Queueing Syst (2010) 66: 155–168
DOI 10.1007/s11134-010-9187-9

Pathwise comparison results for stochastic fluid networks

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Received: 17 December 2009 / Revised: 7 July 2010 / Published online: 25 August 2010

Abstract We prove some new pathwise comparison results for single class stochastic fluid networks. Under fairly general conditions, monotonicity with respect to the (state- and time-dependent) routing matrices is shown. Under more restrictive assumptions, monotonicity with respect to the service rates is shown as well. We conclude by using the comparison results to establish a moment bound, a stability result for stochastic fluid networks with Lévy inputs, and a comparison result for multi-class GPS networks.

Keywords Reflected equations · Comparison theorems · Stochastic networks · Multi-class networks

Mathematics Subject Classification (2000) 60G17 · 90B18 · 68M20 · 60J25

1 Introduction

The analytic analysis of general networks has historically been shown to be a very difficult task. In fact, very few concrete results are known outside of the Markovian setting. In recent years, pathwise analysis has provided invaluable insight into the

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general behavior of various classes of networks, especially for establishing bounds and proving stability.

Stochastic fluid networks (SFNs) are a simple but insightful class of network model for which arrivals are modeled as a fluid and service at the queues can be approximated as a deterministic fluid flow. The particular case of SFNs with constant routing matrix has been extensively studied in a series of papers by Kella [3, 4], Kella and Whitt [5–7] and in the book by Whitt [10]. In particular, the papers of Kella [4] and Kella and Whitt [7] provide stability conditions for SFNs with Lévy and stationary increment inputs respectively, through the use of comparison theorems.

Most of the comparison results for stochastic fluid networks are through their association with reflected equations. Kella and Whitt [7] established several comparison results for reflected equations with constant reflection directions. A comparison result for reflected differential equations, with state- and time-dependent parameters, was proven by Ramasubramanian [9]. Piera and Mazumdar [8] established a similar comparison theorem for reflected diffusions with jumps.

In the first portion of the paper we will prove several pathwise comparison results between the workload processes of single class fluid networks. We will then establish a comparison result on the total workload, for each class, in a multi-class stochastic fluid network under the Generalized Processor Sharing (GPS) service discipline. In the final portion of this paper, we will apply the comparison results to stochastic fluid networks whose inputs are non-decreasing Lévy processes and prove two results. First, we will find bounds for the first moment of the stationary workload process. Secondly, we will prove a stability result for SFNs with state-dependent routing matrices.

2 Assumptions and notation

2.1 The Skorokhod problem

The Skorokhod Oblique Reflection Problem (SP) states that given a càdlàg process $\{X(t); t \geq 0\}$ and an M-matrix R (known as the reflection matrix), there exist unique processes $\{W(t); t \geq 0\}$ (known as the reflected process) and $\{Z(t); t \geq 0\}$ (known as the regulator process) such that $\forall t \geq 0$:

1. $W(t) = X(t) + RZ(t) \geq 0$
2. $Z(0) = 0$ and $dZ(t) \geq 0$
3. $\int_0^t W_i(s)\, dZ_i(s) = 0$.

Furthermore there exists a unique, continuous pair of functions $(\Psi, \Phi) : D[0,\infty) \to D_{\uparrow}[0,\infty) \times D_{+}[0,\infty)$ such that $\Phi(X(\cdot)) = W(\cdot)$ and $\Psi(X(\cdot)) = Z(\cdot)$. $D_{+}[0,\infty)$ denotes the set of non-negative càdlàg process and $D_{\uparrow}[0,\infty)$ denotes the set of non-negative, non-decreasing processes.

A very useful fact is that if there exists another pair of processes $(\hat{W}, \hat{Z})$ that satisfies the first two properties, then $\hat{Z} \geq Z$. This is known as the minimality property of the regulator process. See Chap. 14 of Whitt [10] or Chap. 7 of Chen and Yao [2] for further details.

A generalization of the above classic version of the Skorokhod problem was studied by Ramasubramanian [9]. Given a càdlàg process $\{X(t); t \geq 0\}$ and functions $b : (\Re_+, \Re_+, \Re_+)^N \to \Re^N$, $R : (\Re_+, \Re_+, \Re_+)^{N\times N} \to \Re^{N\times N}$ such that:

– Each component b_i, $1 \leq i \leq N$, is bounded continuous and $(z, w) \mapsto b_i(t, z, w)$ are Lipschitz continuous, uniformly in t.
– Each component R_{ij}, $1 \leq i, j \leq N$, is bounded continuous and $(z, w) \mapsto R_{ij}(t, z, w)$ are Lipschitz continuous, uniformly in t. Moreover, $R_{ii} = 1$.
– There exist a constant $V \in \Re^{N\times N}$ such that $|R_{ij}(t, z, w)| \leq V_{ij}$, for $i \neq j$, and $v_{ii} = 0$. As well, we will assume that $\sigma(V) < 1$ where $\sigma(V)$ denotes the spectral radius of V.

There exists a unique pair of processes (W, Z) such that $\forall t \geq 0$:

1. $W(t) = X(t) + \int_0^t b(s, Z(s^-), W(s^-))\, ds + \int_0^t R(s, Z(s^-), W(s^-))\, dZ(s) \geq 0$
2. $Z(0) = 0$ and $dZ(t) \geq 0$
3. $\int_0^t W_i(s)\, dZ_i(s) = 0$.

2.2 The stochastic fluid network

Stochastic fluid networks are characterized by the 4-tuple $\{J, r, P, W(0)\}$ where $\{J(t); t \geq 0\}$ is the cumulative input process, r is the service rate, P the routing matrix, and $W(0)$ is the initial workload. It is assumed that the queues are work conserving. All stochastic fluid networks in this paper are assumed to be open networks with N queues that each have a single server and infinite capacity.

The cumulative input $\{J_i(t); t \geq 0\}$, $i \in 1 \dots N$ are modeled as a non-decreasing càdlàg processes with $J(0) = 0$.

The routing matrix is assumed to be a function $P : (\Re_+, \Re_+, \Re_+)^{N\times N} \to \Re_+^{N\times N}$, such that for fixed $t, z, w \in \Re_+$:

– Each component of the routing matrix P_{ij}, $1 \leq i, j \leq N$, is bounded continuous and $(z, w) \mapsto P_{ij}(t, z, w)$ are Lipschitz continuous, uniformly in t.
– $P_{ii} = 0$ and $P_{ij} \geq 0$.
– There exist a constant $V \in \Re^{N\times N}$ such that $P_{ij}(t, z, w) \leq V_{ij}$, for $i \neq j$, and $V_{ii} = 0$. As well, $\sigma(V) < 1$ where $\sigma(V)$ denotes the spectral radius of V.
– $P(t, z, w)$ is a substochastic matrix such that $(I - P'(t, z, w))^{-1}$ exists and is non-negative.

Unless otherwise specified, and it will be clear from the context, the service rates are assumed to be a bounded, continuous, non-negative function $r : \Re_+^N \to \Re_+^N$.

In this paper, the reflected process

$$W(t) = W(0) + J(t) - \int_0^t \big(I - P'\big(s, Z(s^-), W(s^-)\big)\big) r(s)\, ds + \int_0^t \big(I - P'\big(s, Z(s^-), W(s^-)\big)\big)\, dZ(s)$$

models the workload process of a fluid network, where I represents the identity matrix.

It will always be assumed that $W(0) \geq 0$ to ensure that $Z(0) = 0$.

3 Comparison theorems

We will begin this section by first stating the Ramasubramanian comparison theorem for general reflected differential equations. This powerful comparison will be used for many of the theorems and lemmas to follow.

Theorem 1 *Let $W^{(1)}$ and $W^{(2)}$ be two solutions to the Skorokhod reflection problem in $\Re^n$ such that*:

$$W^{(1)}(t) = X^{(1)}(t) + \int_0^t b^{(1)}\big(s, Z^{(1)}(s^-), W^{(1)}(s^-)\big)\,ds + \int_0^t R^{(1)}\big(s, Z^{(1)}(s^-), W^{(1)}(s^-)\big)\,dZ^{(1)}(s),$$

$$W^{(2)}(t) = X^{(2)}(t) + \int_0^t b^{(2)}\big(s, Z^{(2)}(s^-), W^{(2)}(s^-)\big)\,ds + \int_0^t R^{(2)}\big(s, Z^{(2)}(s^-), W^{(2)}(s^-)\big)\,dZ^{(2)}(s).$$

If $X^{(1)}(t) - X^{(1)}(s) \le X^{(2)}(t) - X^{(2)}(s)$, $t > s \ge 0$, $X^{(1)}(0) \le X^{(2)}(0)$, $b_i^{(1)}(t, z_1, w_1) \le b_i^{(2)}(t, z_2, w_2)$, $R_{ij}^{(1)}(t, z_1, w_1) \le R_{ij}^{(2)}(t, z_2, w_2) \le 0$, $i \ne j$, *whenever* $w_1 \le w_2$, $z_1 \ge z_2$ *then*:

$$W^{(1)}(t) \le W^{(2)}(t), \quad t \ge 0$$
$$Z^{(1)}(t) \ge Z^{(2)}(t), \quad t \ge 0$$
$$Z^{(1)}(t) - Z^{(1)}(s) \ge Z^{(2)}(t) - Z^{(2)}(s), \quad \forall t > s \ge 0.$$

The inequalities are to be understood componentwise.

The first lemma uses the above comparison theorem to highlight an important relationship between the service rates and the regulator process.

Lemma 1 *Let*

$$W(t) = W(0) + J(t) - \int_0^t (I - P')r(s)\,ds + (I - P')Z(t).$$

Then $\{\int_0^t r(s)\,ds - Z(t) : t \ge 0\}$ *is a non-negative, non-decreasing process.*

Proof Let:

$$W^{(1)}(t) = W(0) + J(t) - \int_0^t (I - P')r(s)\,ds + (I - P')Z^{(1)}(t),$$

$$W^{(2)}(t) = -\int_0^t (I - P')r(s)\,ds + (I - P')Z^{(2)}(t).$$

The solution to the second SP is $(W^{(2)}, Z^{(2)}) = (0, \int_0^{\cdot} r(s)\,ds)$.

The result follows by applying Theorem 1. □

The following theorem establishes a monotonicity result with respect to the initial workload, cumulative input, and the routing matrix.

Theorem 2 *Let*:

$$R^{(i)} = I - P^{(i)'}, \quad i = 1, 2,$$
$$W^{(1)}(t) = W^{(1)}(0) + J^{(1)}(t) - \int_0^t R^{(1)}\big(s, Z^{(1)}(s^-), W^{(1)}(s^-)\big) r(s)\, ds$$
$$+ \int_0^t R^{(1)}\big(s, Z^{(1)}(s^-), W^{(1)}(s^-)\big)\, dZ^{(1)}(s),$$
$$W^{(2)}(t) = W^{(2)}(0) + J^{(2)}(t) - \int_0^t R^{(2)}\big(s, Z^{(2)}(s^-), W^{(2)}(s^-)\big) r(s)\, ds$$
$$+ \int_0^t R^{(2)}\big(s, Z^{(2)}(s^-), W^{(2)}(s^-)\big)\, dZ^{(2)}(s).$$

Also, let $J^{(1)}(t) - J^{(1)}(s) \le J^{(2)}(t) - J^{(2)}(s)$, $t > s \ge 0$, $W^{(1)}(0) \le W^{(2)}(0)$ *and assume that* $P^{(1)}(t, z_1, w_1) \le P^{(2)}(t, z_2, w_2)$ $\forall t \ge 0$, $z_1 \ge z_2$ *and* $w_1 \le w_2$.

Then:

$$W^{(1)}(t) \le W^{(2)}(t), \quad t \ge 0$$
$$Z^{(1)}(t) \ge Z^{(2)}(t), \quad t \ge 0$$
$$Z^{(1)}(t) - Z^{(1)}(s) \ge Z^{(2)}(t) - Z^{(2)}(s), \quad t > s \ge 0.$$

Proof For $(t, z, w) \in \Re_+ \times D_\uparrow[0, \infty) \times D_+[0, \infty)$ define the mappings:

$$X^{(i)}(t, z, w) = W^{(i)}(0) + J^{(i)}(t) - \int_0^t R^{(i)}(s, z_{s^-}, w_{s^-}) r(s)\, ds$$
$$- \int_0^t P^{(i)'}(s, z_{s^-}, w_{s^-})\, dz_s,$$
$$T^{(i)}(t, z, w) = \sup_{0 \le s \le t} \max\big(0, -X^{(i)}(s, z, w)\big)$$

and

$$S^{(i)}(t, z, w) = X^{(i)}(t, z, w) + T^{(i)}(t, z, w)$$

where the sup and max operations are to be applied componentwise and $i = 1, 2$.

Note that the mappings $(T^{(i)}, S^{(i)})$ solves a SP with input $X^{(i)}$ and identity reflection matrix.

Choose processes $(z^{(i)}, w^{(i)}) \in D_\uparrow[0, \infty) \times D_+[0, \infty)$ such that $w^{(2)} \ge w^{(1)}$, $z^{(i)}(0) = 0$,

$$z^{(1)}(t_2) - z^{(1)}(t_1) \ge z^{(2)}(t_2) - z^{(2)}(t_1)$$

and

$$z^{(1)}(t_2) - z^{(1)}(t_1) \le \int_{t_1}^{t_2} r(s)\,ds, \quad \forall t_1 < t_2.$$

From the assumptions it is easy to see that $X^{(2)}(0, z^{(2)}, w^{(2)}) \ge X^{(1)}(0, z^{(1)}, w^{(1)})$ and $X^{(2)}(t_2, z^{(2)}, w^{(2)}) - X^{(2)}(t_1, z^{(2)}, w^{(2)}) \ge X^{(1)}(t_2, z^{(1)}, w^{(1)}) - X^{(1)}(t_1, z^{(1)}, w^{(1)})$ $\forall t_2 > t_1 \ge 0$.

So by Theorem 1:

$$S^{(1)}\big(t, z^{(1)}, w^{(1)}\big) \le S^{(2)}\big(t, z^{(2)}, w^{(2)}\big), \quad t \ge 0$$
$$T^{(1)}\big(t, z^{(1)}, w^{(1)}\big) \ge T^{(2)}\big(t, z^{(2)}, w^{(2)}\big), \quad t \ge 0$$
$$T^{(1)}\big(t_2, z^{(1)}, w^{(1)}\big) - T^{(1)}\big(t_1, z^{(1)}, w^{(1)}\big) \ge T^{(2)}\big(t_2, z^{(2)}, w^{(2)}\big) - T^{(1)}\big(t_1, z^{(2)}, w^{(2)}\big),$$
$$t_2 > t_1 \ge 0.$$

Note that $T^{(1)}(0, z^{(1)}, w^{(1)}) = T^{(2)}(0, z^{(2)}, w^{(2)}) = 0$.

Also by Lemma 1, $T^{(1)}(t_2, z^{(1)}, w^{(1)}) - T^{(1)}(t_1, z^{(1)}, w^{(1)}) \le \int_{t_1}^{t_2} r(s)\,ds$, $\forall t_2 > t_1 \ge 0$.

The result follows since, as was shown in the proof of Theorem 3.7 in [9], the maps $(T^{(i)}, S^{(i)})$ are contraction maps whose unique fixed point is the solution to the SP $(Z^{(i)}, W^{(i)})$. □

The previous result establishes a very intuitive notion. If the fluid leaving the network at each node decreases or a greater amount of fluid arrives at each point in time, then the workload at each queue should, and by Theorem 2 does, increase.

In the previous theorem, monotonicity of the workload with respect to all parameters expect for the service rate was shown. So it is natural to wonder if a similar result with respect to the service rates can be found as well. In general the answer is no, and it is fairly straightforward to find examples of this. But Theorem 1 tells us that, under restrictive conditions, monotonicity can exist.

Lemma 2 *Let*:

$$R = I - P',$$
$$W^{(1)}(t) = W(0) + J(t) - \int_0^t R\big(s, Z^{(1)}(s^-), W^{(1)}(s^-)\big) r^{(1)}(s)\,ds + \int_0^t R\big(s, Z^{(1)}(s^-), W^{(1)}(s^-)\big)\,dZ^{(1)}(s),$$
$$W^{(2)}(t) = W(0) + J(t) - \int_0^t R\big(s, Z^{(2)}(s^-), W^{(2)}(s^-)\big) r^{(2)}(s)\,ds + \int_0^t R\big(s, Z^{(2)}(s^-), W^{(2)}(s^-)\big)\,dZ^{(2)}(s).$$

Assume that $R(t, z_1, w_1) r^{(1)}(t) \ge R(t, z_2, w_2) r^{(2)}(t)$ *and* $R(t, z_1, w_1) \le R(t, z_2, w_2)$ *whenever* $z_1 \ge z_2$ *and* $w_1 \le w_2$.

Then:

$W^{(1)}(t) \le W^{(2)}(t), \quad t \ge 0$

$Z^{(1)}(t) \ge Z^{(2)}(t), \quad t \ge 0$

$Z^{(1)}(t) - Z^{(1)}(s) \ge Z^{(2)}(t) - Z^{(2)}(s), \quad t > s \ge 0.$

Corollary 1 *Let*:

$$W^{(1)}(t) = W(0) + J(t) - (I - P')r^{(1)}t + (I - P')Z^{(1)}(t),$$
$$W^{(2)}(t) = W(0) + J(t) - (I - P')r^{(2)}t + (I - P')Z^{(2)}(t).$$

If $(I - P')r^{(1)} \ge (I - P')r^{(2)}$ *then*:

$W^{(1)}(t) \le W^{(2)}(t), \quad t \ge 0$

$Z^{(1)}(t) \ge Z^{(2)}(t), \quad t \ge 0$

$Z^{(1)}(t) - Z^{(1)}(s) \ge Z^{(2)}(t) - Z^{(2)}(s), \quad t > s \ge 0.$

As mentioned above, increasing the service rates in the network does not necessarily correspond to a decrease in the workload at each queue. But as the remaining comparison theorems will show, they do decrease the total workload in the network.

Theorem 3 *Assume that* $W^{(1)}(0) \le W^{(2)}(0)$,

$$J^{(1)}(t) - J^{(1)}(s) \le J^{(2)}(t) - J^{(2)}(s), \quad t > s \ge 0,$$
$$P^{(1)} \ge P^{(2)} \quad \textit{and} \quad \big(I - P^{(1)'}\big)^{-1}\big(I - P^{(2)'}\big)r^{(2)}(t) \le r^{(1)}(t), \quad \forall t \ge 0.$$

Define:

$$W^{(1)}(t) = W^{(1)}(0) + J^{(1)}(t) - \int_0^t \big(I - P^{(1)'}\big)r^{(1)}(s)\,dt + \big(I - P^{(1)'}\big)Z^{(1)}(t),$$
$$W^{(2)}(t) = W^{(2)}(0) + J^{(2)}(t) - \int_0^t \big(I - P^{(2)'}\big)r^{(2)}(s)\,dt + \big(I - P^{(2)'}\big)Z^{(2)}(t).$$

Then

$$\sum_{j=1\ldots N} W_j^{(1)}(t) \le \sum_{j=1\ldots N} W_j^{(2)}(t), \quad \forall t \ge 0.$$

Proof Assume that $W^{(1)}(0) = W^{(2)}(0) = 0$.

Define

$$Z^*(t) = \big(I - P^{(1)'}\big)^{-1}\big(J^{(2)}(t) - J^{(1)}(t)\big) + \big(I - P^{(1)'}\big)^{-1}\big(I - P^{(2)'}\big)Z^{(2)}(t)$$
$$+ \int_0^t \big(r^{(1)}(s) - \big(I - P^{(1)'}\big)^{-1}\big(I - P^{(2)'}\big)r^{(2)}(s)\big)\,dt.$$

Note that since $P^{(1)} \geq P^{(2)}$, $(I - P^{(1)'})^{-1}(I - P^{(2)'}) \geq I$. Z^* is clearly a non-decreasing process with $Z^*(0) = 0$. So $J^{(1)}(t) - (I - P^{(1)'})r^{(1)}t + (I - P^{(1)'})Z^*(t) = W^{(2)} \geq 0$. Therefore $Z^* \geq Z^{(1)}$ by the minimality property of the regulator process.

This implies that $(I - P^{(1)'})^{-1}W^{(2)} \geq (I - P^{(1)'})^{-1}W^{(1)}$. Since P is substochastic, the result follows by multiplying both sides of the above inequality by $e'(I - P^{(1)'})$, where e is a column vector of ones.

The result follows if $W^{(1)}(0) \leq W^{(2)}(0)$ from the following method: Set $J^{(1)}(t) = J^{(1)}(t) + W^{(1)}(0)$ and $J^{(2)}(t) = J^{(2)}(t) + W^{(2)}(0)$. Shift the starting time from 0 to $-t_0 < 0$ and defining $W^{(1)}(-t_0) = W^{(2)}(-t_0) = 0$ and $J^{(1)}(t) = J^{(2)}(t) = 0$, $\forall t \in [-t_0, 0)$, this implies that $W^{(1)}(t) = W^{(2)}(t) = 0$, $\forall t \in [-t_0, 0)$. The proof follows exactly the same as above but adjusting for the fact the time now starts at $-t_0$ instead of 0. □

The previous theorem (and subsequently the proof) are a generalization of the following corollary which was established in [4] as Lemma 3.1.

Corollary 2 *Assume that* $W^{(1)}(0) \leq W^{(2)}(0)$ *and* $(I - P^{(2)'})r^{(2)} \leq r^{(1)}$.

Define:

$$W^{(1)}(t) = W^{(1)}(0) + J(t) - r^{(1)}t + Z^{(1)}(t),$$
$$W^{(2)}(t) = W^{(2)}(0) + J(t) - \big(I - P^{(2)'}\big)r^{(2)}t + \big(I - P^{(2)'}\big)Z^{(2)}(t).$$

Then

$$\sum_{i=1\ldots N} W_i^{(1)}(t) \leq \sum_{i=1\ldots N} W_i^{(2)}(t), \quad \forall t \geq 0.$$

For any stochastic fluid network, let $A \subset \{1, 2, \ldots, N\}$ and define $\mathcal{P}_A = \{j \in \{1 \ldots N\} : \exists$ a path from the output of queue j to queue i in $A\} \cup A$.

Note that if $j \in \mathcal{P}_A$ and $m \notin \mathcal{P}_A$ then by definition $P_{m,j} = 0$. Consider workload processes $W_{\mathcal{P}_A}$ corresponding to the stochastic fluid networks restricted to queues in $\mathcal{P}_A$, i.e. $\{[J]_{\mathcal{P}_A}, [r]_{\mathcal{P}_A}, [P]_{\mathcal{P}_A}, [W]_{\mathcal{P}_A}(0)\}$, $i = 1, 2$ where $[\cdot]_{\mathcal{P}_A}$ represents the subvector/submatrix restricted to elements in the set $\mathcal{P}_A$. By the uniqueness of the solution to the Skorokhod problem, $W_{\mathcal{P}_A} = [W]_{\mathcal{P}_A}$.

Now we return to the problem of comparing two stochastic fluid networks. In Theorem 3, it was vital that $P^{(1)} \geq P^{(2)}$, as opposed to the much more natural comparison theorem condition that $P^{(1)} \leq P^{(2)}$. The following lemma shows that in the latter case, a little more can be said.

Lemma 3 *Assume that*

$$W^{(1)}(0) \leq W^{(2)}(0),$$
$$J^{(1)}(t) - J^{(1)}(s) \leq J^{(2)}(t) - J^{(2)}(s), \quad t > s \geq 0,$$
$$P^{(1)} \leq P^{(2)} \quad \textit{and} \quad r^{(2)}(t) \leq r^{(1)}(t), \quad \forall t \geq 0.$$

Define:

$$W^{(1)}(t) = W^{(1)}(0) + J^{(1)}(t) - \int_0^t \big(I - P^{(1)'}\big) r^{(1)}(s)\,ds + \big(I - P^{(1)'}\big) Z^{(1)}(t),$$

$$W^{(2)}(t) = W^{(2)}(0) + J^{(2)}(t) - \int_0^t \big(I - P^{(2)'}\big) r^{(2)}(s)\,ds + \big(I - P^{(2)'}\big) Z^{(2)}(t).$$

Then for every set

$$A \subset \{1, 2, \ldots, N\}, \qquad \sum_{j \in \mathcal{P}_A^{(1)}} W_j^{(1)}(t) \le \sum_{j \in \mathcal{P}_A^{(2)}} W_j^{(2)}(t), \quad \forall t \ge 0.$$

Proof Combining Theorems 2 and 3 gives

$$\sum_{j=1\ldots N} W_j^{(1)}(t) \le \sum_{j=1\ldots N} W_j^{(2)}(t), \quad \forall t \ge 0.$$

The condition $P^{(1)} \le P^{(2)}$ implies that $\forall A \subset \{1, 2, \ldots, N\}$ $\mathcal{P}_A^{(1)} \subset \mathcal{P}_A^{(2)}$.

Therefore

$$\sum_{j \in \mathcal{P}_A^{(2)}} W_j^{(2)}(t) \ge \sum_{j \in \mathcal{P}_A^{(2)}} W_j^{(1)}(t) \ge \sum_{j \in \mathcal{P}_A^{(1)}} W_j^{(1)}(t). \qquad \square$$

We will now proceed to prove a comparison results with applications to certain types of multi-class queues which will be discussed in the applications section. This will require a slightly different model than before, which we call the "state process" model.

To simplify the analysis, we will assume that we are dealing with an ON-OFF input and that the system knows the state of the input at time t, i.e. whether $State(J_i(t)) = ON$ or $State(J_i(t)) = OFF$.

Define the state process $S : \Re_+ \to \{0, 1\}^N$

$S_i(t) = 1$ if fluid is flowing out of queue i at time t, otherwise $S_i(t) = 0$.

For the remainder of the section we will assume that the service rates are bounded, positive functions which are now dependent on the state process, i.e. $r : \{0, 1\}^N \to \Re_+^N$.

So the workload process becomes

$$W(t) = J(t) - \int_0^t (I - P') r\big(S(s)\big)\,ds + (I - P') Z(t).$$

From the physics of the fluid network, we will let $S_i = 1$ iff $\exists j \in \mathcal{P}_i$, s.t. $W_j(t^-) > 0$ or $State(J_j(t)) = ON$. The implication of this definition is that if

$S_j(t)=1$, then $S_i(t)=1$. So all permissible states may by strictly smaller than all combinations of $\{0,1\}^{k\times n}$. Denote $\mathcal{S}$ as the set of permissible states of $S(t)$.

Theorem 4 *Assume that*

$$\begin{aligned} &P^{(1)} \le P^{(2)},\\ &W^{(1)}(0) \le W^{(2)}(0),\\ &J^{(1)}(t) - J^{(1)}(s) \le J^{(2)}(t) - J^{(2)}(s), \quad t > s \ge 0. \end{aligned}$$

Given $s^{(1)}, s^{(2)} \in \mathcal{S}$, *if* $s^{(1)} \le s^{(2)}$ *then* $r^{(1)}(s^{(1)}) \ge r^{(2)}(s^{(2)})$.

Define:

$$W^{(1)}(t) = W^{(1)}(0) + J^{(1)}(t) - \int_0^t \big(I - P^{(1)'}\big) r^{(1)}\big(S^{(1)}(s)\big)\,dt + \big(I - P^{(1)'}\big) Z^{(1)}(t),$$

$$W^{(2)}(t) = W^{(2)}(0) + J^{(2)}(t) - \int_0^t \big(I - P^{(2)'}\big) r^{(2)}\big(S^{(2)}(s)\big)\,dt + \big(I - P^{(2)'}\big) Z^{(2)}(t).$$

Then $\forall t \ge 0$, $S^{(1)}(t) \le S^{(2)}(t)$.

Furthermore,

$$\forall A \subset \{1,2,\ldots,N\}, \quad \sum_{j\in\mathcal{P}_A} W_j^{(1)}(t) \le \sum_{j\in\mathcal{P}_A} W_j^{(2)}(t).$$

Proof Before proceeding we require the following fact (Corollary 14.3.5 of [10] and Theorem 3.7 of [9]): $\forall j \in \mathcal{P}_i$ states the processes $W^{(1)}$ and $W^{(2)}$ are continuous at all continuity points of the input processes $J^{(1)}$ and $J^{(2)}$ respectively.

The proof of the main result will follow by contradiction.

From the assumptions, we know that $S^{(1)}(0) \le S^{(2)}(0)$. So assume that there exists a time $T > 0$ such that $S^{(1)}(T) \not\le S^{(2)}(T)$. Let i be a queue that has $S_i^{(1)}(T) = 1$ and $S_i^{(2)}(T) = 0$. Since $S_i^{(2)}(T) = 0$, this implies that $\forall j \in \mathcal{P}_{\{i\}}^{(2)}$ $W_j^{(2)}(T) = 0$ and $state(J_j^{(2)}(T)) = OFF$.

But since $r^{(1)}(S^{(1)}(t)) \ge r^{(2)}(S^{(2)}(t))$ $\forall t \in [0,T)$, we have by Lemma 3 that for all $t \in [0,T)$,

$$\sum_{j\in\mathcal{P}_{\{i\}}^{(2)}} W_j^{(2)}(t) \ge \sum_{j\in\mathcal{P}_{\{i\}}^{(1)}} W_j^{(1)}(t).$$

Non-negativity of the workload process and continuity implies that $\forall j \in \mathcal{P}_{\{i\}}^{(1)}$ $W_j^{(1)}(T) = 0$.

We have a contradiction since $state(J_j^{(2)}(T)) = OFF \Rightarrow state(J_j^{(1)}(T)) = OFF$ and $\forall j \in \mathcal{P}_{\{i\}}^{(1)}$ $W_j^{(1)}(T) = 0$ means that $S_i^{(1)}(T) = 0$. Therefore $\forall t \ge 0$, $S^{(1)}(t) \le S^{(2)}(t)$.

The remainder follows by Lemma 3. □

3.1 Multi-class networks with generalized processor sharing

Multi-class networks are very difficult to analyze analytically. In this section we establish some comparison results for multi-class networks under the Generalized Processor Sharing service discipline. The results are established exploiting the single class state process model, which was defined in the previous section.

Generalized Processor Sharing or GPS is a service discipline that is used to imitate a (weighted) round robin process sharing at each queue. Assume the network has k classes and let $\phi_{c,i} > 0$ denote the "weight" of class $c \in \{1 \dots k\}$ at queue $i \in \{1 \dots N\}$. Without loss of generality, we can assume that

$$\forall i \in \{1 \dots N\}, \qquad \sum_{c \in \{1 \dots k\}} \phi_{c,i} = 1.$$

Each queue has service capacity C_i, and the service rate at time t for each class c at each queue i is

$$r_{c,i}(t) = \frac{S_{c,i}(t)\phi_{c,i}}{\phi_{c,i} + \sum_{\tilde{c} \neq c} S_{\tilde{c},i}(t)\phi_{\tilde{c},i}} C_i,$$

S is defined to be the $k \times N$ dimensional state process.

The workload process vector for all class $c \in \{1 \dots k\}$ is:

$$W_c(t) = W_c(0) + J_c(t) - \int_0^t (I - P'_c) r_c\big(S(s)\big)\, dt + (I - P'_c) Z_c(t).$$

Unfortunately, this model of the workload is not useful for our purposes. Define:

$$\hat{r}_{c,i}(t) = \frac{\phi_{c,i}}{\phi_{c,i} + \sum_{\tilde{c} \neq c} S_{\tilde{c},i}(t)\phi_{\tilde{c},i}} C_i,$$
$$\check{r}_{c,i}(t) = \frac{(1 - S_{c,i}(t))\phi_{c,i}}{\phi_{c,i} + \sum_{\tilde{c} \neq c} S_{\tilde{c},i}(t)\phi_{\tilde{c},i}} C_i,$$
$$\hat{Z}_{c,i}(t) = \int_0^t \check{r}_{c,i}\big(S(s)\big)\, dt + Z_{c,i}(t).$$

It is clear that $r_c(t) = \hat{r}_c(t) - \check{r}_c(t)$, $\hat{Z}_{c,i}(0) = 0$ and $\hat{Z}_{c,i}(t)$ is a non-decreasing process.

Therefore we can now, redefine the workload process as:

$$W_c(t) = W_c(0) + J_c(t) - \int_0^t (I - P'_c) r_c\big(S(s)\big)\, dt + (I - P'_c) Z_c(t),$$
$$W_c(t) = W_c(0) + J_c(t) - \int_0^t (I - P'_c) \hat{r}_c\big(S(s)\big)\, dt + (I - P'_c)\left(\int_0^t \check{r}_c\big(S(s)\big)\, dt + Z_c(t)\right),$$
$$W_c(t) = W_c(0) + J_c(t) - \int_0^t (I - P'_c) \hat{r}_c\big(S(s)\big)\, dt + (I - P'_c) \hat{Z}_c(t).$$

Notice that $\hat{r}_c(S(t))$ has the very useful property that if, $s^{(1)}, s^{(2)} \in \{0,1\}^{k\times N}$ such that $s^{(1)} \le s^{(2)}$, then $\hat{r}_c(s^{(1)}) \ge \hat{r}_c(s^{(2)})$.

For simplicity, from now on we let $r_c \equiv \hat{r}_c$ and $Z_c \equiv \hat{Z}_c$.

Finally, we now convert the multi-class network with k classes and N queues into a larger single class network $\{\tilde{J}, \tilde{r}, \tilde{P}, \tilde{W}(0)\}$ with $k * N$ queues using the following procedure: The multi-class processes $J_c, r_c, W_c(0)$, are mapped to the single class vectorial processes $\tilde{J}, \tilde{r}, \tilde{W}(0)$ using the mapping $(c,i) \to (c-1) * N + i$.

The routing matrix $\tilde{P}$ is a $N * k \times N * k$ block diagonal matrix, with the routing matrices P_c, $c = 1 \dots k$ as the block diagonal elements.

Combining this setup and Theorem 4 establishes the following comparison theorem.

Theorem 5 *Assume that* $\forall c \in \{1 \dots k\}$:

$$\begin{aligned}
&P_c^{(1)} \le P_c^{(2)},\\
&W_c^{(1)}(0) \le W_c^{(2)}(0),\\
&J^{(1)}(t) - J^{(1)}(s) \le J^{(2)}(t) - J^{(2)}(s), \quad t > s \ge 0,\\
&C^{(1)} \ge C^{(2)}.
\end{aligned}$$

Then

$$S^{(1)}(t) \le S^{(2)}(t) \quad \textit{and} \quad \sum_{i=1\dots N} W_{c,i}^{(1)}(t) \le \sum_{i=1\dots N} W_{c,i}^{(2)}(t), \quad \forall t \ge 0.$$

Descriptively the above theorem tells us that increasing congestion in one aspect of the network, increases the total workload in the network for each class. A simple corollary is that increasing the congestion in just one class adversely affects the other classes in terms of workload. Note that this statement is not true in general discrete queueing networks under GPS. In a discrete queueing network, very large arrivals of a certain class could potentially "clog" the routes of that class, which by the very nature of GPS would be beneficial to the other classes.

3.2 Applications

In this section we will apply some of the previous comparison theorems to stochastic fluid networks whose inputs are non-decreasing Lévy processes. We will begin with a bound on the first moment of the stationary workload process and then proceed to prove a stability result.

The processes $\{J_i(t); t \ge 0\}$, $i \in 1 \dots N$ are now independent subordinators (non-decreasing càdlàg Lévy processes) with $J(0) = 0$. We assume $E[W(0)] < \infty$ and that $W(0)$ is independent of $\{J(t); t \ge 0\}$. For ease of notation we will write $J(t) \equiv W(0) + J(t)$. Also, let $\lambda = E[J(1)] < \infty$ and $\sigma^2 = \mathrm{Var}(J(1)) < \infty$. We will assume that service rates are constant for the remainder of the paper.

3.2.1 Bounds on the first moment

Let:

$$W(t) = J(t) - (I - P')rt + (I - P')Z(t),$$
$$W^u(t) = J(t) - (I - P')rt + Z^u(t),$$
$$W^l(t) = J(t) - rt + Z^l(t).$$

We will assume throughout this section that $(I - P')^{-1}\lambda < r$. This stability assumption implies that there exists stationary and limiting distribution for both $\{W^l(t) : t \geq 0\}$ and $\{W(t) : t \geq 0\}$.

Lemma 4 $W^l \leq W \leq W^u$

Proof By Theorem 1, $W \leq W^u$. By Theorem 2, $W \geq W^l$. □

The upper bound in the previous lemma will not be useful for many calculations unless $\lambda < (I - P')r$. So let $W^{u2}(t) = J(t) - \tilde{r}t + Z^{u2}(t)$ where $\tilde{r} > \lambda$ and $(I - P')^{-1}\tilde{r} < r$.

It should be noted that such a $\tilde{r}$ always exists, and that there exists a stationary and limiting distribution for the process $\{W^{u2}(t) : t \geq 0\}$.

Lemma 5

$$e'W^{u2}(t) - e'W^l(t) + W_j^l(t) \geq W_j(t), \quad \forall t \geq 0 \textit{ for each } j = 1 \ldots N$$

where e is a column vector of ones.

Proof By Corollary 2, $e'W^{u2}(t) \geq e'W(t)$. By Theorem 2, $e'W(t) \geq \sum_{i \neq j} W_i^l + W_j$. Combining both inequalities gives the assertion. □

Lemma 6

$$0 \leq E[W_j] - \frac{\sigma_j^2}{2(r_j - \lambda_j)} \leq \sum_{i=1}^{N} \left(\frac{\sigma_i^2}{2(\tilde{r}_i - \lambda_i)} - \frac{\sigma_i^2}{2(r_i - \lambda_i)} \right), \quad j = 1 \ldots N.$$

Additionally suppose that $\lambda < (I - P')r$, *then* $\frac{\sigma^2}{2(r-\lambda)} \leq E[W] \leq \frac{\sigma^2}{2((I-P')r-\lambda)}$ *where the division is to be interpreted componentwise.*

Proof Define the random variables W^l, W^{u2}, W as having the stationary distribution of the respective processes. By [6],

$$E\big[W_j^l\big] = \frac{\sigma_j^2}{2(r_j - \lambda_j)},$$
$$E\big[W_j^{u2}\big] = \frac{\sigma_j^2}{2(\tilde{r}_j - \lambda_j)}.$$

The first result follows from Lemma 5.

Since $\lambda < (I - P')r$, $\{W^u(t) : t \geq 0\}$ has a limiting and stationary distribution, and let W^u be a random with that stationary distribution.

The additional result follows from Lemma 4. □

3.2.2 A stability result

Let $\hat{P}$ be a constant routing matrix and $P(\cdot)$ a state-dependent routing matrix such that $\hat{P} \geq P(w)\ \forall w \in \Re_+^N$.

Let:

$$\hat{W}(t) = J(t) - (I - \hat{P}')rt + (I - \hat{P}')\hat{Z}(t),$$

$$W(t) = J(t) - \int_0^t \big(I - P'\big(W(s^-)\big)\big)r\,ds + \int_0^t \big(I - P'\big(W(s^-)\big)\big)\,dZ(s).$$

Also, assume the stability condition $(I - \hat{P}')^{-1}\lambda < r$.

Lemma 7 *There exists a limiting and stationary distribution for* $\{W(t) : t \geq 0\}$.

Proof By Theorem 6.1 of [9], W is a strong Markov process. By Theorem 2, $\hat{W}(t) \geq W(t)\ \forall t \geq 0$. From the proof of Theorem 3.1 in [4], $\hat{W}$ is a positive recurrent Markov process, with a regenerative atom at 0.

This implies that W is a positive recurrent Markov process with a regenerative atom at 0, as well. The assertion follows by Proposition 3.8 (p. 203) of [1]. □

Acknowledgements This research was supported in part by a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada (NSERC).

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