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ON THE p-MEDIAN POLYTOPE AND THE INTERSECTION PROPERTY: POLYHEDRA AND ALGORITHMS*

MOURAD BAÏOU†, FRANCISCO BARAHONA‡, AND JOSE CORREA§

Abstract. We study a prize-collecting version of the uncapacitated facility location problem and of the p-median problem. We say that the uncapacitated facility location polytope has the intersection property if adding the extra equation that fixes the number of opened facilities does not create any fractional extreme point. We characterize the graphs for which this polytope has the intersection property and give a complete description of the polytope for this class of graphs. This characterization yields a polynomial time cutting plane algorithm for these graphs. We also give a combinatorial polynomial time algorithm to solve the different variants of the p-median and facility location problems studied in this paper.

Key words. uncapacitated facility location, p-median

AMS subject classifications. 90C27, 90C57

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1. Introduction.

The *uncapacitated facility location problem* (UFLP) and the *p-median problem* (pMP) are among the most studied problems in combinatorial optimization. Here we deal with a *prize-collecting* version of them that we denote by UFLP′ and pMP′, respectively. We assume that $G = (U \cup V, A)$ is a bipartite directed graph, not necessarily connected and with no isolated nodes. The arcs are directed from U to V. The nodes in U are called *customers*, and the nodes in V are called *locations*. Each location v has a weight $f(v)$ that corresponds to the revenue obtained by opening a facility at that location, minus the cost of building this facility. Each arc (u, v) has a weight $c(u, v)$ that represents the revenue obtained by assigning the customer u to the opened facility at location v, minus the cost originated by this assignment. The difference between the UFLP and the UFLP′ is that in the first problem each customer *must* be assigned to an opened facility, whereas in the second problem a customer could be not assigned to any facility. If the number of opened facilities is required to be exactly p, we have the pMP and pMP′, respectively.

An integer programming formulation of the UFLP′ is

$$\max \sum_{(u,v)\in A} c(u,v)x(u,v) + \sum_{v\in V} f(v)y(v) \tag{1}$$

$$\sum_{v:(u,v)\in A} x(u,v) \le 1 \quad \forall u \in U, \tag{2}$$

$$x(u,v) \le y(v) \quad \forall (u,v) \in A, \tag{3}$$

$$y(v) \le 1 \quad \forall v \in V, \tag{4}$$

$$x(u,v) \ge 0 \quad \forall (u,v) \in A, \tag{5}$$

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†CNRS and Laboratory LIMOS, University Clermont II, France (baiou@isima.fr).
‡IBM T. J. Watson Research Center, Yorktown Heights, NY 10589 (barahon@us.ibm.com).
§Department of Industrial Engineering, University of Chile, Chile (jcorrea@dii.uchile.cl).

$$y(v) \in \{0,1\} \quad \forall v \in V, \tag{6}$$
$$x(u,v) \in \{0,1\} \quad \forall (u,v) \in A. \tag{7}$$

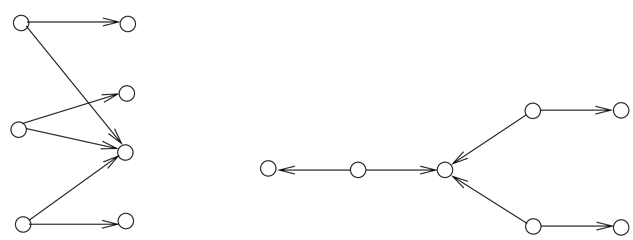
If inequalities (2) are set to equations, then we have a formulation of the UFLP. If we add the equation

$$\sum_{v \in V} y(v) = p \tag{8}$$

to (1)–(7), we have a formulation of the pMP′, and if inequalities (2) are set to equations, we have the pMP.

For a given bipartite graph $G = (U \cup V, A)$, let $UFLP'(G)$ be the convex hull of the solutions of (2)–(7) and $pMP'(G)$ be the convex hull of the solutions of (2)–(8). Analogously we can define the polytopes $UFLP(G)$ and $pMP(G)$. Notice that $UFLP(G)$ is a face of $UFLP'(G)$, and $pMP(G)$ is a face of $pMP'(G)$. Thus a characterization of $pMP'(G)$ and $UFLP'(G)$ yields to a characterization of $pMP(G)$ and $UFLP(G)$. We denote by $P(G)$ the linear relaxation of $UFLP'(G)$ defined by (2)–(5) and by $P_p(G)$ the linear relaxation of $pMP'(G)$ defined by (2)–(5) and (8).

Let F be the graph with node set $\{u_1, u_2, u_3, v_1, v_2, v_3, v_4\}$ and edge set $\{(u_1, v_1), (u_1, v_3), (u_2, v_2), (u_2, v_3), (u_3, v_3), (u_3, v_4)\}$; see Figure 1(a). By convenience the graph F is called a *fork*; this name is due to its representation in Figure 1(b).



(a) (b)

FIG. 1. *The graph F: A fork.*

By setting each variable associated with F to $\frac{1}{2}$, we obtain a fractional extreme point of $P_2(F)$. In general assume that a bipartite graph G contains a fork F. We can set to $\frac{1}{2}$ all variables associated with F and set to zero the remaining variables. This is an extreme point of $P_2(G)$. Such fractional extreme points may be cut off by using a set of valid inequalities for $pMP'(G)$ introduced in [15]. We are going to see that if the graph does not contain a fork, then the $UFLP'(G)$ and $pMP'(G)$ are easy to describe.

In this paper we will consider a set of valid inequalities for $UFLP'(G)$ introduced in [10]. We call them *Cho–Johnson–Padburg–Rao (CJPR)-inequalities*, using the initials of the authors' last names. These inequalities are also valid for $pMP'(G)$ since $pMP'(G) \subseteq UFLP'(G)$. We will show that the addition of these inequalities to $P(G)$ yields an integral polytope when G does not contain a fork.

We say that $UFLP'(G)$ has the *intersection property* with respect to (8) if the intersection of $UFLP'(G)$ with the hyperplane defined by (8) is an integral polytope for every nonnegative integer p. We show that $UFLP'(G)$ has this property if and only if G contains no fork. Based on this we show that the addition of the $CJPR$-inequalities to the system defining $P_p(G)$ gives an integral polytope for every nonnegative integer p if and only if G does not contain a fork. This is the main result of this paper. We also give combinatorial polynomial time algorithms to solve the problems pMP′, UFLP′, pMP, and UFLP when the underlying graph does not contain a fork.

A subclass of graphs with no fork consists of the graphs for which each location has degree at most two. Here we also prove that the UFLP′ is NP-hard if the degree of each location is at most three.

The facets of the uncapacitated facility location polytope have been studied in [18], [14], [10], [11], [8]. In [4] we characterized the graphs for which the natural linear relaxation defines $UFLP(G)$. The UFLP has also been studied from the point of view of approximation algorithms in [25], [12], [26], [6], [27], and others. Other references on this problem are [13] and [20]. The relationship between location polytopes and the stable set polytope has been studied in [14], [10], [11], [16], and others. The facets of $pMP(G)$ have been studied in [1] and [15]. In [2] and [3] the graphs for which the natural linear relaxation is enough to define $pMP(G)$ have been characterized.

This paper is organized as follows. In section 2, we give some notations and definitions and some preliminary results that will be useful all along the paper. Section 3 gives a complete characterization of $UFLP'(G)$ if G has no fork. In section 4, we discuss the intersection of the polytope $UFLP'(G)$ with the hyperplane defined by (8); we also establish $pMP'(G)$ for this class of graphs. Section 5 is devoted to the combinatorial algorithms for these problems.

2. Preliminaries.

2.1. Some definitions and notations. Let $G = (U \cup V, A)$ be a bipartite graph. Denote by $\beta(G)$ the *covering number* of G; that is, the minimum number of locations $v \in V$ needed to cover all customers $u \in U$. Let $F \subseteq A$ be a subset of arcs in A. Denote by $N^-(F)$ (resp., $N^+(F)$) the set of nodes in U (resp., V) incident to an arc in F. Let $G(F) = (N^-(F) \cup N^+(F), F)$ be the bipartite subgraph of G *spanned* by F. Hence $\beta(G(F))$ is the minimum number of nodes in $N^+(F)$ necessary to cover all the nodes in $N^-(F)$ using only arcs in F.

For $S \subseteq U$ and $W \subseteq V$, let $A(S, W)$ denote the set of arcs of A having one endpoint in S and the other in W. Let $\Gamma^+(S)$ (resp., $\Gamma^-(W)$) denote the set of nodes $v \in V$ (resp., $u \in U$) such that there is an arc $(u, v) \in A$ with $u \in S$ (resp., $v \in W$). We denote by $\delta^+(S)$ the set of arcs $(u, v) \in A$ with $u \in S$ and by $\delta^-(W)$ the set of arcs $(u, v) \in A$ with $v \in W$. For a node $u \in U$ (resp., $v \in V$), we write $\delta^+(u)$ (resp., $\delta^-(v)$) instead of $\delta^+(\{u\})$ (resp., $\delta^-(\{v\})$). Usually $d(v)$ denotes the *degree* of a node v in a simple graph, that is, the number of edges incident to v. We keep this notation in our case; that is, $d(u) = |\delta^+(u)|$ for $u \in U$ and $d(v) = |\delta^-(v)|$ for $v \in V$. If there is a risk of confusion, we specify by $d_G(v)$ the degree of the node v with respect to a given graph G. If $A' \subseteq A$ and V' is the set of nodes incident to the arcs of A', we say that $G' = (V', A')$ is the subgraph *spanned* by A'.

If $G = (V, E)$ is an undirected graph, a node set $S \subseteq V$ is called a *stable set* if there is no edge between any pair of nodes in S. A set $K \subseteq V$ is called a *clique* if there is an edge between every pair of nodes in K. We denote by $K_{n,m}$ a graph with node set $\{u_1, \ldots, u_n\} \cup \{v_1, \ldots, v_m\}$ and edge set $\{u_i v_j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$; $K_{n,m}$ is a *complete bipartite graph.* A graph is called *twoconnected* if at least two nodes should be removed to disconnect it. If a and b are two nodes whose removal disconnects the graph, we say that a and b form a *twonode cutset.* If $S_1 \subseteq V$, $S_2 \subseteq V$, and $S_1 \cap S_2 = \emptyset$, we denote by $\delta(S_1, S_2)$ the set of edges with one endnode in S_1 and the other in S_2. We use $\delta(S)$ to denote $\delta(S, V \setminus S)$. For $v \in V$ we write $\delta(v)$ instead of $\delta(\{v\})$.

If $S \subseteq V$, we denote by $E(S)$ the set of edges with both endnodes in S. The graph $H = (S, E(S))$ is the subgraph *induced* by S. If C is a cycle, a *chord* is an edge not in C whose endnodes are in C. An *odd hole* of G is an odd cycle H with no chord.

The set of solutions of a finite system of linear inequalities is called a *polyhedron.* A *polytope* is a bounded polyhedron. An inequality $ax \leq \alpha$ is *valid* for the polytope P if $P \subseteq \{x : ax \leq \alpha\}$. If $ax \leq \alpha$ is a valid inequality for P, then the set $F = \{x \in P : ax = \alpha\}$ is called a *face* of P. The dimension of a polytope P, denoted by $\dim(P)$, is the maximum number of affinely independent points in P minus 1. A polytope in $\mathbb{R}^n$ is *full dimensional* if it is of dimension n. A face of dimension $\dim(P) - 1$ is called a *facet.* A facet of a full-dimensional polytope is defined by a unique linear inequality (up to multiplication by a positive scalar). If P is a full-dimensional polyhedron, then there is a unique (up to multiplication by a positive scalar) nonredundant inequality system $Ax \leq b$ such that $P = \{x : Ax \leq b\}$; moreover there is a natural bijection among the facets of P and the inequalities of that system. An extreme point of P is a face of dimension 0. A polytope is *integral* when each of its extreme points has only integer components. An empty polytope is also integral.

2.2. CPJR-inequalities.

THEOREM 1 (see [10]). *Let G be a bipartite directed graph for any subgraph $G(F)$ of G the inequality*

$$\sum_{(u,v)\in F} x(u,v) - \sum_{v \in N^+(F)} y(v) \leq |N^-(F)| - k \tag{9}$$

is valid for $UFLP'(G)$ if and only if $k \leq \beta(G(F))$.

Let $G(F)$ be a subgraph of G spanned by $F \subseteq A$, where each node in $N^+(F)$ has degree two. In this case $\beta(G(F)) \geq \left\lceil \frac{|N^-(F)|}{2} \right\rceil$. It follows that the inequalities

$$\sum_{(u,v)\in F} x(u,v) - \sum_{v \in N^+(F)} y(v) \leq \left\lfloor \frac{|N^-(F)|}{2} \right\rfloor \tag{10}$$

for all $F \subseteq A$, where $d_{G(F)}(v) = 2$ for all $v \in N^+(F)$, $|N^-(F)| \geq 3$, and odd, are of type (9). Thus inequalities (10) are valid for $UFLP'(G)$. We call them *CJPR-inequalities.* These inequalities are $\{0, \frac{1}{2}\}$–Chvátal–Gomory cuts; cf. [9]. They can be obtained by adding some of the inequalities (2)–(5) multiplied by $1/2$ and by rounding down the right-hand side. An *odd cycle* in a bipartite graph is a cycle having $2(2k+1)$ nodes for some integer $k \geq 1$. When the subgraph $G(F)$ of G is an odd cycle, then inequalities (10) are known as the *odd cycle inequalities*, and their separation can be done in polynomial time; see [4] and [9].

2.3. A reduction to the stable set problem. Let $H = (V, E)$ be a simple undirected graph where each node v has a weight $w(v)$. The *stable set problem* (SSP) consists of finding a stable set S that maximizes $\sum_{v \in S} w(v)$. The stable set polytope, denoted by $SSP(H)$, is the convex hull of incidence vectors of stable sets of H. When considering a polyhedral study for the UFLP, a transformation to the stable set problem is often used. This permits us to derive results for the UFLP by applying known results for the SSP; for instance see [10] and [14]. This transformation is as follows. The variables $\bar{y}(v)$ are exchanged with $1 - y(v)$. Then the integer programming formulation (1)–(7) with respect to a bipartite graph $G = (U \cup V, A)$ becomes the following set packing problem:

$$\max \sum_{(u,v)\in A} c(u,v)x(u,v) - \sum_{v\in V} f(v)\bar{y}(v) + \sum_{v\in V} f(v), \tag{11}$$

$$\sum_{v:(u,v)\in A} x(u,v) \le 1 \quad \forall u \in U, \tag{12}$$

$$x(u,v) + \bar{y}(v) \le 1 \quad \forall (u,v) \in A, \tag{13}$$

$$\bar{y}(v), x(u,v) \in \{0,1\} \quad \forall v \in V \text{ and } \forall (u,v) \in A. \tag{14}$$

Let B be the matrix whose elements are the coefficients of the constraints (12)–(13). The matrix B is an $|U|+|A| \times |U|+|A|$ matrix with 0–1 elements. We call the columns of B $x(u,v)$ for all $(u,v) \in A$ and $\bar{y}(v)$ for all $v \in V$. The *intersection graph* of G denoted by $I(G)$ is constructed by assigning a node to each column $x(u,v)$ and $\bar{y}(v)$. Two nodes are adjacent if their both corresponding columns appear with coefficient 1 in some row of B. Thus the nodes corresponding to the variables that appear in a constraint (12) form a clique, and each node corresponding to $\bar{y}(v)$ is adjacent to a node corresponding to $x(u,v)$ with $(u,v) \in A$; see Figure 2. Problem (11)–(14) with respect to G is equivalent to the SSP with respect to $I(G)$. It follows that the stable set polytope with respect to $I(G)$ may be defined as the convex hull of the solutions of (11)–(14).

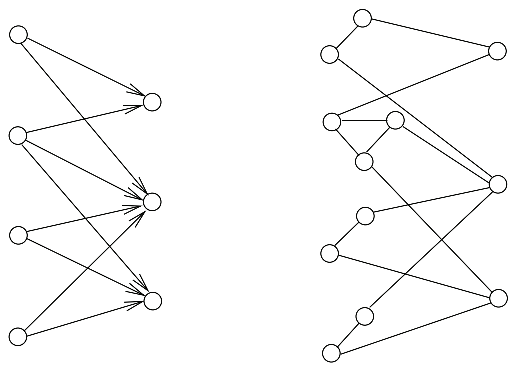


FIG. 2. *The graph G (on the left) with its intersection graph $I(G)$ (on the right).*

2.4. Some properties of the stable set polytope. Let $G = (V,E)$ be an undirected graph. The polytope $SSP(G)$ is full dimensional. The simplest facet defining inequalities of $SSP(G)$ are $x(u) \ge 0$ for all $u \in V$.

THEOREM 2 (see [21]). *If $K \subseteq V$ is a maximal clique, then*

$$\sum_{u\in K} x(u) \le 1$$

defines a facet.

THEOREM 3 (see [12]). *Let $G = (V,E)$ be a graph such that $V = V_1 \cup V_2$, $W = V_1 \cap V_2 \ne \emptyset$, where $(W, E(W))$ is a clique and $E = E(V_1) \cup E(V_2)$. Let $G_1 = (V_1, E(V_1))$, $G_2 = (V_2, E(V_2))$. Then a system of inequalities that defines $SSP(G)$ is obtained by taking the union of the systems that define $SSP(G_1)$ and $SSP(G_2)$ and identifying the variables associated with the nodes in W.*

Let $ax \leq \alpha$ be a facet defining inequality of $SSP(G)$. If a contains at least two nonzero coefficients, we say that $ax \leq \alpha$ defines a nontrivial facet. In this case $a \geq 0$ and $\alpha > 0$. If $ax \leq \alpha$ defines a nontrivial facet of $SSP(G)$, we denote by V_a the set

$$V_a = \{u \mid a_u > 0\}.$$

The subgraph induced by V_a is denoted by G_a, and it is called the *support* of the facet. In the next two remarks and in the next two lemmas we assume that $ax \leq \alpha$ defines a nontrivial facet.

Remark 4. The inequality $ax \leq \alpha$ also defines a facet of $SSP(G_a)$.

Remark 5. If $ax \leq \alpha$ defines a facet of $SSP(G)$, it follows from Theorem 3 and the above remark that G_a is twoconnected.

LEMMA 6 (see [19]). *If $ax \leq \alpha$ defines a facet of $SSP(G)$ and G_a contains a path with nodes p, u, v, q, where u and v have degree two in G_a, then $a_u = a_v$.*

LEMMA 7 (see [19]). *If $ax \leq \alpha$ defines a facet of $SSP(G)$ and G_a is different from an odd hole, then G_a does not contain two paths between any two given nodes p and q such that each node of them different from p, q has degree two in G_a.*

The lemma above will be used to classify the facet defining inequalities of $SSP(G)$, as in the lemma below. Let $G' = (V', E')$ be a graph containing two nodes p and q such that there are two paths p, u, v, q and p, s, t, q between them; the nodes u, v, s, and t have degree two. Let $\bar{V} = V' \setminus \{u, v, s, t\}$.

LEMMA 8. *For the nontrivial facets of $SSP(G')$, their inequalities can be classified in the following types:*

$$\sum_{j \in \bar{V}} a_{ij} x(j) \leq \alpha_i, \;\; i \in I_1 \tag{15}$$

$$\sum_{j \in \bar{V}} a_{ij} x(j) + x(u) + x(v) \leq \alpha_i, \;\; i \in I_2, \tag{16}$$

$$\sum_{j \in \bar{V}} a_{ij} x(j) + x(s) + x(t) \leq \alpha_i, \;\; i \in I_3, \tag{17}$$

$$x(p) + x(u) \leq 1, \tag{18}$$

$$x(u) + x(v) \leq 1, \tag{19}$$

$$x(v) + x(q) \leq 1, \tag{20}$$

$$x(p) + x(s) \leq 1, \tag{21}$$

$$x(s) + x(t) \leq 1, \tag{22}$$

$$x(t) + x(q) \leq 1. \tag{23}$$

Proof. Inequalities (15) correspond to the nontrivial facets whose support contains only nodes in $\bar{V}$; we call I_1 the index set of these.

Inequalities (16) and (17) appear as a consequence of Remark 5 and Lemmas 6 and 7; we call I_2 and I_3 the index sets of these inequalities.

Inequalities (18)–(23) appear as a consequence of Theorem 2. ◻

Remark 9. If

$$\sum_{j \in \bar{V}} b_j x(j) + x(u) + x(v) \leq \alpha$$

defines a facet of $SSP(G')$, then by symmetry

$$\sum_{j \in \bar{V}} b_j x(j) + x(s) + x(t) \leq \alpha$$

also defines a facet.

Remark 10. Let $\bar{G} = G' \setminus \{s, t\}$. It follows from the above discussion that if we have all facet defining inequalities for $SSP(\bar{G})$, then we can obtain the remaining facets defining inequalities of $SSP(G')$ by applying Remark 9 to inequalities (16) and by adding (21)–(23), $x(s) \geq 0$, and $x(t) \geq 0$.

2.5. Decomposition of graphs with no fork. Let $G = (U \cup V, A)$ be a bipartite graph. If there are two nodes u and v in U with $\delta^+(u) = \{(u, w_1), (u, w_2)\}$ and $\delta^+(v) = \{(v, w_1), (v, w_2)\}$, we say that u and v are *twins*. This is an equivalence relation. We are going to use the following decomposition steps.

Step 1. For every equivalence class of U by the relation twin, leave only one node (and remove the others).

Step 2. Remove every node $u \in U$ with degree equal to one.

The graph may consist of several connected components. A component with at most three locations cannot contain a fork. Now we consider a component with at least four locations.

LEMMA 11. *After applying Steps* 1 *and* 2*, a connected component with at least four locations contains a fork if and only if it has a location with degree at least three.*

Proof. If a component has a fork, then it has a location with degree at least three. Now assume that the component has no fork and has at least four locations. Let $u \in V$ be a node with $|\delta^-(u)| \geq 3$. We should have the arcs $(s_1, u), (s_2, u), (s_3, u)$, and the nodes s_1, s_2, and s_3 have degree at least two. Since the remaining graph contains no fork, we have the following two cases.

Case 1. The arcs $(s_1, v), (s_2, v), (s_3, v)$ exist, with $v \neq u$. Since there are no twins and no fork, there must exist two arcs (s_i, w) and (s_j, w) with $i, j \in \{1, 2, 3\}$, $i \neq j$, and w is different from u and v. Now if one of the nodes s_1, s_2, or s_3 is adjacent to a node not in $\{u, v, w\}$, then we must have a fork. Let M be the subgraph induced by $\{u, v, w, s_1, s_2, s_3\}$. Consider the following cases:

- If w is adjacent to a node s_4 not in M and s_4 is adjacent to a node t not in M, we would have a fork.
- The same is true for u and v.

So if any of $\{u, v, w\}$ is adjacent to a node not in M this node is only adjacent to nodes in M. Thus the connected component containing $\{u, v, w\}$ is a bipartite graph with bipartition $\{s_1, \ldots, s_k\}$ and $\{u, v, w\}$. This contradicts the hypothesis.

Case 2. The arcs $(s_i, v), (s_j, v), (s_k, w)$ exist, where i, j, k are in $\{1, 2, 3\}$ and different and $v \neq u$, $w \neq u$, $v \neq w$. Since s_i and s_j are not twins, one of them must be adjacent to a node different from u and v. This node must be w, otherwise a fork is present. Now none of the nodes s_1, s_2, s_3 can be adjacent to a node other than u, v, w because we would have again a fork. An analysis similar to the one in the preceding case implies that if we take the connected component containing $\{u, v, w\}$, we obtain a bipartite graph with bipartition $\{s_1, \ldots, s_k\}$ and $\{u, v, w\}$. Again we have a contradiction. ∎

Lemma 11 gives a simple algorithm to recognize graphs with no fork. First we apply the two steps above, then for each component with at least four locations we check if the degree of every location is at most two.

3. The characterization of $UFLP'(G)$. The main result of this section is the following.

THEOREM 12. *Let $G = (U \cup V, A)$ be a bipartite graph. If G has no fork, then $UFLP'(G)$ is described by inequalities* (2)–(5) *and inequalities* (10).

The proof of this theorem will be given in subsection 3.2. In the following subsection we will first show that Theorem 12 holds for graphs for which each location has degree at most two. Then combining this with the results of subsections 2.4 and 2.5 it will be shown that it also holds for all graphs with no fork.

For the case of three locations, Theorem 6.1 in [10] shows that any facet defining an inequality of $UFLP'(G)$ is among the inequalities (2)–(5) or is an odd cycle inequality, that is, an inequality (10) where $G(F)$ is an odd cycle. Thus we have the following.

THEOREM 13 (see [10]). *If $|V| \leq 3$, then $UFLP'(G)$ is defined by inequalities* (2)–(5) *and inequalities* (10) *that correspond to odd cycle inequalities.*

3.1. The case where each location has degree at most two. First we show the following related NP-hardness result.

THEOREM 14. *The problem $UFLP'$ is NP-hard even when the degree of each customer is at most two and the degree of each location is at most three.*

Proof. We use a reduction from the minimum vertex cover problem when the degree of each node is at most three. Let $H = (W, E)$ be a graph where each node has degree at most three. From H, define the bipartite graph $G = (U \cup V, A)$, where $U = E$ and $V = W$. We have $(u, v) \in A$ if and only if the edge u is incident to the node v in the graph H. Thus by definition, in G, the degree of each node $u \in U$ is two, and the degree of each node $v \in V$ is at most three. Consider an instance of the UFLP$'$ that corresponds to the graph G, where $c(u, v) = M$ for each arc $(u, v) \in A$, where M is a large positive scalar and each node $v \in V$ is associated with a fixed cost $f(v) = -1$. Then clearly the minimum vertex cover problem in H reduces to this instance of the UFLP$'$. ◻

Below we will show that the case where each location has degree at most two reduces to a matching problem. Let $G = (U \cup V, A)$ be a bipartite graph where $d(v) \leq 2$ for all $v \in V$. The graph G contains no isolated node and may or may not be connected. Define from G an undirected graph G' as follows. Split each node $v \in V$ into two nodes v_1 and v_2, call this new set of nodes V'. If v is of degree two, let (u_1, v) and (u_2, v) be the two arcs incident to v; build the edges u_1v_1, u_2v_2, and v_1v_2, and denote this set by E_v. If v is of degree one, let (u, v) be the unique arc incident to v, build the edges uv_1 and v_1v_2, and denote this set by E_v. Let $G' = (W, E)$, where $W = U \cup V'$ and $E = \bigcup_{v \in V} E_v$. Define a weight function w associated with each edge $e \in E$ as follows. For each E_v, let $w(v_1v_2) = -f(v)$, $w(u_1v_1) = c(u_1, v)$, and $w(u_2v_2) = c(u_2, v)$. Notice that when v is of degree one, then there is no edge u_2v_2.

The problem UFLP$'$ with respect to G is equivalent to the following *matching problem* associated to $G' = (W, E)$ and w:

$$\max \sum_{e \in E} w(e)x(e) + \sum_{v \in V} f(v) \tag{24}$$

$$\sum_{e \in \delta(v)} x(e) \leq 1 \quad \forall\, v \in W, \tag{25}$$

$$x(e) \in \{0, 1\} \quad \text{ for } e \in E. \tag{26}$$

For each feasible solution of (24)–(26) there is a feasible solution of (1)–(7) having the same weight and vice versa. In fact, let x^* be a feasible solution of (24)–(26). For each $v \in V$ let $\bar{y}(v) = 1 - x^*(v_1v_2)$. If v is of degree two and (u_1, v) and (u_2, v) are the arcs incident to v, let $\bar{x}(u_1, v) = x^*(u_1v_1)$ and $\bar{x}(u_2, v) = x^*(u_2v_2)$. If v is of degree one and (u, v) is the unique arc incident to v, let $\bar{x}(u, v) = x^*(uv_1)$. It is easy to see that $(\bar{x}, \bar{y})$ is a feasible solution of (1)–(7) having the same weight as x^*. The matching problem

(24)–(26) can be solved in polynomial time [17]. We can also obtain equivalence with the UFLP associated with G. We just have to set inequalities (25) to equations for each node in W that corresponds to a node in U. This problem is also a matching problem and is polynomially solvable. Thus we have the following.

THEOREM 15. *Both problems UFLP and UFLP′ reduce to matching problems when each location has degree at most two.*

Let $MP(G')$ denote the convex hull of the solutions of the matching problem (25)–(26).

THEOREM 16 (see [17]). *For the graph $G' = (W, E)$, $MP(G')$ is defined by the following linear system:*

$$\sum_{e\in\delta(v)} x(e) \leq 1 \quad \textit{for } v \in W, \tag{27}$$

$$\sum_{e\in E(S)} x(e) \leq \left\lfloor \frac{1}{2}|S| \right\rfloor \quad \forall\ S \subseteq W \textit{ with } |S| \geq 3 \textit{ and odd}, \tag{28}$$

$$x(e) \geq 0 \quad \textit{for } e \in E. \tag{29}$$

Remark 17. In inequalities (28) we can assume that

(i) neither v_1 nor v_2 belongs to S when their corresponding node $v \in V$ is of degree one, and

(ii) if for a node $v \in V$ of degree two we have either v_1 or v_2 in S, then both of them with their neighbors in U are in S.

This remark comes from the fact that the graph induced by S must be twoconnected; this is one of the necessary conditions for inequalities (28) to define facets of $MP(G')$ [23], [24]. Theorem 16 is used to prove the following.

THEOREM 18. *Let $G = (U \cup V, A)$ be a bipartite graph where each node in V has degree at most two. Then $UFLP'(G)$ is described by* (2)–(5) *plus inequalities* (10).

Proof. In fact, we will see that the polytope defined by (27)–(29) and the polytope defined by (2)–(5) and (10) are exactly the same polytopes. Rewrite inequalities (27) and (29) as follows:

$$\sum_{e\in\delta(v)} x(e) \leq 1 \quad \text{for } v \in U, \tag{30}$$

$$x(uv_1) + x(v_1v_2) \leq 1 \quad \text{for } v \in V \text{ and } uv_1 \in E, \text{ with } u \in U, \tag{31}$$

$$x(uv_2) + x(v_1v_2) \leq 1 \quad \text{for } v \in V \text{ and } uv_2 \in E, \text{ with } u \in U, \tag{32}$$

$$x(e) \geq 0 \quad \text{for } e \in E. \tag{33}$$

Let $S \subseteq W$ the subset used in an inequality of type (28). Let $S_1 = S \cap U$ and $S_2 = S \setminus S_1$. Let S_2' the set of nodes in $v \in V$ such that v_1 and v_2 are in S_2. By Remark 17, $\bigcup_{v\in S_2'}\{v_1, v_2\} = S_2$. Thus $|S_2|$ is even and $|S_1| \geq 3$ and odd. From this we can rewrite an inequality (28) with respect to S as follows:

$$\sum_{e\in\delta(S_1,S_2)} x(e) + \sum_{v\in S_2'} x(v_1v_2) \leq \left\lfloor \frac{1}{2}(|S_1| + 2|S_2'|) \right\rfloor. \tag{34}$$

For each $v \in V$ replace the variables $x(v_1v_2)$, in (27)–(29), by $1 - y(v)$ and each edge variable $x(uv_i)$ by $x(u, v)$. Then inequality (34) is equivalent to

$$\sum_{(u,v)\in A(S_1,S_2')} x(u, v) - \sum_{v\in S_2'} y(v) \leq \left\lfloor \frac{1}{2}|S_1| \right\rfloor. \tag{35}$$

Inequalities (30) are equivalent to inequalities (2), inequalities (31)–(32) correspond to inequalities (3), and inequalities (33) are equivalent to inequalities (4)–(5). Finally, since any node v in V is at most of degree two, any inequality (35) is of type (10). □

3.2. The proof of Theorem 12. Let $G = (U \cup V, A)$ be a bipartite graph with no fork. Perform Steps 1 and 2 of subsection 2.5, and let $\bar{G}$ be the resulting graph. By Lemma 11, each connected component of $\bar{G}$ consists of either (i) a bipartite graph with three locations, or (ii) a bipartite graph where each location has degree at most two. From Theorems 13 and 18, Theorem 12 holds for $\bar{G}$. Now rebuild G from $\bar{G}$. If we add all the customers with degree one in G and every customer that was a twin of an existing node in $\bar{G}$, then each connected component with three locations in $\bar{G}$ remains a connected component with three locations in G. For this type of graph the result follows from Theorem 13.

Now suppose that we have a connected component in $\bar{G}$, where each location has degree at most two. By reformulating our problem as a set packing problem (11)–(14), proving Theorem 12 is equivalent to showing that the following polytope is integral:

$$\sum_{v:(u,v)\in A} x(u,v) \le 1 \quad \forall u \in U, \tag{36}$$

$$x(u,v) + \bar{y}(v) \le 1 \quad \forall (u,v) \in A, \tag{37}$$

$$\bar{y}(v) \ge 0 \quad \forall v \in V, \tag{38}$$

$$x(u,v) \ge 0 \quad \forall (u,v) \in A, \tag{39}$$

$$\sum_{(u,v)\in F} x(u,v) + \sum_{v\in N^+(F)} \bar{y}(v) \le \left\lfloor \frac{|N^-(F)|}{2} \right\rfloor + |N^+(F)| \quad \forall F \subseteq A, \tag{40}$$

where $|N^-(F)| \ge 3$ and odd, with $d_{G(F)}(v) = 2$, $\forall\ v \in N^+(F)$. Inequalities (40) are inequalities (10) after replacing $y(v)$ by $1 - \bar{y}(v)$ for each $v \in V$. We already know that the polytope above is integral when the degree of each location is at most two. Thus the polytope above is the SSP $\mathrm{SSP}(I(\bar{G}))$.

Now suppose that we add a customer u of degree one that had been removed in Step 2 of subsection 2.5. Let G' be the new graph. The new intersection graph $I(G')$ is obtained by adding a new node $x(u,v)$ to $I(G)$, assuming that u is adjacent to the location v in G. It follows from Theorem 3 that $SSP(I(G'))$ is obtained by adding the variable $x(u,v)$ and the inequalities

$$\begin{aligned} &x(u,v) + \bar{y}(v) \le 1, \\ &x(u,v) \ge 0. \end{aligned}$$

We keep using the same reasoning until all customers of degree one have been added.

Now assume that we add a node u' that was a twin of a node u in $\bar{G}$. We add two new arcs (u', v_1) and (u', v_2) to $\bar{G}$, where (u, v_1) and (u, v_2) already exist in $\bar{G}$. Here we use Remark 10. So for each inequality (40) that contains $x(u, v_1)$ and $x(u, v_2)$ we write a similar inequality but using $x(u', v_1)$ and $x(u', v_2)$ instead. Also the following inequalities are added:

$$\begin{aligned}
&x(u', v_1) + \bar{y}(v_1) \le 1,\\
&x(u', v_2) + \bar{y}(v_2) \le 1,\\
&x(u', v_1) + x(u, v_2) \le 1,\\
&x(u', v_1) \ge 0,\\
&x(u', v_2) \ge 0.
\end{aligned}$$

We keep using the same reasoning until all twins are added.

We obtain exactly the inequalities (36)–(40) with respect to G. Thus $\mathrm{SSP}(I(G))$ is described by inequalities (36)–(40). This completes the proof of Theorem 12.

COROLLARY 19. *Let $G = (U \cup V, A)$ be a bipartite graph. If G has no fork, then $UFLP(G)$ is described by inequalities* (3)–(5)*, inequalities* (10)*, and by setting inequalities* (2) *to equations.*

3.3. The separation of inequalities (10). Let $G = (U \cup V, A)$ be a bipartite graph. Given a vector $(x^*, y^*) \in \mathbb{R}^{|A|+|V|}$, the *separation* problem of inequalities (10) is to decide whether (x^*, y^*) satisfies all these inequalities or to find one of them that is violated.

We show that the separation of these inequalities can be done in polynomial time when the graph has no fork. We may assume that (x^*, y^*) satisfies inequalities (2)–(5). For each connected component of G with three locations the problem is easy to solve, since these inequalities are the odd cycle inequalities and may be separated in polynomial time [9], [4]. Also one can enumerate them since they correspond to odd cycles with three locations and three customers.

Now consider the other components. We can remove all the customers with degree one. In fact, each inequality (10) such that $N^-(F)$ contains a node u of degree one is redundant. Let (u, v) be the unique arc incident to u, and let (u', v) be the other arc incident to v in F. Then we can obtain this inequality by combining inequalities $x(u, v) \le 1$, $x(u', v) \le y(v)$, and the inequality (10) with respect to $F' = F \setminus \{(u, v), (u', v)\}$ that is by itself redundant since $|N^-(F')|$ is even. In other words all such inequalities are satisfied by (x^*, y^*), and all the possible violated inequalities contain only customers with degree at least two.

Now we treat this case. For each set of twins choose the twin node u with $x^*(u, v_1) + x^*(u, v_2)$ maximum and remove the others. At this point we have done exactly Steps 1 and 2 of subsection 2.5; thus from Lemma 11 we know that in the resulting graph, call it $\bar{G}$, the degree of each location is at most 2. We have seen in the proof of Theorem 12 that inequalities (10) with respect to $\bar{G}$ are equivalent to inequalities (28), which are the blossom inequalities introduced in [17]. They can be separated in polynomial time with the algorithm of [22]. Since each inequality (10) that may define a facet contains at most one node from each set of twins, it follows that if one finds a violated inequality (10) with respect to $\bar{G}$, the same inequality is also violated with respect to G. Otherwise there is no violated inequality.

4. The intersection property for $UFLP'(G)$ and the characterization of $pMP'(G)$. Let P be an integral polytope in $\mathbb{R}^n$. Let q be an integer valued row vector in $\mathbb{R}^n$ such that the greatest common divisor of its components is one. For an integer p let $H_p = \{x \in \mathbb{R}^n : qx = p\}$.

We say that P has the *intersection property* with respect to q if for every integer p the polytope $P \cap H_p$ is integral. The following result has been shown in [7].

THEOREM 20. *The stable set polytope of a graph $G = (V, E)$ has the intersection property with respect to $\sum_{v \in V} x(v) = p$ if and only if G is a clawfree graph.*

A *clawfree* graph is a graph that does not contain the bipartite graph $K_{1,3}$ as an induced subgraph. Given a bipartite graph $G = (U \cup V, A)$, we will show in this section that the polytope $UFLP'(G)$ has the intersection property with respect to $\sum_{v \in V} y(v) = p$ if and only if G has no fork. To obtain this result one cannot apply Theorem 20, since the intersection graph of a graph with no fork is not clawfree. Instead we will modify the proof given in [7] to obtain our result.

THEOREM 21. *Let $G = (U \cup V, A)$ be a bipartite graph. $UFLP'(G)$ has the intersection property with respect to $\sum_{v \in V} y(v) = p$ if and only if G has no fork.*

Proof. Necessity. Assume that G contains a fork, and let us show that $Q_p(G) = \{(x, y) \in \mathbb{R}^{|A|+|V|} : (x, y) \in UFLP'(G), \ \sum_{v \in V} y(v) = p\}$ is not integral for some integer p.

Let F be a subgraph of G that is a fork. The nodes of F are $u_1, \ldots, u_3$ and $v_1, \ldots, v_4$, and its arcs are

$$(u_1, v_1), (u_1, v_3), (u_2, v_2), (u_2, v_3), (u_3, v_3), (u_3, v_4).$$

Let $(\bar{x}, \bar{y}) \in \mathbb{R}^{|A|+|V|}$ be the vector defined as follows: $\bar{x}(u, v) = \frac{1}{2}$ if (u, v) is an arc in F and 0 otherwise; $\bar{y}(v) = \frac{1}{2}$ if $v \in \{v_1, \ldots, v_4\}$ and 0 if $v \in V \setminus \{v_1, \ldots, v_4\}$. We will show that, in fact, $(\bar{x}, \bar{y})$ is an extreme point of $Q_2(G)$. Let $(x^1, y^1) \in UFLP'(G)$, where $x^1(u_1, v_3) = x^1(u_2, v_3) = x^1(u_3, v_3) = y(v_3) = 1$ and every other variable takes the value 0. Let $(x^2, y^2) \in UFLP'(G)$, where $x^2(u_1, v_1) = x^2(u_2, v_2) = x^2(u_3, v_4) = y^2(v_1) = y^2(v_2) = y^2(v_4) = 1$ and every other variable takes the value 0. We have $(\bar{x}, \bar{y}) = \frac{1}{2}(x^1, y^1) + \frac{1}{2}(x^2, y^2)$. To show that $(\bar{x}, \bar{y})$ is an extreme point of $Q_2(G)$ we are going to show that (x^1, y^1) and (x^2, y^2) belong to a common one-dimensional face of $UFLP'(G)$. This face is defined as follows:

$$\begin{array}{ll}
x(u, v) = 0 & \text{if } (u, v) \text{ is not in } F, \\
y(v) = 0 & \text{if } v \text{ is not in } F, \\
x(u_1, v_1) + x(u_1, v_3) = 1, & \\
x(u_2, v_2) + x(u_2, v_3) = 1, & \\
x(u_3, v_4) + x(u_3, v_3) = 1, & \\
x(u, v) - y(v) = 0 & \text{for each arc } (u, v) \text{ in } F.
\end{array}$$

Sufficiency. Assume now that G does not contain a fork as a subgraph, and let us show that the polytope $Q_p(G) = \{(x, y) \in \mathbb{R}^{|A|+|J|} : (x, y) \in UFLP'(G), \ \sum_{v \in V} y(v) = p\}$ is integral. A pair (A', J'), $A' \subseteq A$, $J' \subseteq V$, is called a *solution* if each node in U is incident to at most one arc in A' and each arc in A' is incident to a node in J'. Let $(x^{A'}, y^{J'})$ denote the incidence vector of the solution (A', J'). When taking the intersection of $\{y : \sum_{v \in V} y(v) = p\}$ with $UFLP'(G)$, a fractional extreme point lies in a one-dimensional face of $UFLP'(G)$; i.e., it has the form

$$(\bar{x}, \bar{y}) = \lambda(x^{A_1}, y^{J_1}) + (1 - \lambda)(x^{A_2}, y^{J_2}),$$

where $\lambda = \frac{(p - |J_2|)}{|J_1| - |J_2|}$ and $|J_2| + 1 \leq p \leq |J_1| - 1$. We will prove that such a vector is a convex combination of 0-1 vectors in $Q_p(G)$, so it is not an extreme point.

Let $\bar{G}$ be the graph spanned by $A_1 \cup A_2$. We label the nodes and arcs as follows:

- To every arc in $A_1 \cap A_2$ we give the label "*f*" that means fixed. We do the same for every node in $J_1 \cap J_2$.

- Now consider an arc $a = (u, v)$ that has no label. If v has the label f, we add a new node v' with the label "0" and replace (u, v) by $a' = (u, v')$. If $a \in A_1$, we give the label 1 to a (or to a'), otherwise $a \in A_2$, and we give it the label 2. We repeat this for every arc that has not been labeled.
- Now for every node v that has not been labeled, if $v \in J_1$, we give it the label 1, otherwise $v \in J_2$, and we give it the label 2.

After every arc and every node has been labeled, let G' be this new graph. Now build an undirected graph M whose node set are the nodes in G' that have the label 1 or 2. Let v_1 and v_2 be two nodes in M, with the labels 1 and 2, respectively. If the arcs $a_1 = (u, v_1)$ and $a_2 = (u, v_2)$ are in G', we put an edge between v_1 and v_2 in M. All edges in M are obtained in this way. Since G has no fork, the degree of each node in M is at most two. Now we can partition the nodes in M into $B_1, \ldots, B_k$, so that

- every edge in M has both endnodes in the same set B_i,
- for every set B_i, let C_i be the set of nodes in B_i with the label 1 and D_i the set of nodes in B_i with the label 2. Then $|C_i| = |D_i| + 1$. This is possible because the degree of every node in M is at most two.

Let F be the set of nodes in G' with the label f. Also let $C = \cup C_i$ and $D = \cup D_i$. We have $|C| = |D| + k$, and $|J_1| = |F| + |C| = |F| + |D| + k = |J_2| + k$. To build a solution with p nodes from $J_1 \cup J_2$, we can take the nodes in F, the nodes in C_i, $i = 1, \ldots, r$, and the nodes in D_j, $j = r + 1, \cdots, k$, where $r = p - |J_2|$. Now we have to define the set of arcs in the solution. First we have to define what to do with the nodes in G' that have the label 0. Let (u, v) be an arc in G' incident to a node u of degree one, and v has the label 0. If (u, v) has the label 1, then pick arbitrarily a set C_i and assign v to C_i, otherwise pick arbitrarily a set D_i and assign v to it. Let (u, v) and (u, w) be the two arcs incident to a node u that has degree equal to two in G'. We have three cases:

- If v has the label 1 and w has the label 0, then $v \in C_i$ for some index i; we add w to D_i. Notice that w has not been counted (and should not been counted) when computing the cardinality of D_i above.
- If v has the label 2 and w has the label 0, then $v \in D_i$ for some index i; we add w to C_i.
- Finally suppose that (u, v) has the label 1, (u, w) has the label 2, and both v, w have the label 0. In this case we arbitrarily pick a set C_i, and we assign v to C_i and w to D_i.

Remark 22. Note that the assignment of the nodes with label 0 to C_i or D_i must be done before the selection of the sets C_i, $i = 1, \ldots, r$, and the sets D_j, $j = r+1, \cdots, k$. We choose this presentation because the cardinalities of C_i and D_i, before assigning them the nodes with label 0, helps to see that we are taking exactly p facilities in the solution. Now we can describe how to pick the arcs in the solution. Every arc in G' that has the label f should be in the solution. Let $C' = \cup_{i=1}^{i=r} C_i$ and $D' = \cup_{j=r+1}^{j=k} D_j$. For every customer u that is incident to two arcs $a = (u, v)$ and $b = (u, w)$ in G' we proceed as follows. If $v \in C'$, then a is in the solution, otherwise $w \in D'$ and b is in the solution. For a customer u that is incident to one arc $a = (u, v)$, if $v \in C' \cup D'$, then a is in the solution, otherwise it is not.

Call $\mathcal{F}$ the set of all solutions (A^t, J^t) that may be constructed from (A_1, J_1) and (A_2, J_2) as indicated above. From the construction of the solutions in $\mathcal{F}$ above, we have the following:

$$\begin{aligned}\sum_{(A^t,J^t)\in\mathcal{F}}(x^{A^t},y^{J^t}) &= \begin{pmatrix} k-1 \\ p-|J_2|-1 \end{pmatrix}(x^{A_1},y^{J_1}) \\ &\quad + \begin{pmatrix} k-1 \\ k-(p-|J_2|)-1 \end{pmatrix}(x^{A_2},y^{J_2}) \\ &= \frac{\begin{pmatrix} k \\ p-|J_2| \end{pmatrix}(p-|J_2|)}{k}(x^{A_1},y^{J_1}) \\ &\quad + \frac{\begin{pmatrix} k \\ p-|J_2| \end{pmatrix}(k-p+|J_2|)}{k}(x^{A_2},y^{J_2}).\end{aligned}$$

But recall from above that $k = |J_1| - |J_2|$, so

$$\begin{aligned}\sum_{(A^t,J^t)\in\mathcal{F}}(x^{A^t},y^{J^t}) &= \frac{\begin{pmatrix} |J_1|-|J_2| \\ p-|J_2| \end{pmatrix}(p-|J_2|)}{|J_1|-|J_2|}(x^{A_1},y^{J_1}) \\ &\quad + \frac{\begin{pmatrix} |J_1|-|J_2| \\ p-|J_2| \end{pmatrix}(|J_1|-p)}{|J_1|-|J_2|}(x^{A_2},y^{J_2}).\end{aligned}$$

Since $(\bar{x},\bar{y}) = \frac{p-|J_2|}{|J_1|-|J_2|}(x^{A_1},y^{J_1}) + \frac{|J_1|-p}{|J_1|-|J_2|}(x^{A_2},y^{J_2})$, we obtain

$$(\bar{x},\bar{y}) = \frac{1}{\begin{pmatrix} |J_1|-|J_2| \\ p-|J_2| \end{pmatrix}} \sum_{(A^t,J^t)\in\mathcal{F}}(x^{A^t},y^{J^t}).$$

Notice that $\binom{|J_1|-|J_2|}{p-|J_2|}$ is the cardinality of the family $\mathcal{F}$. Thus $(\bar{x},\bar{y})$ is a convex combination of integer vectors in $Q_p(G)$. This implies that the intersection of $\{y : \sum_{j\in V} y(j) = p\}$ with $UFLP'(G)$ does not have fractional extreme points. $\square$

From Theorems 12 and 21, we obtain the main result of this paper.

THEOREM 23. *Let $G = (U \cup V, A)$ be a bipartite graph. The polytope $pMP'(G)$ is described by* (2)–(5), (8), *and* (10) *if and only if G does not contain a fork as a subgraph.*

COROLLARY 24. *Let $G = (U \cup V, A)$ be a bipartite graph. If G has no fork, then $pMP(G)$ is described by* (3)–(5), *inequalities* (2), *transformed into equations,* (8), *and* (10).

Now since the separation of inequalities (10) can be done in polynomial time for graphs with no fork, we have the following.

THEOREM 25. *The problems UFLP′, pMP′, UFLP, and pMP can be solved in polynomial time when the underlying graph does not contain a fork as a subgraph.*

In the next section we give combinatorial algorithms for these problems.

5. A combinatorial algorithm. In this section we give a combinatorial algorithm to solve both problems pMP′ and UFLP′ when the underlying graph has no fork. The problems pMP and UFLP reduce to the problems pMP′ and UFLP′, respectively.

5.1. Solving the problem UFLP′. Let $G = (U \cup V, A)$ be a bipartite graph with no fork. We are going to solve (1)–(7). In order to decide if a graph has no fork we apply the procedure of subsection 2.5; this should give us a graph where each connected component is of two possible types:

- a component with at most three locations, or
- a component where each location has degree at most two.

The case with three locations can be solved by enumeration, so in what follows we deal with a component having four or more locations.

LEMMA 26. *We may assume that each customer has degree at least two.*

Proof. Let u be a customer of degree one. The node u must be adjacent to a location v. If $c(u,v) < 0$, we just ignore u. If $c(u,v) + f(v) \geq 0$, then we remove u and give the weight 0 to v. If γ is the optimal value with the new weights, then $\gamma + c(u,v) + f(v)$ is the optimal value with the original weights. If $c(u,v) + f(v) < 0$, we remove u and give the weight $c(u,v) + f(v)$ to v.

Every customer of degree one is treated in the same way. □

Now we assume that we have a connected component M where each customer has degree at least two. First we are going to use the reduction to the SSP discussed subsection 2.3; later this will be reduced to a matching problem. Thus our problem reduces to finding a stable set in $I(M)$ of maximum weight, where a weight of a node $x(u,v)$ is equal to $c(u,v)$ and the weight of a node $\bar{y}(v)$ is equal to $-f(v)$.

Note that for each set of twins in M, where v_1 and v_2 are the two locations that are adjacent to this set of twins, the corresponding nodes of v_1 and v_2 in $I(M)$ form a twonode cutset. This permit us to use the following decomposition procedure introduced in [5]. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \{a, b\}$, $E_1 \cap E_2 = \emptyset$, and the edge ab does not belong neither to E_1 nor to E_2. Let $G = (V, E)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. Assume that each node u has a weight $w(u)$. Let S_{ab}, $S_{\bar{a}\bar{b}}$, $S_{a\bar{b}}$, and $S_{\bar{a}b}$ be stable sets in G_1 of maximum weight among those which contain, respectively, a and b, nor a nor b, a but not b, b but not a. Let s_{ab}, $s_{\bar{a}\bar{b}}$, $s_{a\bar{b}}$, and $s_{\bar{a}b}$ be their respective weights. Now we define a graph G^* with node-weights w^*, obtained from G_2 as follows. Start with $G^* = G_2$. Let

$$
\begin{aligned}
&w^*(v) = w(v) \text{ if } v \in V_2 \setminus \{a, b\},\\
&\lambda(G_1) = s_{ab} + s_{\bar{a}\bar{b}} - (s_{a\bar{b}} + s_{\bar{a}b}).
\end{aligned}
$$

We have two cases:

- $\lambda(G_1) \geq 0$. In this case we add a node c to G^* and the edges ac and cb. Let

$$
\begin{aligned}
\sigma &= s_{\bar{a}\bar{b}} - \lambda(G_1),\\
w^*(a) &= s_{ab} - s_{\bar{a}b},\\
w^*(b) &= s_{ab} - s_{a\bar{b}},\\
w^*(c) &= \lambda(G_1).
\end{aligned}
$$

- $\lambda(G_1) < 0$. In this case we add two nodes c and d to G^* and the edges ac, cd, and db. Let

$$
\begin{aligned}
\sigma &= s_{\bar{a}\bar{b}} + \lambda(G_1),\\
w^*(a) &= s_{a\bar{b}} - s_{\bar{a}\bar{b}},\\
w^*(b) &= s_{\bar{a}b} - s_{\bar{a}\bar{b}},\\
w^*(c) &= w^*(d) = -\lambda(G_1).
\end{aligned}
$$

THEOREM 27 (see [5]). *Let S^2 be the restriction to G_2 of a maximum weighted stable set S^* of G^* with respect to w^*, and let S be defined as follows:*

- *If* $a \in S^2$ *and* $b \in S^2$, *then* $S = S^2 \cup S_{ab}$.
- *If* $a \in S^2$ *and* $b \notin S^2$, *then* $S = S^2 \cup S_{a\bar{b}}$.
- *If* $a \notin S^2$ *and* $b \in S^2$, *then* $S = S^2 \cup S_{\bar{a}b}$.
- *If* $a \notin S^2$ *and* $b \notin S^2$, *then* $S = S^2 \cup S_{\bar{a}\bar{b}}$.

S *is a maximum weighted stable set of* G, *with respect to* w, *and*

$$w^*(S^*) + \sigma = w(S).$$

Let us apply the above procedure to $I(M)$. Let T be a set of twins, and let a and b be the two locations adjacent to the nodes in T. The set of twins gives a set of disjoint paths with three edges between a and b. This corresponds to the graph G_1 described above. We have to solve four SSPs to define the sets S_{ab}, $S_{\bar{a}\bar{b}}$, $S_{a\bar{b}}$, $S_{\bar{a}b}$, and the value $\lambda(G_1)$ as in the above procedure. Due to the simplicity of G_1, these four problems can be solved by inspection. Then we build the graph G^*, where between a and b we put a path with two or three edges. We repeat this procedure for each set of twins.

Remark that at the end the degree of a node in G^* that corresponds to a location is at most two. Now we have to solve the SSP in G^* with respect to the weights w^* defined by the procedure above. This reduces to a matching problem as follows:

- Split each node v in G^* that correspond to a location into two nodes v_1 and v_2; add an edge between v_1 and v_2 with a weight $w^*(v)$.
  - If v has degree one and uv is the edge incident to v, then add an edge between v_1 and u with a weight $w^*(u)$.
  - If v is of degree two and u_1v and u_2v are the two edges incident to v, add the edges u_1v_1 and u_2v_2 with weights $w^*(u_1)$ and $w^*(u_2)$, respectively.
- Now consider the nodes of G^* that do not correspond to locations in G. These nodes form disjoint cliques.
  - A clique of size one corresponds to the node c added in the construction of G^* in the case where $\lambda(G_1) \geq 0$. Let v and w be the two locations in G^* adjacent to c. After the split operation we have v_1, v_2, w_1, and w_2. Assume that c is adjacent to v_1 and w_1. We remove the node c and add an edge between v_1 and w_1 of weight $w^*(c)$.
  - Consider a clique of size at least two with nodes $c_1, \ldots, c_k$. To each edge c_iv we give the weight $w^*(c_i)$, then we shrink the clique (and replace it by one node).

Solving the SSP in G^* with weights w^* reduces to the maximum matching problem in the graph described above. Using Theorem 27 one can produce the optimal solution of the SSP in $I(M)$. This corresponds to the optimal solution of the problem UFLP′ in the component M of G having four or more locations. A summary of this algorithm is below.

Algorithm for components with more than three locations.

Step 1. Treat customers of degree one as in Lemma 26.
Step 2. Transform into an SSP.
Step 3. Treat sets of twins.
Step 4. Transform into a matching problem and solve it.
Step 5. Recover the solution of the original problem using Theorem 27.

5.2. Solving the problem pMP'. Let $G = (U \cup V, A)$ be a bipartite graph with no fork. In this subsection we give a polynomial combinatorial algorithm to solve the problem pMP' defined by (1)–(8). The main ingredient for this algorithm is the intersection property of the polytope $UFLP'(G)$.

Suppose that the polytope $UFLP'(G)$ is defined by the linear system

$$Ax + By \leq b. \tag{41}$$

In our case, the system above is known and is equivalent to inequalities (2)–(5) and (10); this follows from Theorem 12. We have seen in Theorem 21 that to obtain the polytope $pMP'(G)$, the convex hull of the solutions of the problem (1)–(8), of a graph with no fork it is enough to add the equation

$$\sum_{v \in V} y(v) = p. \tag{42}$$

Thus the problem (1)–(8) is equivalent to

$$\max cx + fy, \tag{43}$$
$$Ax + By \leq b, \tag{44}$$
$$\sum_{v} y(v) = p. \tag{45}$$

We plan to use Lagrangian relaxation to solve this linear program, i.e., we dualize (45). For $\lambda \in \mathbb{R}$, let

$$g(\lambda) = \max \left\{ cx + fy + \lambda \left(p - \sum_{v \in V} y(v) \right) \ \mid Ax + By \leq b \right\}. \tag{46}$$

The function g is convex and piecewise linear. The following lemma is a well-known property of Lagrangian relaxation.

LEMMA 28. *Let γ be the optimal value of* (43)–(45)*. Then*

- $\gamma \leq g(\lambda)$ *for all* λ,
- *and*

$$\min_{\lambda} g(\lambda) = \gamma.$$

Proof. The dual of (43)–(45) is

$$\min \lambda p + \mu b, \tag{47}$$
$$\mu A = c, \tag{48}$$
$$\mu B + \lambda 1 = f. \tag{49}$$

If we fix $\lambda = \bar{\lambda}$, then (47)–(49) becomes

$$\begin{aligned} \min \mu b + \bar{\lambda} p, \\ \mu A = c, \\ \mu B = f - \bar{\lambda} 1, \end{aligned}$$

which is the dual of

$$\begin{aligned} \max cx + (f - \bar{\lambda} 1) y + \bar{\lambda} p, \\ Ax + By \leq b. \end{aligned}$$

This is $g(\bar{\lambda})$. So $g(\bar{\lambda}) \geq \gamma$. If $(\hat{\lambda}, \hat{\mu})$ is an optimal solution of (47)–(49), then $g(\hat{\lambda}) = \gamma$. ∎

In what follows we discuss how to find the minimum of g. Let $M = \sum |c(u,v)| + \sum |d(v)|$. We set $\lambda_1 = M$ and $\lambda_2 = -M$, and we compute $g(\lambda_1)$ and $g(\lambda_2)$. Let (x_1, y_1) and (x_2, y_2) be the corresponding solutions. If the problem is feasible, we should have $\sum y_1(v) \le p$ and $\sum y_2(v) \ge p$. We compute $\bar\lambda$ as the solution of

$$cx_1 + fy_1 + \lambda\left(p - \sum_v y_1(v)\right) = cx_2 + fy_2 + \lambda\left(p - \sum_v y_2(v)\right),$$

and we compute $g(\bar\lambda)$. Let $(\bar x, \bar y)$ be the solution obtained. We have the following two cases:

- $g(\bar\lambda) > cx_1 + fy_1 + \bar\lambda(p - \sum_v y_1(v)) = cx_2 + fy_2 + \bar\lambda(p - \sum_v y_2(v))$. Here we have three subcases:
  - If $\sum \bar y(v) = p$, we stop; then $(\bar x, \bar y)$ is the desired solution.
  - If $\sum \bar y(v) > p$, we replace λ_1 by $\bar\lambda$.
  - If $\sum \bar y(v) < p$, we replace λ_2 by $\bar\lambda$.

  In the last two subcases we repeat the above procedure.
- $g(\bar\lambda) = cx_1 + fy_1 + \bar\lambda(p - \sum_v y_1(v)) = cx_2 + fy_2 + \bar\lambda(p - \sum_v y_2(v))$. Then we stop; we have found the minimum of f. However, we could have $\sum \bar y(v) \ne p$. We discuss this situation below.

LEMMA 29. *Let* $p_1 = \sum y_1(v)$, $p_2 = \sum y_2(v)$ *and assume* $p_1 < p < p_2$. *Let* $\alpha = (p_2 - p)/(p_2 - p_1)$. *Then*

$$(\hat x, \hat y) = \alpha(x_1, y_1) + (1-\alpha)(x_2, y_2)$$

is an optimal solution of (43)-(45).

Proof. Let $\gamma_1 = cx_1 + fy_1$, $\gamma_2 = cx_2 + fy_2$. Then $\gamma_1 + \bar\lambda(p - p_1) = \gamma_2 + \bar\lambda(p - p_2)$, and $\bar\lambda = (\gamma_2 - \gamma_1)/(p_2 - p_1)$. Thus

$$g(\bar\lambda) = \gamma_1 \frac{p_2 - p}{p_2 - p_1} + \gamma_2 \frac{p - p_1}{p_2 - p_1}.$$

Since

$$c\hat x + f\hat y = \gamma_1 \frac{p_2 - p}{p_2 - p_1} + \gamma_2 \frac{p - p_1}{p_2 - p_1},$$

we have that $(\hat x, \hat y)$ is optimal. ∎

Remark that $(\hat x, \hat y)$ is exactly the solution $(\bar x, \bar y)$ defined in the sufficiency part of the proof of Theorem 21. Thus $(\hat x, \hat y)$ can be written as a convex combination of the vectors in the family $\mathcal{F}$ that are feasible for (43)–(45). Then any of these integer vectors is an optimal solution of (43)–(45).

5.3. Solving the problems UFLP and *p*MP. Recall that the problems UFLP and pMP are, respectively, obtained from UFLP′ and pMP′ by replacing inequalities (2) by equalities. To ensure this we add a value "big M" to the weights $c(u,v)$. This number can be

$$M = \max\{|f(v)| : v \in V\} + \max\{|c(u,v)| : (u,v) \in A\} + 1.$$

This will ensure that every customer is assigned to a location with an opened facility if the problem is feasible. Then we apply the algorithms described in this section. If γ is the optimal value with the new weights, then $\gamma - |U|M$ is the optimal value with the original weights.

6. Final remarks. We have shown that $UFLP'(G)$ has the intersection property if and only if G has no fork. When the graph has no odd cycle, then inequalities (2)–(5) define $UFLP'(G)$; cf. [4]. This remains true if in addition G has no fork. If G has odd cycles but no fork, we have to add inequalities (10). Indeed the odd cycle inequalities are not enough as shown in Figure 3.

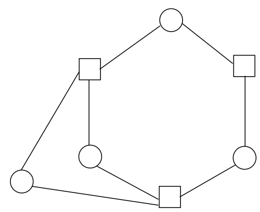


FIG. 3. *A bipartite graph $G = (U \cup V, A)$. The squares (resp., circles) are the nodes in U (resp., V).*

Let $x(u, v) = \frac{1}{3}$ for each arc $(u, v) \in A$ and $y(v) = \frac{1}{3}$ for each node $v \in V$. Then (x, y) satisfies (2)–(5) and satisfies inequalities (10) that correspond to odd cycles. However, it violates an inequality (10) when $F = A$.

Our results show the importance of inequalities (10) for facility location polytopes.

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