

## A continuous framework for open pit mine planning

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**Abstract** This paper proposes a new mathematical framework for the open pit mine planning problem, based on continuous functional analysis. The main challenge for engineers is to determine a sequence of nested profiles maximizing the net present value of the mining operation. The traditional models for this problem have been constructed by using binary decision variables, giving rise to large-scale combinatorial and Mixed Integer Programming problems. Instead, we use a continuous approach which allows for a refined imposition of slope constraints associated with geotechnical stability. The framework introduced here is posed in a suitable functional space, essentially the real-valued functions that are Lipschitz continuous on a given two dimensional bounded region. We derive existence results and investigate qualitative properties of the solutions.

**Keywords** Mine planning · Continuous optimization · Calculus of variations · Functional analysis

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## 1 Introduction and motivation

Generally speaking, three different problems are usually considered by mining engineers for the economic valuation, design and planning of open pit mines as pointed out by [Hustrulid and Kuchta \(2006\)](#). The first one is the Final Open Pit (FOP) problem, also called “Ultimate Pit Limit” problem, UPL for short, which aims to find the region of maximal economic value for exploitation under certain geotechnical stability constraints and assuming infinite extraction capacity. Another more realistic problem is what we call here the Capacitated Final Open Pit (CFOP) which considers an additional constraint on the total capacity for a one-period exploitation. The third problem is a multi-period version of the latter, which we call the Capacitated Dynamic Open Pit (CDOP) problem, with the goal of finding an optimal sequence of extracted volumes in a certain finite time horizon for bounded capacities at each period, the optimality criterion being the total discounted profit.

A common practice for the formulation of these problems consists in describing an ore reserve (copper, for example) via the construction of a three-dimensional block model of the mineralization. Each block corresponds to a unitary volume of extraction characterized by several geologic and economic properties which are estimated from sample data. Block models can be represented as directed graphs where nodes are associated with blocks, while arcs correspond to the precedence of these blocks in the ore reserve. The precedence order is induced by physical and operational constraints as those derived from the geomechanics of slope stability. This discrete approach gives rise naturally to huge combinatorial problems whose mathematical formulations are special large-scale instances of Integer Programming (IP) optimization problems (see, for instance, [Cacetta 2007](#)). This explains why the optimal planning of open pit mines based on block models is usually addressed by approximation methods, heuristics and mixed IP techniques as Linear Programming relaxations of integer variables and branch-and-bound algorithms.

A great number of publications dealing with discrete block modeling for open pit mines have been published since the sixties. A seminal paper by [Lerchs and Grossman \(1965\)](#) proposes a practical procedure to obtain the ultimate pit limit, which have been extensively applied in real mines for many years. The capacitated dynamic problem is more difficult to solve and many methods using discrete optimization techniques have been proposed by [Boland et al. \(2006\)](#), [Cacetta and Hill \(2003\)](#) and [Hochbaum and Chen \(2000\)](#). Some dynamic programming formulations as in [Johnson and Sharp \(1971\)](#) and in [Wilke and Wright \(1984\)](#) give interesting results, but the applicability of these techniques is still not well established for large-scale problems. Metaheuristic and evolutionary algorithms have also been extensively tested by [Denby and Schofield](#) and [Ferland et al. \(2007\)](#).

In this paper we propose an alternative approach to the above mentioned problems based on a continuous framework for the ore reserve as well as the mining activity. The basic idea is to describe the pit contours through a Lipschitz continuous real-valued function, a *profile* which maps each pair of horizontal coordinates to the corresponding

vertical depth. The stability of steep slopes is ensured by a spatially distributed constraint on the local Lipschitz constant of the profile function. The maximal feasible local slope may vary depending on the geotechnical properties of the possibly heterogeneous mineral deposit. The extraction capacity and operational costs are described by a possibly discontinuous *effort* density, a scalar function defined on the three-dimensional region occupied by the ore reserve. In order to quantify the economic value of an extracted volume described by a given profile function, we consider a *gain* density defined on the deposit, which again may be a discontinuous function.

Our goal here is to develop a complete existence theory and investigate qualitative properties of the optimal solutions to the proposed continuous versions of the FOP, CFOP and CDOP problems. The numerical resolution of these problems based on strategies from continuous optimization in functional spaces will be the subject of future research.

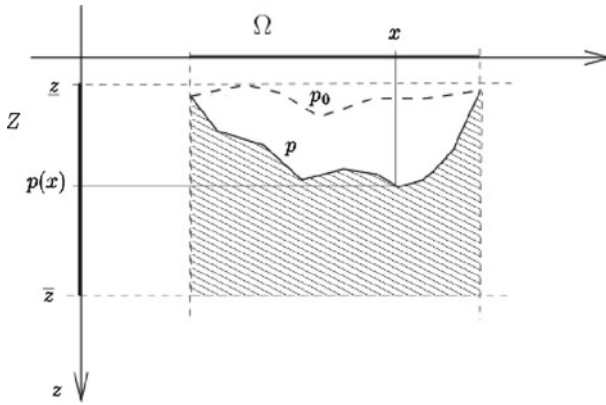
It is worth mentioning that the first documented approach applying continuous functions for a parametrized variant of the CFOP problem seems to be the work by [Matheron \(1975\)](#). He explicitly exploited the underlying lattice structure of the set of feasible profiles for his framework in order to obtain existence results as well as interesting characterizations for optimal solutions but neither a proof of existence nor optimality conditions were given there. More recently, a simple related continuous scheme was introduced by [Morales \(2002\)](#), for underground mines, but no study on existence nor optimality conditions are given there. On the other hand, [Guzmán \(2008\)](#) has proposed a continuous framework for the FOP problem based on shape and topological optimization using level-set techniques, reporting computational results for a very simplified instance of the problem, but again no rigorous existence theory is provided.

This paper is organized as follows. In Sect. 2 we describe the stationary problem in terms of continuous profile functions by introducing nonnegativity, boundary and stability conditions, and we prove that such a set of admissible profiles is compact for the uniform-convergence topology. Furthermore we establish structural properties of this set related to the lattice structure induced by pointwise *min* and *max* operations. In addition, we introduce an effort and a gain function which are related to the capacity constraints and the profit objective function, respectively. In Sect. 3 we state the optimization problems for the stationary case, derive nonconstructive existence results for them and describe qualitative properties of the corresponding optimal solutions by exploiting the lattice structure of the set of feasible profiles. In Sect. 4 we introduce a dynamic planning problem with discount rates, investigate properties of the dynamic feasible set and give an existence result. Finally, in Sect. 5 we briefly summarize the main contributions of this paper and indicate some lines for future research.

## 2 The stationary problem

### 2.1 Continuous profile functions

Throughout this paper  $\Omega$  represents either a bounded connected domain  $\Omega \subset \mathbb{R}^2$  with Lipschitz boundary  $\partial\Omega$ , or a bounded open interval  $\Omega = (a, b) \subset \mathbb{R}$  with  $\partial\Omega = \{a, b\}$ .



**Fig. 1** Example of a continuous profile function on a one dimensional domain

In any case, the state of excavation at any particular time is defined by a function  $p : \overline{\Omega} \rightarrow \mathbb{R}$  called *profile* so that  $z = p(x)$  for  $x \in \overline{\Omega}$ , where the vertical coordinate  $z$  indicates the depth of the pit at point  $x$  (see Fig. 1). In this paper  $p$ , as not stated otherwise, is always an element of the Banach space of continuous real valued functions  $C(\overline{\Omega})$  endowed with the supremum norm given by  $\|p\|_{\infty} \equiv \sup_{x \in \overline{\Omega}} |p(x)|$ . The initial state (profile) is defined by a function  $p_0 \in C(\overline{\Omega})$  so that all admissible profiles  $p$  must satisfy the nonnegativity condition

$$p(x) - p_0(x) \geq 0 \quad \text{for } x \in \overline{\Omega}. \quad (1)$$

Moreover, we assume that no excavation happens on the boundary of  $\Omega$  and thus impose the Dirichlet condition

$$p(x) - p_0(x) = 0 \quad \text{for } x \in \partial\Omega. \quad (2)$$

In other words  $p - p_0$  must belong to the nonnegative orthant of the linear space  $C_0(\overline{\Omega}) \subset C(\overline{\Omega})$  of continuous functions on  $\Omega$  that satisfy homogeneous boundary conditions.

The admissible profiles are not only bounded from below by  $\underline{z} = \min\{p_0(x) | x \in \overline{\Omega}\}$  but also from above by a value  $\overline{z} > \underline{z}$  due to physical and operational conditions. Thus for any admissible profile we assume that

$$p(x) \in Z \equiv [\underline{z}, \overline{z}] \quad \text{for } x \in \overline{\Omega}. \quad (3)$$

Of course, with no loss of generality we may assume  $\underline{z} \geq 0$ . The general situation is sketched in Fig. 1 for the one dimensional case.

### 2.2 Geotechnical stability condition and compactness

In order to measure the local slope associated with a given profile  $p \in C(\overline{\Omega})$ , we define

$$L_p(x) \equiv \limsup_{\tilde{x} \rightarrow x \leftarrow \hat{x}} \frac{|p(\hat{x}) - p(\tilde{x})|}{\|\hat{x} - \tilde{x}\|} \quad \text{for } x \in \overline{\Omega}. \tag{4}$$

Here and throughout  $\|\cdot\|$  denotes the Euclidean norm. One can easily show that for each  $p \in C(\overline{\Omega})$  the corresponding function  $L_p : \overline{\Omega} \rightarrow [0, \infty]$  is upper semi-continuous. Where  $L_p(x)$  is finite it induces sharp local Lipschitz constant for  $p$  around  $x$  in the sense that for all  $\varepsilon > 0$ , there exists for all  $\varepsilon > 0$  a  $\delta > 0$  such that for any  $\tilde{x}, \hat{x}$  with  $\|\tilde{x} - x\| < \delta > \|\hat{x} - x\|$  we have

$$|p(\hat{x}) - p(\tilde{x})| \leq (L_p(x) + \varepsilon)\|\hat{x} - \tilde{x}\|,$$

and if  $L$  is any local Lipschitz constant for  $p$  in a neighborhood of  $x$  then  $L_p(x) \leq L$ .

The key assumption on the admissible profiles  $p$  is the pointwise stability condition

$$L_p(x) \leq \omega(x, p(x)) \quad \text{for } x \in \overline{\Omega}, \tag{5}$$

where  $\omega : \overline{\Omega} \times Z \rightarrow [0, \infty)$  is an upper bound on the limiting local Lipschitz constant of  $p$ . It prescribes the maximal stable local slope and may vary on  $\overline{\Omega} \times Z$  depending on the local geotechnical properties of the material. Rather than assuming continuity of the slope function  $\omega$  we allow for horizontal and vertical jumps, which might be caused by layers of different material. In particular we may have soft and hard material layers next to each other, so that  $\omega$  may jump upwards as one crosses into the harder material. Wherever  $\omega$  is discontinuous we may define it as its upper envelope over a neighborhood and thus we can assume without loss of generality the upper semi-continuity property:

$$\limsup_j \omega(x_j, z_j) \leq \omega(x, z) \tag{6}$$

for all convergent sequences  $(x_j, z_j) \rightarrow (x, z) \in \overline{\Omega} \times Z$ . This assumption immediately implies that  $\omega$  attains a maximum

$$\overline{\omega} \equiv \max_{(x,z) \in \overline{\Omega} \times Z} \omega(x, z). \tag{7}$$

To restrict the feasible profiles to the volume  $\overline{\Omega} \times Z$  we assume without significant loss of generality for notational simplicity that  $\overline{\omega}$  is less than  $0.5|Z|/\text{diam}(\overline{\Omega})$ . Apart from the supremum norm in  $C(\overline{\Omega})$  we will utilize the extended-real valued Lipschitz semi-norm

$$\|p\|_{Lip} = \sup_{x \in \overline{\Omega}} L_p(x) \in [0, \infty] \quad \text{for } p \in C(\overline{\Omega}).$$

For all  $p \in C(\overline{\Omega})$  satisfying (5) we have  $\|p\|_{Lip} \leq \overline{\omega}$  as defined in (7). The linear space of all  $p \in C(\overline{\Omega})$  for which  $\|p\|_{Lip}$  is indeed finite is denoted by  $Lip(\overline{\Omega})$ , which can be endowed with the norm  $\|p\|_{1,\infty} \equiv \|p\|_\infty + \|p\|_{Lip}$  to obtain the standard Banach space of all Lipschitz functions on  $\overline{\Omega}$ . On the subspace  $Lip_0(\overline{\Omega}) \equiv Lip(\overline{\Omega}) \cap C_0(\overline{\Omega})$  the quantity  $\|p\|_{Lip}$  defines a proper norm which is equivalent to  $\|p\|_{1,\infty}$ . In fact, it is not difficult to see that for any  $p \in Lip_0(\overline{\Omega})$  we have that  $\forall x \in \Omega, |p(x)| \leq \|p\|_{Lip} \text{dist}_\Omega(x, \partial\Omega)$ , where  $\text{dist}_\Omega(x, \partial\Omega)$  stands for the distance, relative to  $\Omega$ , from  $x$  to the boundary  $\partial\Omega$  of  $\Omega$ . As  $\Omega$  is bounded, we conclude that for a constant  $C < \infty$  depending only on  $\Omega$  we have that  $\|p\|_\infty \leq C\|p\|_{Lip}$ . In particular, the embedding  $Lip_0(\overline{\Omega}) \rightarrow C_0(\overline{\Omega})$  is completely continuous. It is well-known that the resulting Banach space  $Lip_0(\overline{\Omega})$  is equivalent to the Sobolev space  $W_0^{1,\infty}(\Omega)$  (see Evans 1998).

Furthermore, by virtue of Rademacher’s theorem, every  $p \in Lip(\overline{\Omega})$  is a.e. differentiable in  $\Omega$ . It follows that for every  $p$  satisfying (5) we have that

$$\|\nabla p(x)\| \leq \omega(x, p(x)) \quad \text{for a.e. } x \in \Omega.$$

Throughout, we assume that  $p_0$  satisfies (5). The set

$$\mathcal{P} := \{p \in C(\overline{\Omega}) \mid p \text{ satisfies (1), (2) and (5)}\}$$

is called set of admissible profiles. In fact the bounded set  $\mathcal{P}$  is contained in the affine subspace  $p_0 + Lip_0(\overline{\Omega})$ . Notice that the u.s.c. assumption (6) on  $\omega$  is necessary for the closedness of  $\mathcal{P}$  in  $C(\overline{\Omega})$ . Indeed let us consider the following simple example: Take  $\Omega = (-1, 1)$  and  $\alpha \in [0, 1]$ . Set  $\omega_\alpha(x, z) = 0$  if  $x < 0$ ,  $\alpha$  if  $x = 0$  and 1 if  $x > 0$ . This function is discontinuous at  $x = 0$  and is not u.s.c. if  $\alpha < 1$ . For each  $\varepsilon > 0$ , the profile  $p_\varepsilon(x) = \max(0, x - \varepsilon)$  satisfies (5) with  $\omega = \omega_\alpha$  for any  $\alpha \in [0, 1]$ ; nevertheless, its uniform limit  $p(x) = \max(x, 0)$  is admissible if and only if  $\alpha = 1$ . Furthermore, under (6) we have that  $\mathcal{P}$  is compact in  $C(\overline{\Omega})$  as we will show below.

**Proposition 1** *If  $\omega$  satisfies (6) then  $\mathcal{P}$  is compact and has empty interior in  $C(\overline{\Omega})$ .*

*Proof* As our feasible functions are fixed on  $\partial\Omega$  we know that  $\mathcal{P} - p_0 \subset C_0(\overline{\Omega})$ . The uniform Lipschitz continuity property ensures the equicontinuity of  $\mathcal{P}$ , while all functions in  $\mathcal{P}$  have values in the compact interval  $Z = [\underline{z}, \overline{z}]$ . Hence the asserted compactness of  $\mathcal{P}$  in  $C(\overline{\Omega})$  follows by the well known theorem of Arzela-Ascoli provided we can show that it is closed w.r.t.  $\|\cdot\|_\infty$ .

Let  $p \in p_0 + C_0(\overline{\Omega})$  be a function in the closure of  $\mathcal{P}$  w.r.t.  $\|\cdot\|_\infty$ . By virtue of (6), we have that for given  $x \in \Omega$  and  $\varepsilon > 0$  there exists  $\delta > 0$  such that for any  $\tilde{x} \in \Omega$  and  $q \in \mathcal{P}$  with  $\|\tilde{x} - x\| < \delta > \|q - p\|_\infty$  then

$$\omega(\tilde{x}, q(\tilde{x})) < \omega(x, p(x)) + \varepsilon/4. \tag{8}$$

From this it follows that for all  $\tilde{x} \neq \hat{x}$  in the ball  $B_\delta(x)$  we have

$$\frac{|q(\tilde{x}) - q(\hat{x})|}{\|\tilde{x} - \hat{x}\|} < \omega(x, p(x)) + \frac{\varepsilon}{2}. \tag{9}$$

which will be shown by contradiction. If equality (9) is violated by a couple of points  $\{\tilde{x}, \hat{x} = \tilde{x} + \Delta x\} \in B_\delta(x)$  with  $\Delta x \neq 0$ , then it would be also violated by at least one of the pairs  $\{\tilde{x}, \tilde{x} + \Delta x/2\}$  or  $\{\hat{x} - \Delta x/2, \hat{x}\}$  because of

$$\begin{aligned} \frac{|q(\tilde{x}) - q(\hat{x})|}{\|\tilde{x} - \hat{x}\|} &\leq \frac{1/2(|q(\tilde{x}) - q(\tilde{x} + \Delta x/2)| + |q(\hat{x}) - q(\hat{x} - \Delta x/2)|)}{1/2\|\tilde{x} - \hat{x}\|} \\ &\leq \frac{1/2(2 \max\{|q(\tilde{x}) - q(\tilde{x} + \Delta x/2)|, |q(\hat{x}) - q(\hat{x} - \Delta x/2)|\})}{1/2\|\tilde{x} - \hat{x}\|} \\ &= \frac{\max\{|q(\tilde{x}) - q(\tilde{x} + \Delta x/2)|, |q(\hat{x}) - q(\hat{x} - \Delta x/2)|\}}{1/2\|\tilde{x} - \hat{x}\|} \end{aligned}$$

and  $1/2\|\tilde{x} - \hat{x}\| = \|\tilde{x} - \tilde{x} + \Delta x/2\| = \|\tilde{x} - \tilde{x} + \Delta x/2\|$ . Recursively, we can continue the bisection process to generate a family of nested segments  $\{\tilde{x}_j, \hat{x}_j\}_{j \geq 0}$  of length  $\|\Delta x\|/2^j$ , all of them contained in  $B_\delta(x)$ , such that the corresponding pairs  $\tilde{x}_j, \hat{x}_j$  violates (9) for each  $j$ , and moreover one would have that there exists a limit  $x_* \in [\tilde{x}_j, \hat{x}_j] \subset B_\delta(x)$  such that  $\tilde{x}_j \rightarrow x_*$  and  $\hat{x}_j \rightarrow x_*$  as  $j \rightarrow \infty$ . By construction, using (8) and since  $q \in \mathcal{P}$ , we get:

$$\omega(x_*, q(x_*)) + \frac{\varepsilon}{4} < \omega(x, p(x)) + \frac{\varepsilon}{2} \leq \frac{|q(\tilde{x}_j) - q(\hat{x}_j)|}{\|\tilde{x}_j - \hat{x}_j\|} \leq \omega(x_*, q(x_*)) + R_j(\Delta x)$$

where the remainder  $R_j(\Delta x)$  is of the form  $o(\|\Delta x\|/2^j)/(\|\Delta x\|/2^j)$ . Letting  $j \rightarrow \infty$ , we obtain a contradiction in (8) as  $R_j(\Delta x)$  vanishes. Thus (9) is indeed implied by (8). Let us return to the function  $p$ . Given  $\varepsilon \in (0, 1]$ , let us consider the corresponding  $\delta > 0$  as in (8). For any  $\tilde{x} \neq \hat{x}$  with  $\|\tilde{x} - x\| < \delta$  and  $\|\hat{x} - x\| < \delta$  there exists a profile  $q \in \mathcal{P}$  such that  $\|p - q\|_\infty \leq \|\tilde{x} - \hat{x}\|\varepsilon/4$  and therefore

$$\begin{aligned} \frac{|p(\tilde{x}) - p(\hat{x})|}{\|\tilde{x} - \hat{x}\|} &\leq \frac{|q(\tilde{x}) - q(\hat{x})| + 2\|p - q\|_\infty}{\|\tilde{x} - \hat{x}\|} \leq \omega(x, p(x)) + 2\frac{\varepsilon}{2} \\ &\leq \omega(x, p(x)) + \varepsilon. \end{aligned}$$

Since  $\varepsilon$  may be chosen arbitrary small  $p$  must satisfy the stability condition (5) and is therefore contained in  $\mathcal{P}$ , which is thus closed as asserted. This completes the proof of the compactness of  $\mathcal{P}$ .

To see that  $\mathcal{P}$  has empty interior w.r.t. the  $\|\cdot\|_\infty$  norm we only have to consider a triangle wave function  $p_t$  around an admissible profile  $p \in \mathcal{P}$ . For any  $\varepsilon > 0$ , scaling the amplitude of  $p_t$  we can ensure  $\|p_t - p\|_\infty \leq \varepsilon$ . But letting the wavelength of  $p_t$  go down, the limiting local slope  $L_{p_t}$  increases and thus the stability condition (5) will be violated. Hence in every neighborhood of a feasible profile w.r.t.  $\|\cdot\|_\infty$  there are infeasible profiles. □

As an immediate consequence of Proposition 1 we get that any functional  $F : C(\overline{\Omega}) \rightarrow \mathbb{R}$  which is continuous w.r.t.  $\|\cdot\|_\infty$  attains on  $\mathcal{P}$  a minimum and a maximum. This applies in particular to the distances  $F(p) \equiv \|p - \tilde{p}\|_\infty$  for any fixed  $\tilde{p} \in L^\infty(\Omega) \supset C(\overline{\Omega}) \supset \mathcal{P}$ . Hence we have (non unique) least distance projections from  $L^\infty$  to  $\mathcal{P}$ .

### 2.3 Additional conditions on the slope constraint

One might want to impose two additional conditions on  $\omega(x, z)$  concerning its dependence on the vertical coordinate  $z$ . The first one is that the restrictions  $\omega_x(z) \equiv \omega(x, z)$  be right continuous, i.e.

$$\lim_{\tilde{z} \searrow z} \omega_x(\tilde{z}) = \omega_x(z) \quad \text{for } x \in \overline{\Omega} \quad (10)$$

The physical motivation for this property is the exclusion of certain pathological situations where one may have two regions with soft material lying below the hard one. If  $\omega$  is only u.s.c. with respect to all variables, a profile might be feasible but physically unstable because the maximal slope permitted in the hard material cannot be supported by the soft material below which only allows a milder slope. The assumption (10) of right-continuity w.r.t.  $z$  means that the slope constraint can not simply jump up from a soft layer below a hard one, but must build up gradually.

The second additional condition on  $\omega$  is definitely optional, namely we may require concavity of  $\omega(x, z)$  w.r.t.  $z \in Z$ . This rather strong condition, while clearly not very realistic in the general case, does allow for the possibility of hard material in the middle sandwiched in between soft material on top and below. Of course, one may also consider the case of a concave  $\omega$  which is monotonically increasing or decreasing w.r.t.  $z$  according to the geomechanics of the material.

**Lemma 1** *If  $\omega(x, z)$  is concave w.r.t.  $z$  then  $\mathcal{P}$  is convex.*

*Proof* Take two profiles  $p, q \in \mathcal{P}$  and  $0 < \alpha < 1$ . For any  $x \in \overline{\Omega}$ , we have by definition of  $L_p$  in (4) and the triangle inequality

$$\begin{aligned} L_{(1-\alpha)p+\alpha q}(x) &\leq (1-\alpha)L_p(x) + \alpha L_q(x) \\ &\leq (1-\alpha)\omega(x, p(x)) + \alpha\omega(x, q(x)) \leq \omega(x, (1-\alpha)p(x) + \alpha q(x)) \end{aligned}$$

where the last inequality is a consequence of the concavity property on  $\omega$ . Thus  $(1-\alpha)p + \alpha q \in \mathcal{P}$ .  $\square$

We end this section with a sufficient condition for an admissible profile to be in the interior of  $\mathcal{P}$  in  $Lip(\overline{\Omega})$ .

**Proposition 2** *If  $\omega$  is continuous on  $(\Omega \times Z)$  then any profile  $p \in \mathcal{P}$  for which*

$$\varepsilon \equiv \inf_{x \in \Omega} \{\omega(x, p(x)) - L_p(x)\} > 0 \quad (11)$$

*lies in the interior of  $\mathcal{P}$  in  $Lip(\overline{\Omega})$ .*

*Proof* We have to show that any  $p \in \mathcal{P}$  satisfying (11) has a neighborhood which only contains feasible profiles. Due to the continuity of  $\omega$  on  $\Omega$  there exists for all  $\varepsilon > 0$  a  $\delta > 0$  such that at all  $x \in \Omega$ , we have  $|\omega(x, z) - \omega(x, \tilde{z})| < \varepsilon/2$  if  $|z - \tilde{z}| < \delta$ . Pick any  $q$  with  $\|q - p\|_{Lip} < \varepsilon/8$  and  $\|q - p\|_\infty < \delta$ . The set of such  $q$  is an open



neighborhood of  $p$  in  $\mathcal{P}$  w.r.t. the Lipschitz norm  $\|\cdot\|_{Lip}$ . Any  $x \in \Omega$  is contained in some ball  $B_\delta$  such that on that ball  $p - q$  has a Lipschitz constant of size  $\varepsilon/4$  and  $p$  has a Lipschitz constant of size  $L_p(x) + \varepsilon/4$ . Now we see that for any two points  $\tilde{x}, \hat{x} \in B_\delta(x)$

$$\begin{aligned} |q(\tilde{x}) - q(\hat{x})|/\|\tilde{x} - \hat{x}\| &= |q(\tilde{x}) - p(\tilde{x}) - (q(\hat{x}) - p(\hat{x})) + p(\tilde{x}) - p(\hat{x})|/\|\tilde{x} - \hat{x}\| \\ &\leq |q(\tilde{x}) - p(\tilde{x}) - (q(\hat{x}) - p(\hat{x}))|/\|\tilde{x} - \hat{x}\| + L_p(x) + \varepsilon/4 \\ &\leq L_p(x) + \varepsilon/2 \leq \omega(x, p(x)) - \varepsilon/2 \leq \omega(x, q(x)) \end{aligned}$$

where the last estimation follows from the condition on the difference of  $p$  and  $q$  w.r.t. the supremum norm. Hence  $q$  satisfies the slope constraint in that certainly  $L_q(x) \leq \omega(x, q(x))$  and thus  $q \in \mathcal{P}$ .  $\square$

### 2.4 Structural properties of the admissible profiles set

The next result establishes closedness properties of  $\mathcal{P}$  under pointwise minima and maxima operations. These properties ensure the connectedness of  $\mathcal{P}$  as a subset of  $C(\overline{\Omega})$  even when  $\omega(x, z)$  is not assumed to be concave w.r.t.  $z$  so that  $\mathcal{P}$  can be non-convex.

Moreover we will consider the so called intermediate level profiles

$$q_\tau = \max\{p, \min\{q, \tau\}\} \quad \text{for } \tau \in \mathbb{R} \tag{12}$$

concerning two arbitrary, essentially bounded, ordered functions  $p \leq q \in L^\infty(\overline{\Omega})$ .

**Proposition 3** *Under (6) we have that:*

- (i)  $\mathcal{P}$  is closed with respect to pointwise minima and maxima in that for any subset  $\tilde{\mathcal{P}} \subset \mathcal{P}$  the functions  $\underline{p}(x)$  and  $\overline{p}(x)$  defined by

$$\underline{p}(x) \equiv \inf\{p(x) | p \in \tilde{\mathcal{P}}\} \quad \text{and} \quad \overline{p}(x) \equiv \sup\{p(x) | p \in \tilde{\mathcal{P}}\} \tag{13}$$

also belong to  $\mathcal{P}$ . Thus  $\mathcal{P}$  contains a unique maximal element  $\overline{p}_u \equiv \max_{p \in \mathcal{P}}\{p\}$ .

- (ii) If  $p, q \in \mathcal{P}$  are such that  $p \leq q$  then  $q_\tau \in \mathcal{P}$  for any “level”  $\tau$ . Moreover the path  $\tau \rightarrow q_\tau$  is continuous w.r.t.  $\|\cdot\|_\infty$ .
- (iii) Any two profiles  $p, q \in \mathcal{P}$  are connected via  $\min\{p, q\}$  and  $\max\{p, q\}$ .

*Proof* (i) We will apply induction and a classical diagonalization argument to generalize the claim from the binary case to subsets containing infinitely many profiles. First consider  $\mathit{mathscr{P}} = \{p, q\} \subset \mathcal{P}$ . At points  $x \in \Omega$  where  $r(x) \equiv \max\{p(x), q(x)\} = p(x) > q(x)$  the same is true for all  $\tilde{x} \approx x$  by continuity and we have thus  $L_r(x) = L_p(x) \leq \omega(x, p(x)) = \omega(x, r(x))$ . At points  $x$  where there is a tie  $r(x) = p(x) = q(x)$  we find that for any two

sequences  $y_j \rightarrow x$  and  $z_j \rightarrow x$

$$|r(y_j) - r(z_j)| = |\max\{p(y_j), q(y_j)\} - \max\{p(z_j), q(z_j)\}| \leq \max\{|p(y_j) - p(z_j)|, |q(y_j) - q(z_j)|\}$$

The last inequality follows from the inverse triangle inequality for the supremum norm on  $\mathbb{R}^2$ . After division by  $\|y_j - z_j\|$  and taking the limit  $y_j \rightarrow x \leftarrow z_j$  we find that

$$L_r(x) \leq \max\{L_p(x), L_q(x)\} \leq \max\{\omega(x, p(x)), \omega(x, q(x))\} = \omega(x, r(x)). \tag{14}$$

Thus  $r = \max\{p, q\} \in \mathcal{P}$ .

The same argument applies to  $r = \min\{p, q\} = -\max\{-p, -q\}$  for the slope stability condition. Obviously it follows by induction that maxima and minima of finitely many elements in  $\mathcal{P}$  also belong to  $\mathcal{P}$ .

Now suppose  $\tilde{\mathcal{P}}$  contains infinitely many elements. First of all we note that from  $\bar{p}(x) \equiv \sup\{p(x) \mid p \in \tilde{\mathcal{P}}\}$  it follows that  $\bar{p} \in L^\infty(\bar{\Omega})$ . Now we pick a dense subset  $\{x_j\}_{j \in \mathbb{N}}$  in  $\Omega$ . By induction on  $i$  we now choose sequences  $p_k^{(i)} \in \tilde{\mathcal{P}}$  such that

$$\lim_{k \rightarrow \infty} p_k^{(i)}(x_j) = \bar{p}(x_j) \equiv \sup_{p \in \tilde{\mathcal{P}}} \{p(x_j)\} \quad \text{for } j < i$$

Consider a subsequence  $(\tilde{p}_k) \in \tilde{\mathcal{P}}$  with  $\tilde{p}_k(x_i) \rightarrow \bar{p}(x_i)$  and set  $p_k^{(i+1)} \equiv \max\{p_k^{(i)}, \tilde{p}_k\}$  so that  $p_k^{(i+1)}(x_j) \rightarrow \bar{p}(x_j)$  for  $j \leq i$ . Now we take the diagonal sequence  $p_k^* = p_k^{(k)}$  and get

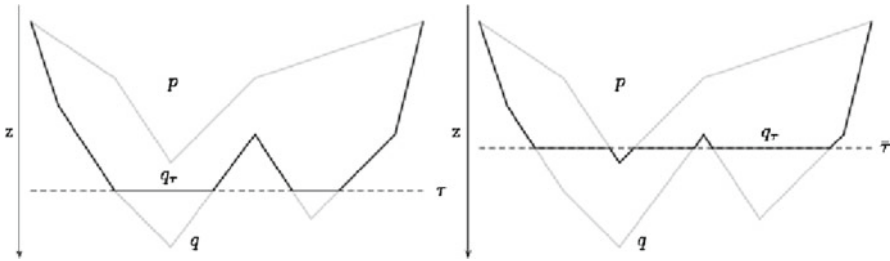
$$\lim_{k \rightarrow \infty} p_k^*(x_j) = \bar{p}(x_j) \quad \text{for } j \leq i \tag{15}$$

We know for  $j \leq k$

$$\bar{p}(x_j) \geq p_k^*(x_j) \geq \underbrace{p_k^{(i+1)}(x_j)}_{\rightarrow \bar{p}(x_j)}$$

and get a subsequence so that for all  $j \in \mathbb{N}$ ,  $\bar{p}(x_j) = \lim_{k \rightarrow \infty} p_k^*(x_j)$  by the convergence in (15). Moreover we can pick a Cauchy subsequence so that without loss of generality  $\|\tilde{p} - p_k^*\|_\infty \rightarrow 0$  for some  $\tilde{p} \in \mathcal{P}$ . Clearly we must have  $\tilde{p} = \bar{p}$  which proves (i).

- (ii) The assertion is again obvious where all three values  $p(x)$ ,  $q(x)$  and  $\tau$  are distinct. When there is a tie between two we may invoke the same argument as in



**Fig. 2** Illustration of level excavations for different levels between two given profiles

(i) and then extend it to a three way tie. Since at all  $x \in \bar{\Omega}$

$$\begin{aligned}
 |q_\tau - q_{\bar{\tau}}| &= |\max\{p, \min\{q, \tau\}\} - \max\{p, \min\{q, \bar{\tau}\}\}| \\
 &\leq |\min\{q, \tau\} - \min\{q, \bar{\tau}\}| \leq |\tau - \bar{\tau}|
 \end{aligned}$$

we have in fact Lipschitz continuity of  $q_\tau$  w.r.t.  $\tau \in \mathbb{R}$ . Consequently,  $\mathcal{P}$  is path connected in  $C(\bar{\Omega})$  since any two  $p, q \in \mathcal{P}$  can be transformed into each other via  $\min\{p, q\}$  or  $\max\{p, q\}$  with the help of a path generated by a varying level  $\tau$  in (12).

(iii) Follows from (i) and (ii). □

The path in (ii) is not continuous w.r.t.  $\|\cdot\|_{Lip}$ . Take for instance  $p \equiv 0$  and  $q(x) = \frac{1}{4} - (x - \frac{1}{2})^2$  on  $\Omega = (0, 1)$  and  $\omega$  sufficiently large, so that we have  $p, q \in \mathcal{P}$  and  $q_\tau(x) = \max\{0, \min\{q(x), \tau\}\}$  belongs to  $\mathcal{P}$ , but  $\|q_\tau - q_0\|_{Lip} = \|q_\tau\|_{Lip} = 1$  for all  $\tau > 0$ .

The profile modifications applying intermediate level profiles used in Proposition 3(ii) will be referred to as “level excavations”. They are depicted in Fig. 2 and make practical sense as material is taken away in horizontal layers. While that does not mean optimality when gains are discounted as discussed in Sect. 4 we note that any monotonic chain of feasible profiles  $p_0 < p_1 < \dots < p_m \in \mathcal{P}$  can be extended to a feasible path from  $p_0$  to  $p_m$  by level excavation between successive profiles  $p_j \leq p_{j+1}$ . The next result on level excavations shows that we can regain feasibility from any bounded  $q \geq p \in \mathcal{P}$ , so there is no danger of getting trapped away from the admissible set during an iterative optimization process.

**Proposition 4** *Under (6) we have that if  $\mathcal{P} \ni p \leq q \in L^\infty(\bar{\Omega})$  then  $q_\tau \in L^\infty(\bar{\Omega})$ , and the set  $V_\tau \equiv \{x \in \Omega \mid L_{q_\tau}(x) > \omega(x, q_\tau(x))\}$  is monotonically growing w.r.t.  $\tau$ .*

*Proof* The proof will be obtained from a case study for the bounds

$$\bar{q} \equiv \limsup_{\bar{x} \rightarrow x} q(\bar{x}) \geq \underline{q} \equiv \liminf_{\bar{x} \rightarrow x} q(\bar{x}) \geq p(x)$$

at any particular point  $x \in \bar{\Omega}$ .

If  $\bar{q} > \underline{q}$  we must have  $L_q(x) = \infty$  and also  $L_{q_\tau}(x) = \infty$  as long as  $\tau > \underline{q}$  which in turn means  $x \in V_\tau$  for  $\tau > \underline{q}$ . On the other hand it follows for  $\tau < \underline{q}$  that  $\min\{\tau, q(\tilde{x})\} = \tau$  for all  $\tilde{x}$  near  $x$  so that clearly  $q_\tau(x) = \max\{p(x), \tau\}$  and thus  $x \notin V_\tau$ . Thus we have monotonicity whether or not  $x \in V_\tau$  for  $\tau = \underline{q}$ .

Now suppose  $\bar{q} = \underline{q}$  which means that  $q(x)$  is continuous at  $x$ . If  $\tau > q(x)$  then for  $\tilde{x} \approx x$  we have that  $q_\tau(\tilde{x}) = q(\tilde{x})$  and if  $\tau < q(x)$  then for  $\tilde{x} \approx x$  we have that  $q_\tau(\tilde{x}) = \max\{p(\tilde{x}), \tau\}$ . Hence we have again  $x \notin V_\tau$  if  $\tau < q(x)$  and for  $\tau > q(x)$  we have  $x \in V_\tau \Leftrightarrow L_q(x) > \omega(x, q(x))$ . Now the only case left to consider is  $L_q(x) \leq \omega(x, q(x))$  where we have to exclude that  $x \in V_\tau$  for  $\tau = q(x)$ . However it follows exactly as in (14) that for  $\tau = q(x)$  we obtain  $L_r(x) \leq \max\{L_p(x), L_q(x)\} = L_q(x)$  so that we have monotonicity in all cases.  $\square$

### 2.5 Effort constraints and gain objective function

In addition to  $\omega$  the modeling of the open pit problem relies on two other given real valued functions, namely

$$e(x, z) \geq e_0 > 0 \quad \text{and} \quad g(x, z) \in \mathbb{R} \quad \text{for} \quad (x, z) \in \Omega \times Z. \tag{16}$$

which are only assumed to be uniformly bounded, i.e.

$$e, g \in L^\infty(\Omega \times Z) \tag{17}$$

Hence it is allowed that  $e$  and  $g$  have jumps due the existence of different types of material in the ore body. For any two given profiles  $q \geq p$  the integral

$$E([p, q]) \equiv \int_{\Omega} \int_{p(x)}^{q(x)} e(x, z) dz dx$$

represents the “effort” to excavate all the material between profile  $p$  and  $q$ , which is expected to be bounded by the capacity of the mine operation. On the other hand, the function

$$G([p, q]) \equiv \int_{\Omega} \int_{p(x)}^{q(x)} g(x, z) dz dx$$

represents the total value or “gain” of the material between  $p$  and  $q$  (without considering a discount rate). Notice that the function  $g(x, z)$  may take negative values. When  $p = p_0$  we abbreviate  $G(q) \equiv G([p_0, q])$  and  $E(q) \equiv E([p_0, q])$ . For an ordered triplet  $p \leq q \leq r$  with  $p, q, r \in \mathcal{P}$  we have additivity in the sense that

$$G([p, r]) = G([p, q]) + G([q, r]) \quad \text{and} \quad E([p, r]) = E([p, q]) + E([q, r]) \tag{18}$$

Now we can give the basic properties of  $G$  and  $E$  as follows.

**Proposition 5** *As a consequence of (17) we have that*

- (i)  $E(p)$  and  $G(p)$  are Lipschitz continuous on  $C(\overline{\Omega})$  with constants  $\|e\|_\infty|\Omega|$  and  $\|g\|_\infty|\Omega|$  respectively, where  $|\Omega|$  denotes the area of  $\Omega$ .
- (ii)  $E(p)$  and  $G(p)$  are Gâteaux differentiable at all  $p \in C(\overline{\Omega}) \setminus A$  where  $A$  is a meager set in the sense of Preiss and Zajíček (2001).
- (iii) If  $e$  (resp.  $g$ ) is continuous on  $\overline{\Omega} \times Z$  then  $E(p)$  (resp.  $G(p)$ ) is everywhere Fréchet differentiable. In particular, for any  $\Delta p \in C(\overline{\Omega})$  we have that

$$\nabla E(p) \cdot \Delta p = \int_{\Omega} e(x, p(x))\Delta p(x)dx. \tag{19}$$

- (iv)  $E$  is convex (resp.  $G$  is concave) if  $e$  is monotonically increasing (resp.  $g$  is monotonically decreasing) w.r.t.  $z$ .

*Proof* We consider throughout only  $E$  without making use of the positivity assumption on  $e$ . Thus the results apply analogously to  $-g(x, z)$ .

- (i) Considering two profiles  $p, \tilde{p} \in \mathcal{P}$  we get

$$|E(\tilde{p}) - E(p)| \leq \int_{\Omega} |\tilde{p}(x) - p(x)| \sup_{z \in Z} |e(x, z)| dz dx \leq |\Omega| \|e\|_\infty \|\tilde{p} - p\|_\infty$$

- (ii) By Preiss and Zajíček (2001, Theorem 12), a generalization of Rademacher’s theorem, every Lipschitz mapping on an open subset  $G$  of a separable Banach space  $X$  to a Banach space  $Y$  with the Radon–Nikodym property is Gâteaux differentiable at all points of  $G$  except those belonging to a meager set  $A$ . In our case  $X = C(\overline{\Omega})$  is separable and  $Y = \mathbb{R}$  has the Radon–Nikodym property, i.e. every Lipschitz map  $\mathbb{R} \rightarrow Y$  is differentiable almost everywhere.
- (iii) We claim that under continuity of  $e$  and  $g$  there is an explicit representation for the Fréchet derivative, which in the case of  $E$  is given by (19). To establish this we rewrite the difference between  $E$  and its proposed linearization as follows:

$$\begin{aligned} & \frac{1}{\|\Delta p\|_\infty} \left| \left( E(p + \Delta p) - E(p) - \int_{\Omega} e(x, p(x))\Delta p(x)dx \right) \right| \\ &= \frac{1}{\|\Delta p\|_\infty} \left| \int_{\Omega} \left\{ \left[ \int_{p(x)}^{p(x)+\Delta p(x)} e(x, z)dz \right] - e(x, p(x))\Delta p(x) \right\} dx \right| \\ &= \frac{1}{\|\Delta p\|_\infty} \left| \int_{\Omega} e(x, p(x) + \eta\Delta p(x))\Delta p(x) - e(x, p(x))\Delta p(x)dx \right| \end{aligned}$$

where  $\eta = \eta(x, p(x), \Delta p(x)) \in [0, 1]$  is obtained by the first mean value theorem for integration. Since  $e(x, z)$  is uniformly continuous on the compact set  $\bar{\Omega} \times Z$  there exist for any  $\varepsilon > 0$  a bound  $\delta > 0$  such that  $\|\Delta p\|_\infty < \delta$  implies  $|e(x, p + \eta\Delta p) - e(x, p)| \leq \varepsilon$  and thus the last expression on the right hand side is bounded above by  $\varepsilon|\Omega|$ , which completes the proof.

- (iv) The monotonicity assumption implies that the antiderivative  $\hat{e}(x, z) \equiv \int_{p_0(x)}^z e(x, \tilde{z})d\tilde{z}$  is convex w.r.t.  $z$ . Hence we find for two profiles  $p, \tilde{p} \in \mathcal{P}$

$$\begin{aligned} E((1 - \alpha)p + \alpha\tilde{p}) &= \int_{\Omega} \int_{p_0(x)}^{(1-\alpha)p(x)+\alpha\tilde{p}(x)} e(x, z)dzdx \\ &= \int_{\Omega} \hat{e}(x, (1 - \alpha)p(x) + \alpha\tilde{p}(x))dx \\ &\leq \int_{\Omega} (1 - \alpha)\hat{e}(x, p(x)) + \alpha\hat{e}(x, \tilde{p}(x))dx \\ &= \int_{\Omega} \left( (1 - \alpha) \int_{p_0(x)}^{p(x)} e(x, z)dz + \alpha \int_{p_0(x)}^{\tilde{p}(x)} e(x, z)dz \right) dx \\ &= (1 - \alpha)E(p) + \alpha E(\tilde{p}) \end{aligned}$$

For  $g$  decreasing we obtain  $G((1 - \alpha)p + \alpha\tilde{p}) \geq (1 - \alpha)G(p) + \alpha G(\tilde{p})$  analogously. □

Note that any of the results of the last proposition apply analogously to  $E([p, q])$  and  $G([p, q])$  when  $p \neq p_0$ . The assumption that the effort rate  $e$  is monotonically increasing w.r.t. the depth  $z$  is natural and realistic; but that the gain rate  $g$  be monotonically decreasing w.r.t.  $z$  would only apply to very particular deposits. Therefore, in general we cannot expect the gain function  $G$  to be concave. Without continuity w.r.t  $z$  no global Fréchet differentiability is attainable even in terms of  $\|\cdot\|_{Lip}$ .

### 3 Optimal stationary profiles

Using the properties of  $\mathcal{P}$ ,  $G$  and  $E$  we have derived in the previous section we can establish existence results for profiles that are optimal in various senses. The continuous formulation we propose for the Final Open Pit problem mentioned in the introduction is the following:

$$(FOP) \quad \max\{G(p) \mid p \in \mathcal{P}\}$$

Similarly, the continuous Capacitated FOP problem is:

$$(CFOP) \quad \max\{G(p) \mid p \in \mathcal{P}, E(p) \leq \bar{E}\}$$

The sets of optimal solutions (global maximizers) for these problems are denoted by  $S(FOP)$  and  $S(CFOP)$ , respectively. The following result establishes a property of the gain and effort functions which in particular is useful to investigate the structure of  $S(FOP)$ .

**Lemma 2** *For all admissible profiles  $p, q \in \mathcal{P}$ , whether ordered, optimal or not, we have that  $G(\max\{p, q\}) = G(p) + G(q) - G(\min\{p, q\})$  and  $E(\max\{p, q\}) = E(p) + E(q) - E(\min\{p, q\})$ .*

*Proof* We have that

$$G([p_0, p]) + G([p_0, q]) = G([p_0, \min\{p, q\}]) + G([p_0, \min\{p, q\}]) + \underbrace{G([\min\{p, q\}, p]) + G([\min\{p, q\}, q])}_{=G([\min\{p, q\}, \max\{p, q\}])}$$

Here everything is done by the decomposition formula for ordered triplet (18) and the fact, that from the minimum of two profiles both excavation and gain are obtained on disjoint areas when we go on excavating to one or the other except the sets where they are the same and thus equivalent to the minimum. □

**Proposition 6** *Under the conditions (6) and (17) we have*

- (i)  $S(FOP)$  is nonempty and contains unique minimal and maximal elements  $\underline{p}_g \leq \overline{p}_g$  so that  $p \in S(FOP) \Rightarrow \underline{p}_g \leq p \leq \overline{p}_g$ .
- (ii) For any bound  $\overline{E} > 0$  there exists at least one global optimizer of (CFOP).

*Proof* (i) The maximum  $G_*$  on  $\mathcal{P}$  of  $G$  is attained due to the continuity of  $G$  and the compactness of  $\mathcal{P}$  by Proposition 1. On the other hand, Lemma 2 implies that the maximum  $\overline{p}_g$  and minimum  $\underline{p}_g$  over all globally optimal profiles are also optimal. The final assertion follows directly from Proposition 3(i).  
 (ii) The existence follows again from the compactness of  $\mathcal{P}$  and the continuity of  $E$  and  $G$ . □

Proposition 3(i) yields that  $\mathcal{P}$  is a complete lattice as there is a maximal element  $\overline{p}$  and a minimal element  $p_0$ , which indeed proofs that it is a bounded lattice. By the existence of (13) for each subset as shown in Proposition 3(i) we have that all subsets have a joint and a meet. In particular the solution set of FOP is a sublattice of  $\mathcal{P}$ . On the other hand,  $S(FOP)$  need not to be connected nor convex unless  $g$  is decreasing w.r.t.  $z$ .

We can use the previous result to obtain a path of optimal profiles subject to excavation constraints. For each  $\lambda \geq 0$  the combined function

$$G_\lambda(p) \equiv G(p) - E(p)/\lambda$$

satisfies all the assumptions we made on  $G(p)$  so far. It is in fact concave if this is true for  $G(p)$  and  $E(p)$  is convex, which follows from  $e(x, z)$  and  $-g(x, z)$  being monotonically growing. Hence  $G_\lambda(p)$  has global minimizers on  $\mathcal{P}$  just like  $G$  and

we obtain a whole path. The next result, for which we provide a proof for the sake of completeness, follows from the general theory of parametric lattice programming; see [Topkis \(1978\)](#).

**Proposition 7** *Under (6) and (17) there exists a path of profiles*

$$p_\lambda \equiv \arg \min\{E(p) \mid p \in \arg \max\{G_\lambda(p) \mid p \in \mathcal{P}\}\} \quad (20)$$

so that  $0 \leq \lambda < \mu \Rightarrow p_\lambda \leq p_\mu$  and  $p_\lambda \in \arg \max\{G(p) \mid p \in \mathcal{P} \wedge E(p) \leq E(p_\lambda)\}$ .

*Proof* The existence of the  $p_\lambda$  follows from Proposition 5(i), namely  $G$  and  $E$  are Lipschitz. Let  $p_\lambda$  be defined as the unique minimal element among the global optimizers of  $G_\lambda$  existing by the fact that  $\mathcal{P}$  is a complete lattice. To prove the monotonicity consider  $q = \min\{p_\lambda, p_\mu\}$  for  $\lambda \leq \mu$ . By (18) we get for  $p_0 \leq q \leq p_\lambda$

$$\begin{aligned} G_\lambda(p_\lambda) &= G(p_\lambda) - E(p_\lambda)/\lambda \\ &= G(q) - E(q)/\lambda + G([q, p_\lambda]) - E([q, p_\lambda])/\lambda \\ &= G_\lambda(q) + G_\lambda([q, p_\lambda]). \end{aligned} \quad (21)$$

Hence, by optimality of  $p_\lambda$ , we have

$$\begin{aligned} 0 \leq G_\lambda([q, p_\lambda]) &= G([q, p_\lambda]) - E([q, p_\lambda])/\lambda \\ &\leq G([q, p_\lambda]) - E([q, p_\lambda])/\mu = G_\mu([q, p_\lambda]) \end{aligned}$$

and using again the disjointness of  $[q, p_\lambda]$  and  $[q, p_\mu]$  we find that

$$\begin{aligned} G(\max\{p_\lambda, p_\mu\}) - E(\max\{p_\lambda, p_\mu\})/\mu &= G_\mu(q) + G_\mu([q, p_\lambda]) + G_\mu([q, p_\mu]) \\ &= G_\mu(p_\mu) + G_\mu([q, p_\lambda]) \\ &\geq G_\mu(p_\mu). \end{aligned}$$

By optimality of  $p_\mu$  for  $G_\mu$  we obtain  $G_\mu([q, p_\lambda]) = 0 \Rightarrow G_\lambda([q, p_\lambda]) = 0$ . Hence Eq. (21) becomes  $G_\lambda(p_\lambda) = G_\lambda(q)$  and by minimality of  $p_\lambda$  we derive  $p_\lambda = q$ . Hence  $p_\lambda \leq p_\mu$ . The last assertion can be checked easily by contradiction.  $\square$

It should be noted that the path  $p_\lambda$  established in Proposition 7 is in general not continuous. That can only be expected in nice cases where  $G$  is strictly concave and  $E$  is strictly convex.

As  $\mathcal{P}$  is bounded the same is true for its image  $I \equiv (E(p), G(p))_{p \in \mathcal{P}} \subset \mathbb{R}^2$  in the configuration space.  $E(p)$  ranges between  $E(p_0) = 0$  and the maximal capacity  $E(p_u)$  that can be sensibly utilized. Here  $p_u$  is the ultimate profile defined in Proposition 3(i). The corresponding gain  $G(p_u)$  will typically be positive but might be zero like  $G(p_0)$  in exceptional cases. Every point  $(E(p), G(p))$  can be reached from the origin by the level excavation path according to Proposition 3(ii) and similar  $(E(p_\infty), G(p_\infty))$ , where  $p_\infty$  denotes an element of the solution set of (FOP), can be reached from it by another level excavation. The slopes of all the paths in the configuration space are bounded by the following result:



**Proposition 8** For any pair  $p \leq q \in \mathcal{P}$  with  $p \neq q$

$$\frac{|G(p) - G(q)|}{E(q) - E(p)} \leq \sigma \equiv \sup \left\{ \frac{|g(x, z)|}{e(x, z)} \mid (x, z) \in \bar{\Omega} \times Z \right\} \leq \frac{\|g\|_\infty}{e_0}$$

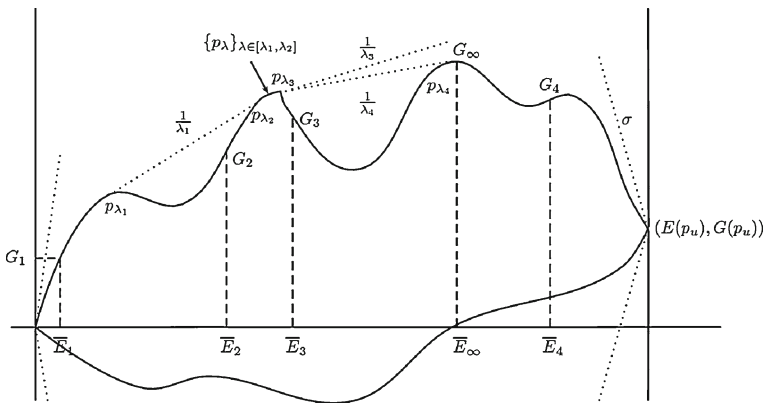
where  $e_0$  is the lower bound on the effort density function as introduced in (16).

*Proof* We have that

$$\begin{aligned} |G(q) - G(p)| &= \left| \int_{\Omega} \int_{p(x)}^{q(x)} g(x, z) dz dx \right| \leq \int_{\Omega} \int_{p(x)}^{q(x)} \left| \frac{g(x, z)}{e(x, z)} \right| e(x, z) dz dx \\ &\leq \sup \left\{ \frac{|g(x, z)|}{e(x, z)} \mid (x, z) \in \bar{\Omega} \times Z \right\} \int_{\Omega} \int_{p(x)}^{q(x)} e(x, z) dz dx \\ &= \sigma(E(q) - E(p)), \end{aligned}$$

which proves the estimate. □

Geometrically  $\sigma$  represents a Lipschitz constant on  $G(p(\tau))$  along any monotone path parametrized such that for  $\tau > \tilde{\tau}$  exactly  $E(p(\tau)) - E(p(\tilde{\tau})) = \tau - \tilde{\tau}$ . In particular  $\sigma$  bounds the slope of the boundary  $\partial I$ , wherever that can be defined at all. Figure 3 illustrates how the pairs  $(E, G)$  may be distributed in configuration space according to Propositions 6, 7 and 8. For each capacity bound  $\bar{E}$  there exists a global solution  $G_*$  of the equality constraint problem  $\max G(p)$  s.t.  $E(p) = \bar{E}$  which represents the upper boundary of the range of feasible configurations  $(E(p), G(p))$  for all  $p \in \mathcal{P}$  with  $E(p) = \bar{E}$ . The existence of these global optima is ascertained analogous to Proposition 6(ii). The vertical dashed lines depict five such pairs at the bounds labeled  $\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_\infty$ , and  $\bar{E}_4$ . The



**Fig. 3** Illustration of the behavior of the pairs  $(E(p), G(p))$  for all the admissible profiles  $p$

one at  $\bar{E}_\infty$  represents the global solution of (FOP), whose existence is ascertained by Proposition 6(i). That particular pair  $(E_\infty, G_\infty)$  and also  $(E_1, G_1)$  but not the other three belong also to the curve of solutions  $(E(p_\lambda), G(p_\lambda))$  with  $p_\lambda$  defined as the global minimizer of  $G_\lambda$ . Its existence and monotonicity is guaranteed by Proposition 7. The images  $(E(p_\lambda), G(p_\lambda))$  form the Northwestern border of the convex hull of the set of configurations I. Let  $1/\lambda_1 > 1/\lambda_4$  denote the slopes of the two convexifying dashed lines. Note that this numbering is independent of that of the previous five pairs. Consequently, for  $\lambda < \lambda_1$  the points  $p_\lambda$  are the unique global minimizers of  $G_\lambda(p)$  and vary continuously as  $\lambda \in (0; \lambda_1]$ . Here  $G_{\lambda_1}$  has at least two global maximizers  $p_{\lambda_1}$  and  $p_{\lambda_2}$  which are quite some distance apart and connected with the first convexifying dashed line. For  $\lambda \in (\lambda_2, \lambda_3]$  the optimal solutions  $p_\lambda$  move continuously along the boundary from  $p_{\lambda_2}$  to  $p_{\lambda_3}$  and afterwards for  $\lambda$  in  $[\lambda_3, \lambda_4]$  the profile  $p_\lambda$  stays constant until there is another jump to  $p_{\lambda_4}$ . Then there is another continuous variation as  $\lambda$  tends to infinity and hence the slope to zero, which is reached at  $p_\infty$ . Note that the original solutions  $(\bar{E}_2, G_2)$ ,  $(\bar{E}_3, G_4)$  and  $(\bar{E}_4, G_4)$  of the equality constraint problem are not reached by the path  $p_\lambda$ . More specifically the pair

$(\bar{E}_1, G_1)$  represents a global maximum of  $G$  subject to the constraint  $E(p) \leq \bar{E}_1$  which occurs also on the path  $p_\lambda$ . The pair  $(\bar{E}_2, G_2)$  represents also a sensible global maximum of (CFOP) but it can not be reached along the path  $p_\lambda$ . The pair  $(\bar{E}_3, G_3)$  represents at best a local maximum of  $G(p)$  s.t.  $E(p) \leq \bar{E}_3$  but not the global maximum. The pair  $(\bar{E}_4, G_4)$  may look like a global maximum but one can do better by simply going to the ultimate gain pit  $p_\infty$  which renders the effort constraint  $E(p) \leq \bar{E}_4$  inactive for this specific illustration.

#### 4 Dynamic trajectory planning

In the previous section we have established the existence of optimal gain profiles with and with excavation constraints. Rather than solving just this stationary problems one is really interested in an trajectory of profiles that gets to the valuable material as fast as possible. In other words we are interested in maximizing the present value based with a certain discount function for future earnings.

We consider paths  $P : [0, T] \mapsto \mathcal{P}$  that are monotonic, i.e.  $s, t \in [0, T]$ , with  $s \leq t$  imply for  $p = P(t)$  and  $q = P(s)$  that  $q(x) \leq p(x)$  for  $x \in \Omega$ . Naturally, the function  $E(P(t))$  must be also monotonically increasing. We assume that there exists an absolutely continuous function  $C : [0, T] \rightarrow \mathbb{R}^+$  with

$$C(t) \equiv \int_0^t c(\tau) d\tau$$

representing the mining capacity in the time interval  $[0, t]$ , with density  $c \in L^\infty(0, T)$  and  $c \geq 0$ . Finally, we impose the capacity condition on  $P$

$$\begin{aligned}
 E([P(s), P(t)]) &= E(P(t)) - E(P(s)) \leq C(t) - C(s) \\
 &= \int_s^t c(\tau) d\tau \text{ for } s \leq t.
 \end{aligned}
 \tag{22}$$

Now we introduce the set of feasible excavation paths:

$$\begin{aligned}
 \mathcal{U} &= \{ P \in C([0, T]; \mathcal{P}) \mid \\
 &\quad p_0 \leq P(s) \leq P(t), \ E([P(s), P(t)]) \leq C(t) - C(s) \text{ for } 0 \leq s \leq t \leq T \}.
 \end{aligned}$$

**Proposition 9** *Under assumption (22), for any  $P \in \mathcal{U}$  we have that*

$$\|P(t) - P(s)\|_\infty \leq \left[ \frac{\|c\|_\infty}{e_0\pi} + 2\bar{\omega} \right] (t - s)^{1/3}.
 \tag{23}$$

Hence the elements of  $\mathcal{U}$  are Hölder equicontinuous and  $\mathcal{U}$  is compact in  $C([0, T]; C(\bar{\Omega}))$  which is identified with  $C([0, T] \times \bar{\Omega})$  endowed with the uniform norm  $\|\cdot\|_\infty$ .

*Proof* For the proof of compactness we will apply again the Arzela-Ascoli theorem. The closedness of  $\mathcal{U}$  is direct from its definition together with the continuity of  $E$  on  $C([0, T]; \mathcal{P})$  endowed with the norm  $\|\cdot\|_\infty$ . In fact, as for all  $t \in [0, T]$  we have  $\{P(t)\}_{P \in \mathcal{P}} \subset \mathcal{P}$  with  $\mathcal{P}$  being compact by Proposition 2, it only remains to prove that  $\mathcal{U}$  is equicontinuous. To this end, fix  $x_0 \in \Omega$  and take  $s, t \in [0, T]$  with  $s < t$ . Let  $P \in \mathcal{U}$  be arbitrary. For any  $x \in \Omega$ , as  $P(s), P(t) \in \mathcal{P}$ , by global Lipschitz continuity of each profile we have that

$$P(t)(x) - P(s)(x) \geq -2\bar{\omega}\|x - x_0\| + P(t)(x_0) - P(s)(x_0).$$

Multiplying by the lower bound  $e_0 > 0$  on  $e(x, z)$  and integrating with respect to  $x$  on the ball  $|x - x_0| \leq \delta$  for some  $\delta > 0$  sufficiently small which we will choose later on, we get

$$\begin{aligned}
 &e_0\pi\delta^2 [-2\bar{\omega}\delta + P(t)(x_0) - P(s)(x_0)] \\
 &\leq \int_\Omega \int_{P(s)(x)}^{P(t)(x)} e(x, z) dz dx \\
 &= E(P(t)) - E(P(s)),
 \end{aligned}$$

where we have supposed that  $\Omega$  is an open subset of  $\mathbb{R}^n$  with  $n = 2$  (the case  $n = 1$  is similar) and that the ball  $B_\delta(x_0)$  is contained in  $\Omega$ . Therefore

$$e_0\pi\delta^2 [-2\bar{\omega}\delta + P(t)(x_0) - P(s)(x_0)] \leq \int_s^t c(\tau) d\tau \leq \|c\|_\infty(t - s),$$

Hence we deduce that for all  $\delta > 0$  small enough, we have

$$0 \leq P(t)(x_0) - P(s)(x_0) \leq \frac{\|c\|_\infty}{e_0\pi} \frac{t - s}{\delta^2} + 2L\delta.$$

If  $t - s > 0$  is small then we can take for instance  $\delta^3 = t - s$  to obtain

$$0 \leq P(t)(x_0) - P(s)(x_0) \leq \left[ \frac{\|c\|_\infty}{e_0\pi} + 2\bar{\omega} \right] (t - s)^{1/3},$$

and since  $x_0 \in \Omega$  is arbitrary, we get (23). This proves the equicontinuity of  $\mathcal{U}$ .  $\square$

Concerning the internal structure of  $\mathcal{U}$ , we firstly note that for arbitrary  $P, Q \in \mathcal{U}$ , neither  $\max\{P, Q\}$  nor  $\min\{P, Q\}$  need to be elements of  $\mathcal{U}$  for all  $t$ , as both may violate the dynamic capacity constraint (22). In fact, it is easy to construct simple examples where such a situation occurs. As a consequence, the lattice structure is lost in the dynamic case. Nevertheless,  $\mathcal{U}$  is path connected as a consequence of the following Lemma.

**Lemma 3** *Let  $P \in \mathcal{U}$  and  $p \in \mathcal{P}$  be given. Then  $\min\{p, P(t)\} \in \mathcal{U} \ni \max\{p, P(t)\}$  for all  $t \in [0, T]$ . Hence any  $P(t)$  is connected to the trivial path  $p_0$  via the path  $P_r(t) \equiv \min\{p_0 + r, P(t)\}$  for  $r \in [0, \bar{z} - \underline{z}]$ .*

*Proof* We have  $p_0 \leq \min\{p, P(s)\}$  and  $p_0 \leq \max\{p, P(t)\}$ . As the path  $P$  is monotonic, both estimations are true for all  $s, t \in [0, T]$  as well. Now consider the path  $\min\{p, P(t)\}$ . We choose  $s \leq t \in [0, T]$  arbitrary and define the following subsets of  $\Omega$ :

$\Omega_1 \equiv \{x \in \Omega | p(x) < P(s)(x) \leq P(t)(x)\}$ ,  $\Omega_2 \equiv \{x \in \Omega | P(s)(x) \leq p(x) \leq P(t)(x)\}$ ,  $\Omega_3 \equiv \{x \in \Omega | P(s)(x) \leq P(t)(x) < p(x)\}$ . Of course  $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ . For (22) we have

$$\begin{aligned} \int_{\Omega} \int_{\min\{p, P(s)\}(x)}^{\min\{p, P(t)\}(x)} e(x, z) dz dx &= \underbrace{\int_{\Omega_1} \int_{p(x)}^{P(s)(x)} e(x, z) dz dx}_{=0} + \underbrace{\int_{\Omega_2} \int_{P(s)(x)}^{p(x)} e(x, z) dz dx}_{\leq \int_{P(s)(x)}^{P(t)(x)} e(x, z) dz dx} + \int_{\Omega_3} \int_{P(s)(x)}^{P(t)(x)} e(x, z) dz dx \\ &\leq \int_{\Omega} \int_{P(s)(x)}^{P(t)(x)} e(x, z) dz dx \leq \int_s^t c(\tau) d\tau \end{aligned}$$

The last estimation holds because  $P$  is a feasible path. For the max operation the argument is analogous, so the proof is complete.  $\square$

Let  $\varphi \in C^1(0, T)$  be a monotonically decreasing discount function starting from  $\varphi(0) = 1$  and ending at  $\varphi(T) < 1$  for some fixed time period  $[0, T]$ . The standard

choice is  $\varphi(t) = e^{-\delta t}$  for some discount rate  $\delta > 0$ . For paths  $P(t, x) \equiv P(t)(x)$  that are smooth in time we define the present value of the gain as

$$\hat{G}(P) \equiv \int_0^T \varphi(t) \int_{\Omega} g(x, P(t, x)) \, dx \, dP(t) = \int_{\Omega} \int_0^T \varphi(t) g(x, P(t, x)) P'(t) \, dt \, dx$$

where the notation  $dP$  suggest that  $P(t, x)$  must be differentiable w.r.t.  $t$ . Integrating by parts we can avoid this requirement and obtain

$$\begin{aligned} \hat{G}(P) &= \left[ \varphi(t) \int_{\Omega} \int_{P(0,x)}^{P(t,x)} g(x, z) \, dz \, dx \right]_0^T + \int_{\Omega} \int_0^T [-\varphi'(t)] \int_{P(0,x)}^{P(t,x)} g(x, z) \, dz \, dt \, dx \\ &= \varphi(T) \left( \int_{\Omega} \hat{g}(x, P(T, x)) - \hat{g}(x, P(0, x)) \, dx \right) \\ &\quad + \int_{\Omega} \int_0^T [-\varphi'(t)] [\hat{g}(x, P(t, x)) - \hat{g}(x, P(0, x))] \, dt \, dx \end{aligned} \tag{24}$$

where  $\hat{g}(x, z) \equiv \int_{p_0(x)}^z g(x, \tau) \, d\tau$  is the antiderivative based on the initial profile which is already familiar from the proof of Proposition 5(iv). This form is feasible even if the Path  $P$  does not satisfy  $P(0, \cdot) = p_0(\cdot)$ . We note that for feasible excavation paths the first term in (24) is the value of the total excavated material of the path discounted by  $\varphi(T)$ . The last term is a correction due to the variation of  $\varphi$  (with  $-\varphi'(t) > 0$ ). This representation of  $\hat{G}(P)$  is well defined for every path in  $C([0, T] \times \Omega)$ .

The optimization problem in the dynamic trajectory planning case, the so called Capacitated Dynamic Open Pit Problem, is the following one

$$(CDOP) \quad \max\{\hat{G}(P) \mid P \in \mathcal{U}, P(0) = p_0\}$$

For the objective function  $\hat{G}$  we obtain the following results.

**Proposition 10** *For arbitrary paths  $P, Q \in \mathcal{U}$  we have*

- (i)  $|\hat{G}(P) - \hat{G}(Q)| \leq 2|\Omega| \|g\|_{\infty} \|P - Q\|_{\infty}$
- (ii) *If  $g(x, z)$  is decreasing w.r.t.  $z$  then  $\hat{G}$  is concave on the feasible set of (CDOP).*
- (iii) *For any  $p \in \mathcal{P}$  we have  $\hat{G}(P) = \hat{G}(\min\{P, p\}) + \hat{G}(\max\{P, p\})$  for any path of the feasible set of (CDOP).*

*Proof* (i) Since  $\|g\|_{\infty}$  is a Lipschitz constant for  $\hat{g}$ , with respect its second argument we can bound the difference in the first term of  $\hat{G}$  in (24) by  $2\varphi(T)|\Omega| \|g\|_{\infty} \|P - Q\|_{\infty}$  as  $\hat{g}(x, P(0, x))$  is not necessarily zero. Since  $-\varphi'(t) \geq 0$  by assumptions we can similarly bound

$$\begin{aligned}
& \left| \int_{\Omega} \int_0^T [-\varphi'(t)] \left( \int_{Q(t,x)}^{P(t,x)} g(x,z) dz - \int_{Q(0,x)}^{P(0,x)} g(x,z) dz \right) dt dx \right| \\
& \leq \int_{\Omega} \int_0^T [-\varphi'(t)] (|P(t,x) - Q(t,x)| \|g\|_{\infty} + |P(0,x) - Q(0,x)| \|g\|_{\infty}) dt dx \\
& \leq \int_{\Omega} \int_0^T [-\varphi'(t)] 2 \|P - Q\|_{\infty} \|g\|_{\infty} dt dx \\
& = \int_{\Omega} 2 \|g\|_{\infty} \|P - Q\|_{\infty} (\varphi(0) - \varphi(T)) dx
\end{aligned}$$

This is equal to  $(1 - \varphi(T))2|\Omega| \|g\|_{\infty} \|P - Q\|_{\infty}$ . Summing the terms cancels out  $\varphi(T)$  and the Lipschitz constant becomes completely independent of the discount function.

- (ii) This is analogous to the proof of Proposition 5(ii) by taking into account that for any feasible path of (CDOP)  $\hat{g}(x, P(0, x)) = 0$  holds. We omit the details.
- (iii) Let  $\Omega_p(t) \equiv \{x \in \Omega \mid p \leq P(t)\}$  be the subset of the domain where the path is pointwise greater than the given profile. We have

$$\begin{aligned}
\hat{G}(P) &= \varphi(T) \left( \int_{\Omega_p(T)} \left[ \hat{g}(x, p(x)) + \int_{p(x)}^{P(T,x)} g(x, \zeta) d\zeta \right] dx \right. \\
&\quad \left. + \int_{\overline{\Omega_p(T)}} \hat{g}(x, P(T, x)) dx \right) \\
&\quad + \int_0^T [\varphi'(t)] \left( \int_{\Omega_p(t)} \hat{g}(x, p(x)) dx + \int_{\Omega_p(t)} \int_{p(x)}^{P(t,x)} g(x, \zeta) d\zeta dx \right. \\
&\quad \left. + \int_{\overline{\Omega_p(t)}} \hat{g}(x, P(T, x)) dx \right) dt
\end{aligned}$$

This yields to

$$\hat{G}(P) = \varphi(T) \left( \int_{\Omega_p(T)} \hat{g}(x, p(x)) dx + \int_{\overline{\Omega_p(T)}} \hat{g}(x, P(T, x)) dx \right)$$

$$\begin{aligned}
 & + \int_0^T [\varphi'(t)] \left( \int_{\Omega_p(t)} \hat{g}(x, p(x)) dx + \int_{\overline{\Omega_p(t)}} \hat{g}(x, P(T, x)) dx \right) dt \\
 & + \varphi(T) \int_{\Omega_p(T)} \int_{p(x)}^{P(T,x)} g(x, \zeta) d\zeta dx + \int_0^T [\varphi'(t)] \int_{\Omega_p(t)} \int_{p(x)}^{P(t,x)} g(x, \zeta) d\zeta dx dt \\
 & = \hat{G}(\min\{P, p\}) + \hat{G}(\max\{P, p\})
 \end{aligned}$$

The first two lines of the sum indeed represent the discounted gain of the path  $P$  “stopped” at the profile  $p$ . □

By combining Propositions 9 and 10 we conclude immediately that  $\hat{G}$  attains global maximizer on  $\mathcal{U}$ . Without any additional assumption and for a limited time horizon  $T < \infty$  one can quite easily construct examples where the set of global minimizers is disconnected and may jump around violently when the gain density is slightly perturbed. One only has to think of two areas with high gain in separate places that have nearly identical values and excavation costs.

Due to the Lipschitz continuity of  $\hat{G}$  on the Banach space  $C([0, T] \times \overline{\Omega})$ , it has similar differentiability properties as those stated in Proposition 5 for  $G$ . However as it is not possible to guarantee that the optimizers are not elements of the meager set of exceptional points where one may not even have Gâteaux differentiability, optimality conditions cannot be formulated in terms of classical derivatives. For example the minimization of the maximum of finitely many distinct affine functions in one variable will almost certainly lead to a point where two of them tie. There zero is in the subdifferential, i.e. the convex hull of the two adjacent slopes but is not a proper derivative. Only where the maximum of the affine functions happens to be attained by a constant function can there be differentiability at a minimizer.

We conclude with a relation to the stationary problem (FOP). Recall that  $p_\infty$  is a solution of FOP accordingly with (20).

**Proposition 11** *Let  $\varphi \in C^1([0, T])$  be monotonically decreasing. Then*

- (i) *If  $\hat{G}(P) > 0$  for some  $P \in \mathcal{U}$  with  $P(0) = p_0$  then  $G_\infty \equiv G(p_\infty) > 0$ .*
- (ii) *If  $\hat{G}_*$  is the optimal value of (CDOP) then  $\hat{G}_* = \max\{\hat{G}(P) \mid P \in \mathcal{U}_\infty\}$ , where  $\mathcal{U}_\infty \equiv \mathcal{U} \cap C([0, T]; \mathcal{P}_\infty)$  for  $\mathcal{P}_\infty \equiv \{p \in \mathcal{P} \mid p \leq p_\infty\}$ .*

*Proof* (i) Suppose  $G_\infty = G(p_\infty) \leq 0$  and choose  $P \in \mathcal{U}$  arbitrary. Then we have for any  $t$  also  $G(P(t)) = \int_\Omega \hat{g}(x, P(t, x)) dx \leq 0$ . Otherwise we would have  $G(P(t)) > G_\infty$  which contradicts the definition of  $G_\infty$ . Substituting this relation into the integrated form (24) immediately yields  $\hat{G}(P) \leq 0$  which completes the proof of (i).

- (ii) Consider a  $P \in \mathcal{U}$  with  $\Omega_{p_\infty}(t) \neq \emptyset$  from a certain time  $t < T$ . By Proposition 10

(iii) we know we can decompose the objective in the following way.

$$\begin{aligned} \hat{G}(P) &= \hat{G}(\min\{P, p_\infty\}) + \varphi(T) \int_{\Omega_p(T)} \int_{p_\infty(x)}^{P(T,x)} g(x, \zeta) d\zeta dx \\ &+ \int_0^T [\hat{\varphi}'(t)] \int_{\Omega_p(t)} \int_{p_\infty(x)}^{P(t,x)} g(x, \zeta) d\zeta dx dt \end{aligned}$$

The last two terms, representing the gain the path yields between  $p_\infty$  and  $P(T)$ , are an integral over the part where the path is pointwise larger than  $p_\infty$ . Hence, because  $p_\infty$  is the solution of (FOP), they can not be positive and for the stopped path we have  $\hat{G}(\min\{P, p_\infty\}) \geq \hat{G}(P)$ . Thus any maximizer of  $\hat{G}$  is bounded by the solutions of (FOP).  $\square$

By virtue of Prop. 11(ii) for maximizing the discounted gain it suffices to consider feasible profiles such that  $p \leq p_\infty$ . This is a continuous analogue of a well known property of standard binary formulations for these type of problems, namely that the optimal Final Open Pit is an upper bound on the region to be considered of interest for the nested sequence of profiles, a property which is used to reduce the size of the original block model.

## 5 Concluding remarks

As we mentioned in the introduction, to the best of our knowledge, the first attempts to give rigorous mathematical formulations of open pit mine planning problems based on continuous functions date back to the works by Matheron in the 1970s. For instance, in [Matheron \(1975\)](#) is developed a general measure-theoretic approach to a parametrized version of the CFOP problem, obtaining the analogous to Propositions 3(i), 6(i) and 7 of this paper. No significant theoretical contribution seems to have been made. A complete comparison with Matheron's approach is out of the scope of this paper but will be addressed in a forthcoming work currently under preparation.

In our approach, the optimization problems are posed in functional spaces and a complete existence theory holds. In fact, under realistic assumptions, the feasible set of profiles in the stationary case as well as the set of paths of them in the dynamical case are compact sets in suitable Banach spaces of real valued functions, which implies the existence of solutions as the objective functions and constraints are proven to be (Lipschitz) continuous. In addition, we have provided structural properties of feasible sets and optimal solutions in the stationary case, and we have developed a parametric qualitative analysis of the behavior in the effort-gain configuration space. We have also obtained sufficient regularity conditions for differentiability of effort and gain functions, under the assumption that the effort and gain densities are continuous.

The optimization problems formulated in this paper are examples of variational problems in Banach spaces with a special structure. We have focused on the existence



theory, and there remain plenty of interesting open questions. The first one is how to obtain useful necessary optimality conditions under weak regularity assumptions. As the reference Banach space is non-reflexive, optimality conditions as those presented in [Bazaraa et al. \(1976\)](#) are not valid. Also we lack Slater-type conditions, which require the feasible set to have interior points. To avoid these requirements one has to generalize the concepts of convexity and interior points; see, for instance, [Borwein and Goebel \(2003\)](#), [Borwein and Lewis \(1992\)](#) while a short introduction can be found in [Daniele et al. \(1994\)](#).

The approximate numerical solution of the continuous problems is beyond the scope of this paper. Instead of trying to resolve optimality conditions on function space once they are formulated, a natural idea is to consider a so-called direct method, i.e. to discretize first. For example one may approximate the admissible profiles by piecewise linear-affine functions or more general polynomials, thus obtaining finite dimensional NLP problems. One complication is that due to the presumed discontinuity of the data one cannot restrict the approximations to the solution to be continuous either. On the other hand, if the original infinite dimensional problem is convex, a suitable discretization scheme may preserve this property so that the resulting finite dimensional optimization problem may be solved by algorithms for convex NLP (see [Boyd and Vandenberghe 2004](#)). Moreover, one might exploit some of the established structural properties for algorithmic purposes. Indeed, the parametric characterization of the stationary optimal profiles given in (20) leads naturally to a finite dimensional Bilevel Programming problem. Finally, if one were able to design a discretization method which preserves the lattice structure of the stationary admissible set, one might adapt to this setting some efficient algorithms coming from discrete optimization (see [McCormick 2006](#)). We plan to investigate these possibilities in future research.

Finally, we would like to make clear that the continuous approach proposed here is not intended to substitute the more traditional discrete techniques, but to supplement them with additional tools coming from continuous optimization in functional spaces. On one hand, as the underlying discrete block models become larger in terms of the number of unitary extraction blocks, the continuous formulations may be viewed as a sort of limiting averaged model, which should provide qualitative and quantitative information about the behavior of optimal solutions from a macroscopic point of view. In particular, continuous optimization is potentially well suited for investigating the sensibility w.r.t. discount rates or extraction effort capacities, which are very interesting issues for future research. On the other hand, the approximate resolution of a macroscopic continuous framework might be useful to obtain insight on how to construct good starting points for discrete formulations based on namely microscopic block models.

The connections between the continuous and discrete approaches, in terms of theoretical and algorithmic aspects, and the question on how we might use them from a practical point of view, are clearly key points that will be addressed in our future research.

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## References

- Bazaraa M, Shetty C, Goode J, Nashed M (1976) Nonlinear programming without differentiability in Banach spaces: necessary and sufficient constraint qualification. *Appl Anal* 5:165–173
- Boland N, Fricke C, Froyland G (2006) A strengthened formulation for the open pit mine production scheduled problem. Preprint, University of Melbourne, Parkville, VIC 3010, Australia
- Borwein J, Goebel R (2003) Notions of relative interior in Banach spaces. *J Math Sci* 115:2542–2553
- Borwein J, Lewis A (1992) Partially finite convex programming, Part I: quasi relative interiors and duality theory. *Math Progr* 57:15–48
- Boyd S, Vandenberghe L (2004) *Convex optimization*. Cambridge University Press, Cambridge
- Cacetta L, Hill SP (2003) An application of branch and cut to open pit mine scheduling. *J Glob Optim* 27:349–365
- Cacetta L (2007) Application of optimisation techniques in open-pit mining. In: Weintraub A et al (ed) *Handbook of operations research in natural resources*. Springer, New York
- Daniele P, Giuffrè S, Idone G, Maugeri A (1994) Infinite dimensional duality and applications. *Math Ann* 339:221–239 (2007)
- Denby B, Schofield D, Open pit design and scheduling by use of genetic algorithms. *Trans Inst Min Met (Sec. A: Min. Industry)* 103:A2–A26
- Evans L (1998) *Partial differential equations*. Graduate Studies in Mathematics, vol 19. American Mathematical Society, Providence
- Ferland J, Amaya J, Djuimo MS (2007) Particle swarm procedure for the capacitated open pit mining problem. In: *Autonomous Robot and Agents*. Studies in Computational Intelligence, Book Series. Springer, Berlin
- Guzmán JI (2008) Ultimate Pit Limit determination: a new formulation for an old (and poorly specified) problem. Communication in the Workshop on Operations Research in Mining, December 10–12, Viña del Mar, Chile
- Hochbaum D, Chen A (2000) Performance analysis and best implementation of old and new algorithms for open-pit mining problem. *Oper Res* 48:894–914
- Hustrulid W, Kuchta M (2006) *Open pit mine planning and design, vol 1: Fundamentals*. Taylor & Francis, London
- Johnson TB, Sharp WR (1971) A Three-dimensional dynamic programming method for optimal ultimate open pit design. Report of Investigation 7553, U.S. Bureau of Mines
- Lerchs H, Grossman IF (1965) Optimum design of open pit mines. *Trans CIM* 58:47–54
- Matheron G (1975) Parametrag de contours optimaux, Note Geostat. No. 128. Centre de Geostatistique, Fontainebleau
- McCormick ST (2006) Submodular function minimization, Chap. 7. In: K Aardal, G Nemhauser, R Weismantel (eds) *The handbook on discrete optimization*. Elsevier, Amsterdam pp 321–391
- Morales N (2002) *Modelos Matemáticos para planificación minera*. Engineering Thesis, Universidad de Chile, Santiago, Chile
- Preiss D, Zajíček L (2001) Directional derivatives of Lipschitz functions. *Isr J Math* 125:1–27
- Topkis DM (1978) Minimizing a submodular function on a lattice. *Oper Res* 26:305–321
- Wilke FL, Wright EA (1984) Determining the optimal ultimate pit design for hard rock open pit mines using dynamic programming. *Erzmet* 37:139–144