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Solutions of some simple boundary value problems within the context of a new class of elastic materials

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A R T I C L E I N F O

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A B S T R A C T

Some simple boundary value problems are studied, for a new class of elastic materials, wherein deformations are expressed as non-linear functions of the stresses. Problems involving 'homogeneous' stress distributions and one-dimensional stress distributions are considered. For such problems, deformations are calculated corresponding to the assumed stress distributions. In some of the situations, it is found that non-unique solutions are possible. Interestingly, non-monotonic response of the deformation is possible corresponding to monotonic increase in loading. For a subclass of models, the strain–stress relationship leads to a pronounced strain-gradient concentration domain in the body in that the strains increase tremendously with the stress for small range of the stress (or put differently, the gradient of the strain with respect to the stress is very large in a narrow domain), and they remain practically constant as the stress increases further. Most importantly, we find that for a large subclass of the models considered, the strain remains bounded as the stresses become arbitrarily large, an impossibility in the case of the classical linearized elastic model. This last result has relevance to important problems in which singularities in stresses develop, such as fracture mechanics and other problems involving the application of concentrated loads.

1. Introduction

It has been shown recently that if by elastic bodies we mean bodies that are incapable of dissipating energy in any process to which these bodies are subject to, then this class is far larger than Cauchy elastic or Green elastic bodies (see Rajagopal [1,2], Rajagopal and Srinivasa [3,4] for a detailed discussion of the rationale for such models). Elastic bodies need not only be characterized by providing an explicit expression for the stress as a function of the deformation gradient (Cauchy elastic bodies), or by assuming that there exists a stored energy that depends on the deformation gradient (Green elastic bodies). It is possible that elastic bodies could be defined by implicit constitutive relationships between the stress and the deformation gradient, and the stored energy could be a function of both the stresses and the deformation gradient. In fact, the models could be even more general in that one need not even define a deformation gradient in order to define elastic bodies. For instance, Rajagopal and Srinivasa [4] provide rate equations wherein the symmetric part of the velocity gradient and the rate of the Green–Saint Venant strain are related by implicit relations.

Based on the work of Rajagopal [1,2], Bustamante and Rajagopal [5] developed equations governing plane stress and plane strain for constitutive relations wherein the linearized strain bears a non-linear relationship to the stress. Such a model can be rigorously justified within the context of models, where the stretch tensor is a non-linear function of the stress or within the context of theories wherein the stress and the stretch are implicitly related (see Rajagopal [1]). Bustamante and Rajagopal [5] showed that even for constitutive models where the linearized strain depends non-linearly on the stress, one could introduce an Airy stress function which automatically satisfies the equations of equilibrium and the compatibility equation reduces to a non-linear partial differential equation for the Airy stress function. In three-dimensional problems one could introduce the stress potential and the compatibility equations would then reduce to a system of non-linear partial differential equations for the stress potential (see Finzi [6], Truesdell and Toupin [7]).

Even in two-dimensional problems, as the governing equations are very complicated and not amenable to analysis, Bustamante and Rajagopal [5] developed a weak formulation within which numerical calculations could be carried out. Bustamante [8] subsequently extended the analysis in Bustamante and Rajagopal [5], both within the context of developing governing equations as well as numerical analysis. However, in none of the above studies involving this new class of elastic materials were any specific

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boundary value problems solved. Recently, Rajagopal [9] considered a specific model that exhibits limiting strain in the presence of arbitrarily large stresses and solved some very simple boundary value problems, namely extension, shear, torsion, circumferential shear and telescopic shear as well as certain combinations of the same.

In this paper, we shall study several one-dimensional problems where the stresses are both homogeneous and inhomogeneous, within the context of this new class of elastic materials. Even in the case of simple uniaxial extension, for a subclass of models, the strain–stress relationship leads to a pronounced strain-gradient concentration domain, a region where the strains increase tremendously with respect to the stress. More importantly, for a large subclass of models, it is possible that the strain will remain bounded as the stress increases indefinitely. Such a response characteristic has bearing on important technological problems involving singularities in the stresses such as fracture mechanics and the application of concentrated loads. We also study the problem of biaxial extension when the stress fields are homogeneous. We consider four subclasses of constitutive relations wherein the linearized strain is a non-linear function of the stress. The most interesting feature of our study is the finding that even when the stress vary monotonically, in some models the strains vary non-monotonically. Moreover, we also find that for some problems, and in some models, the solution may be non-unique. The manifestation of such features in even one-dimensional problems warns us that the numerical resolution of complicated boundary value problems have to be carried out with a great deal of care. The numerical study of two- and three-dimensional problems, with the aim towards gaining an understanding of problems such as crack propagation is the final goal of our studies. As the models always present finite strain, limited to, however, small a value we choose it to be a priori, it presents a very appealing methodology to study fracture mechanics. It is apparent from the above response characteristics exhibited by the new class of models, that they greatly widen the scope of responses that can be described thereby enlarging the capability of the modeler to characterize the non-linear response of elastic bodies. We thus feel that it is worthwhile investigating the response characteristics of such bodies in some detail and this study is aimed at enhancing our understanding of this new class of elastic bodies.

The arrangement of the paper is as follows. In the next section, we provide a brief review of the basic framework and introduce the kinematics. We also introduce the counterpart to Green elastic materials of the classical theory of elasticity (see Bustamante [8] for a detailed discussion of such materials) and then develop the governing equations. In Section 3 we first introduce several different homogeneous states of stresses corresponding to different boundary value problems, namely those of simple uniaxial extension, simple shear and biaxial extension. In the case of the simple problem of uniaxial extension of a cylinder, it is possible within the context of the most general class of models, for the relationship between the stretch and the stress to be non-monotone. Such a response of course depends on the choice of the constitutive equation. While such non-monotone behavior might seem undesirable, it is possible that such behavior is indeed exhibited by real materials that show unstable response characteristics. Such behavior could also be associated with the fact that different types of response correspond to different phases of the material.[1] We also find that more than one solution for the strain could be found for the same state of stress. However, it is imperative to recognize that such instabilities or non-uniqueness relies intimately on the choice of the constitutive relation. It is possible to pick models that belong to the new class of elastic bodies that do not exhibit such non-monotone behavior or non-uniqueness of solutions. We follow this study of homogeneous states of stress with an analysis of some inhomogeneous stress states in one-dimensional problems and axially symmetric problems, in Section 4. We end the paper with some concluding remarks in Section 5.

[1] One should not think of the response as 'phase transition' as the phenomenon of 'phase transition' is entropy producing and an elastic body is incapable of producing entropy. Unfortunately there is a considerable amount of confusion in the field; the response of a mixture of two elastic bodies is being confused with the notion of 'phase transition', which is a dynamic process.

2. Basic equations

2.1. Kinematics

Let $\mathbf{X} \in \kappa_R(\mathcal{B})$ denote a particle belonging to body $\mathcal{B}$ in the reference configuration $\kappa_R(\mathcal{B})$, and let $\mathbf{x} \in \kappa_t(\mathcal{B})$ denote the position of the same particle in the current configuration $\kappa_t(\mathcal{B})$ at time t. We shall assume the mapping $\boldsymbol{\chi}$ which assigns the position $\mathbf{x}$ at time t, i.e., $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X},t)$ is sufficiently smooth to make all the derivatives that are taken, meaningful. The displacement $\mathbf{u}$ and the deformation gradient $\mathbf{F}$ are defined through $\mathbf{u}=\mathbf{x}-\mathbf{X}$ and $\mathbf{F} = \partial\boldsymbol{\chi}/\partial\mathbf{X}$. The Cauchy–Green stretch tensors $\mathbf{B}$ and $\mathbf{C}$ are defined through $\mathbf{B}=\mathbf{F}\mathbf{F}^{\mathrm{T}}$, $\mathbf{C}=\mathbf{F}^{\mathrm{T}}\mathbf{F}$, and the Green–Saint Venant strain $\mathbf{E}$ and the linearized strain $\boldsymbol{\varepsilon}$ are defined through

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{C}-\mathbf{I}), \tag{1}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right) + \left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^{\mathrm{T}}\right]. \tag{2}$$

The above kinematical definitions are sufficient for our purpose. Within the context of the classical linearized theory of elasticity, which presumes that the displacement gradient is small, it does not matter whether we use (X,Y,Z) or (x,y,z) as variables, to within the order of the approximation being used.

2.2. Constitutive equations

Rajagopal and coworkers have demonstrated that the class of elastic solids is larger than the class of Cauchy and Green elastic materials (see [1–3]). These new models include constitutive equations or relationships of the forms

$$\mathbf{B} = \boldsymbol{f}(\mathbf{T}), \quad \mathbf{g}(\mathbf{T},\mathbf{B}) = \mathbf{0}, \tag{3}$$

where in general $(3)_1$ might not be invertible and $(3)_2$ cannot be reduced to the class of models defined as Cauchy elastic.

Rajagopal [2] showed that when we consider small gradients for the displacement field, i.e., $\max_{\substack{X\in\mathcal{B}\\ t\in\mathbb{R}}}\|\nabla\mathbf{u}\| = O(\delta)$, $\delta \ll 1$, then from $(3)_1$ the appropriate approximation when we consider small strains but arbitrarily large stresses must be of the form

$$\boldsymbol{\varepsilon} = \boldsymbol{h}(\mathbf{T}). \tag{4}$$

In the case of an isotropic material we have (see [5])

$$\boldsymbol{\varepsilon} = \gamma_0\mathbf{I} + \gamma_1\mathbf{T} + \gamma_2\mathbf{T}^2, \tag{5}$$

where γ_0, γ_1 and γ_2 are scalar functions that depend, for example, on the invariants

$$I_T = \mathrm{tr}\mathbf{T}, \quad II_T = \tfrac{1}{2}[(\mathrm{tr}\mathbf{T})^2 - \mathrm{tr}(\mathbf{T}^2)], \quad III_T = \det\mathbf{T}. \tag{6}$$

Bustamante [8] showed that there is a subclass of materials for which there exists a scalar function $W = W(\mathbf{T})$ such that

$$\boldsymbol{\varepsilon} = \frac{\partial W}{\partial\mathbf{T}}. \tag{7}$$

In this paper we will consider constitutive equations of the form (5) and in Section 3.3.2 we will also consider constitutive equations of the type (7). We shall consider some special subclasses of materials belonging to the class (5) to highlight the difference between the response of such bodies and that exhibited by the classical linearized elastic body. We will consider the following four models, belonging to the class defined by (5) given by

$$\textbf{(a)}\quad \gamma_0=-\alpha I_T,\quad \gamma_1=\beta+f(I_T),\quad \gamma_2=0, \tag{8}$$

$$\textbf{(b)}\quad \gamma_0=-\alpha\left[1-\frac{1}{(1+\beta I_T)}\right],\quad \gamma_1=\frac{\alpha\gamma}{\sqrt{1+\iota(I_T^2-2II_T)}},\quad \gamma_2=0, \tag{9}$$

$$\textbf{(c)}\quad \gamma_0=-\alpha[1-e^{-\beta I_T}],\quad \gamma_1=\frac{\alpha\gamma}{1+\iota\sqrt{I_T^2-2II_T}},\quad \gamma_2=0, \tag{10}$$

$$\textbf{(d)}\quad \gamma_0=-\alpha I_T,\quad \gamma_1=\beta,\quad \gamma_2=\gamma, \tag{11}$$

where α,β,γ and ι are constants and $f=f(I_T)$ is a sufficiently smooth function that depends only on the first invariant. The different constants that appear above have different values for the four different constitutive equations. We consider the set of values shown in Table 1 for the cases (b) and (c).

The four different models have been chosen to illustrate and highlight the possibility that one could have interesting response features exhibited by the new class of models. For instance, models (b) and (c) exhibit limiting strains and strain boundary layers for the problems considered and for the range of the parameter values, while the model (a) does not. In Section 4.3 we consider the constitutive model (d), and we find that one can have non-unique solutions for certain simple inhomogeneous states of stresses, a situation that is not possible in the other models.

2.3. Boundary value problem

If the linearized strains are given as (usually non-invertible) functions of the stresses, then to have continuous displacement fields the linearized strain tensor must satisfy the compatibility equations (see [7]). For plane problems there is only one equation to be satisfied, namely

$$\varepsilon_{11|22}+\varepsilon_{22|11}-2\varepsilon_{12|12}=0. \tag{12}$$

The equilibrium equation for the stress tensor without body forces is $T^{ij}_{,i}=0$, and for plane problems they are automatically satisfied if we use the Airy stress function Φ defined through (see [10])

$$T^{11}=\Phi_{,22},\quad T^{22}=\Phi_{,11},\quad T^{12}=-\Phi_{,12}. \tag{13}$$

It follows from (5) and (13) that (12) reduces to a (in general non-linear) partial differential equation for Φ. In Cartesian coordinates the full form for this equation in terms of the components of the stress are given in Section 3.3 of [5].

We now provide Eqs. (12) and (13) for the case of Cartesian and polar coordinates. The appropriate equations can be found, for instance in [11,12].

Table 1
Values for the constants that appear in the models (b) and (c).

α	10^{-3}			
β	10^{-7}	10^{-5}	10^{-3}	10^{-1}
γ	10^{-3}	10^{-2}	10^{-1}	1
ι	10^{-2}	10^{-1}	1	10^2

Here α is dimensionless and the unit for the constants β and γ is (Pa^{-1}), and for ι is (Pa^{-2}) in model (b) and (Pa^{-1}) in model (c).

Cartesian coordinates: The compatibility equation (12) and the representations (13) become

$$\frac{\partial^2\varepsilon_{xx}}{\partial y^2}+\frac{\partial^2\varepsilon_{yy}}{\partial x^2}=2\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y}, \tag{14}$$

and

$$T_{xx}=\frac{\partial^2\Phi}{\partial y^2},\quad T_{yy}=\frac{\partial^2\Phi}{\partial x^2},\quad T_{xy}=-\frac{\partial^2\Phi}{\partial x\partial y}, \tag{15}$$

respectively.

Polar coordinates: In case we use polar coordinates (r,θ), (12) and (13) become

$$\frac{\partial^2\varepsilon_{rr}}{\partial\theta^2}+r^2\frac{\partial^2\varepsilon_{\theta\theta}}{\partial r^2}-2r\frac{\partial\varepsilon^2_{r\theta}}{\partial r\partial\theta}-2\frac{\partial\varepsilon_{r\theta}}{\partial\theta}+2r\frac{\partial\varepsilon_{\theta\theta}}{\partial r}-r\frac{\partial\varepsilon_{rr}}{\partial r}=0. \tag{16}$$

and

$$T_{rr}=\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2},\quad T_{\theta\theta}=\frac{\partial^2\Phi}{\partial r^2},\quad T_{r\theta}=\frac{1}{r^2}\frac{\partial\Phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\theta}. \tag{17}$$

Remark. In Section 4 we consider plane stress problems in Cartesian and polar coordinates. We assume that the body under consideration is a very thin flat plate in the z-direction. The external loads do not have components in the direction z, as a result $T_{zz}=0$ (and the same happens with the shear components of the stress in the z-direction). From (5) we see that the components of the stress tensor being zero in the z-direction does not necessarily mean that the strain $\varepsilon_{zz}=0$. However, since we work with very thin bodies, we assume that ε_{zz} is very small, and as a result only (14) and (16) need to be solved in our analysis. Similar assumptions are made within the context of the linearized theory of elasticity, for example, see Section 3.2 of [5] for a similar discussion.

3. Homogeneous deformations

As indicated in the Introduction, before attempting to solve complicated two-dimensional problems, we start by studying the predictions of the constitutive equations that we have chosen within the context of some simple problems. By analyzing the behavior of the deformation for these simple problems, we hope to identify difficulties such as 'instability' and/or non-uniqueness of solutions for a given problem when the physical problem presents no such behavior. This knowledge may help in developing the appropriate numerical methods for two-dimensional problems for these models, as we will be solving non-linear differential equations using iterative schemes, which in some cases may not converge.

3.1. Uniform tension in a cylinder

The first 'homogenous' stress state we consider corresponds to the case of a cylinder under 'uniform' tension. In this case, using a cylindrical coordinate system, we assume that the only non-zero component of the stress tensor is the normal axial component $T_{zz}=\sigma$. In this case the invariants (6) are given by

$$I_T=\sigma,\quad II_T=0,\quad III_T=0. \tag{18}$$

From (5) the axial component of the strain tensor ε_{zz} is

$$\varepsilon_{zz}=\gamma_0(\sigma)+\gamma_1(\sigma)\sigma+\gamma_2(\sigma)\sigma^2. \tag{19}$$

Eq. (19) is the most general expression for the z-component of the strain for this boundary value problem. Since ε_{zz} and the other components of the strain tensor are constant, the compatibility equations are satisfied automatically.

It could be possible that for some functions γ_i, $i=0,1,2$ the function ε_{zz} may not be a monotonically increasing quantity, which would mean that for some values of σ, smaller values for the axial deformation could be obtained for larger values of the external loads, which is somewhat contrary to physical expectation. However, such a situation might be possible due to some instability associated with the body's response.

From (19) we have

$$\frac{d\varepsilon_{zz}}{d\sigma}=\gamma_1(\sigma)+\frac{d\gamma_0}{d\sigma}(\sigma)+\left[\frac{d\gamma_1}{d\sigma}(\sigma)+2\gamma_2(\sigma)\right]\sigma+\frac{d\gamma_2}{d\sigma}\sigma^2. \quad (20)$$

Remark. Within the context of our theory, the linearized theory of elasticity would be a special case of (5), where the stress bears a linear relation to the linearized strain. It is the non-dimensional version of the stress that is small. That is we can define a dimensionless quantity $\mathbf{T}/T_0$, where T_0 would have physical units of stress, and could be used as a parameter to compare, for example, with the stresses such that the linearization of (4) would be valid.

Thus, when $\sigma \to 0$ Eq. (20) would become the same as in the classical linearized theory, and so evaluating (20) at $\sigma=0$ we can assume that

$$\frac{d\varepsilon_{zz}}{d\sigma}(0)=\frac{d\gamma_0}{d\sigma}(0)+\gamma_1(0)>0. \quad (21)$$

If γ_0, γ_1 and γ_2 are continuous functions with continuous derivatives at $\sigma=0$, then there exists an interval for σ around 0 such that $d\varepsilon_{zz}/d\sigma>0$. We would like now to investigate if there exists a 'critical' value $\sigma_{cr}>0$ such that

$$\frac{d\varepsilon_{zz}}{d\sigma}(\sigma_{cr})=\gamma_1(\sigma_{cr})+\frac{d\gamma_0}{d\sigma}(\sigma_{cr})+\left[\frac{d\gamma_1}{d\sigma}(\sigma_{cr})+2\gamma_2(\sigma_{cr})\right]\sigma_{cr}+\frac{d\gamma_2}{d\sigma}\sigma_{cr}^2=0. \quad (22)$$

For the three prototypical constitutive equations (8)–(10) we shall investigate the consequence of the inequality (21) and the above condition (22).

Model (*a*): In this case from (8) we have $\gamma_0=-\alpha\sigma$ and $\gamma_1=\beta+f(\sigma)$ and (21) holds if

$$-\alpha+\beta+f(0)>0. \quad (23)$$

In the linearized theory we have $\alpha=\nu/E$, $\beta=(1+\nu)/E$ and $f=0$, where E and ν are the Young and Poisson moduli. In this case (23) simply implies $\nu/E>0$, which holds trivially.

Regarding (22) this becomes

$$\beta+f(\sigma_{cr})-\alpha+\frac{df}{d\sigma}(\sigma_{cr})\sigma_{cr}=0. \quad (24)$$

Model (*b*): In this case we have $\gamma_0=\alpha[1-1/(1+\beta\sigma)]$, $\gamma_1=\alpha\gamma/\sqrt{1+\iota\sigma^2}$ and $\gamma_2=0$, then from (19) we obtain that

$$\varepsilon_{zz}(\sigma)=\alpha\left[1-\frac{1}{(1+\beta\sigma)}+\frac{\gamma\sigma}{\sqrt{1+\iota\sigma^2}}\right], \quad (25)$$

and $d\gamma_0/d\sigma=\alpha\beta/(1+\beta\sigma)^2$, $d\gamma_1/d\sigma=-\alpha\gamma\iota\sigma/(1+\iota\sigma^2)^{3/2}$, and as a result the inequality (21) implies that $\alpha(\beta+\gamma)>0$, and if $\alpha>0$ it would imply that

$$\beta+\gamma>0. \quad (26)$$

As for (22) if $\alpha>0$ then after some manipulations we obtain that

$$g(\sigma_{cr})=\frac{\beta}{(1+\beta\sigma_{cr})^2}+\frac{\gamma}{(1+\iota\sigma_{cr}^2)^{3/2}}=0. \quad (27)$$

It is more illustrative to consider the behavior of $\varepsilon_{zz}(\sigma)$ and $g(\sigma)$ for a number of different values for the parameters than to study (27) analytically. Let us consider the values for (b) given in Table 1. In such a case, from (25) and (27) we obtain Figs. 1 and 2.

From Fig. 1 we see that for the two cases considered, for moderately large stresses, the strain tends to become constant.

The deformation ε_{zz} is small, and unlike the classical linearized theory, we notice that the normal strain in the z direction remains bounded as the stress increases. We would need to also verify if $\|\nabla \mathbf{u}\| \ll 1$ (see Section 2(c) of [8]).

From Fig. 2 we see that at least for the range considered for σ, and the different values for the constants, the function $g(\sigma)$ does not have any zero.

Model (*c*): For this last case, from (18) and (10) we have $\gamma_0=\alpha(1-e^{-\beta\sigma})$ and $\gamma_1=\alpha\gamma/(1+\iota|\sigma|)$, as a result from (19) we

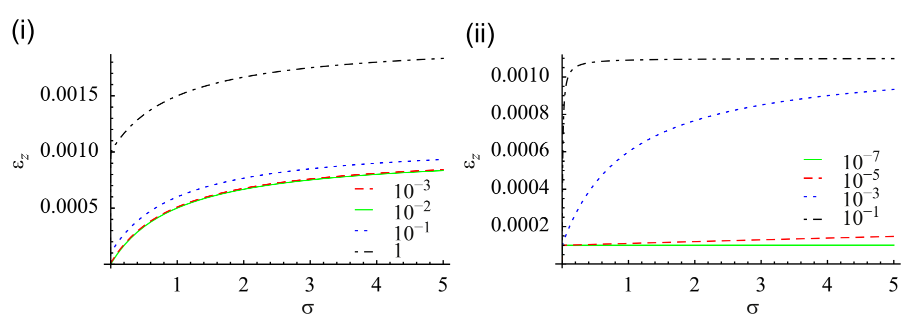

Fig. 1. Axial component of the strain for model (b), as a function of the uniform axial stress for different values for the constants in (25). The stress σ is in (kPa), for the three cases $\alpha=10^{-3}$. Case (i): $\beta=10^{-3}$ Pa^{-1}, $\iota=1$ Pa^{-2} and different values of γ (Pa^{-1}). Case (ii): $\gamma=10$ Pa^{-1}, $\iota=1$ Pa^{-2} and different values of β (Pa^{-1}).

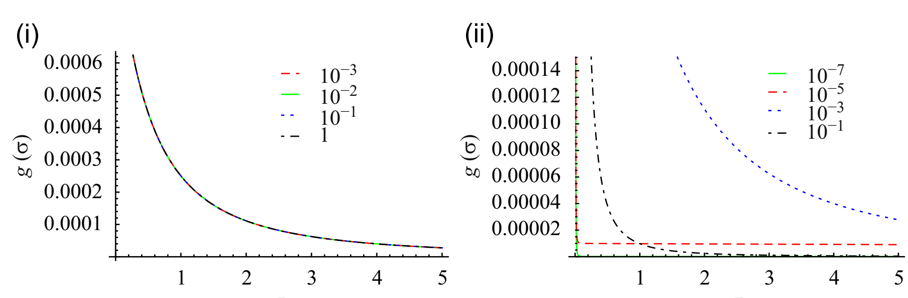

Fig. 2. Function $g(\sigma)$ for model (b), for different values for the constants in (27). The stress σ is in (kPa). Case (i): $\beta=10^{-3}$ Pa^{-1}, $\iota=1$ Pa^{-2} and different values of γ (Pa^{-1}). Case (ii): $\gamma=10$ Pa^{-1}, $\iota=1$ Pa^{-2} and different values of β (Pa^{-1}).

obtain that

$$\varepsilon_{zz}=\alpha\left[1-e^{-\beta\sigma}+\frac{\gamma\sigma}{1+\iota|\sigma|}\right]. \quad (28)$$

From (21) we obtain the same condition (26) if $\alpha>0$. Considering that the first derivative of γ_1 does not exist at $\sigma=0$ we assume $\sigma>0$, and in such a case we have $\mathrm{d}\gamma_1/\mathrm{d}\sigma=-\alpha\gamma\iota/(1+\iota\sigma)^2$ and from (22) we have

$$g(\sigma_{cr})=\beta e^{-\beta\sigma_{cr}}+\frac{\gamma}{(1+\iota\sigma_{cr})^2}=0. \quad (29)$$

As in the previous case, we plot the axial component of the strain (28) and the function $g(\sigma)$ for a number of cases for the parameters extracted from Table 1, the results are shown in Figs. 3 and 4.

The behavior exhibited here resembles the previous case closely with regards to the influence of the different constants, and in particular the non-existence of a critical value for σ.

3.2. Uniform shear stress distribution

The second 'homogeneous' stress state we consider is a uniform shear applied, for example, to a slab. In Cartesian coordinates we assume the stress distribution to be of the form

$$T_{xy}=\tau \quad \text{constant}, \quad T_{xy}=T_{yy}=0. \quad (30)$$

In this case from (6) we have $I_T=0$, $II_T=-\tau^2$ and $III_T=0$, and from (5) we obtain that

$$\varepsilon_{xy}=\gamma_1(\tau)\tau, \quad \varepsilon_{xx}=\varepsilon_{yy}=\gamma_0(\tau)+\gamma_2(\tau)\tau^2. \quad (31)$$

As we know for this kind of material the state of pure shear stress does not necessarily mean a state of pure shear deformation [5].

As in Section 3.1, we can assume that for $\tau\to 0$ the material should behave as a linearized isotropic solid. For such a material the derivative $\mathrm{d}\varepsilon_{xy}/\mathrm{d}\tau(0)$ should be positive. From $(31)_1$ we have the restriction

$$\gamma_1(0)>0. \quad (32)$$

We see here that in the linearized case $\gamma_1(0)$ would be simply the shear modulus. As for γ_0, a requirement would be $\gamma_0(0)=0$, so that we have only shear deformation in such a case.

If (32) holds, there is an interval for τ such that $\mathrm{d}\varepsilon_{xy}/\mathrm{d}\tau(\tau)>0$, and there could be a 'critical' shear stress τ_{cr} such that

$$\frac{\mathrm{d}\varepsilon_{xy}}{\mathrm{d}\tau}(\tau_{cr})=\frac{\mathrm{d}\gamma_1}{\mathrm{d}\tau}(\tau_{cr})\tau_{cr}+\gamma_1(\tau_{cr})=0. \quad (33)$$

We explore now the consequences of (32) and (33) for our three prototypical constitutive equations (8)–(10).

Model (*a*): In this case from (8) we have γ_0 and $\gamma_1=\beta+f(0)$ and (32) implies

$$\beta+f(0)>0. \quad (34)$$

With regards to (33) we see that there is no critical value, and if (34) holds then it holds for any τ.

Model (*b*): In this case from (30) and (9) we have $\gamma_0=0$ and $\gamma_1=\alpha\gamma/\sqrt{1+2\iota\tau^2}$, and for the shear component of the deformation we have

$$\varepsilon_{xy}=\frac{\alpha\gamma\tau}{\sqrt{1+2\iota\tau^2}}. \quad (35)$$

Fig. 5 shows how ε_{xy} varies with the shear stress for different values of $\alpha\gamma$ and ι. We once again notice that for sufficiently small values of τ the shear strain increases exceedingly sharply as a function of the shear stress, but then remains essentially constant, i.e., we have domains of steep gradients in the strain with respect to the stress.

Regarding (32), this inequality holds for $\alpha\gamma>0$, and from (35) and (33) we find that $-2\alpha\gamma\iota\tau_{cr}^2/(1+2\iota\tau_{cr}^2)^{3/2}+\alpha\gamma/\sqrt{1+2\iota\tau_{cr}^2}=0$, and this does not have solution.

Model (*c*): Finally from (10) we have $\gamma_0=0$ and $\gamma_1=\alpha\gamma/(1+\iota\sqrt{2}|\tau|)$, and we obtain

$$\varepsilon_{xy}=\frac{\alpha\gamma\tau}{1+\iota\sqrt{2}}|\tau|. \quad (36)$$

In Fig. 6 we display the variation of ε_{xy} with the stress and once again we find that for certain values of the parameters, we

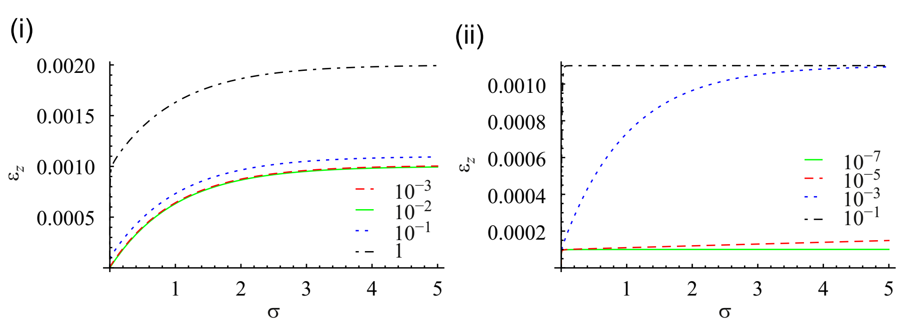

Fig. 3. Axial component of the strain for model (c), as a function of the uniform axial stress for different values for the constants in (28). The stress σ is in (kPa), for the three cases $\alpha=10^{-3}$. Case (i): $\beta=10^{-3}$ Pa^{-1}, $\iota=1$ Pa^{-1} and different values of γ (Pa^{-1}). Case (ii): $\gamma=10$ Pa^{-1}, $\iota=1$ Pa^{-1} and different values of β (Pa^{-1}).

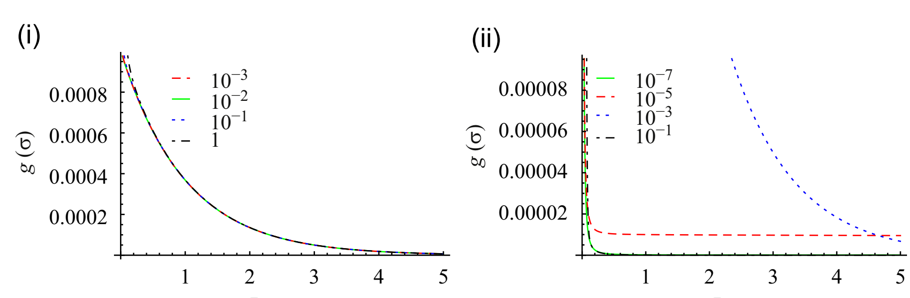

Fig. 4. Function $g(\sigma)$ for model (c), for different values for the constants in (29). The stress σ is in (kPa). Case (i): $\beta=10^{-3}$ Pa^{-1}, $\iota=1$ Pa^{-1} and different values of γ (Pa^{-1}). Case (ii): $\gamma=10$ Pa^{-1}, $\iota=1$ Pa^{-1} and different values of β (Pa^{-1}).

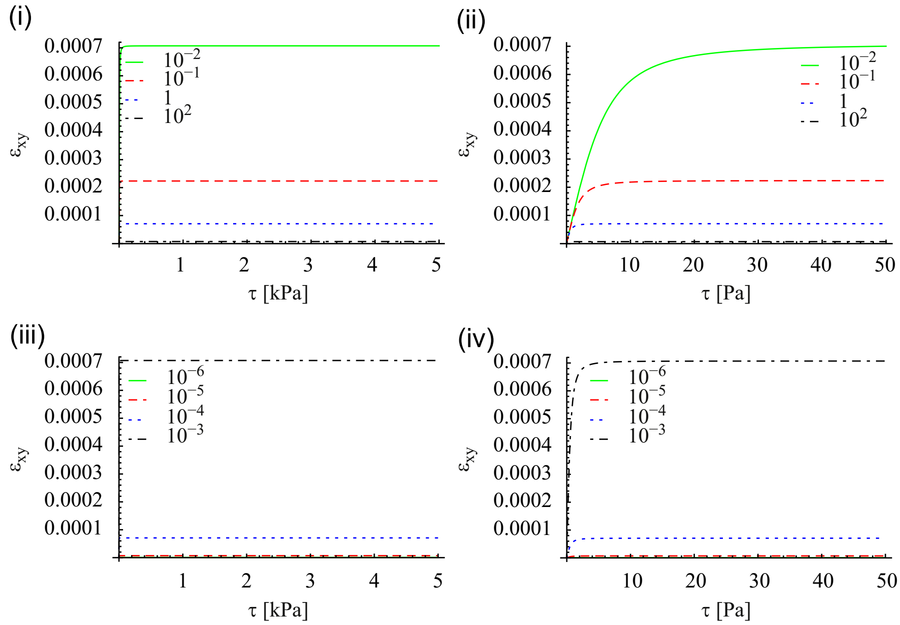

Fig. 5. Shear strain corresponding to uniform shear stress for model (b). (i) and (ii) are plots for different ranges for τ, when $\alpha\gamma = 10^{-4}$ Pa^{-1} for four different values of $\imath$ (Pa^{-2}). (iii) and (iv) are plots for different ranges for τ, when $\imath = 1$ Pa^{-2} for four different values of $\alpha\gamma$ (Pa^{-1}).

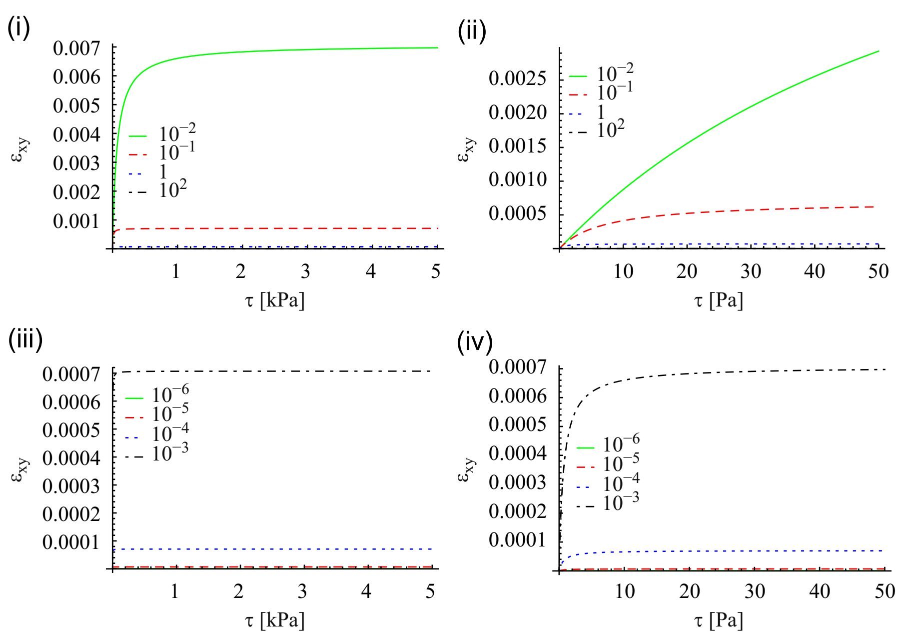

Fig. 6. Shear strain corresponding to uniform shear stress for model (c). (i) and (ii) are plots for different ranges for τ, when $\alpha\gamma = 10^{-4}$ Pa^{-1} for four different values of $\imath$ (Pa^{-1}). (iii) and (iv) are plots for different ranges for τ, when $\imath = 1$ Pa^{-1} for four different values of $\alpha\gamma$ (Pa^{-1}).

can have pronounced domains of strain gradient with respect to stress.

3.3. Biaxial extension

The biaxial extension of a thin plate, has been used in the classical theory of non-linear elasticity as a valuable tool to determine the stored energy function for isotropic hyperelastic materials (see, for example, Chapter 10 of [13]). It has also been used to study the instabilities that are manifested by some constitutive equations. In this section we explore these two issues for our new class of elastic materials. We assume that the plate is very thin in the z direction, within the context of a Cartesian coordinate system.

3.3.1. Behavior under uniform tension

Consider the case when we have the same external load applied in the x and y directions. We have

$$T_{xx} = T_{yy} = \sigma, \quad T_{xy} = 0. \tag{37}$$

For this problem it follows from (6) that $I_T = 2\sigma$ and $II_T = \sigma^2$. Therefore from (5), the components of the strain are

$$\varepsilon_{xx} = \varepsilon_{yy} = \gamma_0(\sigma) + \gamma_1(\sigma)\sigma + \gamma_2(\sigma)\sigma^2, \quad \varepsilon_{xy} = 0. \tag{38}$$

As in the case of the previous two boundary value problems, we shall assess the behavior dictated by (8)–(10), and in particular study situations when for increasing external load σ the deformations ε_{xx}, ε_{yy} may not be increasing functions.

Without loss of generality let us consider only the component ε_{xx}. From $(38)_1$ we have

$$\frac{d\varepsilon_{xx}}{d\sigma}(\sigma)=\frac{d\gamma_0}{d\sigma}(\sigma)+\frac{d\gamma_1}{d\sigma}(\sigma)\sigma+\gamma_1(\sigma)+\frac{d\gamma_2}{d\sigma}(\sigma)\sigma^2+2\gamma_2(\sigma)\sigma. \tag{39}$$

Using arguments similar to those used in the case of the simple tension of a cylinder and the shear problem, we assume that when the 'limit' $\sigma\to 0$ the material should behave as a classical linearized elastic body, and in this case we impose the condition $d\varepsilon_{xx}/d\sigma(0)>0$ (we are assuming the plate can be only in tension), therefore from $(38)_1$ we have the restriction

$$\frac{d\gamma_0}{d\sigma}(0)+\gamma_1(0)>0. \tag{40}$$

A critical point (if any), for which $d\varepsilon_{xx}/d\sigma$ ceases to be an increasing function of σ, can be found from (39) as

$$\frac{d\gamma_0}{d\sigma}(\sigma_{cr})+\frac{d\gamma_1}{d\sigma}(\sigma_{cr})\sigma_{cr}+\gamma_1(\sigma_{cr})+\frac{d\gamma_2}{d\sigma}(\sigma_{cr})\sigma_{cr}^2+2\gamma_2(\sigma_{cr})\sigma_{cr}=0. \tag{41}$$

Model (*a*): For this constitutive equation from (39) and (37) we have $\gamma_0=-2\alpha\sigma$ and $\gamma_1=\beta+f(2\sigma)$. As a result, (40) holds if and only if

$$-2\alpha+\beta+f(0)>0, \tag{42}$$

while a 'critical' value for the stress σ_{cr} (if any) is found from (41) as

$$\frac{d}{d\sigma}(\sigma f(2\sigma))\bigg|_{\sigma=\sigma_{cr}}=2\alpha-\beta. \tag{43}$$

Model (*b*): For this constitutive equation, it follows from (9) and (37) that $\gamma_0=\alpha[1-1/(1+\beta\sigma)]$ and $\gamma_1=\alpha\gamma/\sqrt{1+2\imath\sigma^2}$, and for the strain we have

$$\varepsilon_{xx}=\alpha\left[1-\frac{1}{(1+\beta\sigma)}+\frac{\gamma\sigma}{\sqrt{1+2\imath\sigma^2}}\right]. \tag{44}$$

The restriction (40) for this constitutive equation leads to $\alpha(2\beta+\gamma)>0$, and if $\alpha>0$ then this implies that

$$2\beta+\gamma>0. \tag{45}$$

We can compare this inequality with (26).

The condition concerning the existence of a critical value of the stress (39), for this problem becomes $2\beta/(1+2\beta\sigma_{cr})^2+\alpha\gamma/(1+2\imath\sigma_{cr}^2)^{3/2}=0$. Since this last expression and (44) are very similar in form to (25) and (27), we do not provide figures for the same.

Model (*c*): For this constitutive equation it follows from (10) that $\gamma_0=\alpha[1-e^{-2\beta\sigma}]$ and $\gamma_1=\alpha\gamma/(1+\sqrt{2}\imath|\sigma|)$, as a result

$$\varepsilon_{xx}=\alpha\left[1-e^{-2\beta\sigma}+\frac{\gamma\sigma}{1+\sqrt{2}}\imath|\sigma|\right]. \tag{46}$$

Restriction (40) implies the same inequality as before, namely (45), under the same assumption $\alpha>0$ (see Section 3.1(c)). A critical value for the stress σ_{cr} is found from (39) as $2\beta e^{-2\beta\sigma_{cr}}+\gamma/(1+\sqrt{2}\imath\sigma_{cr})-\sqrt{2}\gamma\imath/(1+\sqrt{2}\imath\sigma_{cr})^2=0$.

As in the previous case (b), here we do not plot the above expression or the deformation (46), because they resemble closely that provided by in Figs. 3 and 4 in Section 3.1.

3.3.2. *Remarks on the constitutive functions and the biaxial extension problem*

In analogy to the question that we ask in the classical non-linear theory of elasticity, here we ask ourselves the question: Can we find completely all the functions that appear in a constitutive equation of the form $\boldsymbol{\varepsilon}=\alpha_0\mathbf{I}+\alpha_1\mathbf{T}+\alpha_2\mathbf{T}^2$ using the biaxial test?

We consider a thin plate subjected now to two (in general different) uniform tensile loads, say

$$T_{xx}=\sigma_x,\quad T_{yy}=\sigma_y,\quad \text{and}\quad T_{xy}=0, \tag{47}$$

where σ_x and σ_y are constants. Instead of working with a constitutive equation of the form (5), we consider the Green-like elastic material proposed by Bustamante [8]. We have $\boldsymbol{\varepsilon}=\partial W/\partial\mathbf{T}$, and for an isotropic material $W=W(\tilde{I}_T,\tilde{II}_T,\tilde{III}_T)$, where

$$\tilde{I}_T=\mathrm{tr}\mathbf{T},\quad \tilde{II}_T=\tfrac{1}{2}\,\mathrm{tr}\mathbf{T}^2,\quad \tilde{III}_T=\tfrac{1}{3}\mathrm{tr}\mathbf{T}^3. \tag{48}$$

It can be shown that (see Section 4 of [8])

$$\boldsymbol{\varepsilon}=W_1\mathbf{I}+W_2\mathbf{T}+W_3\mathbf{T}^2, \tag{49}$$

where $W_1=\partial W/\partial\tilde{I}_T$, $W_2=\partial W/\partial\tilde{II}_T$ and $W_3=\partial W/\partial\tilde{III}_T$.

From (47) and (48) we have $\tilde{I}_T=\sigma_x+\sigma_y$, $\tilde{II}_T=\frac{1}{2}(\sigma_x^2+\sigma_y^2)$ and $\tilde{III}_T=\frac{1}{3}(\sigma_x^3+\sigma_y^3)$. From (49) on using (47) we obtain that

$$\varepsilon_{xx}=W_1+W_2\sigma_x+W_3\sigma_x^2, \tag{50}$$

$$\varepsilon_{yy}=W_1+W_2\sigma_y+W_3\sigma_y^2, \tag{51}$$

and $\varepsilon_{xy}=0$. We see that we have three unknowns, namely W_1, W_2 and W_3, and only two Eqs. (50) and (51) to find them, which is not enough in general.

Bustamante [8, Section 5] found a condition for a function W such that a material of this class would be incompressible. In such a case we need to work with a W such that (see Section 2.2)

$$W=\overline{W}(I_T^*,II_T^*), \tag{52}$$

where $I_T^*=\tilde{II}_T-\frac{1}{6}\tilde{I}_T$ and $II_T^*=\tilde{III}_T+\frac{2}{27}\tilde{I}_T^3-\frac{2}{3}\tilde{I}_T\tilde{II}_T$. As a result we obtain

$$W_1=-\frac{1}{6}\frac{\partial\overline{W}}{\partial I_T^*}+\frac{2}{3}\left(\frac{1}{3}\tilde{I}_T^2-\tilde{II}_T\right)\frac{\partial\overline{W}}{\partial II_T^*}, \tag{53}$$

$$W_2=\frac{\partial\overline{W}}{\partial I_T^*}-\frac{2}{3}\tilde{I}_T\frac{1}{6}\frac{\partial\overline{W}}{\partial II_T^*}, \tag{54}$$

$$W_3=\frac{\partial\overline{W}}{\partial II_T^*}. \tag{55}$$

Now, the three derivatives W_1, W_2 and W_3 that appear in (50) and (51) depend only on two unknown functions $\partial\overline{W}/\partial I_T^*$ and $\partial\overline{W}/\partial II_T^*$, which can be found from the experiment in an unique way. Note the resemblance we have here with what happens in the classical non-linear theory of elasticity, when we work with incompressible materials for the biaxial extension problem [13].

4. One-dimensional problems. Non-homogeneous stress distributions

In the previous section we analyzed the response corresponding to a number of constitutive equations to some very simple homogeneous problems. The main conclusion was that at least for the problems considered, the linearized strain did not show any 'unusual' behavior, such as the strain decreasing while the stress increases.

In this section we continue assessing this new class of elastic bodies, but now for some problems with 'non-uniform' stress distributions. We study whether in some cases we could have more than one solution for a given problem, and investigate the implications of the same within the physical point of view.

As we expect fully three-dimensional problems corresponding to the models of interest to us lead to a very complicated non-linear partial differential equation (12), we only study problems where

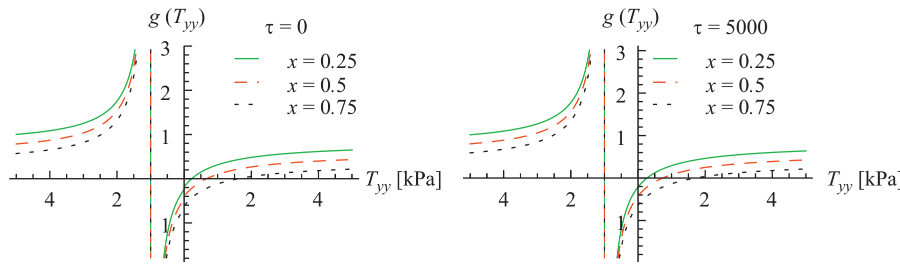

Fig. 7. Variation of the function $g(T_{yy})$ versus T_{yy} in the case of the constitutive function (9).

the stress is a function of one variable. Though simple, they are yet very interesting because in some cases we can obtain exact solutions for some particular constitutive equations.

We study two different classes of problems: those in Cartesian coordinates and in polar coordinates.

4.1. Problems involving Cartesian coordinates

Let us introduce the Airy stress function Φ, and study the case when this function is such that all the components of the stress (plane stress case) are only functions of x. Since $T_{xx}=T_{xx}(x)$, $T_{yy}=T_{yy}(x)$ and $T_{xy}=T_{xy}(x)$, from (15) we have $T_{xx}=\partial^2\Phi/\partial y^2$, $T_{yy}=\partial^2\Phi/\partial x^2$ and $T_{xy}=-\partial^2\Phi/\partial x\partial y$, we can prove that

$$\Phi=\varphi(x)+\tau xy, \tag{56}$$

where τ is constant. Therefore

$$T_{xx}=0,\quad T_{yy}=T_{yy}(x)=\frac{d^2\varphi}{dx^2},\quad T_{xy}=-\tau. \tag{57}$$

The compatibility equation (14) in this case reduces to

$$\frac{d^2\varepsilon_{yy}}{dx^2}=0, \tag{58}$$

whose solution is $\varepsilon_{yy}=c_1x+c_0$, where c_1 and c_0 are constants. From (5) we have $\varepsilon_{yy}=\gamma_0+\gamma_1T_{yy}(x)+\gamma_2[\tau^2+T_{yy}(x)^2]$ and so we obtain the non-linear algebraic equation for $T_{yy}(x)$

$$\gamma_0+\gamma_1T_{yy}(x)+\gamma_2[\tau^2+T_{yy}(x)^2]=c_1x+c_0, \tag{59}$$

where we recall that γ_0, γ_1 and γ_2 are in this case functions of $T_{yy}(x)$ and τ. From (6) we also have

$$I_T=T_{yy}(x),\quad II_T=-\tau^2,\quad III_T=0. \tag{60}$$

Note that for the linear case where we have $\gamma_0=-(\nu/E)\mathrm{tr}\mathbf{T}=-(\nu/E)T_{yy}(x)$, $\gamma_1=(1+\nu)/E$ and $\gamma_2=0$, Eq. (59) becomes

$$-\frac{\nu}{E}T_{yy}(x)+\frac{1+\nu}{E}T_{yy}(x)=c_1x+c_0, \tag{61}$$

which implies $T_{yy}(x)=E(c_1x+c_0)$. Let us consider a plate defined by $0\le x\le A$ and $y=\pm B/2$ and the boundary conditions $T_{yy}(0)=T_{yy}(A)=T_0$. From (61) we infer that $c_1=0$ and $c_0=T_0/E$, which is a trivial result.

We now explore the solutions for (59) for the three constitutive equations given in (8)–(10) for the same plate defined earlier. We will study two boundary conditions, the one already mentioned $T_{yy}(0)=T_{yy}(A)=T_0$ and $T_{yy}(0)=0$, $T_{yy}(A)=T_0$.

For the above stress distribution we consider the additional example (11), for which a non-unique solution is found in Section 4.3.

Model (*a*): We need to solve (59) in the particular case, where from (8) and (60) we obtain that $\gamma_0=-\alpha T_{yy}(x)$ and $\gamma_1=\beta+f(T_{yy}(x))$, and as a result (59) becomes

$$f(T_{yy})T_{yy}+(\beta-\alpha)T_{yy}=c_1x+c_0. \tag{62}$$

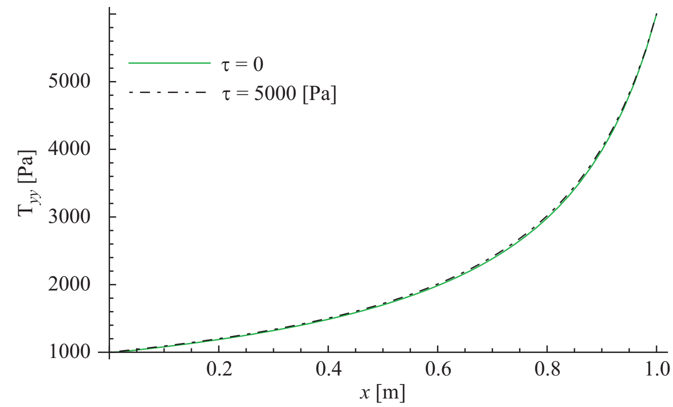

Fig. 8. Solution for the case of inhomogeneous stress distribution in a plate for the constitutive function (9).

For the given boundary conditions and function f we could have more than one solution for the previous equation, but without a specific form for f we can not say anything else.

Model (*b*): For this problem from (9) and (60) we have $\gamma_0=\alpha[1-1/(1+\beta T_{yy})]$ and $\gamma_1=\alpha\gamma/\sqrt{1+\iota(T_{yy}^2+2\tau^2)}$, and it follows from (59) that we need to solve[2]

$$g(T_{yy})=1-\frac{1}{(1+\beta T_{yy})}+\frac{\gamma T_{yy}}{\sqrt{1+\iota(T_{yy}^2+2\tau^2)}}-\frac{1}{\alpha}(c_1x+c_0)=0. \tag{63}$$

For the case $T_{yy}(0)=0$, $T_{yy}(A)=T_0$ we obtain a numerical solution of (63). First, let us explore if for some parameters that appear in the constitutive equation (9) there is one or more solutions to (63). To do so, let us assume

$$\alpha=10^{-3},\quad \beta=10^{-3}\ \mathrm{Pa}^{-1},\quad \gamma=10^{-2}\ \mathrm{Pa}^{-1},\quad \iota=10^{-1}\ \mathrm{Pa}^{-2}$$

and $A=1$ m, $T_0=5000$ Pa and two values for τ, say 0 and 5000 Pa. In Fig. 7 we have plotted the function $g(T_{yy})$ for three values for x, say $x=0.25$, 0.5 and 0.75 m, to see how many solutions we have for T_{yy} for (63). From Fig. 7 we observe that at least for the values for x that has been explored, and for the range of T_{yy} considered, there is only one solution for (63).[3] We assume now (even though the above numerical example does not necessarily offer a proof of it) that for any $0<x<A$ there is one solution for (63). In Fig. 8 we have the solution of (63) for two cases with and without shear stress τ.

From Fig. 8 we first note that there is not much difference between the two cases $\tau=0$ or 5000 Pa. If we compare these results with the solution in the linearized case wherein we would have a linear distribution for T_{yy}, for the present model under

[2] Here function g has a different meaning than that in Eqs. (27) and (29).

[3] Note that the function $g(T_{yy})$ not only has one zero in the interval considered, but also a discontinuity at $T_{yy}=-1$ kPa, which is a consequence of the second term on the left hand side of (63), when $1+\beta T_{yy}=0$.

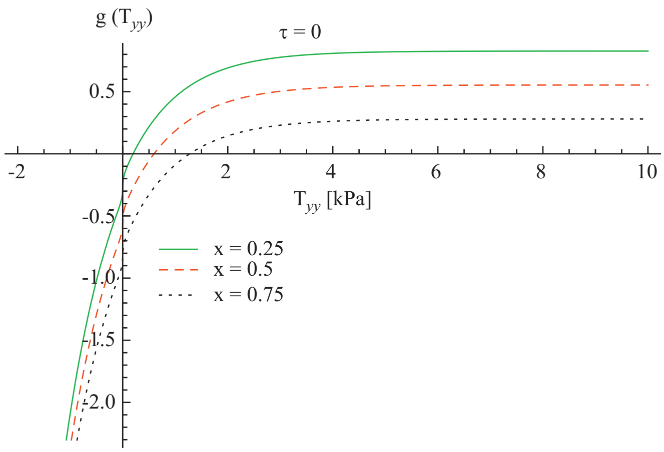

Fig. 9. Variation of $g(T_{yy})$ for the constitutive equation (10).

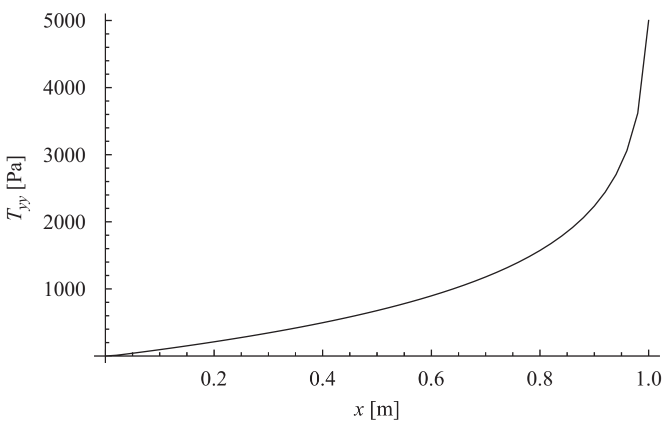

Fig. 10. Solution corresponding to Eq. (64).

consideration, we find that though the deformations are small, the solutions are non-linear.

Model (*c*): In this last case, from (10) and (60) we have $\gamma_0=\alpha[1-e^{-\beta T_{yy}}]$ and $\gamma_1=\alpha\gamma/(1+\iota\sqrt{T_{yy}^2+2\tau^2})$, and as a result from (59) we need to solve

$$g(T_{yy})=1-e^{-\beta T_{yy}}+\frac{\gamma T_{yy}}{1+\iota\sqrt{T_{yy}^2+2\tau^2}}-\frac{1}{\alpha}(c_1x+c_0)=0. \tag{64}$$

In Fig. 9 we can see the behavior of $g(T_{yy})$ for the case $\tau=0$ and for three different positions x. As the solutions are similar to the behavior of the function for the case $\tau=5000$ Pa, we do not provide figures for the same. We use the same values for the constants and other parameters as in case (b). We observe that for an ample range for T_{yy} the behavior of $g(T_{yy})$ is such that we have only one solution for (64). In Fig. 10 we see the behavior of $T_{yy}(x)$ for this constitutive equation when $\tau=0$.

4.2. Problems in polar coordinates

Let us consider the simplified case when $\Phi=\varphi(r)$ in (17), then $T_{rr}=(1/r)\mathrm{d}\varphi/\mathrm{d}r$, $T_{r\theta}=0$, $T_{\theta\theta}=\mathrm{d}^2\varphi/\mathrm{d}r^2$ and so

$$T_{\theta\theta}=T_{rr}+r\frac{\mathrm{d}T_{rr}}{\mathrm{d}r}, \tag{65}$$

which is equivalent to the well-known equation $\mathrm{d}T_{rr}/\mathrm{d}r=(T_{\theta\theta}-T_{rr})/r$. We assume now that the stress depends only on the radial position. In this case the invariants I_T and II_T (6) also depend on the radial position and so $\gamma_0=\gamma_0(r)$, $\gamma_1=\gamma_1(r)$ and $\gamma_2=\gamma_2(r)$. From (5) and the above results for the components of the deformation tensor we have

$$\varepsilon_{rr}=\gamma_0(r)+\gamma_1(r)T_{rr}(r)+\gamma_2(r)T_{rr}(r)^2, \tag{66}$$

$$\varepsilon_{\theta\theta}=\gamma_0(r)+\gamma_1(r)T_{\theta\theta}(r)+\gamma_2(r)T_{\theta\theta}(r)^2, \tag{67}$$

while $\varepsilon_{r\theta}=0$. For the above components, Eq. (16) becomes $r^2\mathrm{d}^2\varepsilon_{\theta\theta}/\mathrm{d}r^2+2r\mathrm{d}\varepsilon_{\theta\theta}/\mathrm{d}r-r\mathrm{d}\varepsilon_{rr}/\mathrm{d}r=0$, which is equivalent to

$$r\frac{\mathrm{d}\varepsilon_{\theta\theta}}{\mathrm{d}r}+\varepsilon_{\theta\theta}-\varepsilon_{rr}=c \quad \text{where } c \text{ is a constant.} \tag{68}$$

For the linearized case where $\gamma_0=-(\nu/E)(T_{rr}-T_{\theta\theta})$ and $\gamma_1=(1+\nu)/E$, the solution of (68) is simply $T_{rr}(r)=(cE/2)\ln r+c_0+c_1/r^2$.

Let us next consider the non-linear case. Now, the invariants $(6)_{1,2}$ are

$$I_T=T_{rr}+T_{\theta\theta}=2T_{rr}+r\frac{\mathrm{d}T_{rr}}{\mathrm{d}r},\quad II_T=T_{rr}T_{\theta\theta}=T_{rr}\frac{\mathrm{d}}{\mathrm{d}r}(rT_{rr}). \tag{69}$$

On substituting (60), (67) in (68), a lengthy but straightforward calculation leads to the following non-linear ordinary differential equation:

$$\begin{aligned}&r^2\gamma_1\frac{\mathrm{d}^2T_{rr}}{\mathrm{d}r^2}+2r^2\gamma_2T_{rr}\frac{\mathrm{d}^2T_{rr}}{\mathrm{d}r^2}+2r^3\gamma_2\frac{\mathrm{d}T_{rr}}{\mathrm{d}r}\frac{\mathrm{d}^2T_{rr}}{\mathrm{d}r^2}\\&\quad+\left(r^2\frac{\mathrm{d}\gamma_1}{\mathrm{d}r}+3r\gamma_1\right)\frac{\mathrm{d}T_{rr}}{\mathrm{d}r}+\left(2r^2\frac{\mathrm{d}\gamma_2}{\mathrm{d}r}+6r\gamma_2\right)T_{rr}\frac{\mathrm{d}T_{rr}}{\mathrm{d}r}\\&\quad+\left(r^3\frac{\mathrm{d}\gamma_2}{\mathrm{d}r}+5r^2\gamma_2\right)\left(\frac{\mathrm{d}T_{rr}}{\mathrm{d}r}\right)^2+r\frac{\mathrm{d}\gamma_1}{\mathrm{d}r}T_{rr}+r\frac{\mathrm{d}\gamma_2}{\mathrm{d}r}T_{rr}^2+r\frac{\mathrm{d}\gamma_0}{\mathrm{d}r}=c.\end{aligned} \tag{70}$$

We can see that it is highly unlikely that we can obtain an explicit analytical solution to the above non-linear equation.

Let us consider the two constitutive equations (9) and (10) for the above problem. We do not study the constitutive equation (a) since there is not much to do when one has to deal with a general function $f(I_T)$.

Model (*b*): For this case, by virtue of (9) and (69) we have

$$\gamma_0=\alpha\left\{1-\frac{1}{\left[1+\beta\left(2T_{rr}+r\frac{\mathrm{d}T_{rr}}{\mathrm{d}r}\right)\right]}\right\}, \tag{71}$$

$$\gamma_1=\frac{\alpha\gamma}{\sqrt{1+\iota\left[\left(2T_{rr}+r\frac{\mathrm{d}T_{rr}}{\mathrm{d}r}\right)^2-2T_{rr}\left(T_{rr}+r\frac{\mathrm{d}T_{rr}}{\mathrm{d}r}\right)\right]}}. \tag{72}$$

The numerical solution of (70) is depicted in Fig. 11.

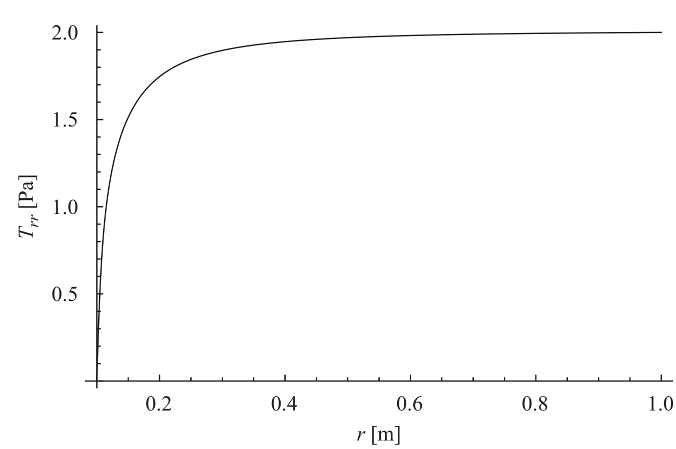

Fig. 11. Solution to (70) for the constitutive equation (9).

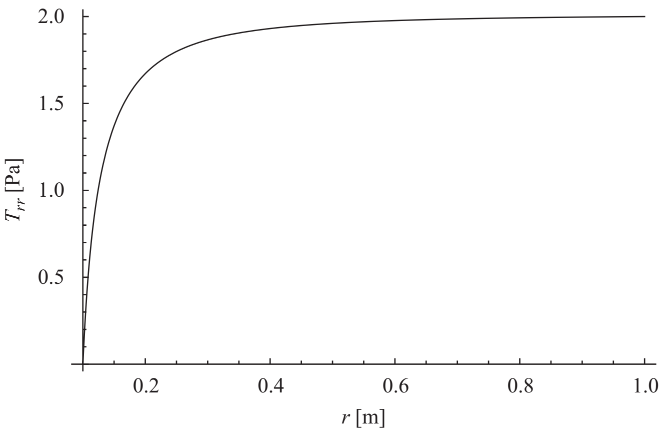

Fig. 12. Solution to (70) for the constitutive equation (10).

For this problem, as well as in the next case, we use $\alpha = 10^{-3}$, $\beta = 10^{-3}$ Pa^{-1}, $\gamma = 10^{-1}$ Pa^{-1} and $\imath = 1$ Pa^{-2}. The body under consideration is a thin circular plate with a circular hole at its centre. The inner and outer radii are 0.1 and 1 m, respectively, and we are applying a radial tension on the outer surface of the plate, while the inner surface is free of stress. The tension applied on the outer radius is 2 Pa. When the external load is, for example, 1000 Pa the numerical scheme for (70) using (71) and (72) does not converge (considering that in Section 4.1, when working in Cartesian coordinates we considered an external load of 5000 Pa). This difficulty with regards to obtaining a meaningful numerical solution which converges has important ramifications, especially if we intend to use (9) in more complex two-dimensional or three-dimensional problems.

Model (*c*): When we consider the constitutive equation (10) we obtain that

$$\gamma_0 = \alpha[1 - e^{-\beta(2T_{rr} + r\mathrm{d}T_{rr}/\mathrm{d}r)}], \tag{73}$$

$$\gamma_1 = \frac{\alpha\gamma}{\left[1 + \imath\sqrt{\left(2T_{rr} + r\dfrac{\mathrm{d}T_{rr}}{\mathrm{d}r}\right)^2 - 2T_{rr}\left(T_{rr} + r\dfrac{\mathrm{d}T_{rr}}{\mathrm{d}r}\right)}\right]}, \tag{74}$$

and the numerical solution of (70) is portrayed in Fig. 12.

The boundary value problem that is solved is the same as in (b), and the numerical values of the different constants in (73) and (74) are the same. In this problem, as in the previous case, we had the same numerical problems working with 'large' loads at the outer radius, and this is the reason we considered an external radial load of 2 Pa for the outer surface of the plate, and the stress free state for the inner surface. From Figs. 11 and 12 we can notice the behavior of T_{rr} is very similar for the models (b) and (c).

4.3. An example of non-uniqueness

It has been mentioned that for the class of materials described in this paper non-unique solutions are possible. No non-unique solutions were found so far with the examples of constitutive functions (8)–(10) with regard to the previous boundary value problems, and so in this section we would like to show this feature with a different specific example. Let us consider Eq. (5) for the particular case of model (d) presented in (11). Let us consider the non-homogeneous stress distribution described in Section 4.1. From (11) and (60) we have $\gamma_0 = -\alpha T_{yy}$, $\gamma_1 = \beta$ and $\gamma_2 = \gamma$. As a result (59) becomes

$$\gamma T_{yy}^2 + (\beta - \alpha)T_{yy} + \gamma\tau^2 - c_1 x - c_0 = 0, \tag{75}$$

which is a quadratic equation for T_{yy} with two solutions

$$T_{yy}(x) = \frac{1}{2\gamma}[\alpha - \beta \pm \sqrt{(\beta-\alpha)^2 - 4\gamma(\gamma\tau^2 - c_1 x - c_0)}]. \tag{76}$$

An interesting question is: What happens when $(\beta-\alpha)^2 - 4\gamma(\gamma\tau^2 - c_1 x - c_0) < 0$? Depending on the values of α, β, γ and τ, and also on the boundary conditions, we might have a situation where there could be no solution for the problem, if the above inequality holds for some x.

If we assume the boundary conditions $T_{yy}(0) = T_{yy}(A) = T_0$, then for the two solutions (76) we have $c_0 = \gamma(T_0^2 - \tau^2) + T_0(\beta - \alpha)$ and $c_1 = 0$. This case would be interesting since $c_1 = 0$ means T_{yy} would be constant with respect to x.

Finally, let us study the additional boundary condition $T_{yy}(0) = 0$, $T_{yy}(A) = T_0$. In such a case for both solutions (76) we have $c_0 = \alpha\tau^2$ and $c_1 = (T_0/A)(-\alpha + \beta + \gamma T_0)$.

5. Final remarks

In this paper we have studied some very simple homogeneous and inhomogeneous deformations concerning a new class of elastic bodies within the context of boundary value problems that depend on one spatial variable. This new class of elastic bodies hold much promise in the field of fracture mechanics and crack propagation as the strains that come into play are always small, a value that can be fixed a priori. The reason for looking at such simple problems that depend on one space variable is because we expect the non-linear constitutive relations to lead to governing equations that are very daunting. As expected, even in the case of the simple problems that we have considered, we found that the solutions are quite counter-intuitive. It is possible to find solutions to the case of simple uniaxial extension, where the strain varies non-monotonically with respect to the stress. We also found non-unique solutions within the context of other boundary value problems that we studied. The problems that we have studied portend several numerical challenges for problems in two and three dimensions for we could not obtain convergence when, for instance, the loads are reasonably large, even within the one-dimensional context.

The solutions that we found, in of themselves, offer some very interesting insights. We find that there are regions of concentration of strain gradients with respect to the stress, i.e., strain boundary layers. They are qualitatively markedly different from the solutions that are obtained in the case of linearized elastic solids. In view of the potential of the new class of models offering the possibility of resolving the inherent contradictions within the classical linearized elastic model in problems involving singular stresses, a more systematic and detailed study of two- and three-dimensional problems within the context of such models is warranted.

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