm{d}y_j \right\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R})} \leq C \, \varepsilon^\alpha \, \|\phi\|_{\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \,.$$

Finally, using the expansion of the Laplacian in Fermi coordinates, we check that (reducing α if this is necessary)

$$\left\| \int_{\mathbb{R}} \rho_j \, H_j' \, (\Delta - \partial_{x_j}^2 - \partial_{y_j}^2) \, \phi \, \mathrm{d}y_j \right\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} \leq C \, \varepsilon^\alpha \, \|\phi\|_{\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \,.$$

Collecting these estimates completes the proof of the claim.

Step 2 - We now assume that $\phi \in \mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)$ and $\kappa \in \mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)$ satisfy $\mathbb{L}(\phi, \kappa) = 0$. We prove that $\phi = 0$ and $\kappa = 0$ provided ε is close to 0. The proof is by contradiction and close to the proof of Lemma 3.3. Assume that for a sequence ε_n tending to 0 there exist $\phi_n \neq 0$ and κ_n solution of $\mathbb{L}(\phi_n;\kappa_n) = 0$. We normalize ϕ_n so that

$$\|W^{-1}_{\sigma,\varepsilon\tau} \, \phi_n\|_{L^\infty(\mathbb{R}^2)} = 1 \,.$$

We pick up a point $(x_n, y_n) \in \mathbb{R}^2$ such that $W^{-1}_{\sigma,\varepsilon\tau}(x_n, y_n) \, \phi_n(x_n, y_n) \geq \frac{1}{2}$. We define the sequence $\tilde{\phi}_n$ by

$$\tilde{\phi}_n(x, y) := W^{-1}_{\sigma,\varepsilon\tau}(x_n, y_n) \, \phi_n(x - x_n, y - y_n).$$

Using elliptic estimates together with Ascoli-Arzela's theorem, we can assume that (up to a subsequence) the sequence $\tilde{\phi}_n$ converges uniformly, as n tends to $+\infty$, to some function $\tilde{\phi}$ on compacts of $\mathbb{R}^2$. The choice of the point (x_n, y_n) implies that $\tilde{\phi}(0,0) \geq \frac{1}{2}$ and hence is not identically equl to 0. To identify the equation satisfied by $\tilde{\phi}$, we distinguish two cases according to the behavior of the sequence (x_n, y_n).

If, for some subsequence, (x_n, y_n) stays at finite distance from any curve $\bar\Gamma_j$, then $\tilde\phi$ satisfies

$$(\Delta + 1 - 3\,H^2(\cdot - y_0))\,\tilde\phi = 0\,,$$

for some $y_0 \in \mathbb{R}$. Moreover

$$\int_{\mathbb{R}} \tilde\phi\, H'(\cdot - y_0)\,\mathrm{d}y = 0\,.$$

Finally, $|\tilde\phi| \leq C\,(\cosh y)^{-\sigma}$ in $\mathbb{R}^2$. However, the result of Lemma 3.2 shows that $\tilde\phi = 0$, which is a contradiction.

If, for no subsequence (x_n, y_n) stays at finite distance from the curves $\bar\Gamma_j$, then $\tilde\phi$ satisfies

$$(\Delta - 2)\,\tilde\phi = 0\,.$$

Finally, either $|\tilde\phi| \leq C\,(\cosh y)^{\sigma}$ (or $|\tilde\phi| \leq C\,e^{\sigma y}$ or $|\tilde\phi| \leq C\,e^{-\sigma y}$) in $\mathbb{R}^2$. We then consider the function $\tilde W_{a,b}(x,y) := \cosh(a\,x)\,\cosh(b\,y)$ which satisfies $(\Delta - 2)\,\tilde W_{a,b} = -(2 - a^2 - b^2)\,\tilde W_{a,b}$. Taking $a \in (\sigma, \sqrt{2})$ and $b > 0$ such that $a^2 + b^2 < 2$, we can use $\tilde W_{a,b}$ as a barrier to prove that $|\tilde\phi| \leq \nu\,\tilde W_{a,b}$ for all $\nu > 0$. Letting ν tend to 0 we conclude that $\phi \equiv 0$ which is again a contradiction.

Having reached a contradiction in all cases, the proof of the claim is complete. □

Observe that, thanks to the uniqueness result, one can also obtain $\mathbb{G}(\varepsilon, \mathbf{v}, \tilde{\mathbf{h}}; \cdot)$ from $\mathbb{G}(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$ using a perturbation argument as in the proof of Lemma 5.3. Hence we obtain the :

Corollary 5.1. — *There exists a constant $C > 0$ (independent of ε, α_1 and δ_1) such that,*

$$\|\mathbb{G}(\varepsilon, \mathbf{v}, \tilde{\mathbf{h}}; f) - \mathbb{G}(\varepsilon, \mathbf{v}, \mathbf{h}; f)\|_{\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \leq C\,\|\tilde{\mathbf{h}} - \mathbf{h}\|_{\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R},\mathbb{R}^k)}\,\|f\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)}\,,$$

provided $\varepsilon > 0$ is small enough.

5.3. Estimates. — We now measure how far the function $\bar u = \bar u(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$ is from a genuine solution of (1.1). To do so, we analyze the nonlinear operator $Q(\varepsilon, \mathbf{v}, \mathbf{h}; 0)$ which has been defined in (4.50). Recall that

$$Q(\varepsilon, \mathbf{v}, \mathbf{h}; 0) = -(\Delta \bar u + \bar u - \bar u^3)$$

where $\bar u = \bar u(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$. The following result is close to the corresponding analysis performed in [11].

Proposition 5.2. — *Assume that $\sigma \in (0, \sqrt{2}]$ and $\tau > 0$ are fixed so that*

$$\tau < \frac{\tau_0}{\sqrt{2}}.$$

Further assume that δ_1 and α_1 (defined in (5.54)) are fixed. Then, there exists a constant $C > 0$ independent of ε, : $alpha_1$ and δ_1 and there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, we have :

$$\|Q(\varepsilon, \mathbf{v}, \mathbf{0}; 0)\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \leq C\,\varepsilon^{2 - \frac{\sigma}{\sqrt{2}}}\,, \tag{5.63}$$

and

(5.64) $$\|Q(\varepsilon, \mathbf{v}, \tilde{\mathbf{h}}; 0) - Q(\varepsilon, \mathbf{v}, \mathbf{h}; 0)\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \leq C\,\varepsilon^{2-\frac{\sigma}{\sqrt{2}}}\,\|\tilde{\mathbf{h}} - \mathbf{h}\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)}.$$

Proof. — The proof is fairly technical and, in order to enlighten the key points and the ideas involved as clearly as possible, we will assume that $k = 2$. The estimates in the general case follow from similar considerations but notations are more involved.

We first derive the estimates where the cutoff function $\eta_\varepsilon = 1$. In this case, we simply have

$$\bar{u} = H_1 - H_2 - 1\,,$$

and, we can reorganize $Q(\varepsilon, \mathbf{v}, \mathbf{h}; 0)$ as follows

$$\begin{aligned}\Delta\,\bar{u} + (1 - \bar{u}^2)\,\bar{u} \;&=\; (\Delta H_1 + H_1 - H_1^3) - (\Delta H_2 + H_2 - H_2^3)\\ &-\; (H_1 - H_2 - 1)^3 + H_1^3 - H_2^3 - 1.\end{aligned}$$

We now restrict our attention to the subregion V_- in V where $y_1 + y_2 \leq 0$ (similar estimates are available in the region where $y_1 + y_2 \geq 0$). In V_-, we write

$$(H_1 - H_2 - 1)^3 - H_1^3 + H_2^3 + 1 = 3(H_2 + 1)^2\,(H_1 - 1) + 3\,\sqrt{2}\,H_1'\,(H_2 + 1)$$

since $1 - H_1^2 = \sqrt{2}\,H_1'$. Taking advantage of the fact that $H'' + H - H^3 = 0$, and using the expansion of the Laplacian in Fermi coordinates, we realize that

(5.65)

$$\begin{aligned}\Delta\,\bar{u} + (1 - \bar{u}^2)\,\bar{u} \;&=\; \left(\frac{1}{2}\frac{\partial_{y_1} A_1}{A_1} - \varepsilon^2\,\frac{h_1''}{A_1} - 3\,\sqrt{2}\,(H_2 + 1) + \frac{1}{2}\frac{\partial_{x_1} A_1}{A_1^2}\,h_1'\right)\,H_1'\\ &-\; \left(\frac{1}{2}\,\frac{\partial_{y_2} A_2}{A_2} - \varepsilon^2\,\frac{h_2''}{A_2} + \frac{1}{2}\,\frac{\partial_{x_2} A_2}{A_2^2}\,h_2'\right)\,H_2'\\ &-\; 3\,(H_2 + 1)^2\,(H_1 - 1) + \varepsilon^2\,\left(\frac{1}{A_1}\,(h_1')^2\,H_1'' - \frac{1}{A_2}\,(h_2')^2\,H_2''\right)\,,\end{aligned}$$

where we have defined

$$X^*_{j,\varepsilon}\,H_j'(x_j, y_j) := H'(y_j) \qquad \text{and} \qquad X^*_{j,\varepsilon}\,H_j''(x_j, y_j) := H''(y_j)\,.$$

To evaluate these terms, we will use the following facts

$$H_2' = \mathcal{O}_{\mathcal{C}^\infty((-\infty,0))}(e^{\sqrt{2}\,y_2}) \quad \text{and} \quad H_2 + 1 = \mathcal{O}_{\mathcal{C}^\infty(-\infty,0)}(e^{\sqrt{2}\,y_2})$$

while

$$H_1 - 1 = \mathcal{O}_{\mathcal{C}^\infty(0,\infty)}(e^{-\sqrt{2}\,y_1}) \qquad \text{and} \qquad H_1 - 1 = \mathcal{O}_{\mathcal{C}^\infty(-\infty,0)}(1)\,.$$

And we also make use of (5.51) which gives y_2 in terms of y_1 and x_1

(5.66)

$$y_2 = (1 + \mathcal{O}_{\mathcal{C}^\infty(V)}(\varepsilon^2))\,y_1 + \varepsilon\,(a_1^\pm - a_2^\pm + \mathcal{O}_{\mathcal{C}^\infty(V)}(\delta_1\,\varepsilon^{\alpha_1}))\,|x_1| + \sqrt{2}\,\log\varepsilon + \mathcal{O}_{\mathcal{C}^\infty(V)}(1)$$

with $\pm$ according to wether $x_1 \geq 0$ or $x_1 \leq 0$ (remember that $\alpha_1 \in (0, 2)$). We find with some work

(5.67) $$\sup_{\mathbf{x}\in V} W^{-1}_{\sigma,\varepsilon\,\tau}\,\|\Delta\,\bar{u}_\varepsilon + (1 - \bar{u}_\varepsilon^2)\,\bar{u}_\varepsilon\|_{\mathcal{C}^{0,\mu}(B_1(\mathbf{x}))} \leq C\,\varepsilon^{2-\frac{\sigma}{\sqrt{2}}}\,.$$

Let us now explain where the estimate comes from. It turns out that the parameters σ and τ which define the weights have to be chosen so that $\sigma \in (0, \sqrt{2})$ and also

$\tau \in (0, \frac{\tau_0}{\sqrt{2}})$. This is needed to ensure that the function we evaluate has the appropriate decay in both the x and y directions so that its weighted norm is finite. With this choice, a quick inspection of the structure of $\Delta \bar{u}_\varepsilon + (1-\bar{u}_\varepsilon^2)\, \bar{u}_\varepsilon$ shows that, to estimate the norm of this function, there are two region of interest (namely regions where the norm is actually achieved) : the region close to the curve defined by $y_1 = 0$ (namely the curve $\bar{\Gamma}_1$) and the region close to the curve defined by $y_1 + y_2 = 0$. It turns out that the estimate comes from the evaluation of the term $(H_2+1)^2\, H_1'$ along the curve $y_1 + y_2 = 0$. Indeed, we have

$$W_{\sigma,\varepsilon\tau}^{-1}(H_2+1)^2\, H_1' \sim e^{\sqrt{2} y_2} (\cosh y_1)^{\sigma-\sqrt{2}}\, (\cosh x_1)^{\varepsilon\tau}\,,$$

when $y_1 + y_2 \leq 0$. Therefore, we find that

$$W_{\sigma,\varepsilon\tau}^{-1}\, (H_2+1)^2\, H_1' \sim e^{(\sigma - 2\sqrt{2}) y_1}\, (\cosh x_1)^{\varepsilon\tau}\,,$$

when $y_1 + y_2 = 0$. Now, along this curve, we have from (5.67)

$$y_1 = \frac{\varepsilon}{2}\, (a_2^\pm - a_1^\pm + \mathcal{O}(\delta_1 \varepsilon^{\alpha_1}))\, |x_1| - \frac{1}{\sqrt{2}}\, \log \varepsilon + \mathcal{O}(1)\,.$$

again with $\pm$ according to wether $x_1 \geq 0$ or $x_1 \leq 0$. Therefore, we conclude that

$$\sup_{y_1+y_2=0} W_{\sigma,\varepsilon\tau}^{-1}\, (H_2+1)^2\, H_1' \leq C\, \varepsilon^{2-\frac{\sigma}{\sqrt{2}}}\,.$$

Observe that we have implicitly used the fact that

$$\tau < \left(\sqrt{2} - \frac{\sigma}{2}\right)\, (a_2^\pm - a_1^\pm)\,,$$

so that the above supremum is finite. Since, by definition of τ_0 we have $a_2^\pm - a_1^\pm \geq \tau_0$ and since we assume that $\sigma \in (0, \sqrt{2})$, then one can check that this inequality holds provided $\tau < \frac{\tau_0}{\sqrt{2}}$.

Using similar arguments, we find that the terms $\frac{\partial_{y_j} A_j}{A_j}\, H_j'$ contribute to the estimate by at most a constant times ε^2 and the term $(H_2+1)^2\, (H_1 - 1)$ contributes to the estimate by at most a constant times ε^2. All other quantities involving the functions h_j give a contribution of size a constant (depending on δ_1) times $\varepsilon^{2+\alpha_1}$ to the estimate, and hence this contribution can be absorbed into $C\, \varepsilon^{2-\frac{\sigma}{\sqrt{2}}}$ provided ε is chosen small enough.

We finally have to take into account the effect of the cutoff function η_ε. We denote by $\bar{V} \subset V$ the set where η_ε is not equal to either 0 or 1. It is easy to check that

$$\sup_{\mathbf{x} \in \bar{V}} W_{\sigma,\varepsilon\,\tau}^{-1}\, \|\Delta\, \bar{u}_\varepsilon + \bar{u}_\varepsilon\, (1 - \bar{u}_\varepsilon^2)\|_{\mathcal{C}^{0,\mu}(B_1(\mathbf{x}))} \leq C\, \varepsilon^2\,. \tag{5.68}$$

The estimate then follows from (5.68) and (5.69). □

We are now interested in the estimates of the functions

$$F_j(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot) := \int_{\mathbb{R}} (\Delta \bar{u} + \bar{u} - \bar{u}^3)\, \rho_j\, H_j'\, \mathrm{d}y_j\,,$$

as functions of x (or x_j). As we will see in the proof of the next result, there exists $\beta > 0$ such that

$$F_j(\varepsilon, \mathbf{0}, \mathbf{0}; x) = -\varepsilon^2 \left(c_* \, q_j'' + c^* \left(e^{\sqrt{2}\,(q_j - q_{j+1})} - e^{\sqrt{2}(q_{j-1} - q_j)} \right) (\varepsilon x) + \mathcal{O}(\varepsilon^{2+\beta}) \right),$$

on any compact of $\mathbb{R}$. Here the constants c^* and c_* are given by

$$c^* := 6\sqrt{2} \int_{\mathbb{R}} e^{\sqrt{2}t} (H'(t))^2 \, dt = 12 \int_{\mathbb{R}} e^{2t} \, (\cosh t)^{-4} \, dt = 32 \, ,$$

and

$$c_* := \int_{\mathbb{R}} (H'(t))^2 \, dt = \sqrt{2} \int_{\mathbb{R}} (\cosh t)^{-4} \, dt = \frac{4}{3}\sqrt{2} \, .$$

The estimate we have obtained in the previous proposition is quite general and does not use the fact that the functions q_j are required to be solutions to the Toda system (1.7). In contrast, this expansion shows that the estimates of F_j strongly relies on this assumption and indeed, $F_j(\varepsilon, \mathbf{0}, \mathbf{0}; \cdot) = \mathcal{O}(\varepsilon^{2+\beta})$ if $\mathbf{q}$ is a solution of (1.7).

It will be convenient to define

$$F_j^0(\varepsilon, \mathbf{v}, \mathbf{h}; x) := -\varepsilon^2 \Big(c_* \, (v_j + h_j)'' + c^* \, \sqrt{2} \left(e^{\sqrt{2}\,(q_j - q_{j+1})} (v_j + h_j - v_{j+1} - h_{j+1}) \right.$$
$$\left. - \, e^{\sqrt{2}(q_{j-1} - q_j)} (v_{j-1} + h_{j-1} - v_j - h_j) \right) \Big) (\varepsilon x) \, ,$$

and we finally define $\mathring{\mathbf{F}} := (\mathring{F}_1, \ldots, \mathring{F}_k)$ where

$$\mathring{F}_j := F_j - F_j^0 \, .$$

We have the :

Proposition 5.3. — *Assume that $\sigma \in (0, \sqrt{2})$ and $\tau \in (0, \tau_0)$ are fixed. Further assume that α_1 and δ_1 are fixed. Then, there exists $\beta_1 \in (0,1)$ and $C > 0$ (which neither depend on ε, α_1 and δ_1, nor on σ and τ) such that the following estimates hold*

$$\|\mathring{\mathbf{F}}(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)\|_{\mathcal{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} \leq C \, \varepsilon^{2+\beta_1} \, ,$$

and

$$\|\mathring{\mathbf{F}}(\varepsilon, \mathbf{v}, \tilde{\mathbf{h}}; \cdot) - \mathring{\mathbf{F}}(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)\|_{\mathcal{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} \leq C \, \varepsilon^{2+\beta_1} \, \|\tilde{\mathbf{h}} - \mathbf{h}\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} \, ,$$

for all ε small enough and provided $\mathbf{v}$, $\mathbf{h}$ and $\tilde{\mathbf{h}}$ satisfy (5.54).

Proof. — Again, we only consider the case where $k = 1$ since this simplifies the notations.

The starting point if the formula (5.66) which was obtained in the proof of the previous Proposition. The result then follows at once from the integration of this formula against $\rho_1 \, H_1'$. Let us mention the most important aspects of this computation. For brevity we will denote $\tilde{\mathbf{q}} = \mathbf{q} + \mathbf{v}$.

We can write

$$\int_{\mathbb{R}} \frac{1}{2} \frac{\partial_{y_j} A_j}{A_j} (H_j')^2 \rho_j \, dy_j = - \frac{\varepsilon^2 \tilde{q}_j''}{(1+\varepsilon^2 (\tilde{q}_j')^2)^{1/2}} \int_{\mathbb{R}} \frac{1}{A_j} (H_j')^2 \rho_j \, dy_j + \frac{\varepsilon^4 (\tilde{q}_j'')^2}{(1+\varepsilon^2(\tilde{q}_j')^2)^2} \int_{\mathbb{R}} \frac{1}{A_j} y_j (H_j')^2 \rho_j \, dy_j .$$

Since A_j is close to 1, we can estimate

$$\int_{\mathbb{R}} \frac{1}{2} \frac{\partial_{y_j} A_j}{A_j} (H_j')^2 \rho_j \, dy_j = -\varepsilon^2 \tilde{q}_j'' \int_{\mathbb{R}} (H_j')^2 \rho_j \, dy_j + \mathcal{O}(\varepsilon^4 (\cosh x_1)^{-2\varepsilon\tau_0}) .$$

Now

$$\int_{\mathbb{R}} (H_j')^2 \rho_j \, dy_j = \int_{\mathbb{R}} (H')^2 \, dy + \mathcal{O}(\varepsilon^\beta) ,$$

where $\beta > 0$ is fixed (and in fact depends on the definition of the cutoff function ρ_ε see (5.52)). Hence, reducing β if this is necessary, we conclude that

$$\int_{\mathbb{R}} \frac{1}{2} \frac{\partial_{y_1} A_j}{A_j} (H_j')^2 \rho_j \, dy_j = -\varepsilon^2 \tilde{q}_j'' \int_{\mathbb{R}} (H')^2 \, dy + \mathcal{O}(\varepsilon^{2+\beta} (\cosh x_j)^{-\varepsilon\tau}) . \tag{5.69}$$

Similarly, we have

$$-\varepsilon^2 h_j'' \int_{\mathbb{R}} \frac{1}{A_j} (H_j')^2 \rho_j \, dy_j = -\varepsilon^2 h_j'' \int_{\mathbb{R}} (H')^2 \, dy + \mathcal{O}(\varepsilon^{2+\beta} (\cosh x_1)^{-\varepsilon\tau}) , \tag{5.70}$$

for some $\beta > 0$.

Considering in (5.66) the remaining terms which carry the factor H_1', we have

$$\int_{\mathbb{R}} (H_2+1) (H_1')^2 \rho_1 \, dy_1 = \int_{\mathbb{R}} \big(H(y_2 - h_2(\varepsilon x_2)) + 1\big) \big(H'(y_1 - h_1(\varepsilon x_1))\big)^2 \rho_1 \, dy_1$$

Elementary geometry and the fact that $\tilde{q}_j \sim \varepsilon x_j$ at $\pm\infty$ yields the following estimates (please compare with (5.51))

$$y_2 = \tilde{q}_1(\varepsilon x_1) - \tilde{q}_2(\varepsilon x_1) + \sqrt{2} \log \varepsilon + y_1 (1 + \mathcal{O}(\varepsilon^2)) + \mathcal{O}\big(\varepsilon^3 |x_1|\big),$$

and

$$x_2 = (1 + \mathcal{O}(\varepsilon^2)) \, x_1 + \mathcal{O}(\varepsilon \, y_1) + \mathcal{O}\big(\varepsilon \, \log \tfrac{1}{\varepsilon}\big),$$

in the region where $y_1 + y_2 \le 0$. Using this together with the estimate $H_2 + 1 \sim 2 \, e^{\sqrt{2} y_2}$, which holds in a tubular neighborhood of $\bar{\Gamma}_1$, we conclude that

$$\begin{aligned} &\int_{\mathbb{R}} \big(H(y_2 - h_2(\varepsilon x_1)) + 1\big) \big(H'(y_1 - h_1(\varepsilon x_1))\big)^2 \rho_1 \, dy_1 \\ &= -2 \, \varepsilon^2 \int_{\mathbb{R}} e^{\sqrt{2} y} (H')^2 \, dy \, e^{\sqrt{2}(q_1 - q_2)(\varepsilon x_1)} \\ &\quad - 2 \sqrt{2} \, \varepsilon^2 \int_{\mathbb{R}} e^{\sqrt{2} y} (H')^2 \, dy \, e^{\sqrt{2}(q_1 - q_2)(\varepsilon x_1)} (v_1 + h_1 - v_2 - h_2)(\varepsilon x_1) \\ &\quad + \mathcal{O}(\varepsilon^{2+\beta} (\cosh x_1)^{-\varepsilon\tau}) , \end{aligned} \tag{5.71}$$

for some constant $\beta > 0$.

As already mentioned, the fact that q_j is a solution of the Toda system implies that the leading parts in (5.70), (5.71) and (5.72) cancel. The other terms resulting from multiplication of (5.66) by $H_1'\rho_1$ can easily be estimated by $\mathcal{O}(\varepsilon^{2+\beta}\,(\cosh x_1)^{-\varepsilon\tau})$ and similar estimates can be obtained for the Hölder derivatives, completing the proof of the first estimate. The other estimate follows using similar arguments. □

5.4. Solvability of the nonlinear problem. — We are now in a position to apply a first fixed point theorem, to find, close to the approximate solution $\bar u$ a solution of (1.1) which has the desired features. First, we assume that we are given $\mathbf{v} \in \mathcal{E}$ and $\mathbf{h} \in \mathcal{C}^{2,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)$ satisfying (5.54) and we look for a function $\phi = \phi(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$ solution of

$$\mathbb{L}(\varepsilon, \mathbf{v}, \mathbf{h}; \phi, \kappa) = Q(\varepsilon, \mathbf{v}, \mathbf{h}; \phi) \tag{5.72}$$

Thanks to the result of Proposition 5.1, this equation can be rewritten as a fixed point problem

$$(\phi, \kappa) = \mathbb{G}(\varepsilon, \mathbf{v}, \mathbf{h}; Q(\varepsilon, \mathbf{v}, \mathbf{h}; \phi)) \tag{5.73}$$

We choose $\sigma \in (0, \sqrt{2})$ and $\tau \in (0, \frac{\tau_0}{\sqrt{2}})$ so that the results of the previous sections apply for ε small enough. Collecting the results of the previous sections, we prove the :

Proposition 5.4. — *Assume that $\sigma \in (0, \sqrt{2})$ and $\tau \in (0, \frac{\tau_0}{\sqrt{2}})$ are fixed. Further assume that α_1 and δ_1 are fixed. Then, there exists $C_0 > 0$ (independent of the choice of α_1 and δ_1) and there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, there exists a unique $(\phi, \kappa) \in \mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2) \times \mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)$ solution of (5.73) which satisfies*

$$\|\phi\|_{\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} + \|\kappa\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} \leq C_0\, \varepsilon^{2-\frac{\sigma}{\sqrt{2}}}\,.$$

Proof. — The result of Proposition 5.2 and Proposition 5.1 show that

$$\|\mathbb{G}(\varepsilon, \mathbf{v}, \mathbf{h}; Q(\varepsilon, \mathbf{v}, \mathbf{h}; 0))\|_{\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)\times\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} \leq \bar C\, \varepsilon^{2-\frac{\sigma}{\sqrt{2}}}$$

for some constant $\bar C > 0$ which does not depend on ε. We now choose $C_0 = 2\,\bar C$. Next, observe that the nonlinearity with respect to ϕ in Q is simply given by $\phi^3 + 3\,\bar u\,\phi^2$ and it is easy to check that

$$\|Q(\varepsilon, \mathbf{v}, \mathbf{h}; \tilde\phi) - Q(\varepsilon, \mathbf{v}, \mathbf{h}; \phi)\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \leq C\, \varepsilon^{2-\frac{\sigma}{\sqrt{2}}}\, \|\tilde\phi - \phi\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)}\,,$$

provided $\tilde\phi, \phi$ are both in the ball of radius $C_0\,\varepsilon^{2-\frac{\sigma}{\sqrt{2}}}$ in $\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)$. It is now standard to prove that, provided ε is chosen small enough, (5.74) has a solution which can be obtained as a fixed point for contraction mapping in this ball. □

The solution we have obtained in the previous proposition will be denoted by $(\phi(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot), \kappa(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot))$. It is standard to check that, reducing ε_0 if this is necessary, ϕ depends smoothly on the parameter $\mathbf{h}$ and, in some sense to be made precise, also depends continuously on $\mathbf{v}$. However, more will be needed and, with little work, we can estimate the Lipschitz dependence of this solution with respect to $\mathbf{h}$. This is the content of the following :

Lemma 5.4. — *Under the assumptions of the previous Proposition, there exists* $C > 0$ *such that the following estimate holds*

$$\|\phi(\varepsilon, \mathbf{v}, \tilde{\mathbf{h}}; \cdot) - \phi(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)\|_{\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \le C\, \varepsilon^{2-\sqrt{2}\,\sigma}\, \|\tilde{\mathbf{h}} - \mathbf{h}\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)}.$$

Proof. — To distinguish the operators depending on different values of $\mathbf{h}$ we will adorn the operators and functions with a subscript $\mathbf{h}$, writing $\mathbb{L}_{\mathbf{h}}$, $\bar{u}_{\mathbf{h}}$, ... instead of $\mathbb{L}$, $\bar{u}$, ... We also write $Q_{\mathbf{h}} = Q(\varepsilon, \mathbf{v}, \mathbf{h}; \phi_{\mathbf{h}})$.

Taking the difference between the equation satisfied by $\phi_{\mathbf{h}}$ and the equation satisfied by $\phi_{\tilde{\mathbf{h}}}$, we find

$$(\phi_{\tilde{\mathbf{h}}} - \phi_{\mathbf{h}}, \kappa_{\tilde{\mathbf{h}}} - \kappa_{\mathbf{h}}) = \mathbb{G}(\varepsilon, \mathbf{v}, \mathbf{h}; Q_{\mathbf{h}} - Q_{\tilde{\mathbf{h}}}) + \mathbb{G}(\varepsilon, \mathbf{v}, \mathbf{h}; Q_{\tilde{\mathbf{h}}}) - \mathbb{G}(\varepsilon, \mathbf{v}, \tilde{\mathbf{h}}; Q_{\tilde{\mathbf{h}}}).$$

We have

$$Q_{\mathbf{h}} - Q_{\tilde{\mathbf{h}}} = \left(\Delta\bar{u}_{\mathbf{h}} + \bar{u}_{\mathbf{h}} - \bar{u}_{\mathbf{h}}^3 - \Delta\bar{u}_{\tilde{\mathbf{h}}} - \bar{u}_{\tilde{\mathbf{h}}} + \bar{u}_{\tilde{\mathbf{h}}}^3\right)$$
$$+ \phi_{\tilde{\mathbf{h}}}^3 - \phi_{\mathbf{h}}^3 + 3\,\bar{u}_{\mathbf{h}}\,(\phi_{\tilde{\mathbf{h}}}^2 - \phi_{\mathbf{h}}^2) + 3\,(\bar{u}_{\tilde{\mathbf{h}}}^2 - \bar{u}_{\mathbf{h}}^2)\,\phi_{\tilde{\mathbf{h}}}^2.$$

We evaluate each term on the right hand side. Making use of the bound of the solutions of (5.73) which have been obtained in Proposition 5.4, we can write

$$\|3\,(\bar{u}_{\tilde{\mathbf{h}}}^2 - \bar{u}_{\mathbf{h}}^2)\,\phi_{\tilde{\mathbf{h}}}^2\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \le C\,\varepsilon^{2-\frac{\sigma}{\sqrt{2}}}\,\|\tilde{\mathbf{h}} - \mathbf{h}\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)}.$$

Similarly, we get

$$\|\phi_{\tilde{\mathbf{h}}}^3 - \phi_{\mathbf{h}}^3 + 3\,\bar{u}_{\mathbf{h}}\,(\phi_{\tilde{\mathbf{h}}}^2 - \phi_{\mathbf{h}}^2)\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \le C\,\varepsilon^{2-\frac{\sigma}{\sqrt{2}}}\,\|\phi_{\mathbf{h}} - \phi_{\tilde{\mathbf{h}}}\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)}.$$

Finally, Proposition 5.2 yields

$$\|\Delta\bar{u}_{\mathbf{h}} + \bar{u}_{\mathbf{h}} - \bar{u}_{\mathbf{h}}^3 - \Delta\bar{u}_{\tilde{\mathbf{h}}} - \bar{u}_{\tilde{\mathbf{h}}} + \bar{u}_{\tilde{\mathbf{h}}}^3\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \le C\,\varepsilon^{2-\frac{\sigma}{\sqrt{2}}}\,\|\tilde{\mathbf{h}} - \mathbf{h}\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)}.$$

Therefore, we conclude that

$$\|Q_{\mathbf{h}} - Q_{\tilde{\mathbf{h}}}\|_{\mathcal{C}^{0,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \le C\,\varepsilon^{2-\frac{\sigma}{\sqrt{2}}}\,\left(\|\phi_{\mathbf{h}} - \phi_{\tilde{\mathbf{h}}}\|_{\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} + \|\tilde{\mathbf{h}} - \mathbf{h}\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)}\right).$$

On the other hand, using Corollary 5.1 and Proposition 5.2, we get

$$\|\mathbb{G}(\varepsilon, \mathbf{v}, \mathbf{h}; Q_{\tilde{\mathbf{h}}}) - \mathbb{G}(\varepsilon, \mathbf{v}, \tilde{\mathbf{h}}; Q_{\tilde{\mathbf{h}}})\|_{\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)\times\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} \le C\,\varepsilon^{2-\frac{\sigma}{\sqrt{2}}}\,\|\tilde{\mathbf{h}} - \mathbf{h}\|_{\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R},\mathbb{R}^k)}.$$

Summarizing the above we have:

$$\|\phi_{\mathbf{h}} - \phi_{\tilde{\mathbf{h}}}\|_{\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\tau}(\mathbb{R}^2)} \le C\,\varepsilon^{2-\frac{\sigma}{\sqrt{2}}}\,\left(\|\phi_{\mathbf{h}} - \phi_{\tilde{\mathbf{h}}}\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} + \|\tilde{\mathbf{h}} - \mathbf{h}\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)}\right).$$

The desired estimate follows by taking ε small enough. □

We now explain in which sense the solution $\phi(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)$ depends continuously on $\mathbf{v}$. To this aim, let us denote by $X_{j,\mathbf{v}}$ instead of X_j the parameterization defined in (4.42) so that its dependence with respect to $\mathbf{v}$ becomes apparent. Similarly, we will write $\rho_{j,\mathbf{v}}$, instead of ρ_j, $H'_{j,\mathbf{v}}$ instead of H'_j, ... We define a family of diffeormorphism

$Y_{\mathbf{v}}$ smoothly depending on $\mathbf{v} \in \mathcal{E}$ (satisfying (4.46)) and designed in such a way that $Y_{\mathbf{0}} \equiv \mathrm{Id}$ and that

$$\|\nabla (Y_{\mathbf{v}} - \mathrm{Id})\|_{\mathcal{C}^\infty(\mathbb{R}^2)} \leq C\, \varepsilon\, \|\mathbf{v}\|_{\mathcal{E}}\,,$$

and, for all $j = 1, \ldots, k$,

$$Y_{\mathbf{v}}(X_{j,\mathbf{v}}(x_j, y_j)) = X_{j,\mathbf{0}}(x_j, y_j)\,,$$

for all (x_j, y_j) such that $|y_j| \leq \frac{4\sqrt{2}}{3} \log \frac{1}{\varepsilon}$. Observe that, with this choice

$$\rho_{j,\mathbf{v}} = \rho_{j,\mathbf{0}} \circ Y_{\mathbf{v}}\,,$$

and

$$H'_{j,\mathbf{v}} = H'_{j,\mathbf{0}} \circ Y_{\mathbf{v}}\,,$$

on the support of ρ_j.

Lemma 5.5. — *The mapping*

$$\mathbf{v} \in \mathcal{E} \longmapsto \phi_{\mathbf{v}} \circ Y_{\mathbf{v}}^{-1} \in \mathcal{C}^{2,\mu}_{\sigma, \varepsilon\, \tau}(\mathbb{R}^2)\,,$$

is continuous (beware that the weighted space of the right had side is the one corresponding to $\mathbf{v} = \mathbf{0}$).

Proof. — We denote by $\phi_{\mathbf{v}}$ the solution obtained in Proposition 5.73. We can write

$$(\Delta + 1 - 3\bar u_{\mathbf{v}}^2)\, \phi_{\mathbf{v}} = - \left(\Delta \bar u_{\mathbf{v}} + u_{\mathbf{v}} - u_{\mathbf{v}}^3\right) + \phi_{\mathbf{v}}^3 + 3\, \bar u_{\mathbf{v}}\, \phi_{\mathbf{v}}^2\,.$$

We can write

$$\phi_{\mathbf{v}} = \tilde\phi_{\mathbf{v}} \circ Y_{\mathbf{v}}$$

and, composing with $Y_{\mathbf{v}}^{-1}$, we can write the equation satisfied by $\tilde\phi_{\mathbf{v}}$ as

(5.74)
$$\begin{aligned}(\Delta + 1 - 3\, \bar u_{\mathbf{0}}^2)\, \tilde\phi_{\mathbf{v}} \;&=\; - \left(\Delta \bar u_{\mathbf{v}} + u_{\mathbf{v}} - u_{\mathbf{v}}^3\right) \circ Y_{\mathbf{v}}^{-1} + \tilde\phi_{\mathbf{v}}^3 + 3\, \bar u_{\mathbf{v}} \circ Y_{\mathbf{v}}^{-1}\, \tilde\phi_{\mathbf{v}}^2 \\ &+\; 3\, (\bar u_{\mathbf{v}}^2 \circ Y_{\mathbf{v}}^{-1} - \bar u_{\mathbf{0}}^2)\, \tilde\phi_{\mathbf{v}} + \left(\Delta(\tilde\phi_{\mathbf{v}} \circ Y_{\mathbf{v}}) \circ Y_{\mathbf{v}}^{-1} - \Delta \tilde\phi_{\mathbf{v}}\right).\end{aligned}$$

By definition of $Y_{\mathbf{v}}$, we see that $\tilde\phi_{\mathbf{v}}$ satisfies the orthogonality condition (5.55) with $\mathbf{v} = \mathbf{0}$. It is easy to check that $\tilde\phi_{\mathbf{v}}$ is also the unique solution of (5.75) whose norm is bounded by a constant times $\varepsilon^{2-\frac{\sigma}{\sqrt{2}}}$ and which can be obtained as a fixed point for contraction mapping (this implicitly uses the uniqueness result of Lemma 5.3). Observe that we are now working in a fixed function space $\mathcal{C}^{2,\mu}_{\sigma,\varepsilon\, \tau}(\mathbb{R}^2)$ whose definition corresponds to $\mathbf{v} = \mathbf{0}$. Using this formulation we can check that the mapping $\mathbf{v} \in \mathcal{E} \longmapsto \phi_{\mathbf{v}} \circ Y_{\mathbf{v}}^{-1}$ is continuous. □

We now explain how to choose $\mathbf{v}$ and $\mathbf{h}$ so that

(5.75)
$$\kappa(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot) = 0\,.$$

Observe that, multiplying the equation (5.73) by $\rho_j\, H'_j$ we see that the equation $\kappa(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot) = 0$ can be written as

$$\int_{\mathbb{R}} (\Delta \bar u + \bar u - \bar u^3)\, \rho_j\, H'_j\, \mathrm{d}y_j + \int_{\mathbb{R}} (\Delta \phi + \phi - 3\, \bar u^2\, \phi) \rho_j\, H'_j\, \mathrm{d}y_j = \int_{\mathbb{R}} (\phi^3 + 3\, \bar u\, \phi^2)\, \rho_j\, H'_j\, \mathrm{d}y_j\,,$$

for $j = 1, \ldots, k$. It is worth mentioning that both $\bar u$ and ϕ depend on ε, $\mathbf{v}$ and $\mathbf{h}$. We study each of the three terms which compose this equation. First we observe that in Proposition 5.3 we have already derived an estimate for $\mathbf{F} = (F_1, \ldots, F_k)$, where

$$F_j(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot) := \int_{\mathbb{R}} (\Delta \bar u + \bar u - \bar u^3)\, \rho_j\, H_j'\, \mathrm{d}y_j\,.$$

Next, let us define $\mathbf{E} := (E_1, \ldots, E_k)$ where

$$E_j(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot) := \int_{\mathbb{R}} (\Delta \phi + \phi - 3\, \bar u^2\, \phi) \rho_j\, H_j'\, \mathrm{d}y_j\,.$$

We have the following :

Lemma 5.6. — *Assume that $\sigma \in (0, \sqrt{2})$ and $\tau \in (0, \frac{\tau_0}{\sqrt{2}})$ are fixed. Further assume that α_1 and δ_1 are fixed. Then, there exist a constant $\beta_2 > 0$ (which does not depend on ε, σ, α_1 and δ_1) and a constant $C > 0$ such that*

$$\|\mathbf{E}(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)\|_{\mathcal{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} \leq C\, \varepsilon^{2 - \frac{\sigma}{\sqrt{2}} + \beta_2}\,,$$

and

$$\|\mathbf{E}(\varepsilon, \mathbf{v}, \tilde{\mathbf{h}}; \cdot) - \mathbf{E}(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)\|_{\mathcal{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} \leq C\, \varepsilon^{2 - \frac{\sigma}{\sqrt{2}} + \beta_2}\, \|\tilde{\mathbf{h}} - \mathbf{h}\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)}\,,$$

for all ε small enough, provided $\mathbf{v}$, $\mathbf{h}$ and $\tilde{\mathbf{h}}$ satisfy (5.54).

Proof. — The proof is very close to the analysis we have a already performed in the proof of Lemma 5.3. Indeed, following a similar computation we can rewrite E_j as

$$\begin{aligned}
E_j &= 2\, \varepsilon\, h_j' \int_{\mathbb{R}} \left(\rho_j'\, H_j' + \rho_j\, H_j''\right) \partial_{x_j} \phi\, \mathrm{d}y_j + \varepsilon^2\, h_j'' \int_{\mathbb{R}} \left(\rho_j'\, H_j' + \rho_j\, H_j''\right) \phi\, \mathrm{d}y_j \\
&- \varepsilon^2\, (h_j')^2 \int_{\mathbb{R}} \left(\rho_j''\, H_j' + 2\, \rho_j'\, H_j'' + \rho_j\, H_j'''\right) \phi\, \mathrm{d}y_j \\
&+ \int_{\mathbb{R}} (\rho_j''\, H_j' + 2\, \rho_j'\, H_j'')\, \phi\, \mathrm{d}y_j + 3 \int_{\mathbb{R}} \rho_j\, H_j'\, (H_j^2 - \bar u^2)\, \phi\, \mathrm{d}y_j \\
&+ \int_{\mathbb{R}} \rho_j\, H_j'\, (\Delta - \partial^2_{x_j} - \partial^2_{y_j})\, \phi\, \mathrm{d}y_j.
\end{aligned}$$

Instead of going through a technical proof, we simply explain where the estimate comes from. We observe that, thanks to the result of Proposition 5.4, all the terms which carry a factor of $h_j', (h_j')^2$ or h_j'' in front can be estimated by a constant times $\varepsilon^{3 - \frac{\sigma}{\sqrt{2}} + \alpha_1}$.

Using the definition of the cutoff function ρ_j and the exponential decay of functions $1 \pm H$, H' and H'', we can estimate

$$\left\| \int_{\mathbb{R}} (\rho_j''\, H_j' + 2\, \rho_j'\, H_j'')\, \phi\, \mathrm{d}y_j + 3 \int_{\mathbb{R}} \rho_j\, H_j'\, (H_j^2 - \bar u^2)\, \phi\, \mathrm{d}y_j \right\|_{\mathcal{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} \leq C\, \varepsilon^{2 - \frac{\sigma}{\sqrt{2}} + \beta}\,,$$

where $\beta > 0$ only depends on the definition of ρ_ε given in (5.52). The estimate for the last term in the expression for E_j is straightforward using the expression of the Laplacian in Fermi coordinates.

The second estimate follows from similar consideration together with the result of Lemma 5.4. We leave the details to the reader. □

Let us define $\bar{\mathbf{E}} := (\bar{E}_1, \ldots, \bar{E}_k)$ where

$$\bar{E}_j(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot) := \int_{\mathbb{R}} (\phi^3 + 3\,\bar{u}\,\phi^2)\,\rho_j\,H_j'\,\mathrm{d}y_j.$$

We have the:

Lemma 5.7. — *Under the assumptions of the previous Lemma, there exists $C > 0$ such that the following estimates hold*

$$\|\bar{\mathbf{E}}(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)\|_{\mathcal{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} \leq C\,\varepsilon^{4-\sqrt{2}\,\sigma},$$

and

$$\|\bar{\mathbf{E}}(\varepsilon, \mathbf{v}, \tilde{\mathbf{h}}; \cdot) - \bar{\mathbf{E}}(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)\|_{\mathcal{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} \leq C\,\varepsilon^{4-\sqrt{2}\,\sigma}\,\|\tilde{\mathbf{h}} - \mathbf{h}\|_{\mathcal{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)},$$

for all ε small enough, provided $\mathbf{v}$, $\mathbf{h}$ and $\tilde{\mathbf{h}}$ satisfy (5.54).

Proof. — The proof follows at once from Lemma 5.4 and the estimates for the solutions of (5.73) provided by Proposition 5.4. □

We are now in a position to explain how the constant α_1, which was used in (5.54), is fixed. We fist assume that $\sigma \in (0, \sqrt{2})$ and $\mu \in (0,1)$ are chosen so that $2 - \sqrt{2} - \mu > 0$, $-\frac{\sigma}{\sqrt{2}} + \beta_2 - \mu > 0$ and $\beta_1 - \mu > 0$, where β_1 and β_2 are the constants which appear in the last Lemmas. Observe that it is crucial that $\beta_2 > 0$ could be chosen not to depend on σ. Then we define

$$\alpha_1 = \min\left\{2 - \sqrt{2} - \mu, -\tfrac{\sigma}{\sqrt{2}} + \beta_2 - \mu, \beta_1 - \mu\right\}.$$

With this choice, it follows from Proposition 5.3, Lemma 5.6 and Lemma 5.7 that the condition (5.76) is equivalent to

$$\varepsilon^2\,(c_0(\mathbf{v}+\mathbf{h})'' + \mathbf{N}(\mathbf{v}+\mathbf{h}))_j = \hat{E}_j(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot) \tag{5.76}$$

where $\mathbf{N}$ is the matrix of the linearized Toda system associated to linearization of (2.15) about the solution $\mathbf{q}$ (see (2.24)), and where $\hat{\mathbf{E}} := (\hat{E}_1, \ldots, \hat{E}_k)$ satisfies

Lemma 5.8. — *Assume that σ, μ and τ are fixed as above. Then, there exists a constant $C_1 > 0$ (independent of ε and δ_1) such that the following estimates hold*

$$\|\hat{\mathbf{E}}(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)\|_{\mathcal{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} \leq C_1\,\varepsilon^{2+\alpha_1+\mu},$$

and

$$\|\hat{\mathbf{E}}(\varepsilon, \mathbf{v}, \tilde{\mathbf{h}}; \cdot) - \hat{\mathbf{E}}(\varepsilon, \mathbf{v}, \mathbf{h}; \cdot)\|_{\mathcal{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} \leq C_1\,\varepsilon^{2+\alpha_1+\mu}\,\|\tilde{\mathbf{h}} - \mathbf{h}\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)}$$

for all ε small enough, provided $\mathbf{v}$, $\mathbf{h}$ and $\tilde{\mathbf{h}}$ satisfy (5.54).

In view of the result of Lemma 2.3 and the previous Lemma, it is natural to solve (5.77) in the space $\mathcal{C}^{2,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k) \oplus \mathcal{E}$. At this point, it is worth mentioning that for a function $\mathbf{g}\colon \mathbb{R} \to \mathbb{R}^k$ we have the obvious estimate

$$\|\mathbf{g}(\varepsilon\,\cdot)\|_{\mathcal{C}^{\ell,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)} \leq C\,\|\mathbf{g}\|_{\mathcal{C}^{\ell,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)},$$

while on the other hand we have

$$\|\mathbf{g}\|_{\mathcal{C}^{\ell,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)} \leq C\,\varepsilon^{-\ell-\mu}\,\|\mathbf{g}(\varepsilon\,\cdot)\|_{\mathcal{C}^{\ell,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)}\,.$$

Collecting the previous analysis, it is easy to check that :

Lemma 5.9. — *There exists $\delta_1 > 0$ such that, for all $\mathbf{v} \in \mathcal{E}$ satisfying $\|\mathbf{v}\|_{\mathcal{E}} \leq \delta_1\,\varepsilon^{\alpha_1}$ and for all $\varepsilon > 0$ small enough, there exists a unique $\mathbf{h} \in \mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R};\mathbb{R}^k)$ and $\bar{\mathbf{v}} \in \mathcal{E}$ satisfying*

$$\varepsilon^2\left(c_0(\bar{\mathbf{v}}+\mathbf{h})'' + \mathbf{N}(\bar{\mathbf{v}}+\mathbf{h})\right) = \hat{\mathbf{E}}(\varepsilon,\mathbf{v},\mathbf{h};\cdot)\,, \tag{5.77}$$

and

$$\|\bar{\mathbf{v}}+\mathbf{h}\|_{\mathcal{C}^{2,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)\oplus\mathcal{E}} \leq \tfrac{\delta_1}{2}\varepsilon^{\alpha_1}\,.$$

Moreover $\bar{\mathbf{v}}$ depends continuously on $\mathbf{v}$.

Proof. — The proof of this lemma follows immediately from the theory developed in section 2 and more specifically Lemma 2.3, the result of Lemma 5.8 and the use of a fixed point theorem for contraction mapping.

Using the result of Lemma 2.3, we can rewrite the equation we want to solve as

$$\bar{\mathbf{v}}+\mathbf{h} = \varepsilon^{-2}\,T^{-1}\left(\hat{\mathbf{E}}(\varepsilon,\mathbf{v},\mathbf{h};\cdot)\right)\,,$$

Thanks to the result of Lemma 5.8 and the above remark, we can estimate for all $\varepsilon > 0$ small enough

$$\|\varepsilon^{-2}\,T^{-1}\left(\hat{\mathbf{E}}(\varepsilon,\mathbf{v},\mathbf{0};\cdot)\right)\|_{\mathcal{C}^{2,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)} \leq \bar{C}_1\,\varepsilon^{\alpha_1}\,,$$

for some constant $C_1 > 0$ which does not depend on the choice of δ_1. In particular, we can choose, we can choose $\delta_1 = 4\,\bar{C}_1$ and the previous estimate will be valid provided we take $\varepsilon > 0$ small enough. Let us denote by Π the projection

$$\Pi : \mathcal{C}^{2,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)\oplus\mathcal{E} \longmapsto \mathcal{C}^{2,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)\,.$$

Using Lemma 5.8 together with a fixed point theorem for contraction mapping, we get the existence of a (unique) fixed point $\mathbf{h}$, for the mapping

$$\tilde{\mathbf{h}} \longmapsto \varepsilon^{-2}\,\Pi\,T^{-1}\left(\hat{\mathbf{E}}(\varepsilon,\mathbf{v},\tilde{\mathbf{h}};\cdot)\right)\,,$$

in the ball of radius $\frac{\delta_1}{2}\,\varepsilon^{\alpha_1}$ in $\mathcal{C}^{2,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)$. This fixed point $\mathbf{h}$ then induces a (unique) $\bar{\mathbf{v}} \in \mathcal{E}$ by the identify

$$\bar{\mathbf{v}} := \varepsilon^{-2}\,T^{-1}\left(\hat{\mathbf{E}}(\varepsilon,\mathbf{v},\mathbf{h};\cdot)\right) - \mathbf{h}\,.$$

We clearly $\bar{\mathbf{v}}+\mathbf{h}$ is a solution of (5.78) and we have the estimate

$$\|\bar{\mathbf{v}}+\mathbf{h}\|_{\mathcal{C}^{2,\mu}_{\tau}(\mathbb{R};\mathbb{R}^k)\oplus\mathcal{E}} \leq \tfrac{\delta_1}{2}\varepsilon^{\alpha_1}\,.$$

This completes the proof of the result. Continuity with respect to $\mathbf{v}$ follows from Lemma 5.5. □

We will write $\bar{\mathbf{v}} = \bar{\mathbf{v}}(\varepsilon, \mathbf{v})$ for the element of $\mathcal{E}$ which is given by the previous Lemma. Therefore, in order to complete the proof of the result it remains to find $\mathbf{v}$ such that

$$\mathbf{v} = \bar{\mathbf{v}}(\varepsilon, \mathbf{v}).$$

This can be easily achieved by using Browder's fixed point theorem in the ball of radius $\delta_1 \varepsilon^{\alpha_1}$ in $\mathcal{E}$. Observe that we do not apply a fixed point theorem for contraction mapping to determine $\mathbf{v}$ since this would require to prove Lipshitz dependence of all solutions with respect to $\mathbf{v}$. Even though this Lipshitz dependence holds, it would require some extra work and will complicate the notations. Therefore, we have chosen to solve this last equation using some topological fixed point result instead of a fixed point theorem for contraction mapping.

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