Applied Mathematics Letters 25 (2012) 352–356

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Threshold condition for global existence and blow-up to a radially symmetric drift–diffusion system☆

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ARTICLE INFO

Article history:
Received 25 October 2010
Received in revised form 25 August 2011
Accepted 7 September 2011

Keywords:
Blow-up
Drift–diffusion
Threshold condition
Parabolic equations

ABSTRACT

For a class of drift–diffusion systems Kurokiba et al. [M. Kurokiba, T. Nagai, T. Ogawa, The uniform boundedness and threshold for the global existence of the radial solution to a drift–diffusion system, Commun. Pure Appl. Anal. 5 (2006) 97–106.] proved global existence and uniform boundedness of the radial solutions when the L_1-norm of the initial data satisfies a threshold condition. We prove in this letter that this result prescribes a region in the plane of masses which is sharp in the sense that if the drift–diffusion system is initiated outside the threshold region of global existence, then blow-up is possible: suitable initial data can be built up in such a way that the corresponding solution blows up in a finite time.

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1. Introduction

A mathematical model for particles interacting via the gravitational potential is the following system of PDE's:

$$\begin{cases} \partial_t n - \Delta n + \nabla \cdot (n\nabla\psi) = 0 & t > 0,\ x \in \mathbb{R}^2 \\ \partial_t p - \Delta p - \nabla \cdot (p\nabla\psi) = 0 & t > 0,\ x \in \mathbb{R}^2 \\ -\Delta\psi = -(p-n) \quad x \in \mathbb{R}^2 \\ n(0,x) = n_0(x) \geq 0, \qquad p(0,x) = p_0(x) \geq 0 \quad x \in \mathbb{R}^2. \end{cases} \tag{1}$$

The initial value problem (1) is one of the most representative systems of the so-called drift–diffusion models. Kurokiba and Ogawa proved in [1] for system (1) local well-posedness, positiveness of the variables n and p, mass conservation and the following blow-up result.

Theorem 1 (*Blow-up in Finite Time*)**.** *Let $s > 1$ and*

$$L^2_s(\mathbb{R}^2) = \{f \in L^1_{loc}(\mathbb{R}^2);\ (1+|x|^2)^{s/2} f(x) \in L^2(\mathbb{R}^2)\}.$$

Let n_0 and p_0 be given in $L^2_s(\mathbb{R}^2)$ with $n_0, p_0 \geq 0$ everywhere, and satisfying

$$\frac{\left(\int_{\mathbb{R}^2}(n_0 - p_0)dx\right)^2}{\int_{\mathbb{R}^2}(n_0+p_0)dx} > 8\pi. \tag{2}$$

Then the solution of (1) blows up in a finite time.

☆ The authors acknowledge the Chilean & French Governments through a Conicyt-Inria Grant.

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doi:10.1016/j.aml.2011.09.013

The possibility of having global existence in time for system (1) whenever

$$\frac{\left(\int_{\mathbb{R}^2}(n_0 - p_0)dx\right)^2}{\int_{\mathbb{R}^2}(n_0 + p_0)dx} < 8\pi \tag{3}$$

was suggested by Kurokiba and Ogawa in [1] and it was partially proved by Kurokiba et al. in [2]. Specifically, they proved in particular the following.

Theorem 2 (*Boundedness and Global Existence*). *Let $s > 1$. Suppose that the initial data n_0 and $p_0 \in L^2_s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ are positive radially symmetric functions. If*

$$\|n_0\|_{L^1(\mathbb{R}^2)}, \qquad \|p_0\|_{L^1(\mathbb{R}^2)} < 8\pi, \tag{4}$$

then the corresponding radially symmetric solution $(n(t), p(t))$ exists globally in $C(0, \infty; L^2_s(\mathbb{R}^2))$. Moreover, there exists a constant C such that

$$\sup_{t\geq 0}\|n(t)\|_{L^\infty(\mathbb{R}^2)} \leq C, \qquad \sup_{t\geq 0}\|p(t)\|_{L^\infty(\mathbb{R}^2)} \leq C.$$

However, condition (3) results not to be sufficient to guarantee global existence as was proved by Espejo et al. in [3]. The aim of this letter is to show that although condition (3) does not represent a sufficient condition for global existence in time, the condition of global existence (4) is optimal (in the radial case). Precisely, we show that if θ_1 and θ_2 are arbitrary positive parameters satisfying

$$\theta_1 > 8\pi \quad \text{or} \quad \theta_2 > 8\pi,$$

then initial data n_0 and p_0 can be constructed in such a way that

$$\theta_1 = \int_{\mathbb{R}^2} n_0 dx, \qquad \theta_2 = \int_{\mathbb{R}^2} p_0 dx,$$

and system (1) blows up.

We find it worth pointing out that the blow-up result of Theorem 1 is valid even in the non-radial case; meanwhile Theorem 2 holds true only under radially symmetric conditions on the initial data. As already mentioned, condition (4) involves the constant 8π which results to be sharp. The problem of finding a similar optimal threshold condition under non-radial initial conditions remains open. Threshold-type conditions for a similar system, mainly for the Keller–Segel model of two species with non-radial initial conditions, are currently being investigated by Conca et al. (see [4], for example).

Notation. We denote $M_1(r, t) = \int_{B(0,r)} ndx = 2\pi\int_0^r n\rho d\rho$, $M_2(r, t) = \int_{B(0,r)} pdx = 2\pi\int_0^r p\rho d\rho$, $\theta_1 = \int_{\mathbb{R}^2} n_0 dx$ and $\theta_2 = \int_{\mathbb{R}^2} p_0 dx$. In terms of M_1 and M_2, system (1) reduces to

$$\left.\begin{aligned} \partial_t M_1 &= r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial M_1}{\partial r}\right) - \frac{M_2 - M_1}{2\pi r}\frac{\partial M_1}{\partial r} \\ \partial_t M_2 &= r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial M_2}{\partial r}\right) + \frac{M_2 - M_1}{2\pi r}\frac{\partial M_2}{\partial r} \end{aligned}\right\}. \tag{5}$$

2. Optimization of the blow-up condition

In order to simplify system (5) we prove first that under suitable conditions on the initial data, then either $M_1 \leq M_2$ or $M_1 \geq M_2$. This result will allow us to reduce our analysis to only one equation and then obtain our main result concerning blow-up.

Theorem 3 (*Mass Comparison*). *Suppose that the initial data n_0, $p_0 \in L^2_s(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$ of (1) are positive radially symmetric functions. If*

$$M_1(0, r) = \int_{\mathbb{R}^2} n_0 dx \geq \int_{\mathbb{R}^2} p_0 dx = M_2(0, r), \tag{6}$$

then for any solution in $C([0, T); L^2_s(\mathbb{R}^2)) \cap C([0, T); C^2(\mathbb{R}^2))$,

$$M_1(t, r) \geq M_2(t, r).$$

Proof. The idea of the proof is to formulate a suitable parabolic equation in the variable $R := \int_0^r (n - p)\rho d\rho$ and then apply the maximum principle to show that $R \geq 0$. With this end in mind, we first define the following variables:

$$v(t, x) = n(t, x) + p(t, x)$$
$$w(t, x) = n(t, x) - p(t, x).$$

It follows that (v, w) satisfies the following parabollic–elliptic system:

$$\partial_t v - \Delta v + \nabla(w\nabla\psi) = 0 \quad t > 0,\ x \in \mathbb{R}^2$$
$$\partial_t w - \Delta w + \nabla(v\nabla\psi) = 0 \quad t > 0,\ x \in \mathbb{R}^2$$
$$-\Delta\psi = w \quad x \in \mathbb{R}^2$$
$$v(0, x) = n_0(x) + p(x), \qquad w(0, x) = n_0(x) - p_0(x).$$

We change now to polar coordinates and integrate on $(0, r)$. Denoting $S = \int_0^r v\rho d\rho$, $R = \int_0^r w\rho d\rho$ and using $v = \frac{1}{r}\frac{\partial}{\partial r}\int_0^r v\rho d\rho = \frac{1}{r}\frac{\partial S}{\partial r}$ and $w = \frac{1}{r}\frac{\partial}{\partial r}\int_0^r w\rho d\rho = \frac{1}{r}\frac{\partial R}{\partial r}$ we get after simplyfing the following reduced system:

$$\partial_t S - r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial S}{\partial r}\right) - \frac{1}{r}\frac{\partial R}{\partial r}R = 0 \tag{7}$$

$$\partial_t R - r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial R}{\partial r}\right) - \frac{1}{r}\frac{\partial S}{\partial r}R = 0. \tag{8}$$

By hypothesis in $t = 0$ we have $R \geq 0$ and $R = 0$ on $r = 0$, in addition the coefficient of R is negative. By means of the change of variables $R = \overline{R}e^t$ in (8) we get

$$\partial_t \overline{R} - r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \overline{R}}{\partial r}\right) + \left(-\frac{1}{r}\frac{\partial S}{\partial r} - 1\right)\overline{R} = 0.$$

Now if the minimum of $\overline{R}$ were negative at this point we would have $\partial_t\overline{R} \leq 0$, $\frac{\partial\overline{R}}{\partial r} = 0$ and $\frac{\partial^2\overline{R}}{\partial r^2} \geq 0$ and therefore $\partial_t\overline{R} - r\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\overline{R}}{\partial r}) = \partial_t\overline{R} - \frac{\partial^2\overline{R}}{\partial r^2} + \frac{1}{r}\frac{\partial\overline{R}}{\partial r} \leq 0$ and $(-\frac{1}{r}\frac{\partial S}{\partial r} - 1)\overline{R} > 0$ getting a contradiction with (8). It follows that $\overline{R} \geq 0$ and consequently $R \geq 0$. Using that

$$R = \int_0^r w\rho d\rho = \int_0^r (n - p)\rho d\rho$$
$$= \frac{1}{2\pi}\left(\int_{B(0,r)} n\rho d\rho - \int_{B(0,r)} p\rho d\rho\right) = \frac{1}{2\pi}(M_1 - M_2)$$

we conclude that

$$M_1 \geq M_2. \tag{9}$$

Therefore, variables M_1 and M_2 are comparable. □

The following result was proved in [[3], Th. 3] and it plays an essential role for our considerations. For the sake of completeness we outline the proof here.

Theorem 4 (*Conditions for the Boundedness of p*)**.** *Suppose that the initial data $n_0, p_0 \in L^2_s(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$ of (1) are positive radially symmetric functions. If the initial data of (1) satisfy*

$$n(r, 0) \geq p(r, 0), \tag{10}$$

then for any solution in $C([0, T); L^2_s(\mathbb{R}^2)) \cap C([0, T); C^2(\mathbb{R}^2))$ there exists a constant C such that

$$p(r, t) \leq C \quad \forall t > 0, x \in \mathbb{R}^2.$$

Proof. From (5) and (9) it follows that

$$\partial_t M_2 \leq r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial M_2}{\partial r}\right). \tag{11}$$

At the time $t = 0$, the variable p_0 is bounded for some constant C, then we have $M_2(r, 0) = \frac{1}{2\pi}\int_{B(0,r)} p_0\rho d\rho \leq \frac{1}{2\pi}C\int_{B(0,r)}\rho d\rho = Cr^2$. Introducing the transformation $\overline{M}(r, t) = M_2(r, t) - Cr^2$, it follows from (11) that $\overline{M}$ satisfies

$$\begin{aligned} &\partial_t\overline{M} \leq \frac{\partial^2\overline{M}}{\partial r^2} - \frac{1}{r}\frac{\partial\overline{M}}{\partial r} \\ &\overline{M}(r, 0) \leq 0, \qquad \overline{M}(0, t) = 0, \qquad \overline{M}(R, t) \leq 0. \end{aligned} \tag{12}$$

Thus the maximum principle yields

$$\overline{M}(r, t) = M_2(r, t) - Cr^2 \leq 0$$

and, hence, $M_2(r,t) = 2\pi \int_0^r p\rho d\rho \le Cr^2$. Using regularity theory for parabolic equations (see [5]), we then obtain the bound $p = \frac{1}{r}\frac{\partial M_2}{\partial r} \le C$, for a suitable constant $C > 0$. □

The last theorem will allow us to simplify system (5) and apply the moments technique (see, e.g. [6–8]) to prove blow-up. This final result shows that in the radial case condition (2) for blow-up can be improved and even more it shows that conditions (4) for global existence (2) are optimal.

Theorem 5 (*Finite Time of Existence for p*). *Suppose that the initial data $n_0, p_0 \in L^2_s(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$ are positive radially symmetric functions and that the inequality $n(r,0) \ge p(r,0)$ is satisfied. If*

$$\frac{\theta_1}{2\pi}(8\pi - \theta_1) + Cm_1(0) < 0, \tag{13}$$

where

$$\theta_1 = \int_{\mathbb{R}^2} n_0 dx \qquad \theta_2 = \int_{\mathbb{R}^2} p_0 dx,$$

is fulfilled, then we have $T_{\max} < \infty$, where $T_{\max}$ is the maximum time of existence of solution n.

Proof. Let $m_1(t) = \int_{\mathbb{R}^2} |x|^2 n(x,t)dx$. Multiplying the first equation of (1) by $|x|^2$ and integrating the resulting relation over $\mathbb{R}^2$, we obtain

$$\partial_t \int_{\mathbb{R}^2} n|x|^2 dx = \int_{\mathbb{R}^2} |x|^2 \Delta n dx - \int_{\mathbb{R}^2} |x|^2 \nabla \cdot (n\nabla\psi)dx. \tag{14}$$

From Green's identity we get

$$\begin{aligned}\partial_t \int_{\mathbb{R}^2} n|x|^2 dx &= \int_{\mathbb{R}^2} (\Delta|x|^2) n dx - \int_{\mathbb{R}^2} |x|^2 \nabla \cdot (n\nabla\psi)dx = 4\int_{\mathbb{R}^2} n dx + \int_{\mathbb{R}^2} \nabla(|x|^2)\cdot(n\nabla\psi)dx \\ &= 4\int_{\mathbb{R}^2} n dx + 2\int_{\mathbb{R}^2} n(x\cdot\nabla\psi)dx.\end{aligned}$$

From $\frac{\partial\psi}{\partial r} = \frac{M_2 - M_1}{2\pi r}$ and the identity for radial symmetric functions $x\cdot\nabla\psi = r\frac{\partial\psi}{\partial r}$ we get

$$\begin{aligned}\int_{\mathbb{R}^2} n(x\cdot\nabla\psi)dx &= 2\pi\int_0^\infty nr\frac{\partial\psi}{\partial r} r dr = 2\pi\int_0^\infty n\left(\frac{M_2 - M_1}{2\pi}\right) r dr \\ &= \frac{1}{2\pi}\int_0^\infty M_2 n r dr - \int_0^\infty M_1 n r dr.\end{aligned} \tag{15}$$

From Theorem 4 we know that p is bounded. Thus we obtain the estimate $M_2 \le cr^2$. It follows from (15) that

$$\begin{aligned}\int_{\mathbb{R}^2} n(x\cdot\nabla\psi)dx &\le C\int_0^\infty nr^3 dr - \frac{1}{2\pi}\int_0^\infty M_1\frac{\partial M_1}{\partial r}dr \\ &= \frac{C}{2\pi}\int_{\mathbb{R}^2} n|x|^2 dx - \frac{1}{4\pi}\theta_1^2.\end{aligned}$$

From (14) it follows that

$$\begin{aligned}\frac{d}{dt}m_1(t) &\le 4\theta_1 + 2\left(-\frac{1}{4\pi}\theta_1^2\right) + Cm_1(t) \\ &= 4\theta_1 - \frac{1}{2\pi}\theta_1^2 + Cm_1(t) \\ &= \frac{\theta_1}{2\pi}(8\pi - \theta_1) + Cm_1(t).\end{aligned}$$

Suppose

$$\frac{\theta_1}{2\pi}(8\pi - \theta_1) + Cm_1(0) < 0.$$

In consequence,

$$0 \le m_1(t) < m_1(0) + \left(\frac{\theta_1}{2\pi}(8\pi - \theta_1) + Cm_1(0)\right)t.$$

Thus there exists $T_0 \in (0, \infty)$ such that

$$m_1(t) \to 0 \quad \text{as } t \to T_0.$$

Therefore $T_{\max} \leq T_0 < \infty$. □

In a similar way we obtain the following result.

Theorem 6 (*Finite Time of Existence for n*)**.** *Suppose that the initial data n_0 and $p_0 \in C_0^\infty(\mathbb{R}^2)$ and*

$$p(r, 0) \geq n(r, 0).$$

If

$$\frac{\theta_2}{2\pi}(8\pi - \theta_2) + Cm_2(0) < 0, \tag{16}$$

where

$$\theta_1 = \int_{\mathbb{R}^2} n_0 dx \qquad \theta_2 = \int_{\mathbb{R}^2} p_0 dx,$$

is fulfilled, then we have $T_{\max} < \infty$, where $T_{\max}$ is the maximum time of existence of solution p.

Theorems 5 and 6 show that if θ_1 and θ_2 are arbitrary positive parameters satisfying

$$\theta_1 > 8\pi \quad \text{or} \quad \theta_2 > 8\pi,$$

then we can construct initial data n_0 and p_0 such that

$$\theta_1 = \int_{\mathbb{R}^2} n_0 dx, \qquad \theta_2 = \int_{\mathbb{R}^2} p_0 dx,$$

and system (1) blows up. For example, take n_0, p_0 satisfying (6) with $\theta_1 = \int_{\mathbb{R}^2} n_0 dx > 8\pi$ together with an initial moment $m_1(0)$ small enough such that inequality (13) holds, then Theorem 5 implies blow-up for system (1). Consequently, the optimal blow-up region should be the square found by Kurokiba et al. in [2].

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