Existence of solutions for the equations modeling the motion of rigid bodies in an ideal fluid

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Abstract

In this paper, we study the motion of rigid bodies in a perfect incompressible fluid. The rigid-fluid system fils a bounded domain in $\mathbb{R}^3$. Adapting the strategy from Bourguignon and Brezis [1], we use the stream lines of the fluid and we eliminate the pressure by solving a Neumann problem. In this way, the system is reduced to an ordinary differential equation on a closed infinite dimensional manifold. Using this formulation, we prove the local in time existence and uniqueness of strong solutions.

Notation. Throughout this paper Ω denotes an open bounded and connected subset of $\mathbb{R}^3$ and S_0 is a closed set with nonempty interior and with smooth boundary such that $S_0 \subset \Omega$. We denote as usual by $SO_3(\mathbb{R})$ the special orthogonal group on $\mathbb{R}^3$. We will often use functions defined from a time interval to $\mathbb{R}^3$ or to $SO_3(\mathbb{R})$. these functions will be denoted using bold characters, such as $\mathbf{h} : [0, T] \to \mathbb{R}^3$ or $\mathbf{R} : [0, T] \to SO_3(\mathbb{R})$. The same kind of notation will be used for three other time dependent vector fields $\mathbf{k}$, $\boldsymbol{\omega}$, $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ which will be defined in the sequel. The five time dependent fields mentioned above will define the state $\mathbf{z}$ of the fluid-solid system. A vector from $\mathbb{R}^3$ or a matrix from $SO_3(\mathbb{R})$ will be denoted by h or by R, respectively. The transposed of a matrix will be denoted by * so that the column vector of components a and b is denoted either $\begin{pmatrix} a \\ b \end{pmatrix}$ or by $(a, b)^*$. Differentiation with respect to time is often denoted a dot. The vector, respectively the inner, product of $v,\ w \in \mathbb{R}^3$ will be denoted by $v \wedge w$ and $v \cdot w$, respectively . The Jacobian matrix of a vector field $y \mapsto f(y)$ defined on an open subset of $\mathbb{R}^3$ will be denoted by $\mathrm{D}_y f$ or simply by $\mathrm{D}f$.

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1 Introduction

The interaction of rigid bodies and of ideal fluids is a topic which was probably first tackled by d'Alembert, Kelvin and Kirchhoff, who considered the case of a potential fluid (sometimes called inviscid fluid), with the solid-fluid system filling the whole space. In this case the governing equations can be reduced to system of ODE's on a finite dimensional manifold. We refer to the book of Lamb [9, chapter 6] for a detailed presentation of this early contributions and to Kanso, Marsden, Rowley and Melli-Huber [8] for the application of the above theory to self-propelled motions of solids in an inviscid fluid. Recently Houot and Munnier in [7] used shape sensitivity analysis techniques to deal with either bounded or unbounded domains. They also tackled the special case of a cylinder in a half space. They showed in particular that, unlike the case of a viscous fluid (see San Martin, Starovoitov and Tucsnak [15], Hillairet [6], Hesla [5]), the cylinder can touch the wall in finite time with non zero velocity. The damping effect of the wall on the cylinder is also studied.

In the general case the system is genuinely infinite dimensional, so it cannot be reduced to ODE's on finite dimensional manifolds. As usual in fluid-solid interaction problems, a major difficulty comes from the fact that the equations for the fluid (Euler's equations in our case) hold in a time dependent domain, so that we have a free boundary value problem. As far as we know, the first papers tackling the case of a non potential flow are Ortega, Rosier and Takahashi [12] and [13]. The main result in these works asserts the existence and uniqueness of classical solutions in two space dimensions and with the rigid-fluid system filling the whole space. More recently, Rosier and Rosier in [14] proved the existence of strong solutions in the case in which the solid is a ball, with the fluid-rigid system filling $\mathbb{R}^n$, with $n \geqslant 2$.

The aim of the present work is to prove the existence an uniqueness of strong solutions in three space dimensions, with a bounded fluid-rigid domain and with the possibility of considering more than one solid. An idea which seems attractive, since it yields a transformed problem written in a fixed domain, is the use of groups of diffeomorphisms as proposed in Ebin and Marsden [3]. Our approach, based on this idea, follows more closely Bourguignon and Brezis [1]. The first new difficulty we need to tackle is that, the fluid domain being variable and the normal velocity of the fluid being different from zero on the fluid-solid interface we are not able to apply the Leray projector. Therefore, in order to eliminate the pressure we need to solve non-homogeneous Neumann problems for the Laplacian. The second difficulty consists in the fact that we need to compare solutions of these Neumann problems in different domains and to show that they depend smoothly on some geometric parameters.

To be more precise, the motion of the fluid is described by the classical Euler equations, whereas the motion of the rigid bodies is governed by the balance equations for linear and angular momentum (Newton's laws). For the sake of simplicity we state and prove our results in the case of a single rigid body, but our methods can be easily be adapted to the case of several rigid bodies. Assume that the system fluid-rigid body fills the domain Ω in $\mathbb{R}^3$ and that at $t = 0$ the solid is located at S_0 (see the paragraph on notation from the beginning of the paper for the properties of Ω and S_0). The position of the solid at instant $t \geqslant 0$ is denoted by $S(t)$. We assume that the solid is surrounded by a perfect homogeneous incompressible fluid filling, for each $t \geqslant 0$, the domain $F(t) = \Omega \setminus S(t)$. In this work we study the following initial and boundary value problem:

$$\rho_F \frac{\partial u}{\partial t} + \rho_F (u \cdot \nabla) u + \nabla p = 0 \qquad x \in F(t),\ t \geqslant 0, \tag{1.1a}$$

$$\operatorname{div} u = 0 \qquad x \in F(t),\ t \geqslant 0, \tag{1.1b}$$

$$u \cdot n = 0 \qquad x \in \partial\Omega,\ t \geqslant 0, \tag{1.1c}$$

$$u \cdot n = (\dot{\mathbf{h}} + \boldsymbol{\omega} \wedge (x - \mathbf{h})) \cdot n \qquad x \in \partial S(t),\ t \geqslant 0, \tag{1.1d}$$

$$m_s \ddot{\mathbf{h}} = \int_{\partial S(t)} pn \,\mathrm{d}x, \qquad t \geqslant 0, \tag{1.1e}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(J\boldsymbol{\omega}) = \int_{\partial S(t)} p(x - \mathbf{h}) \wedge n \,\mathrm{d}x, \qquad t \geqslant 0, \tag{1.1f}$$

$$\dot{\mathbf{R}}(t) = A(\boldsymbol{\omega}(t))\mathbf{R}(t), \qquad t \geqslant 0, \tag{1.1g}$$

$$u(0, x) = u_0(x) \qquad x \in F(0), \tag{1.1h}$$

$$\mathbf{h}(0) = h_0,\ \dot{\mathbf{h}}(0) = k_0,\ \mathbf{R}(0) = Id_{\mathcal{M}_3},\ \boldsymbol{\omega}(0) = \omega_0, \tag{1.1i}$$

where the unknowns are u (the Eulerian velocity field of the fluid), p (the pressure of the fluid), $\mathbf{h}$ (the trajectory of the mass center of the rigid body), $\mathbf{R}$ (the time variation of the orthogonal matrix giving the orientation of the solid) and $\boldsymbol{\omega}$ (the time variation of the angular velocity of the rigid body). The density of the fluid ρ_F is supposed to be a constant. The fluid occupies, at $t = 0$, the domain $F_0 = \Omega \setminus S_0$. The domain $S(t)$ is defined by

$$S(t) = \{\mathbf{h}(t) + \mathbf{R}(t)(y - h_0) \ \mid\ y \in S_0,\ t \geqslant 0\}.$$

The domain occupied by the fluid at instant t is $F(t) = \Omega \setminus S(t)$. The skew-symmetric matrix $A(\omega)$ is given by

$$A(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \qquad (\omega \in \mathbb{R}^3). \tag{1.2}$$

The notation m_s stands for the mass of the solid and $J(t)$ designs its inertia matrix defined by

$$J_{i,j}(t) = \rho_s \int_{S(t)} [(x - \mathbf{h}(t)) \wedge e_i] \cdot [(x - \mathbf{h}(t)) \wedge e_j] \,\mathrm{d}x \qquad (i, j \in \{1, 2, 3\}), \tag{1.3}$$

where the constant ρ_s stands for the density of the solid and $(e_k)_{k=1,2,3}$ is the canonical basis in $\mathbb{R}^3$. It is easy to check that $J(t) = \mathbf{R}(t)J_0\mathbf{R}^*(t)$ for every $t \geqslant 0$, where J_0 is the matrix defined by

$$(J_0)_{i,j} = \rho_S \int_{S_0} [(y - h_0) \wedge e_i] \cdot [(y - h_0) \wedge e_j] \, \mathrm{d}y, \tag{1.4}$$

for every $i, j \in \{1, 2, 3\}$. Notice that the matrix J_0 does not depend on the position of the solid and that the last formula easily implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}(J\boldsymbol{\omega}) = \boldsymbol{\omega} \wedge (J\boldsymbol{\omega}) + J\dot{\boldsymbol{\omega}}. \tag{1.5}$$

Moreover, we have denoted by $\partial S(t)$ the boundary of the rigid body at instant t and by $n(t, x)$ the unit normal to $\partial S(t)$ at the point x directed to the interior of the rigid body.

Throughout this paper we assume that the considered boundaries are smooth in the sense that there exist the functions $\delta_0, \delta_1 \in C^\infty(\mathbb{R}^3, \mathbb{R})$ such that

$$\partial\Omega = \left\{x \in \mathbb{R}^3 \,|\, \delta_0(x) = 0\right\}, \qquad \partial S(0) = \left\{x \in \mathbb{R}^3 \,|\, \delta_1(x) = 0\right\}, \tag{1.6}$$

$$n(x) = -\nabla\delta_0(x), \qquad x \in \partial\Omega, \qquad n(x) = -\nabla\delta_1(x), \qquad x \in \partial S(0). \tag{1.7}$$

An important role in this work will be played by the set $P(\Omega, S_0)$, defined as follows:

Definition 1.1. *The set of all admissible solid configurations from the solid position S_0, denoted $P(\Omega, S_0)$, is the set of all pairs $\begin{pmatrix} h_1 \\ R_1 \end{pmatrix} \in \mathbb{R}^3 \times SO_3(\mathbb{R})$ such that there exist functions*

$$\mathbf{h} \in C([0, 1]; \mathbb{R}^3), \quad \mathbf{R} \in C([0, 1]; SO_3(\mathbb{R})),$$

with

$$\mathbf{h}(0) = h_0, \quad \mathbf{h}(1) = h_1, \quad \mathbf{R}(0) = Id_3, \quad \mathbf{R}(1) = R_1,$$
$$\mathbf{h}(t) + \mathbf{R}(t)(y - h_0) \in \Omega \qquad (t \in [0, 1], \; y \in S_0).$$

Remark 1.2. *For each $t \geqslant 0$ the position of the solid and the domain filled by the fluid are completely described by the pair $(\mathbf{h}(t), \mathbf{R}(t))^* \in P(\Omega, S_0)$. Therefore, the evolution of the domains $F(t)$ and $S(t)$ is totally described by the function $\mathbf{q} \in \mathcal{C}^2([0, T], \mathcal{P}(\Omega, S_0))$ defined by $\mathbf{q}(t) = (\mathbf{h}(t), \mathbf{R}(t))^*$. Consequently, in the remaining part of this work, we use the notation $F_{\mathbf{q}(t)}$ and $S_{\mathbf{q}(t)}$ instead of $F(t)$ and $S(t)$. We also denote $q_0 = (h_0, Id_{\mathcal{M}_3})^* = \mathbf{q}(0)$. More generally, for every $q = (h, R)^* \in P(\Omega, S_0)$ we denote*

$$S_q = \{h + R(y - h_0) \;\;|\;\; y \in S_0\}, \qquad F_q = \Omega \setminus S_q. \tag{1.8}$$

In order to give a precise statement of our main result we first introduce some spaces. For an open set $\mathcal{O} \subset \mathbb{R}^3$ we denote

$$N^m(\mathcal{O}) = \left\{ q \in H^m(\mathcal{O}) \;\mid\; \int_{\mathcal{O}} q(x)\mathrm{d}x = 0 \right\}. \tag{1.9}$$

We next defines some spaces of functions defined on time variable domains. Let $\mathbf{q} \in C^2([0,\infty), \mathcal{P}(\Omega, S_0))$ and let $\Psi : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ be such that for every $t \in [0,T]$, the map $x \mapsto \Psi(t,x)$ is a C^∞ diffeomorphism from F_0 to $F_{\mathbf{q}(t)}$.

Let $v(t,\cdot)$, $t \geq 0$ be a family of functions with $v(t,\cdot) : F_{\mathbf{q}(t)} \to \mathbb{R}^3$ for every $t \geqslant 0$. Denote $v_\Psi(t,y) = v(t,\Psi(t,y))$, for all $t \geqslant 0$ and for all $y \in F_0$. With the above notation we introduce the following function spaces:

$$\begin{aligned} C^k([0,T], H^m(F_{\mathbf{q}})) &= \{v \mid v_\Psi \in C^k([0,T], H^m(F_0))\}, \\ C^k([0,T], N^m(F_{\mathbf{q}})) &= \{v \mid v_\Psi \in C^k([0,T], N^m(F_0)\}, \end{aligned}$$

where $k \in \{0,1\}$, $m \geqslant 0$ is an integer and H^m are the usual Sobolev spaces. It is not difficult to check that the above definitions are independent of the choice of the diffeomorphism Ψ.

We can now state the main result in this paper.

Theorem 1.3. *Let $m \geqslant 3$ be an integer. Let $S_0 \subset \Omega$ be as in the notational preamble of this work and let $h_0 = \frac{1}{\mathrm{vol}\,(S_0)} \int_{S_0} x\,\mathrm{d}x$. Let $k_0 \in \mathbb{R}^3$, $\omega_0 \in \mathbb{R}^3$ and $u_0 \in H^m(F_0, \mathbb{R}^3)$ satisfy:*

$$\begin{aligned} \mathrm{div}\, u_0(x) &= 0 && x \in F_0, \\ u_0(x) \cdot n(x) &= (k_0 + \omega_0 \wedge (x - h_0)) \cdot n(x) && x \in \partial S_0, \\ u_0(x) \cdot n(x) &= 0 && x \in \partial\Omega. \end{aligned}$$

Then there exists $T_0 > 0$ such that (1.1) admits a unique solution $(\mathbf{q}, u, p)$ with

$$\mathbf{q} \in C^2\big([0,T_0), P(\Omega, S_0)\big), \tag{1.10}$$
$$u \in C\big([0,T_0), H^m(F_{\mathbf{q}})\big) \cap C^1\big([0,T_0), H^{m-1}(F_{\mathbf{q}})\big), \tag{1.11}$$
$$p \in C\big([0,T_0), N^{m+1}(F_{\mathbf{q}})\big). \tag{1.12}$$

2 Idea of the proof of Theorem 1.3

As already mentioned, the basic idea of the proof, borrowed from Bourguignon and Brezis [1], consists in reducing (1.1) to an ODE on an infinite dimensional manifold. In this section we briefly describe this reduction process and we give the main steps of the proof of Theorem 1.3. Let $\mathbf{q} = \begin{pmatrix} \mathbf{h} \\ \mathbf{R} \end{pmatrix}$ and u be functions satisfying (1.10) and (1.11) for some $T_0 > 0$, with $\mathrm{div}\, u = 0$. We introduce the flow $\boldsymbol{\eta}$ associated to u, which is defined as the solution of

$$\frac{\partial \boldsymbol{\eta}}{\partial t}(t,y) = u(t, \boldsymbol{\eta}(t,y)), \quad \boldsymbol{\eta}(0,y) = y \text{ for all } y \in F_0. \tag{2.1}$$

By the Cauchy-Lipschitz Theorem $\boldsymbol{\eta}(t,\cdot)$ is a diffeomorphism from F_0 onto $F_{\mathbf{q}(t)}$. Moreover, since $\operatorname{div} u = 0$, by Liouville's Theorem (see, for instance, Hartman [4, p.96]), we have

$$\det[D_y \boldsymbol{\eta}(t,y)] = 1, \qquad (t \in [0,T_0), \quad y \in F_0).$$

Moreover, we set

$$\frac{\partial \boldsymbol{\eta}}{\partial t}(t,y) = \boldsymbol{\xi}(t,y) \qquad (t \geqslant 0, y \in F_0). \tag{2.2}$$

Notice that u can be expressed in terms of $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ by

$$u(t,x) = \boldsymbol{\xi}(t, \boldsymbol{\eta}^{-1}(t,x)) \qquad (t \in [0,T_0), \quad y \in F_0). \tag{2.3}$$

In order to express (1.1) as a first-order ordinary differential equation we note that from the formula

$$\frac{\partial \boldsymbol{\xi}}{\partial t}(t,y) = \frac{\partial u}{\partial t}(t, \boldsymbol{\eta}(t,y)) + (u \cdot \nabla) u(t, \boldsymbol{\eta}(t,y)),$$

it follows that u satisfies (1.1a) iff

$$\frac{\partial \boldsymbol{\xi}}{\partial t}(t,y) = -\nabla p(t, \boldsymbol{\eta}(t,y)), \qquad (y \in F_0,\ t \in [0,T_0)). \tag{2.4}$$

Consider the function $\mathbf{k} \in C^1([0,T_0), \mathbb{R}^3)$ defined by

$$\dot{\mathbf{h}}(t) = \mathbf{k}(t) \qquad (t \in [0,T_0)). \tag{2.5}$$

Define $\boldsymbol{\omega} \in C^1([0,T_0), \mathbb{R}^3)$ by

$$\dot{\mathbf{R}}(t) = A(\boldsymbol{\omega}(t)) \mathbf{R}(t) \qquad (t \in [0,T_0)). \tag{2.6}$$

As it will be shown in Sections 3 and 4, by solving appropriate Neuman problems, the pressure p can be expressed, for each $t \in [0,T_0)$ as a function of $\mathbf{z} = (\boldsymbol{\eta}, \mathbf{q}, \boldsymbol{\xi}, \mathbf{k}, \boldsymbol{\omega})^*$, so that, using (2.4), (2.5) and (2.6), the system (1.1) can be written in the equivalent form

$$\dot{\mathbf{z}}(t) = \mathcal{L}(\mathbf{z}(t)), \qquad \mathbf{z}(0) = z_0,$$

where $\mathcal{L} : F^m \to E^m$, with

$$E^m = H^m(F_0, \mathbb{R}^3) \times \mathbb{R}^3 \times \mathcal{M}_3(\mathbb{R}) \times H^m(F_0, \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3, \tag{2.7}$$

and F^m is a closed subset of E^m.

The norm of $z \in E^m$ is defined by

$$\|z\|^2_{E^m} = \|\eta\|^2_{H^m(F_0,\mathbb{R}^3)} + \|h\|^2 + \|R\|^2 + \|\xi\|^2_{H^m(F_0,\mathbb{R}^3)} + \|k\|^2 + \|\omega\|^2,$$

where $\|\cdot\|$ stand for the Euclidean norm on $\mathbb{R}^n$. Endowed with this norm E^m is a Hilbert space.

For $(\mathbf{q}, u, p)$ satisfying (1.10)-(1.12) we define

$$\mathbf{z}(t) = \begin{pmatrix} \boldsymbol{\eta}(t,\cdot) \\ \mathbf{q}(t) \\ \boldsymbol{\xi}(t,\cdot) \\ \mathbf{k}(t) \\ \boldsymbol{\omega}(t) \end{pmatrix}, \tag{2.8}$$

where $\boldsymbol{\xi}(t,\cdot)$, $\boldsymbol{\eta}(t,\cdot)$, $\mathbf{k}(t)$ and $\boldsymbol{\omega}(t)$ are defined by (2.1), (2.2), (2.5) and (2.6), respectively.

To define F^m we introduce, for every $q \in P(\Omega, S_0)$, the sets

$$\begin{aligned}\mathrm{Diff}^m(F_0, F_q) = \big\{ \eta : \overline{F_0} \to \overline{F_q} \,\big|\, &\eta \text{ invertible}, \eta \in H^m(F_0, \mathbb{R}^3), \\ &\eta^{-1} \in H^m(F_q, \mathbb{R}^3) \text{ and } \det[D_y(\eta(y))] = 1 \big\}, \end{aligned} \tag{2.9}$$

$$\Sigma^m(\Omega, S_0) = \left\{ \sigma = \begin{pmatrix} \eta \\ q \end{pmatrix} \,\middle|\, q \in P(\Omega, S_0) \text{ and } \eta \in \mathrm{Diff}^m(F_0, F_q) \right\}, \tag{2.10}$$

where $P(\Omega, S_0)$ has been defined in Definition 1.1 and F_q is given in (1.8). The set $\Sigma^m(\Omega, S_0)$, simply denoted by Σ^m in the sequel, is formed by the admissible positions of the system. The set of admissible velocities from a position $\sigma = \begin{pmatrix} q \\ \eta \end{pmatrix}$ describes the tangent space to Σ^m at the point σ, which is given by

$$\begin{aligned}\mathrm{T}_\sigma \Sigma^m = \big\{ (\xi, k, \omega)^* \in H^m(F_0, \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3 \,\big|\, u = \xi \circ (\eta^{-1}) \in H^m(F_q, \mathbb{R}^3), \\ \operatorname{div} u = 0 \text{ in } F_q,\ u \cdot n = 0 \text{ on } \partial\Omega,\ u \cdot n = [k + \omega \wedge (x - h)] \cdot n \text{ on } \partial S_q \big\}. \end{aligned} \tag{2.11}$$

The subset F^m of E^m is defined by

$$F^m = \{ z \in E^m \quad | \quad \sigma = (\eta, q)^* \in \Sigma^m,\ (\xi, k, \omega)^* \in T_\sigma \Sigma^m \}. \tag{2.12}$$

It is not difficult to check that F^m is a locally closed subset of E^m, in the sense that for every $z \in F^m$ there exists a closed ball B of F^m centered at z such that $F^m \cap B$ is a closed subset of E^m. Moreover, as it will be shown in Section 6, Σ^m is an infinite-dimensional manifold and F^m is its tangent bundle.

The precise definition of $\mathcal{L}$ requires some preparation, so it is postponed to Sections 3 and 4. In order to prove the main result we show in Section 5 that $\mathcal{L}$ satisfies the assumptions of the following version of the Cauchy-Lipschitz theorem, which is a particular case of Theorem 2 from Martin [11].

Proposition 2.1. *Let F be a locally closed subset of a Hilbert space E and let $\mathcal{L} : [0, T) \times F \to E$ be such that* *a)* $\mathcal{L}$ *is a locally Lipschitz in* z *and continuous in* t*;*

b) $$\lim_{s\to 0^+}\frac{1}{s}\,\mathrm{dist}\left(z+s\mathcal{L}\begin{pmatrix} t\\ z\end{pmatrix};F\right)=0 \qquad \left(\begin{pmatrix} t\\ z\end{pmatrix}\in[0,T)\times F\right).$$

Then for every $z_0\in F$ *there exists* $T_0>0$ *such that the equation*

$$\dot{\mathbf{z}}(t)=\mathcal{L}(t,\mathbf{z}(t)),\qquad \mathbf{z}(0)=z_0$$

admits a unique solution $\mathbf{z}\in C^1([0,T_0),F)$*.*

3 Study of the pressure

The study of the pressure p is the key point in order to reduce (1.1) to a system of ordinary differential equations. In this section we write the pressure as the sum of two terms, each of them satisfying a Neumann problem for the Laplacian. We first introduce some function spaces and we recall classical results related to Neumann problems. Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\mathcal{O}$, $m\in\mathbb{N}$ and $n\in\mathbb{N}$. Recall the definition of $N^m(\mathcal{O})$ from (1.9) and let $V^m(\mathcal{O})$ the space defined by

$$V^m(\mathcal{O})=\big\{(f,g)^*\in H^m(\mathcal{O})\times H^{m+1/2}(\partial\mathcal{O})\big| \int_{\mathcal{O}} f(x)\mathrm{d}x+\int_{\partial\mathcal{O}} g(x)\mathrm{d}\sigma_x=0\Big\}. \quad (3.1)$$

The following classical result on the wellposedness of the Neumann problem for the Laplace operator can be found in the book of Lions and Magenes [10, Chapter 5].

Theorem 3.1. *Let* $m\in\mathbb{N}$*. Then, for every* $\begin{pmatrix} f\\ g\end{pmatrix}\in V^m(\mathcal{O})$*, the boundary value problem*

$$\begin{aligned} -\Delta\varphi(x) &= f(x) \qquad x\in\mathcal{O},\\ \frac{\partial\varphi}{\partial n}(x) &= g(x) \qquad x\in\partial\mathcal{O}, \end{aligned}$$

admits a unique solution $\varphi\in N^{m+2}(\mathcal{O})$*. Moreover,* φ *satisfies*

$$\int_{\mathcal{O}}\nabla\varphi\cdot\nabla\psi\,\mathrm{d}x=\int_{\mathcal{O}} f\psi\,\mathrm{d}x+\int_{\partial\mathcal{O}} g\psi\,d\sigma_x \qquad (\psi\in H^{m+2}(\mathcal{O})), \qquad (3.2)$$

and there exists a constant C *(depending only on* $\mathcal{O}$ *and* m*) such that*

$$||\nabla\varphi||_{H^{m+1}(\mathcal{O})}\le C(||f||_{H^m(\mathcal{O}}+||g||_{H^{m+1/2}(\partial\mathcal{O})}) \qquad \left(\begin{pmatrix} f\\ g\end{pmatrix}\in V^m(\mathcal{O})\right).$$

In order to prove that the boundary value problem for the pressure is well posed, we need several technical results. Let $q \in P(\Omega, S_0)$. We first note that, thanks to the smoothness of ∂F_q, the map $x \mapsto n(x)$, defined on ∂F_q, can be extended to $\overline{F_q}$ by a function in $H^m(F_q)$. This extension is not unique so that the partial derivatives of n on ∂F_q are not uniquely determined. However, it can be easily checked that for every vector field τ which is tangent to ∂F_q, the quantity $\sum_{j=1}^3 \tau_j \frac{\partial n_i}{\partial x_j}$, with $i \in \{1,2,3\}$ does not depend on the choice of the extension.

Proposition 3.2. *Let $m \geq 3$ be an integer, let $q \in P(\Omega, S_0)$ and assume that $w \in H^m(F_q, \mathbb{R}^3)$, $w \cdot n = 0$ on ∂F_q. Then the function $x \mapsto \sum_{i,j} \frac{\partial w_j}{\partial x_i} \frac{\partial w_i}{\partial x_j}$, is in $H^{m-1}(F_q)$ whereas the function $x \mapsto \sum_{i,j} w_i w_j \frac{\partial n_i}{\partial x_j}$, is in $H^{m-1/2}(\Gamma)$, where Γ is either $\partial\Omega$ or ∂S_q.*

Proof. The first property follows from the fact that, under our assumptions, $H^{m-1}(F_q)$ is an algebra.

To prove the second property we notice that it suffices to use the fact that $H^m(F_q)$ is an algebra, the smoothness of the map $x \mapsto \frac{\partial n_i}{\partial x_j}(x)$ defined on F_q and the trace theorem. □

The above lemma allows us to introduce, for every $q \in P(\Omega, S_0)$, the operators:

$$\mathcal{F}_q : H^m(F_q, \mathbb{R}^3) \to H^{m-1}(F_q), \qquad \mathcal{F}_q(u) = \sum_{i,j} \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j}, \tag{3.3}$$

$$\mathcal{G}_q : H^m(F_q, \mathbb{R}^3) \to H^{m-1/2}(\Gamma), \qquad \mathcal{G}_q(u) = \sum_{i,j} u_i u_j \frac{\partial n_i}{\partial x_j}|_\Gamma, \tag{3.4}$$

where Γ is either $\partial\Omega$ or $\partial S_{\mathbf{q}}$.

An important ingredient allowing to write (1.1) as an ordinary differential equation is the following result:

Proposition 3.3. *Let $T_0 > 0$, let $m \geq 3$ be an integer, let*

$$\begin{aligned}
\begin{pmatrix} \mathbf{h} \\ \mathbf{R} \end{pmatrix} = \mathbf{q} \ &\in \ C^2\big([0,T_0], \mathcal{P}(\Omega, S_0)\big), \\
u \ &\in \ C\big([0,T_0), H^m(F_{\mathbf{q}})\big) \cap C^1\big([0,T_0), H^{m-1}(F_{\mathbf{q}})\big), \\
p \ &\in \ C\big([0,T_0), N^{m+1}(F_{\mathbf{q}})\big).
\end{aligned}$$

Assume that u satisfies

$$\begin{aligned}
(\operatorname{div} u)(t,x) &= 0 && x \in F_{\mathbf{q}(t)},\ t \in [0,T_0), \\
(u \cdot n)(t,x) &= 0 && x \in \partial\Omega,\ t \in [0,T_0), \\
(u \cdot n)(t,x) &= v(t,x) \cdot n(t,x) && x \in \partial S_{\mathbf{q}(t)},\ t \in [0,T_0),
\end{aligned}$$

where

$$v(t,x) = \dot{\mathbf{h}}(t) + \boldsymbol{\omega}(t) \wedge (x - \mathbf{h}(t)), \qquad \textit{for all } x \in F_{\mathbf{q}(t)},\ t \in [0,+\infty). \tag{3.5}$$

Moreover, assume that u*,* p *and* $\mathbf{q}$ *satisfy* (1.1a)*. Then, for very* $t \in [0, T_0)$*, we have*

$$-\Delta p(t,x) = \rho_F \mathcal{F}_{\mathbf{q}(t)}(u)(t,x) \qquad (x \in F_{\mathbf{q}(t)}), \tag{3.6}$$

$$\frac{\partial p}{\partial n}(t,x) = \rho_F \mathcal{G}_{\mathbf{q}(t)}(u)(t,x) \qquad (x \in \partial\Omega), \tag{3.7}$$

$$\frac{\partial p}{\partial n}(t,x) = \rho_F \mathcal{G}_{\mathbf{q}(t)}(u-v)(t,x) + 2\rho_F (u-v)\cdot(\boldsymbol{\omega}(t)\wedge n(t,x))$$
$$-\rho_F \left[\ddot{\mathbf{h}}(t) + \dot{\boldsymbol{\omega}}(t)\wedge(x-\mathbf{h}(t)) + \boldsymbol{\omega}(t)\wedge(\boldsymbol{\omega}(t)\wedge(x-\mathbf{h}(t)))\right]\cdot n(t,x) \quad (x \in \partial S_{\mathbf{q}(t)}), \tag{3.8}$$

where $\mathcal{F}_q$ *and* $\mathcal{G}_q$ *are defined by* (3.3) *and* (3.4) *and* v *stands for the velocity of the solid defined by* (3.5)*.*

Proof. Assume that u, p, $\mathbf{q}$ satisfy (1.1a). By applying the div operator to (1.1a) we get that p satisfies, for every $t \in [0, T_0)$,

$$-\Delta p(t,x) = \rho_F \mathrm{div}\left[\frac{\partial u}{\partial t}(t,x) + (u(t,x)\cdot\nabla)u(t,x)\right] \qquad (x \in F_{\mathbf{q}(t)}). \tag{3.9}$$

By using the fact that $\mathrm{div}\, u \equiv 0$, the right-hand side of the above relation can be expressed as

$$\begin{aligned}\mathrm{div}\left[\frac{\partial u}{\partial t}(t,x) + (u\cdot\nabla)u(t,x)\right] &= \sum_{i,j}\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}(t,x) + u.\nabla \mathrm{div}\,(u)(t,x)\\ &= \sum_{i,j}\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}(t,x).\end{aligned}$$

The above formula and (3.9) imply (3.6).

On the other hand, by taking normal traces of all the terms in (1.1a) we obtain

$$\frac{\partial p}{\partial n}(t,x) = \rho_F\left[-\frac{\partial u}{\partial t}(t,x) - (u\cdot\nabla)u(t,x)\right]\cdot n(t,x) \qquad (x \in \Gamma), \tag{3.10}$$

where $\Gamma = \partial\Omega$ or $\Gamma = S_{\mathbf{q}(t)}$. The above boundary conditions can be expressed in terms of the velocity and of the position of the solid. First note that

$$n(t,x) = n(0,x) \qquad (x \in \partial\Omega,\ t \in [0,T_0)).$$

Additionally, note that, for every $t \in [0, T_0)$ and $y \in \partial S_0$, we have

$$n(t, \Psi(t, y)) = \mathbf{R}(t) n(0, y),$$

and

$$u(t, \Psi(t, y)) \cdot (\mathbf{R}(t) n(0, y)) = v(t, \Psi(t, y)) \cdot (\mathbf{R}(t) n(0, y)), \tag{3.11}$$

where

$$x = \Psi(t, y) = \mathbf{h}(t) + \mathbf{R}(t)(y - h_0) \qquad (y \in \partial S_0),$$

and v is the solid velocity given in (3.5). By taking the derivative with respect to t of the two sides of (3.11), we obtain that for every $t \in [0, T_0)$ and $x \in \partial S_{\mathbf{q}(t)}$ we have:

$$\begin{aligned}\left\{\frac{\partial u}{\partial t}(t, x) + [(v \cdot \nabla) u(t, x)]\right\} \cdot n(t, x) + u(t, x) \cdot (\boldsymbol{\omega}(t) \wedge n(t, x)) \\ = \left\{\frac{\partial v}{\partial t}(t, x) + [(v \cdot \nabla) v(t, x)]\right\} \cdot n(t, x) + v(t, x) \cdot (\boldsymbol{\omega}(t) \wedge n(t, x)).\end{aligned} \tag{3.12}$$

Using in the above formula the fact (easy to check) that

$$[(v(t, x) \cdot \nabla) v(t, x)] \cdot n(t, x) = -v(t, x) \cdot (\boldsymbol{\omega}(t) \wedge n(t, x)),$$

we obtain that for every $t \in [0, T_0)$ and $x \in \partial S_{\mathbf{q}(t)}$ we have:

$$\begin{aligned}\left\{\left[\frac{\partial u}{\partial t} + (u \cdot \nabla) u\right] \cdot n\right\}(t, x) = [(u - v) \cdot \nabla](u - v) \cdot n(t, x) \\ - 2(u - v) \cdot (\boldsymbol{\omega} \wedge n(t, x)) + \left[(v \cdot \nabla) v + \frac{\partial v}{\partial t}\right] \cdot n(t, x)\end{aligned} \tag{3.13}$$

Using again the relation $(u - v) \cdot n = 0$ on $S_{\mathbf{q}(t)}$, we have

$$[(u - v) \cdot \nabla](u - v) \cdot n(t, x) = -\mathcal{G}_{\mathbf{q}(t)}(u - v)(t, x). \tag{3.14}$$

By combining (3.13) and (3.14) and (3.10) we obtain (3.8).

To obtain (3.7) it suffices to apply (3.10) with $v = 0$ (so that $\boldsymbol{\omega} = 0$). $\square$

From Proposition 3.3 (more precisely from (3.8)) we note that the pressure depends on $\ddot{\mathbf{h}}$ and on $\dot{\boldsymbol{\omega}}$. In order to make this dependence more precise we introduce, for every $q \in P(\Omega, S_0)$, the potential functions Φ_i for $i = 1, \ldots, 6$ which are solutions of the Neumann problems:

$$-\Delta \Phi_i(q; x) = 0 \qquad x \in F_q, \tag{3.15a}$$

$$\frac{\partial \Phi_i}{\partial n}(q; x) = 0 \qquad x \in \partial\Omega, \tag{3.15b}$$

$$\frac{\partial \Phi_i}{\partial n}(q; x) = K_i(q; x) \qquad x \in \partial S_q, \tag{3.15c}$$

where

$$\begin{aligned} K_i(q;x) &= n_i(x) && \text{for } i = 1,2,3, \\ K_i(q;x) &= [(x-h)\wedge n(x)]_{i-3} && \text{for } i = 4,5,6. \end{aligned} \tag{3.16}$$

Denote $\Phi = (\Phi_1, \cdots, \Phi_6)^*$.

These functions have been introduced in the book of Lamb [9] and they were used, in particular, in the work of Houot and Munnier [7] to describe the motion of rigid bodies in a perfect fluid undergoing a potential flow. From Theorem 3.1 on the Neumann problem, it is easy to check that $\Phi \in C^\infty(F_q;\mathbb{R}^6)$. Moreover, the following properties are proved in [7].

Proposition 3.4. *For every $q_0 \in P(\Omega, S_0)$, there exists a neighborhood $\mathcal{O}$ of q_0 of such that*

- *the mapping $q \mapsto \Phi(q;\cdot)$ from $\mathcal{O}$ to $C^\infty(F_q;\mathbb{R}^6)$ is of class C^2;*
- *for all $i, j \in \{1, \ldots, 6\}$ the mappings*

$$q \mapsto I_{i,j}(q) = \int_{F_q} \nabla\Phi_i(q;x)\cdot\nabla\Phi_j(q;x)\mathrm{d}x,$$

are of class C^2 on $\mathcal{O}$.

We also need a potential μ, defined as follows. For $z = (\eta, q, \xi, k, \omega)^* \in F^m$, where F^m has been defined in (2.12), we set $u(z;x) = \xi(\eta^{-1}(x))$, with $x \in F_q$. The potential μ is defined as the solution of the boundary value problem

$$-\Delta\mu(z;x) = \mathcal{F}_q(u(z;x)) \qquad (x \in F_q), \tag{3.17a}$$

$$\frac{\partial\mu}{\partial n}(z;x) = \mathcal{G}_q(u(z;x)) \qquad (x \in \partial\Omega), \tag{3.17b}$$

$$\begin{aligned} \frac{\partial\mu}{\partial n}(z;x) &= \mathcal{G}_q(u-v)(x) + 2(u(z;x)-v(z;x))\cdot(\omega\wedge n(x)) \\ &\quad - [\omega\wedge(\omega\wedge(x-h))]\cdot n(x) \qquad (x \in \partial S_q), \end{aligned} \tag{3.17c}$$

where

$$v(z;x) = k + \omega\wedge(x-h),$$

and $\mathcal{F}_q$, $\mathcal{G}_q$ are defined in (3.3), (3.4).

Remark 3.5. *With the above notation for Φ and μ, if $(u, p, \mathbf{q})$ satisfy (3.6)–(3.8) and $\mathbf{z}(t)$ is defined by (2.8), then the pressure can be written*

$$p(\mathbf{z}(t);x) = \rho_F\mu(\mathbf{z}(t);x) - \rho_F\Phi(\mathbf{q}(t);x)\cdot(\ddot{\mathbf{h}}(t), \dot{\boldsymbol{\omega}}(t))^*, \tag{3.18}$$

where $\cdot$ stands for the inner product in $\mathbb{R}^6$.

4 An equivalent form of the governing equations

Throughout this section we assume that $m \geqslant 3$ and

$$\begin{aligned} \mathbf{q} &\in C^2([0,T),\mathcal{P}(\Omega,S_0)),\\ u &\in C([0,T),H^m(F_{\mathbf{q}(\cdot)})\cap C^1([0,T),H^{m-1}(F_{\mathbf{q}(\cdot)}),\\ p &\in C([0,T),N^{m+1}(F_{\mathbf{q}(t)})). \end{aligned}$$

At this point we need the virtual mass of the solid (see, for instance, [7]) which is the six by six matrix $\mathcal{K}(q)$ defined, for every for every $q \in P(\Omega, S_0)$, by

$$\mathcal{K}(q) = \mathcal{K}_S(q) + \mathcal{K}_F(q), \qquad \mathcal{K}_S(q) = \begin{pmatrix} m_s Id_3 & 0 \\ 0 & J \end{pmatrix},$$

$$\mathcal{K}_F(q) = \left(\rho_F \int_{F_q} \nabla\Phi_i(q;x)\cdot\nabla\Phi_j(q;x)\mathrm{d}x\right)_{1\leq i,j\leq 6}, \quad (4.1)$$

where $J = J(q)$ is the inertia matrix of the solid (1.3). It is easy to check that $\mathcal{K}_S(q)$ is strictly positive and $\mathcal{K}_F(q)$ is positive so that $\mathcal{K}(q)$ is invertible.

The result below shows that equations (1.1e) and (1.1f) can be rewritten as equations giving $\ddot{\mathbf{h}}(t)$ and $\dot{\boldsymbol{\omega}}(t)$, in terms of $\mathbf{z}(t)$ defined in (2.8).

Proposition 4.1. *Assume that $u, p, \mathbf{q}$ satisfy* (1.1). *Then*

$$\begin{pmatrix}\ddot{\mathbf{h}}(t)\\ \dot{\boldsymbol{\omega}}(t)\end{pmatrix} = [\mathcal{K}(\mathbf{q}(t))]^{-1}\left[\begin{pmatrix}0\\ (J(t)\boldsymbol{\omega}(t))\wedge\boldsymbol{\omega}(t)\end{pmatrix} + \rho_F\int_{F_{\mathbf{q}(t)}}\nabla\mu(\mathbf{z}(t);x)\cdot\nabla\Phi(\mathbf{q}(t);x)\mathrm{d}x\right]. \quad (4.2)$$

In the above formula, the notation $\nabla\mu\cdot\nabla\Phi$ stands for the six dimensional vector of components $(\nabla\mu\cdot\nabla\Phi_i)_{1\leqslant i\leqslant 6}$, where μ is the solution of (3.17) *and $(\Phi_i)_{1\leqslant i\leqslant 6}$ are defined by* (3.15).

Proof. The decomposition of the pressure (3.18), the formulae (1.1e) and (1.1f) imply that, for every $t \in [0, T_0)$ we have

$$\begin{aligned} m_s\ddot{\mathbf{h}}_j(t) = \rho_F &\int_{\partial S_{\mathbf{q}(t)}} \mu(\mathbf{z}(t);x)K_j(\mathbf{q}(t);x)\mathrm{d}\sigma_x \\ &- \rho_F\sum_{i=1}^{3}\ddot{\mathbf{h}}_i(t)\int_{\partial S_{\mathbf{q}(t)}}\Phi_i(\mathbf{q}(t);x)K_j(\mathbf{q}(t);x)\mathrm{d}\sigma_x \\ &- \rho_F\sum_{i=1}^{3}\dot{\boldsymbol{\omega}}_i(t)\int_{\partial S_{\mathbf{q}(t)}}\Phi_{i+3}(\mathbf{q}(t);x)K_j(\mathbf{q}(t);x)\mathrm{d}\sigma_x, \end{aligned} \quad (4.3)$$

$$\sum_{i=1}^{3} J_{i,j}(t)\dot{\boldsymbol{\omega}}_i(t) = (J(t)\boldsymbol{\omega}(t) \wedge \boldsymbol{\omega}(t))_j + \rho_F \int_{\partial S_{\mathbf{q}(t)}} \mu(\mathbf{z}(t); x) K_{j+3}(\mathbf{q}(t); x) \mathrm{d}\sigma_x$$
$$- \rho_F \sum_{i=1}^{3} \ddot{\mathbf{h}}_i(t) \int_{\partial S_{\mathbf{q}(t)}} \Phi_i(\mathbf{q}(t); x) K_{j+3}(\mathbf{q}(t); x) \mathrm{d}\sigma_x$$
$$- \rho_F \sum_{i=1}^{3} \ddot{\boldsymbol{\omega}}_i(t) \int_{\partial S_{\mathbf{q}(t)}} \Phi_{i+3}(\mathbf{q}(t); x) K_{j+3}(\mathbf{q}(t); x) \mathrm{d}\sigma_x, \quad (4.4)$$

where K_j have been defined in (3.16).

On the other hand, using (3.15), (3.17) and Green's formula we get

$$\int_{\partial S_{\mathbf{q}(t)}} \Phi_i(\mathbf{q}(t); x) K_j(\mathbf{q}(t); x) \mathrm{d}\sigma_x = \int_{\partial S_{\mathbf{q}(t)}} \Phi_i(\mathbf{q}(t); x) \frac{\partial \phi_j}{\partial n}(x) \mathrm{d}\sigma_x,$$
$$= \int_{F_{\mathbf{q}(t)}} \nabla \Phi_i(\mathbf{q}(t); x) \cdot \nabla \Phi_j(t, x) \mathrm{d}x,$$

$$\int_{\partial S_{\mathbf{q}(t)}} \mu(\mathbf{z}(t); x) K_j(\mathbf{q}(t); x) \mathrm{d}\sigma_x = \int_{\partial S_{\mathbf{q}(t)}} \mu(\mathbf{z}(t); x) \frac{\partial \phi_j}{\partial n}(x) \mathrm{d}\sigma_x,$$
$$= \int_{F_{\mathbf{q}(t)}} \nabla \mu(\mathbf{z}(t); x) \cdot \nabla \Phi_j(\mathbf{q}(t); x) \mathrm{d}x,$$

Using the last two formulas in (4.3) and (4.4) we obtain the conclusion (4.2). □

Recall the definition of E^m and F^m from (2.7) and (2.12), respectively, and let $\mathcal{L}_F : F^m \to H^m(\Omega, \mathbb{R}^3)$ be defined by

$$\mathcal{L}_F(z)(y) = \rho_F \nabla \Phi(q; \eta) \cdot \mathcal{L}_S(z)(y) - \rho_F \nabla \mu(z; \eta(y)) \qquad (y \in F_0), \qquad (4.5)$$

for every $z = (\eta, q, \xi, k, \omega)^* \in F^m$, where Φ is the solution of the Neumann problem (3.15), μ is solution of (3.17) and

$$\mathcal{L}_S(z) = [\mathcal{K}(q)]^{-1} \left[\begin{pmatrix} 0 \\ (J(\omega) \wedge \omega \end{pmatrix} + \rho_F \int_{F_q} \nabla \mu(z; x) \cdot \nabla \Phi(q; x) \mathrm{d}x \right]. \qquad (4.6)$$

Let $\mathcal{L} : F^m \to E^m$ be defined by

$$\mathcal{L}(z) = \begin{pmatrix} \xi \\ k \\ A(\omega) R \\ \mathcal{L}_F(z) \\ \mathcal{L}_S(z) \end{pmatrix}. \qquad (4.7)$$

In the last part of this section we show that the system (1.1) is equivalent to the ordinary differential equation

$$\frac{d\mathbf{z}}{dt}(t) = \mathcal{L}(\mathbf{z}(t)), \quad \mathbf{z}(0) = (Id_{F_{q_0}}, h_0, Id_3, u_0, k_0, \omega_0)^*. \tag{4.8}$$

In the following Proposition we prove that every solution of (1.1) generates a solution of (4.8).

Proposition 4.2. *Let $m \geqslant 3$ an integer, assume that $(h_0, Id_3)^* \in P(\Omega, S_0)$ and $(u_0, k_0, \omega_0)^* \in T_{\sigma_0}\Sigma$ where $\sigma_0 = (Id_{F_{q_0}}, h_0, Id_3)$. Moreover, assume that*

$$\begin{aligned} \mathbf{q} &\in C^2\big([0,T), P(\Omega, S_0)\big), \\ u &\in C\big([0,T), H^m(F_{\mathbf{q}})\big) \cap C^1\big([0,T), H^{m-1}(F_{\mathbf{q}})\big), \\ p &\in C\big([0,T), N^{m+1}(F_{\mathbf{q}})\big) \end{aligned}$$

satisfy the system (1.1)*. Then* $\mathbf{z}$ *defined by* (2.8) *satisfies* (4.8)*.*

Proof. The equations for $\boldsymbol{\eta}, \mathbf{h}, \mathbf{R}$ in (4.8) are nothing else but the definitions of $\boldsymbol{\xi}$, $\mathbf{k}$ and $\boldsymbol{\omega}$ from (2.2), (2.5) and (2.6), respectively. The fact that the equations for $\boldsymbol{\xi}$, $\mathbf{k}$ and $\boldsymbol{\omega}$ hold follows from (2.4), Proposition 4.1 and from (3.18). □

We still have to show that every solution of (4.8) generates a strong solution of (1.1).

Proposition 4.3. *Let $m \geqslant 3$ an integer, assume that $(h_0, Id_3)^* \in P(\Omega, S_0)$ and $(u_0, k_0, \omega_0)^* \in T_{\sigma_0}\Sigma$ where $\sigma_0 = (Id_{F_{q_0}}, h_0, Id_3)^*$. Moreover, assume that*

$$\mathbf{z} = (\boldsymbol{\eta}, \mathbf{h}, \mathbf{R}, \boldsymbol{\xi}, \mathbf{k}, \boldsymbol{\omega})^* \in C([0,T_0); F^m) \cap C^1([0,T_0); E^m),$$

is a solution of (4.8)*. Let $\mathbf{q}, u, p$ be defined by $\mathbf{q} = (\mathbf{h}, \mathbf{R})^*$,*

$$u(t,x) = \boldsymbol{\xi}(t, \boldsymbol{\eta}^{-1}(t,x)), \qquad t \in [0,T_0),\ x \in F_{\mathbf{q}(t)},$$

and let the pressure p be defined by (3.18)*. Then $\mathbf{q}, u, p$ satisfy the smoothness conditions* (1.10)-(1.12) *and the system* (1.1)*.*

Proof. First remark that, since $\mathbf{z} \in C([0,T); F^m) \cap C^1([0,T); E^m)$, we have

$$\begin{aligned} (\operatorname{div} u)(t,x) &= 0 \qquad (t \in [0,T_0),\ x \in F_{\mathbf{q}(t)}), \\ u(t,x) \cdot n(t,x) &= 0 \qquad (t \in [0,T_0),\ x \in \partial\Omega), \\ u(t,x) \cdot n(t,x) &= v(t,x) \cdot n(t,x) \qquad (t \in [0,T_0),\ x \in \partial S_{\mathbf{q}(t)}), \end{aligned}$$

so that equations (1.1b), (1.1c), (1.1d) are satisfied. From the definition (4.8) of $\mathcal{L}$ we obtain that $\dot{\mathbf{R}} = A(\boldsymbol{\omega})\mathbf{R}$ and

$$\begin{aligned} \begin{pmatrix} \ddot{\mathbf{h}}(t) \\ \dot{\boldsymbol{\omega}}(t) \end{pmatrix} &= [\mathcal{K}(\mathbf{q}(t)]^{-1} \left[\begin{pmatrix} 0 \\ (J(t)\boldsymbol{\omega}(t)) \wedge \boldsymbol{\omega}(t) \end{pmatrix} + \rho_F \int_{F_{\mathbf{q}}(t)} \nabla\mu(t,x) \cdot \nabla\Phi(t,x) \mathrm{d}x \right], \\ \dot{\xi}(t) &= \rho_F \nabla\Phi(t, \eta(t,y)) \cdot \mathcal{L}_S(t, \mathbf{z}(t)) - \rho_F \nabla\mu(t, \eta(t,y)), \end{aligned}$$

where $\mathcal{K}(\mathbf{q})$ is given by (4.1). The Newton's laws (1.1e) and (1.1f) come from the definition of the pressure (3.18) in the same way as in the proof of Proposition 4.1. Finally, using the relation $\boldsymbol{\xi} = u \circ \boldsymbol{\eta}$ and (3.18), we obtain that (1.1a) also holds. □

5 Locally Lipschitz property of $\mathcal{L}$

In this section we tackle a key point of our approach, which consists in proving that the map $\mathcal{L}$ is locally Lipschitz. We frequently use below results and methods from [1] combined with techniques specific to our problem, which require to compare functions defined on two different open sets.

Recall that the manifold F^m is defined by

$$F^m = \left\{ z = \begin{pmatrix} \sigma \\ \nu \end{pmatrix} \in E^m \;\middle|\; \sigma = \begin{pmatrix} \eta \\ q \end{pmatrix} \in \Sigma^m, \; \begin{pmatrix} \xi \\ k \\ \omega \end{pmatrix} \in T_\sigma \Sigma^m \right\},$$

where Σ^m is defined by (2.10) and $T_\sigma\Sigma^m$ by (2.11). For an element $z \in F^m$, the first three components $\sigma = (\eta, h, R)^* \in \Sigma^m$ define the "position" of the system whereas $\nu = (\xi, k, \omega)^* \in T_\sigma\Sigma^m$ defines the velocity. The key point in this section is the study of the application μ from (3.17). Recall the notation $q = \begin{pmatrix} h \\ R \end{pmatrix}$.

The main new issue we need to tackle is the study the dependence of the solution μ of (3.17) with respect to the geometric parameter q. The dependence of μ with respect to ξ and η is studied using the ideas in [1].

We first introduce several functions which are useful for the remaining part of this section. Let $\alpha, \beta_0,\ \beta$ and τ be the mappings on F^m defined by

$$\alpha(z; y) = \mathcal{F}_q(\xi \circ \eta^{-1})(\eta(y)) \qquad (y \in F_0), \tag{5.1}$$

$$\beta_0(z; y) = \mathcal{G}_q(\xi \circ \eta^{-1})(\eta(y)) \qquad (y \in \partial\Omega), \tag{5.2}$$

$$\beta(z; y) = \mathcal{G}_q(\xi \circ \eta^{-1} - v)(\eta(y)) \quad (y \in \partial S_0), \tag{5.3}$$

$$\begin{aligned} \tau(z; y) = {} & 2(\xi(y) - v(\eta(y))) \cdot \omega \wedge RN(\widetilde{y}) \\ & - [(v \cdot \nabla)v(\eta(y)) - \omega \wedge k] \cdot RN(\widetilde{y}) \quad (y \in \partial S_0), \end{aligned} \tag{5.4}$$

where $\widetilde{y} = h_0 + R^*(\eta(y) - h)$, $\mathcal{F}_q$ and $\mathcal{G}_q$ have been defined in (3.3) and (3.4), whereas N is a smooth extension of the unit normal vector of ∂F_0 to $\overline{F_0}$.

Remark 5.1. *According to a result from Takahashi [17] and Cumsille and Tucsnak [2], for every* $q = (h, R)^* \in P(\Omega, S_0)$ *and* $\varepsilon > 0$ *small enough there exists a* C^∞ *diffeomorphism* $\Psi_q : \overline{F_0} \to \overline{F_q}$ *such that* $\det[D\Psi_q(y)] = 1$ *for all* $y \in F_0$ *and*

$$\Psi_q(y) = y \quad \text{if } d(y, \partial\Omega) \leqslant \varepsilon, \qquad \Psi_q(y) = h + R(y - h_0) \quad \text{if } d(y, \partial S_q) \leqslant \varepsilon. \tag{5.5}$$

Using Ψ_q the unit normal vector field on ∂F_q can be extended to $\overline{F_q}$ such that

$$Dn(x) = DN(x) \qquad x \in \partial\Omega, \tag{5.6}$$
$$Dn(x) = RDN(h_0 + R^*(x-h))R^* \qquad x \in \partial S_q. \tag{5.7}$$

Moreover, the construction of Ψ in [17] shows that Ψ is C^∞ with respect to q.

Proposition 5.2. *Let $m \geqslant 3$ be an integer. Then the mappings α, β_0, β and τ are locally Lipschitz (with respect to z) from F^m to $H^{m-1}(F_0)$, $H^{m-1/2}(\partial\Omega)$, $H^{m-1/2}(\partial S_0)$ and $H^{m-1/2}(\partial S_0)$, respectively.*

Proof. Let $z_0 = (\sigma_0, \nu_0)^* \in F^m$ with

$$\sigma_0 = \begin{pmatrix} Id_{F_0} \\ h_0 \\ Id_3 \end{pmatrix} \in \Sigma^m, \qquad \nu_0 = \begin{pmatrix} u_0 \\ k_0 \\ \omega_0 \end{pmatrix} \in T_{\sigma_0}\Sigma^m.$$

For $r > 0$ we define $B^m(r) \subset F^m$ by

$$B^m(r) = \{z \in F^m \mid \|z - z_0\|_{E^m} \leqslant r\}.$$

We first note that, by the chain rule, we have

$$\alpha(z;y) = \operatorname{tr}\left\{([D\xi(y)][D\eta(y)]^{-1})^2\right\} \qquad (y \in F_0).$$

Since $H^{m-1}(F_0)$ is a Banach algebra, to show that α is Lipschitz on $B^m(r)$ it suffices to check that the maps

$$z \mapsto \mathrm{D}\xi, \qquad z \mapsto [\mathrm{D}\eta]^{-1},$$

are Lipschitz from $B^m(r)$ to $[H^{m-1}(F_0)]^9$. The first map above is obviously Lipschitz whereas for the second one it suffices to use the fact that for every 3×3 matrices A, B of determinant equal to 1 we have

$$A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1}, \quad A^{-1} = \mathrm{cof}(A)^t,$$

where $\mathrm{cof}(A)$ is the signed cofactors matrix of A.

For β_0 we remark that, using again the chain rule combined with (5.6), we have

$$\beta_0(z;y) = [DN(\eta(y)))]\xi(y) \cdot \xi(y) \qquad (y \in \partial\Omega).$$

By applying Lemma A.3 from [1] it follows that the mapping $z \mapsto DN \circ \eta$ is Lipschitz $B^m(r)$ to $[H^m(F_0)]^9$. Using again the fact that $H^m(F_0)$ is a Banach algebra it follows that β_0 is Lipschitz from $B^m(r)$ to $H^m(F_0)$. By the trace theorem it follows that β_0 is Lipschitz from $B^m(r)$ to $H^{m-1/2}(\partial\Omega)$.

For β we note that

$$\beta(z;y) = R[DN(\eta(y))]R^*(\xi(y) - v(\eta(y))) \cdot (\xi(y) - v(\eta(y))) \quad (y \in F_0).$$

The fact that β is Lipschitz from $B^m(r)$ to $H^{m-1/2}(\partial\Omega)$ can now be proved in the same way as for β_0.

Finally for τ we notice that for every $x \in F_q$ we have

$$(v \cdot \nabla)v(x) - \omega \wedge k = \omega \wedge [\omega \wedge (x - h)].$$

Inserting the above formula in (5.4) and applying again Lemma A.3 from [1], the claimed Lipschitz property of τ easily follows. □

We also need the following classical result (see, for instance, [1, Lemma 5]):

Proposition 5.3. *Let $m \geqslant 3$ be an integer and let Ω be bounded domain of $\mathbb{R}^3$ with smooth boundary. Then for every $u \in H^m(\Omega, \mathbb{R}^3)$ there exists a constant K, which depends on m and on Ω, such that*

$$\|u\|_{H^m(\Omega,\mathbb{R}^3)} \leq K \left[\|\operatorname{div} u\|_{H^{m-1}(\Omega,\mathbb{R}^3)} + \|\operatorname{curl}(u)\|_{H^{m-1}(\Omega,\mathbb{R}^3)} \right.$$
$$\left. + \|u \cdot n\|_{H^{m-1/2}(\partial\Omega,\mathbb{R}^3)} + \|u\|_{H^{m-1}(\Omega,\mathbb{R}^3)} \right].$$

where

$$(\operatorname{curl} u)_{i,j} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \qquad (i, j \in \{1, 2, 3\}).$$

Moreover if $|||\cdot|||$ is a norm on $H^{m-1}(\Omega, \mathbb{R}^3)$ such that

$$|||u||| \leq C\|u\|_{H^{m-1}(F_0,\mathbb{R}^3)} \qquad \left(u \in H^{m-1}(F_0, \mathbb{R}^3)\right) \tag{5.8}$$

for some $C > 0$ then there exists a constant $K > 0$ such that

$$\|u\|_{H^m(\Omega,\mathbb{R}^3)} \leq K \left[\|\operatorname{div} u\|_{H^{m-1}(\Omega,\mathbb{R}^3)} + \|\operatorname{curl} u\|_{H^{m-1}(\Omega,\mathbb{R}^3)} \right.$$
$$\left. + \|u \cdot n\|_{H^{m-1/2}(\partial\Omega,\mathbb{R}^3)} + |||u||| \right].$$

We are now in a position to give the main ingredient needed to prove that $\mathcal{L}$ is locally Lipschitz. This result concerns the potential μ introduced in (3.17).

Proposition 5.4. *For every integer $m \geqslant 3$, the function χ defined on F^m by*

$$\chi(z)(y) = \nabla\mu(z)(\eta(y)) \qquad (y \in F_0), \tag{5.9}$$

is locally Lipschitz from F^m to $H^{m-1}(F_0, \mathbb{R}^3)$.

Proof. Let $z_0 = (\sigma_0, \nu_0)^* \in F^m$ with

$$\sigma_0 = (Id_{F_0}, h_0, Id_3)^* \in \Sigma^m, \quad \nu_0 = (u_0, k_0, w_0)^* \in T_{\sigma_0}\Sigma^m.$$

We use again the notation from the proof of Proposition 5.2, i.e., for $r > 0$ we set

$$B^m(r) = \{z \in F^m \ | \ \|z - z_0\|_{E^m} \leq r\}.$$

In the the remaining part of this proof, z_1 and z_2 are generic points in $B^m(r)$ and we denote by $K(r)$ any Lipschitz constant obtained in Proposition 5.2. With the notation from this section, it is not difficult to check that the Neumann problem (3.17) can be rewritten as:

$$-\Delta\mu(z;x) = \alpha(z;\eta^{-1}(x)) \qquad (x \in F_q), \tag{5.10a}$$
$$\frac{\partial\mu}{\partial n}(z;x) = \beta_0(z;\eta^{-1}(x)) \qquad (x \in \partial\Omega), \tag{5.10b}$$
$$\frac{\partial\mu}{\partial n}(z;x) = \beta(z;\eta^{-1}(x)) + \tau(z;\eta^{-1}(x)) \qquad (x \in \partial S_q), \tag{5.10c}$$

where α, β_0, β and τ have been defined in (5.1), (5.2), (5.3) and (5.4). The main difficulty consists in the fact that the functions $\mu_1 = \mu(z_1;\cdot)$ and $\mu_2 = \mu(z_2;\cdot)$ are not defined on the same domain. Using Lemma A.4 from [1] we have

$$\begin{aligned}\|\chi(z_1) - \chi(z_2)\|_{H^m(F_0,\mathbb{R}^3)} &= \|\nabla\mu_1\circ\eta_1 - \nabla\mu_2\circ\eta_2\|_{H^m(F_0,\mathbb{R}^3)} \\ &\leqslant K(r)\|\nabla\mu_1\circ\eta - \nabla\mu_2\|_{H^m(F_{q_2},\mathbb{R}^3)},\end{aligned}$$

where $\eta = \eta_1 \circ \eta_2^{-1}$. By applying Proposition 5.3 we obtain

$$\|\nabla\mu_1\circ\eta - \nabla\mu_2\|_{H^m(F_{q_2},\mathbb{R}^3)} \leq K(r)(I_1 + I_2 + I_3 + I_4),$$

where I_i, with $i \in \{1,2,3,4\}$, are given by

$$\begin{aligned} I_1 &= \|\operatorname{div}(\nabla\mu_1\circ\eta - \nabla\mu_2)\|_{H^{m-1}(F_{q_2})}, \\ I_2 &= \|\operatorname{curl}(\nabla\mu_1\circ\eta - \nabla\mu_2)\|_{H^{m-1}(F_{q_2},\mathcal{M}_3(\mathbb{R}))}, \\ I_3 &= \|(\nabla\mu_1\circ\eta - \nabla\mu_2)\cdot n\|_{H^{m-1/2}(\partial F_{q_2})}, \\ I_4 &= |||\nabla\mu_1\circ\eta - \nabla\mu_2|||, \end{aligned}$$

where $|||\cdot|||$ is the norm on $H^{m-1}(F_{\mathbf{q}_2},\mathbb{R}^3)$ defined by

$$\begin{aligned}|||u||| = \sup\Bigg\{\int_{F_{q_2}} u(x)\cdot\gamma(x)\mathrm{d}x \,|\, \gamma \in C^m(\overline{F_{q_2}},\ \mathbb{R}^3), \\ \gamma(x) = 0 \text{ for all } x \in \partial F_{q_2},\ \|\gamma\|_{C^m(\overline{F_{q_2}},\mathbb{R}^3)} \leqslant 1\Bigg\},\end{aligned} \tag{5.11}$$

which clearly satisfies (5.8). Using Lemma A.4 from [1] and (5.10) we have

$$I_1 \leq K(r)\left(\|\operatorname{div}(\nabla\mu_1\circ\eta) - \Delta\mu_1\circ\eta\|_{H^{m-1}(F_{q_2},\mathbb{R}^3)} + \|\alpha(z_1) - \alpha(z_2)\|_{H^{m-1}(F_{q_0},\mathbb{R}^3)}\right).$$

Using Lemmas A.4 and 4 from [1] we obtain

$$\begin{aligned}\|\operatorname{div}(\nabla\mu_1\circ\eta) - \Delta\mu_1\circ\eta\|_{H^{m-1}(F_{q_2})} &\leqslant K(r)\left\|\eta - Id_{F_{q_2}}\right\|_{H^m(F_{q_2})}\|\nabla\mu_1\|_{H^m(F_{q_1})}, \\ &\leqslant K(r)\|\eta_1 - \eta_2\|_{H^m(F_0)}\|\nabla\mu_1\|_{H^m(F_{q_1})}.\end{aligned}$$

On the other hand, using again Lemma A.4 and Theorem 3.1 we have

$$\|\nabla\mu_1\|_{H^m(F_{q_1})} \leq K(r)(\|\alpha(z_1)\|_{H^{m-1}(F_0)} + \|\beta_0(z_1)\|_{H^{m-1/2}(\partial\Omega)} + \|\beta(z_1)+\tau(z_1)\|_{H^{m-1/2}(\partial S_{q_0})} \leqslant \widetilde{K}(r) \qquad (z_1 \in B^m(r)). \tag{5.12}$$

The last two estimates and the locally Lipschitz property of α proved in Proposition 5.2 imply that

$$I_1 \leq K(r)\|z_1 - z_2\|_{E^m} \qquad (z_1,\ z_2 \in B^m(r)). \tag{5.13}$$

Using the fact that $\mathrm{curl}(\nabla f) = 0$ together with arguments completely similar to those used for I_1, we obtain a constant $K(r)$ such that

$$I_2 \leq K(r)\|z_1 - z_2\|_{E^m} \qquad (z_1,\ z_2 \in B^m(r)). \tag{5.14}$$

To tackle I_3, let n^i the unit normal vector to ∂F_{q_i}, with $i \in \{1,2\}$. We have

$$I_3 \leq \left[\left\|(\nabla\mu_1\circ\eta)\cdot\left(n^2 - n^1\circ\eta\right)\right\|_{H^{m-1/2}(\partial F_{q_2})} + \left\|\frac{\partial\mu_1}{\partial n^1}\circ\eta - \frac{\partial\mu_2}{\partial n^2}\right\|_{H^{m-1/2}(\partial F_{q_2})}\right].$$

Using trace inequalities, estimate (5.12) and Lemma A.4 from [1] we obtain that

$$\begin{aligned} I_3 &\leq K(r)\left[\left\|n^2(\eta_2) - n^1(\eta_1)\right\|_{H^{m-1/2}(\partial F_0)} + \left\|\frac{\partial\mu_1}{\partial n^1}\circ\eta_1 - \frac{\partial\mu_2}{\partial n^2}\circ\eta_2\right\|_{H^{m-1/2}(\partial F_0)}\right] \\ &= K(r)\Bigg[\left\|n^2(\eta_2) - n^1(\eta_1)\right\|_{H^{m-1/2}(\partial F_0)} + \|\beta_0(z_1) - \beta_0(z_2)\|_{H^{m-1/2}(\partial\Omega)} \\ &\qquad + \|\beta(z_1)+\tau(z_1) - \beta(z_2) - \tau(z_2)\|_{H^{m-1/2}(\partial S_0)}\Bigg]. \end{aligned}$$

Applying Lemma A.3 from [1] to the extensions of n^i to F_{q_i} (these extensions have been defined in Remark 5.1, it follows that

$$\left\|n^2(\eta_2) - n^1(\eta_1)\right\|_{H^{m-1/2}(\partial F_0,\mathbb{R}^3)} \leq K(r)\|z_1 - z_2\|_{E^m} \qquad (z_1,\ z_2 \in B^m(r)).$$

The last two estimates and the Lipschitz properties of $\alpha,\ \beta_0,\ \beta$ and τ imply that

$$I_3 \leq K(r)\|z_1 - z_2\|_{E^m} \qquad (z_1,\ z_2 \in B^m(r)). \tag{5.15}$$

To study I_4 we first note that, for every $\gamma \in C^m(\overline{F_{q_2}},\mathbb{R}^3)$ with $\gamma = 0$ of ∂F_{q_2} we have

$$\int_{F_{q_2}} [\nabla\mu_1\circ\eta(x) - \nabla\mu_2(x)]\cdot\gamma(x)\mathrm{d}x = \int_{F_{q_1}} \nabla\mu_1\cdot\gamma\circ\eta^{-1}(x)\mathrm{d}x - \int_{F_{q_2}} \nabla\mu_2(x)\cdot\gamma(x)\mathrm{d}x. \tag{5.16}$$

Consider the functions $\psi_k : \overline{F_{q_k}} \to \mathbb{R}$, defined as the solutions of the Neumann problems:

$$-\Delta\psi_1 = -\mathrm{div}\,(\gamma \circ \eta^{-1}), \qquad \text{in} \ \ F_{q_1}, \tag{5.17a}$$

$$\frac{\partial \psi_1}{\partial n} = 0 \qquad \text{on} \ \ \partial F_{q_1}. \tag{5.17b}$$

$$-\Delta\psi_2 = -\mathrm{div}\,\gamma, \qquad \text{in} \ \ F_{q_2}, \tag{5.18a}$$

$$\frac{\partial \psi_2}{\partial n} = 0 \qquad \text{on} \ \ \partial F_{q_2}. \tag{5.18b}$$

Taking the inner product in $L^2(F_{q_1})$ (respectively in $L^2(F_{q_2})$) of the first equation in (5.17) (respectively in (5.18)) by μ_1 (respectively by μ_2) and the subtracting side by side, we obtain that

$$\int_{F_{q_1}} \nabla\psi_1 \cdot \nabla\mu_1 \,\mathrm{d}x - \int_{F_{q_2}} \nabla\psi_2 \cdot \nabla\mu_2 \,\mathrm{d}x = \int_{F_{q_1}} \nabla\mu_1 \cdot \gamma \circ \eta^{-1}(x)\mathrm{d}x - \int_{F_{q_2}} \nabla\mu_2(x) \cdot \gamma(x)\mathrm{d}x.$$

The above formula and (5.16) yield that

$$\int_{F_{q_2}} [\nabla\mu_1 \circ \eta(x) - \nabla\mu_2(x)] \cdot \gamma(x)\mathrm{d}x = \int_{F_{q_1}} \nabla\psi_1 \cdot \nabla\mu_1 \,\mathrm{d}x - \int_{F_{q_2}} \nabla\psi_2 \cdot \nabla\mu_2 \,\mathrm{d}x.$$

Using the variational formulation of the Neumann problem (5.10) we obtain that, for $i \in \{1, 2\}$, we have

$$\begin{aligned}\int_{F_{q_i}} \nabla\mu_i \cdot \nabla\psi_i \,\mathrm{d}x = \int_{F_{q_i}} \alpha(z_i; \eta_i^{-1}(x))\psi_i(x)\mathrm{d}x + \int_{\partial\Omega} \beta_0(z_i; \eta_i^{-1}(x))\psi_i(x)\,\mathrm{d}\sigma_x \\ + \int_{\partial S_{q_i}} \left[\beta(z_i; \eta_i^{-1}(x)) + \tau(z_i; \eta_i^{-1}(x))\right] \psi_i(x)\mathrm{d}\sigma_x.\end{aligned}$$

The last two formulas imply that

$$\begin{aligned}\int_{F_{q_2}} &[\nabla\mu_1 \circ \eta(x) - \nabla\mu_2(x)] \cdot \gamma(x)\mathrm{d}x \\ &= \int_{F_{q_1}} \alpha(z_1; \eta_1^{-1}(x))\psi_1(x)\mathrm{d}x - \int_{F_{q_2}} \alpha(z_2; \eta_2^{-1}(x))\psi_2(x)\mathrm{d}x \\ &+ \int_{\partial\Omega} \beta_0(z_1; \eta_1^{-1}(x))\psi_1(x)\,\mathrm{d}\sigma_x - \int_{\partial\Omega} \beta_0(z_2; \eta_2^{-1}(x))\psi_2(x)\,\mathrm{d}\sigma_x \\ &+ \int_{\partial S_{q_1}} \beta(z_1; \eta_1^{-1}(x))\,\mathrm{d}\sigma_x - \int_{\partial S_{q_2}} \beta(z_2; \eta_2^{-1}(x))\,\mathrm{d}\sigma_x \\ &+ \int_{\partial S_{q_1}} \tau(z_1; \eta_1^{-1}(x))\psi_1(x)\mathrm{d}\sigma_x - \int_{\partial S_{q_2}} \tau(z_2; \eta_2^{-1}(x))\psi_2(x)\mathrm{d}\sigma_x.\end{aligned} \tag{5.19}$$

To estimate the difference of the first two terms in the right-hand side of the above formula, we note that

$$\int_{F_{q_1}} \alpha(z_1;\eta_1^{-1}(x))\psi_1(x)\mathrm{d}x - \int_{F_{q_2}} \alpha(z_2;\eta_2^{-1}(x))\psi_2(x)\mathrm{d}x$$
$$= \int_{F_{q_0}} [\alpha(z_1;y)(\psi_1\circ\eta_1)(y)\mathrm{d}y - \alpha(z_2;y)(\psi_2\circ\eta_2)(y)]\,\mathrm{d}y$$
$$\leqslant \|\alpha(z_1;\cdot)-\alpha(z_2;\cdot)\|\;\|\psi_1\circ\eta_1\| + \|\alpha(z_2;\cdot)\|\;\|\psi_1\circ\eta_1 - \psi_2\circ\eta_2\|, \quad (5.20)$$

where all the norms above are in $L^2(F_{q_0})$. The first term in the right-hand side of the above relation is readily estimated by using Proposition 5.2 to get

$$\|\alpha(z_1;\cdot)-\alpha(z_2;\cdot)\|\;\|\psi_1\circ\eta_1\| \leqslant K(r)\|z_1-z_2\|_{E^m}\;\|\gamma\|_{C^m(\overline{F_{\mathbf{q}_2}};\mathbb{R}^3)}, \qquad (5.21)$$

for every $z_1,\ z_2\in B^m(r)$. To estimate the second term in the right-hand side of (5.20) we remark that, using the variational formulations of (5.17) and (5.18) and a simple change of variables we have, for $k\in\{1,2\}$,

$$\int_{F_{q_0}} (\nabla\psi_k\circ\eta_k)\cdot(\nabla\varphi_k\circ\eta_k)\,\mathrm{d}y = \int_{F_{q_0}} (\gamma\circ\eta_2)\cdot(\nabla\varphi_k\circ\eta_k)\,\mathrm{d}y \quad (\varphi_k\in H^m(F_{q_k})).$$

Denoting $\widetilde{\psi_k}=\psi_k\circ\eta_k$ the last formula becomes

$$\int_{F_{q_0}} (D\eta_k^{-1})(D\eta_k^{-1})^*\nabla\widetilde{\psi_k}\cdot\nabla\varphi\,\mathrm{d}y = \int_{F_{q_0}} (D\eta_k^{-1})(\gamma\circ\eta_2)\cdot\nabla\varphi\,\mathrm{d}y, \qquad (\varphi\in H^m(F_{q_0})).$$

Subtracting side by side the formulas corresponding to $k=1$ and $k=2$ it is not difficult to see that, for every

$$\gamma\in C^m(\overline{F_{q_2}},\ \mathbb{R}^3), \qquad \gamma=0 \text{ on } \partial F_{q_2},$$

we have

$$\int_{F_{q_0}} |\nabla\widetilde{\psi_1}-\nabla\widetilde{\psi_2}|^2\,\mathrm{d}y \leqslant K(r)\|z_1-z_2\|^2_{E^m} \qquad (z_1,\ z_2\in B^m(r)).$$

The above estimate, combined to (5.20) and (5.21) imply that

$$\int_{F_{q_1}} \alpha(z_1;\eta_1^{-1}(x))\psi_1(x)\mathrm{d}x - \int_{F_{q_2}} \alpha(z_2;\eta_2^{-1}(x))\psi_2(x)\mathrm{d}x$$
$$\leqslant K(r)\|z_1-z_2\|^2_{E^m} \qquad (z_1,\ z_2\in B^m(r)).$$

The other terms in the right-hand side of (5.19) can be estimated in a similar way. In order to keep this paper of reasonable length we skip the proof of the corresponding estimates. □

We are now in position to prove that $\mathcal{L}$ is locally Lipschitz.

Proposition 5.5. *The mappings $\mathcal{L}_S$, $\mathcal{L}_F$ and $\mathcal{L}$ are locally Lipschitz on F^m.*

Proof. We begin by showing that $\mathcal{L}_S$ is locally Lipschitz in $B^m(r)$ for a given r. From Proposition 3.4 the mapping $q \to \mathcal{K}(q)$ is C^2 from $P(\Omega, S)$ to $\mathcal{M}_6(\mathbb{R})$ (recall that $\mathcal{K}(q)$ is the virtual mass matrix defined in (4.1)). Using Proposition 3.4 together with Lemmas A.2 and A.3 from [1], it follows that the mapping $z \mapsto \nabla\Phi \circ \eta$ is Lipschitz from $B^m(r)$ to $H^m(F_0, \mathcal{M}_{3\times 6}(\mathbb{R}))$, where $(\Phi_k)_{k\in\{1,\ldots,6\}}$ satisfy (3.15). Moreover, the mapping $z \mapsto (0_3, (J\omega)\wedge\omega)^*$ is Lipschitz from $B^m(r)$ to $\mathbb{R}^6$. Using the notation in Proposition 4.1, the last term in the right-hand side of (4.2) writes

$$\int_{F_q} \nabla\mu(x)\cdot\nabla\Phi(x)\,\mathrm{d}x = \int_{F_0} \nabla\mu(\eta(y))\cdot\nabla\Phi(\eta(y))\,\mathrm{d}y,$$

so that, using Proposition 5.4 and Proposition 3.4, we obtain that that this term defines a Lipschitz function from $B^m(r)$ to $\mathbb{R}^6$. Using next the smoothness of the map $q \mapsto \mathcal{K}(q)$, it follows that $\mathcal{L}_S$ is Lipschitz from $B^m(r)$ to $\mathbb{R}^6$.

Finally, the fact that $\mathcal{L}_F$ is locally Lipschitz readily follows from (4.5) and the corresponding properties of Φ, $\mathcal{L}_S$ and μ. □

6 $\mathcal{L}$ is tangent to F^m

In this section we show that the vector field defined by the operator $\mathcal{L}$ from (4.7) is tangent to the closed set F^m which has been defined in (2.12). More precisely, the main result of this section is

Proposition 6.1. *Let $m \geqslant 3$ an integer and let $z_0 \in F^m$. Then*

$$\lim_{r\to 0}\frac{1}{r}\mathrm{dist}(z_0 + r\mathcal{L}(z_0); F^m) = 0.$$

In order to prove the above proposition we need some notation and several auxiliary results. Throughout this section e_0 denotes the identity map on F_0 and $q_0 = \begin{pmatrix} h_0 \\ Id_3 \end{pmatrix}$. Moreover $z_0 = \begin{pmatrix} \sigma_0 \\ \nu_0 \end{pmatrix}$ denotes a generic element of F^m, where $\sigma_0 = \begin{pmatrix} \eta_0 \\ h_0 \\ R_0 \end{pmatrix} \in \Sigma^m$ and $\nu_0 = \begin{pmatrix} \xi_0 \\ k_0 \\ \omega_0 \end{pmatrix} \in T_{\sigma_0}\Sigma^m$. Let

$$\sigma = \begin{pmatrix} \eta \\ h \\ R \end{pmatrix} \in H^m(F_0, \mathbb{R}^3) \times P(\Omega, S_0),$$

and recall the properties (1.6) and (1.7) of $\partial\Omega$. We define the map

$$\vartheta(\sigma) = \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix} (\sigma) \qquad \big((\sigma \in H^m(F_0, \mathbb{R}^3) \times P(\Omega, S_0)\big), \tag{6.1}$$

where

$$\vartheta_1(\sigma)(y) = \det(D\eta)(y) - \frac{1}{|F_0|}\int_{F_0} \det(D\eta(x))\,\mathrm{d}x - \frac{1}{|F_0|}\int_{\partial\Omega} \delta_0(\eta(x))\,\mathrm{d}\sigma_x$$
$$- \frac{1}{|F_0|}\int_{\partial S_0} \delta(h_0 + R^*(\eta(y) - h))\,\mathrm{d}\sigma_x \qquad \big(\sigma \in H^m(F_0, \mathbb{R}^3) \times P(\Omega, S_0),\ y \in F_0\big), \tag{6.2}$$

$$\vartheta_2(\sigma)(y) = \begin{cases} \delta_0(\eta(y)) & (\sigma \in H^m(F_0, \mathbb{R}^3) \times P(\Omega, S_0),\ y \in \partial\Omega), \\ \delta(h_0 + R^*(\eta(x) - h)) & (\sigma \in H^m(F_0, \mathbb{R}^3) \times P(\Omega, S_0),\ y \in \partial S_0). \end{cases} \tag{6.3}$$

Since we obviously have

$$\int_{F_0} \vartheta_1\,\mathrm{d}y + \int_{\partial F_0} \vartheta_2\,\mathrm{d}\sigma_y = 0,$$

it follows that ϑ maps $H^m(F_0, \mathbb{R}^3) \times P(\Omega, S_0)$ into $V^{m-1}(F_0)$ (see (3.1) for the definition of this space).

Lemma 6.2. *The function ϑ defined above is of class C^k for all integer $k \geq 1$ and we have*

$$\partial_\eta\vartheta \begin{pmatrix} e_0 \\ q_0 \end{pmatrix} (V) = \begin{pmatrix} \operatorname{div} V \\ -V \cdot n_{|\partial F_0} \end{pmatrix} \qquad \big(V \in H^m(F_0, \mathbb{R}^3)\big). \tag{6.4}$$

Moreover, $\partial_\eta\vartheta \begin{pmatrix} e_0 \\ q_0 \end{pmatrix}$ maps $H^m(F_0, \mathbb{R}^3)$ onto $V^{m-1}(F_0)$.

Proof. Since $m \geq 3$, it follows that H^{m-1} is an algebra so that the map $\eta \mapsto \det(D\eta)$ is of class C^k from $H^m(F_0, \mathbb{R}^3)$ to $H^{m-1}(F_0, \mathbb{R})$ for every $k \geq 1$. It is easy to check that the other terms in the definition of ϑ_2 and are smooth functions so that ϑ is of class C^k for every $k \geq 1$.

Using (1.7) it follows that, for every $\sigma \in H^m(F_0, \mathbb{R}^3) \times P(\Omega, S_0)$, $V \in H^m(F_0, \mathbb{R}^3)$, we have

$$(\partial_\eta\vartheta_2)(\sigma)(V)(y) = \begin{cases} -V(y) \cdot n(\eta(y)) & (y \in \partial\Omega), \\ -R^*V(y) \cdot n(h_0 + R^*(\eta(y) - h)) & (y \in \partial S_0). \end{cases} \tag{6.5}$$

On the other hand, using the fact that the differential of $A \mapsto \det(A)$ is the linear map $H \mapsto \operatorname{tr}(\operatorname{cof}(A)H)$, where $\operatorname{cof}(A)$ is the signed cofactors matrix of A, we obtain that, for every $\sigma \in H^m(F_0, \mathbb{R}^3) \times P(\Omega, S_0)$, $V \in H^m(F_0, \mathbb{R}^3)$, we have

$$\partial_\eta \vartheta_1(\sigma)(V)(y) = \mathrm{tr}(\mathrm{cof}(D\eta)DV)(y) - \frac{1}{|F_0|}\int_{F_0} \mathrm{tr}(\mathrm{cof}(D\eta)DV))(y)\,\mathrm{d}y$$
$$+ \frac{1}{|F_0|}\int_{\partial\Omega} V(y)\cdot n(\eta(y))\mathrm{d}\sigma_y + \frac{1}{|F_0|}\int_{\partial S_0} R^* V(y)\cdot n(h_0 + R^*(\eta(y)-h))\,\mathrm{d}\sigma_y.$$

Taking $h = h_0$, $R = Id_3$ and $\eta = e_0$ in the above formula and by using (6.5) we obtain (6.4). Finally, the fact that the right-hand side of (6.4) defines a map from $H^m(F_0, \mathbb{R}^3)$ onto $V^{m-1}(F_0)$ is classical, see for instance, Lemma 2.4.1 in Sohr [16, page 79]. □

The above lemma can be used, in particular, to show that Σ^m is an infinite-dimensional manifold over $H^m(F_0, \mathbb{R}^3) \times \mathbb{R}^3 \times \mathcal{M}_3(\mathbb{R})$ and to compute its tangent space at σ_0.

Proposition 6.3. *We have*

$$\Sigma^m = \left\{ \begin{pmatrix} \eta \\ q \end{pmatrix} \in H^m(F_0, \mathbb{R}^3) \times P(\Omega, S_0) \;\middle|\; \vartheta\begin{pmatrix} q \\ \eta \end{pmatrix} = 0 \right\}. \tag{6.6}$$

Moreover, the tangent space to Σ^m at every $\sigma = \begin{pmatrix} \eta \\ q \end{pmatrix} \in \mathcal{O} \cap \Sigma^m(\Omega, S_0)$ is the space $T_\sigma\Sigma^m$ defined in (2.11).

Proof. The fact that the set in the left-hand side of (6.6) is a subset of the set int the right-hand side is obvious. To prove the converse inclusion, we first note from $\vartheta_1(q, \eta) = 0$ it follows that $\det(D\eta)$ is constant in F_0. On the other hand, from $\vartheta_2(q, \eta) = 0$ it follows that $\eta(q)(\partial F_0) \subset \partial F_q$. These assertions, combined to the fact that F_0 and F_q have the same volume, imply that $\det(D\eta) = 1$ in F_0. Moreover, the above properties enable us to apply the global inverse mapping theorem of Caccioppoli (see, for instance, Zeidler [18, Theorem 4.G. page 174]) to obtain that $\eta \in \mathrm{Diff}^m(F_0, F_q)$. This concludes the proof of (6.6).

In order to prove the second assertion in the proposition, we first note that for every $\sigma = (\eta, h, R)^* \in \Sigma^m$ and every $(\xi, k\,\omega)^* \in H^m(F_0, \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3$, we have

$$\mathrm{D}\vartheta_1(\sigma)\begin{pmatrix} \xi \\ k \\ \omega \end{pmatrix}(y) = \mathrm{tr}\left(\mathrm{D}\eta^{-1}\,\mathrm{D}\xi\right) \qquad (y \in F_0),$$

$$\mathrm{D}\vartheta_2(\sigma)\begin{pmatrix} \xi \\ k \\ \omega \end{pmatrix}(y)$$
$$= \begin{cases} -\xi\cdot n(\eta(y)) & (y \in \partial\Omega)\,, \\ -R^*\left[\xi(y) - k - A(\omega)(\eta(y)-h)\right]\cdot n(h_0 + R^*(\eta(y)-h)) & (y \in \partial S_0)\,, \end{cases}$$

where $A(\omega)$ has been defined in (1.2). Since η is a diffeomorphism from F_0 to F_q, denoting $u = \xi \circ \eta^{-1}$ and making the change of variable $x = \eta(y)$, we obtain

$$\mathrm{D}\vartheta_1(\sigma)\begin{pmatrix}\xi\\ k\\ \omega\end{pmatrix}(\eta^{-1}(x)) = (\operatorname{div} u)(x) \qquad (x \in F_q),$$

$$\mathrm{D}\vartheta_2(\sigma)\begin{pmatrix}\xi\\ k\\ \omega\end{pmatrix}(\eta^{-1}(x)) = \begin{cases}-u\cdot n(x) & (x\in\partial\Omega)\,,\\ -\left[u(x)-k-\omega\wedge(x-h)\right]\cdot n(x) & (x\in\partial S_q)\,.\end{cases}$$

From the above formulas it follows that the kernel of $D\phi(\sigma)$ is $T_\sigma\Sigma^m$ so that we obtain the second assertion in the proposition. □

Proposition 6.4. *Let* $\sigma_0 = \begin{pmatrix}e_0\\ h_0\\ Id_3\end{pmatrix} \in \Sigma^m, \quad \nu_0 = \begin{pmatrix}u_0\\ k_0\\ \omega_0\end{pmatrix} \in T_{\sigma_0}\Sigma^m$ *and let*

$$\gamma_0 = \begin{pmatrix}\Gamma\\ L\\ M\end{pmatrix} \in H^m(F(0),\mathbb{R}^3)\times\mathbb{R}^3\times\mathbb{R}^3$$

such that

$$\operatorname{div}(\Gamma)(y) = \mathcal{F}(u_0(y)) \qquad (y\in F_0), \tag{6.7a}$$
$$\Gamma\cdot n(y) = -\mathcal{G}(u_0(y)) \qquad (y\in\partial\Omega), \tag{6.7b}$$
$$\begin{aligned}\Gamma\cdot n(y) = &-\mathcal{G}(u_0(y)-v_0(y)) - 2(u_0-v_0)\cdot(\omega_0\wedge n)\\ &+\left[L+M\wedge(y-h_0)+\omega_0\wedge(w_0\wedge(y-h_0))\right]\cdot n \qquad (y\in\partial S_0),\end{aligned} \tag{6.7c}$$

where $v_0(y) = k_0 + \omega_0\wedge(y-h_0)$. *Then there exists* $\varepsilon > 0$ *and a curve*

$$\sigma = \begin{pmatrix}\boldsymbol{\eta}\\ \mathbf{h}\\ \mathbf{R}\end{pmatrix} \in C^2([0,\varepsilon];\Sigma^m(\Omega,S_0))$$

satisfying

$$\sigma(0)=\sigma_0, \qquad \dot\sigma(0) = \begin{pmatrix}u_0\\ k_0\\ A(\omega_0)\end{pmatrix} \qquad \ddot\sigma(0) = \begin{pmatrix}\Gamma\\ L\\ A(M)+[A(\omega_0)]^2\end{pmatrix}. \tag{6.8}$$

Proof. According to Proposition 6.3 the curve σ is contained in Σ^m iff $\vartheta(\sigma(t)) = 0$ for every $t\in[0,\varepsilon]$.

We begin by constructing the "rigid displacement part" $\begin{pmatrix}\mathbf{h}\\ \mathbf{R}\end{pmatrix}$ of the curve σ. To do that, we define $\mathbf{h},\ \boldsymbol{\omega}:\mathbb{R}\to\mathbb{R}^3$

$$\mathbf{h}(t) = h_0 + tk_0 + \frac{t^2}{2}L, \quad \boldsymbol{\omega}(t) = \omega_0 + tM \qquad t\in\mathbb{R}, \tag{6.9}$$

and $\mathbf{R} : \mathbb{R} \to SO_3(\mathbb{R})$ is defined as the solution of the initial value problem

$$\mathbf{R}(0) = Id_3, \quad \dot{\mathbf{R}}(t) = A(\boldsymbol{\omega}(t))\mathbf{R}(t), \tag{6.10}$$

where $A(\boldsymbol{\omega})$ is the skew-adjoint matrix defined in (1.2). Note that

$$\dot{\mathbf{R}}(0) = A(\omega_0), \qquad \ddot{\mathbf{R}}(0) = A(M) + [A(\omega_0)]^2. \tag{6.11}$$

The above functions being continuous, it follows that there exists $\varepsilon' > 0$ such that

$$\mathbf{q}(t) = \begin{pmatrix} \mathbf{h}(t) \\ \mathbf{R}(t) \end{pmatrix} \in P(\Omega, S_0) \qquad t \in [0, \varepsilon'].$$

In order to construct the "fluid part" η of the curve σ we first note that, since $\begin{pmatrix} e_0 \\ \mathbf{q}(0) \end{pmatrix} \in \Sigma^m$, we have $\vartheta \begin{pmatrix} e_0 \\ \mathbf{q}(0) \end{pmatrix} = 0$. Therefore, by combining Lemma 6.2 with a version of the implicit function theorem (see, for instance, Zeidler [18, Theorem 4.H. page 171]) it follows that there exists $\varepsilon \in (0, \varepsilon']$ and a function $\boldsymbol{\eta} : [0, \varepsilon] \to H^m(F_0, \mathbb{R}^3)$ such that $\eta(0) = e_0$ and

$$\vartheta \begin{pmatrix} \boldsymbol{\eta}(t) \\ \mathbf{q}(t) \end{pmatrix} = 0, \qquad \mathcal{P}\left(\boldsymbol{\eta}(t) - tu_0 - \frac{t^2}{2}\Gamma\right) = 0 \qquad (t \in [0, \varepsilon]), \tag{6.12}$$

where $\mathcal{P}$ is the orthogonal projector from $H^m(F_0, \mathbb{R}^3)$ onto $\operatorname{Ker} \partial_n \vartheta \begin{pmatrix} e_0 \\ \mathbf{q}(0) \end{pmatrix}$. Note that, according to Lemma 6.2, we have

$$\operatorname{Ker} \partial_n \vartheta \begin{pmatrix} e_0 \\ \mathbf{q}(0) \end{pmatrix} = \left\{ u \in H^m(F_0, \mathbb{R}^3) \ | \ \operatorname{div} u = 0 \quad u \cdot n = 0 \text{ on } \partial F_0 \right\} \tag{6.13}$$

In the remaining part of the proof we show that, with the above choice of $\boldsymbol{\eta}$, $\mathbf{h}$ and $\mathbf{R}$, the curve

$$\sigma(t) = \begin{pmatrix} \boldsymbol{\eta}(t) \\ \mathbf{h}(t) \\ \mathbf{R}(t) \end{pmatrix} \qquad (t \in [0, \varepsilon])$$

satisfies (6.8). We first note that from (6.9) it follows that

$$\mathbf{h}(0) = h_0, \quad \dot{\mathbf{h}}(0) = k_0, \quad \ddot{\mathbf{h}}(0) = L.$$

From the above formula combined with (6.10) and (6.11) we see that, in order to prove (6.8), we have only to check that

$$\dot{\boldsymbol{\eta}}(0) = u_0, \qquad \ddot{\boldsymbol{\eta}}(0) = \Gamma. \tag{6.14}$$

Taking the derivative with respect to t of the formula $\det(\mathrm{D}\boldsymbol{\eta}(t)) = 1$ we obtain that

$$\operatorname{tr}((\mathrm{D}\boldsymbol{\eta}(t))^{-1}\,\mathrm{D}\dot{\boldsymbol{\eta}}(t)) = 0 \qquad (t \in [0, \varepsilon]). \tag{6.15}$$

Using next the fact that $\delta_0(\boldsymbol{\eta}(t)) = 0$ on $\partial\Omega$ it follows that

$$\dot{\boldsymbol{\eta}}(t) \cdot n(\boldsymbol{\eta}(t)) = 0 \qquad (\text{on } \ \partial\Omega). \tag{6.16}$$

Moreover, since $\delta(h_0 + \mathbf{R}^*(t)(\boldsymbol{\eta}(t) - \mathbf{h}(t))) = 0$ on ∂S_0, we have

$$(\dot{\boldsymbol{\eta}}(t) - \dot{\mathbf{h}}(t) - \boldsymbol{\omega}(t) \wedge (\boldsymbol{\eta}(t) - \mathbf{h}(t))) \cdot \mathbf{R}(t) n(h_0 + \mathbf{R}^*(t)(\boldsymbol{\eta}(t) - \mathbf{h}(t))) = 0 \quad (\text{on } \ \partial S_0). \tag{6.17}$$

On the other hand, taking the derivative of the second formula in (6.12) with respect to t we obtain

$$\mathcal{P}(\dot{\boldsymbol{\eta}}(t) - u_0 - t\Gamma) = 0 \qquad (t \in [0, \varepsilon]). \tag{6.18}$$

Taking $t = 0$ in (6.15)–(6.18) we obtain

$$\begin{array}{rcl} \operatorname{div}(\dot{\boldsymbol{\eta}}(0)) &=& 0, \qquad (\text{in } \ F_0), \\ \dot{\boldsymbol{\eta}}(0) \cdot n &=& 0 \qquad (\text{on } \ \partial\Omega), \\ \dot{\boldsymbol{\eta}}(0) \cdot n &=& (k_0 + \omega_0 \wedge (y - h_0)) \qquad (y \in \partial S_0), \\ \mathcal{P}(\dot{\boldsymbol{\eta}}(0)) &=& \mathcal{P}(u_0). \end{array}$$

The above relations clearly imply that the first equality in (6.14) holds.

In order to prove the second equality in (6.14) we take the derivative of (6.15)–(6.18) and then we make $t = 0$. In this way we obtain

$$\begin{array}{rcl} \operatorname{div} \ddot{\boldsymbol{\eta}}(0) &=& \mathcal{F}(u_0) \qquad (\text{in } \ F_0), \\ \ddot{\boldsymbol{\eta}}(0) \cdot n &=& -\mathcal{G}(u_0) \qquad (\text{on } \ \partial\Omega), \\ \ddot{\boldsymbol{\eta}}(0) \cdot n &=& -\mathcal{G}(u_0 - v_0) - 2(u_0 - v_0) \cdot (\omega_0 \wedge n) \\ && + [L + M \wedge (y - h_0) + \omega_0 \wedge (\omega_0 \wedge (y - h_0))] \cdot n \qquad (y \in \partial S_0), \\ \mathcal{P}(\ddot{\boldsymbol{\eta}}(0)) &=& \mathcal{P}(\Gamma), \end{array}$$

where v_0 has been defined in the statement of this proposition. Using (6.7) it follows that the second equality in (6.14) also holds. □

We are now in a position to prove the main result of this section.

Proof of Proposition 6.1. Recall the notation for z_0 from the beginning of this section. We first note (by using an appropriate change of variables) that it suffices to prove the result for $\eta_0 = e_0$ and $R_0 = Id_3$. This will be done by constructing a curve $\mathbf{Z}(\cdot)$ in F^m such that

$$\lim_{r \to 0} \frac{1}{r} \operatorname{dist}(z_0 + r\mathcal{L}(z_0); \mathbf{Z}(r)) = 0. \tag{6.19}$$

The main tool of the proof is Proposition 6.4, with an appropriate choice of Γ, L and M. More precisely, u_0, k_0 and ω_0 are chosen to be those in (1.1) and we take

$$\Gamma = \mathcal{L}_F(z_0), \qquad \begin{pmatrix} L \\ M \end{pmatrix} = \mathcal{L}_S(z_0), \tag{6.20}$$

where $\mathcal{L}_F$ and $\mathcal{L}_S$ have been defined in (4.5) and (4.6), respectively. The fact that Γ, L and M chosen above satisfy the assumptions in Proposition 6.4 follows from (3.15) and (3.17). Define

$$\mathbf{Z}(t) = \begin{pmatrix} \boldsymbol{\sigma}(t) \\ \dot{\boldsymbol{\eta}}(t) \\ \dot{\mathbf{h}}(t) \\ \omega_0 + tM \end{pmatrix},$$

where $\boldsymbol{\sigma}(t) = (\boldsymbol{\eta}(t), \mathbf{h}(t), \mathbf{R}(t))^*$ is the curve constructed in Proposition 6.4. By combining (6.8) and (6.20) it follows that

$$\mathbf{Z}(0) = z_0, \qquad \dot{\mathbf{Z}}(0) = \mathcal{L}(z_0),$$

which imply (6.19). □

The proof of our main result in Theorem 1.3 can be now written as follows.

Proof of Theorem 1.3. The assumptions in Theorem 1.3 imply that

$$z_0 = (e_0, h_0, Id_3, u_0, k_0, \omega_0)^* \in F^m.$$

Therefore, we can combine Propositions 5.5, 6.1 and 2.1 to obtain that the initial value problem

$$\dot{\mathbf{z}} = \mathcal{L}(\mathbf{z}), \quad \mathbf{z}(0) = z_0, \tag{6.21}$$

admits an unique solution

$$\mathbf{z} = (\boldsymbol{\eta}, \mathbf{h}, \mathbf{R}, \boldsymbol{\xi}, \mathbf{k}, \boldsymbol{\omega})^* \in C^0\big([0, T_0); F^m)\big) \cap C^1\big([0, T_0); E^m)\big).$$

According to Proposition 4.3, $\mathbf{q}$, u defined by $\mathbf{q} = (\mathbf{h}, \mathbf{R})^*$,

$$u(t, x) = \xi(t, \boldsymbol{\eta}^{-1}(t, x)) \qquad (t \in [0, T_0),\ x \in F_{\mathbf{q}(t)}),$$

and the pressure p defined by (3.18) define a strong solution of (1.1). We have thus shown the announced existence result.

To prove the uniqueness, it suffices to note that, according to Proposition 4.2, any strong solution of (1.1) defines a solution of (6.21) and to apply Proposition 2.1. □

References

[1] J. P. Bourguignon and H. Brezis, *Remarks on the Euler equation*, J. Functional Analysis, 15 (1974), pp. 341–363.

[2] P. Cumsille and M. Tucsnak, *Wellposedness for the Navier-Stokes flow in the exterior of a rotating obstacle*, Math. Methods Appl. Sci., 29 (2006), pp. 595–623.

[3] D. G. Ebin and J. Marsden, *Groups of diffeomorphisms and the notion of an incompressible fluid.*, Ann. of Math. (2), 92 (1970), pp. 102–163.

[4] P. Hartman, *Ordinary differential equations*, John Wiley & Sons Inc., New York, 1964.

[5] T. I. Hesla, *Collisions of smooth bodies in viscous fluids, a mathematical investigation*, PhD thesis, University of Minnesota, May 2005.

[6] M. Hillairet, *Lack of collision between solid bodies in a 2D incompressible viscous flow*, Comm. Partial Differential Equations, 32 (2007), pp. 1345–1371.

[7] J.-G. Houot and A. Munnier, *On the motion and collisions of rigid bodies in an ideal fluid.* Private communication, 2006.

[8] E. Kanso, J. E. Marsden, C. W. Rowley, and J. B. Melli-Huber, *Locomotion of articulated bodies in a perfect fluid*, J. Nonlinear Sci., 15 (2005), pp. 255–289.

[9] H. Lamb, *Hydrodynamics*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, sixth ed., 1993. With a foreword by R. A. Caflisch [Russel E. Caflisch].

[10] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes. II*, Ann. Inst. Fourier (Grenoble), 11 (1961), pp. 137–178.

[11] R. H. Martin, Jr., *Differential equations on closed subsets of a Banach space*, Trans. Amer. Math. Soc., 179 (1973), pp. 399–414.

[12] J. H. Ortega, L. Rosier, and T. Takahashi, *Classical solutions for the equations modelling the motion of a ball in a bidimensional incompressible perfect fluid*, M2AN Math. Model. Numer. Anal., 39 (2005), pp. 79–108.

[13] ——, *On the motion of a rigid body immersed in a bidimensional incompressible perfect fluid*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 24 (2007), pp. 139–165.

[14] C. Rosier and L. Rosier, *Smooth solutions for the motion of a ball in an incompressible perfect fluid*, J. Funct. Anal., 256 (2009), pp. 1618–1641.

[15] J. A. San Martín, V. Starovoitov, and M. Tucsnak, *Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid*, Arch. Ration. Mech. Anal., 161 (2002), pp. 113–147.

[16] H. Sohr, *The Navier-Stokes equations*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2001. An elementary functional analytic approach.

[17] T. Takahashi, *Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain*, Adv. Differential Equations, 8 (2003), pp. 1499–1532.

[18] E. Zeidler, *Nonlinear functional analysis and its applications. I*, Springer-Verlag, New York, 1986. Fixed-point theorems, Translated from the German by Peter R. Wadsack.