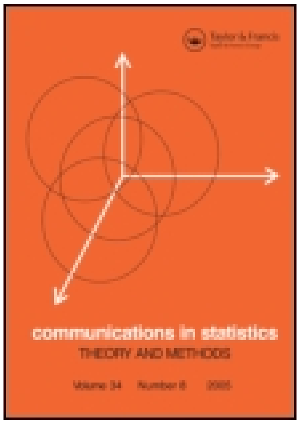

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Central Limit Theorem for the Number of Near-Records

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Near-records in a sequence of random variables $X_n, n \geq 1$, are observations within a fixed distance of the current maximum. More precisely, as defined by Balakrishnan et al. (2005), X_n is a near-record if $X_n \in (M_{n-1} - a, M_{n-1}]$, where $M_n = \max\{X_1, \ldots, X_n\}$ and $a > 0$ is fixed. In this article we establish the asymptotic normality of $D_n = \sum_{i=1}^{n} \mathbf{1}_{\{X_i \in (M_{i-1}-a, M_{i-1}]\}}$, the number of near-records among the first n observations, when the underlying random variables are independent and identically distributed, with common continuous distribution.

Keywords Central limit theorem; Near-record.

Mathematics Subject Classification 60G70; 60F05.

1. Introduction

Let $X_n, n \geq 1$, be a sequence of non negative random variables, $M_n = \max\{X_1, \ldots, X_n\}$, for $n \geq 1$, and $M_0 = 0$. For a fixed parameter $a > 0$, let $I_n = \mathbf{1}_{\{X_n \in (M_{n-1}-a, M_{n-1}]\}}$, $n \geq 1$, be the indicators of near-record observations. The number of near-records among the first n observations is $D_n = \sum_{i=1}^{n} I_i$, $n \geq 1$. In this article, we prove asymptotic normality for D_n under the assumption that the X_n are independent and identically distributed, with absolutely continuous distribution F, concentrated on $[0, \infty)$.

Near-records are a natural generalization of records and may have applications in actuarial mathematics, as they can be considered as insurance claims with values close to record claims. In particular, Balakrishnan et al. (2005) pointed out that the sum of near-record observations is a quantity of interest to insurance companies. On the other hand, Teugels (1982) described a procedure based on records to assess

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the validity of a model for insurance claims, although he warns that refinements are necessary due to the scarcity of records. We believe that the number and values of near-records can be used, together with records, to improve on such procedure; see also Hashorva (2003) and Hashorva and Hüsler (2005), for applications closely related to near-records. For theory and applications of records, see Arnold et al. (1998) and Nevzorov (2001).

Another field where near-records can be useful is industrial quality control, in the so-called destructive stress testing; see Glick (1978) and Gulati and Padgett (2003) for discussion and statistical procedures related to stress testing. In these experiments, a common procedure is to observe the sample sequentially and stress each element only up to the minimum level that some previous element broke at, thus obtaining a sequence of lower-record values. Again, given the small number of records relative to the number of trials, an improvement over this scheme can be achieved by stressing the elements a bit further than the previous record value (by a fixed factor $c > 1$, say), which yields a sequence of "multiplicative" near-records, together with the sequence of lower-records. Finally, taking minus log of the observations, we obtain the usual records and near-records. Maximum likelihood estimation using this scheme is introduced in Gouet et al. (2012), with substantially better performance than that obtained using only records. The results of the present article are relevant for this type of statistical procedure since they allow to estimate the number of near-records present in the sample as a function of the calibration parameter c.

Theoretical studies have focused on the random variable $\xi_n(a)$, which counts near-records associated with the n-th record value. It is shown in Balakrishnan et al. (2005) that, under mild conditions on F, $\xi_n(a)$ converges weakly to a random variable, which is either 0, geometrically distributed or ∞, depending on the tail behavior of F. Additional results and refinements are given in Pakes (2007), where the window size a is allowed to vary with n and limits are related to the maximal domains of attraction. More recent asymptotic results for D_n can be found in Gouet et al. (in press). An interesting phenomenon reported there is the a.s. convergence of D_n to a finite limit D_∞, for heavy-tailed distributions, with square-integrable hazard function. For such distributions, the convergence in probability of $\xi_n(a)$ to 0 (obtained in Balakrishnan et al., 2005) is strengthened to $\xi_n(a) = 0$ a.s., for sufficiently large n. Also, weak and strong laws of large numbers are presented in Gouet et al. (in press), for distributions with divergent D_n.

Our goal here is to complement the laws of large numbers with corresponding central limit theorems. As in Gouet et al. (in press), we work with sequences of nonnegative, independent and identically distributed random variables $X_n, n \geq 1$, with common distribution F and density function f, continuous on $(0, \infty)$. We assume that $r_F := \sup\{x \geq 0 : F(x) < 1\} = \infty$ because, otherwise, D_n behaves ultimately as a binomial random variable, with success probability $1 - F(r_F - a)$ and so, the asymptotic normality of D_n is readily obtained. Finally we suppose that f is strictly decreasing on $(0, \infty)$, which implies f positive and F strictly increasing on $(0, \infty)$. Observe that, because upper extremes diverge to $r_F = \infty$, our results remain valid for distributions with ultimately decreasing density.

Our notation is explained below. Convergence in probability and in distribution are denoted by the superscripted arrows $\xrightarrow{P}$ and $\xrightarrow{D}$ respectively. The equivalence $a_n \sim b_n$ between real sequences means that either both a_n and b_n tend to infinity or zero, with $\lim_n a_n/b_n = 1$ or both converge to non zero finite limits. The equivalence

$\sim$ between functions is interpreted analogously and, when applied to random sequences, is understood in the almost sure (a.s.) sense. Inequalities among random variables are also in the a.s. sense. The survival, hazard, cumulative hazard, and quantile functions associated to a distribution F are, respectively, defined by $\overline{F}(t) = 1 - F(t)$, $\lambda(t) = f(t)/\overline{F}(t)$, $\Lambda(t) = \int_0^t \lambda(s)ds$, for $t \geq 0$, and $m(t) = \overline{F}^{-1}(1/t)$, for $t \geq 1$. Observe that $\overline{F}(t) = 1$, $f(t) = 0$, and $\lambda(t) = 0$, for $t < 0$.

In the next section, we present the main result of the article and examples. Proofs and intermediate results are given in Sec. 3. The Appendix contains technical lemmas and two important theorems, which are central in our approach and are presented for the sake of completeness.

2. Main Result and Example

Theorem 2.1. *Let*

$$a_n = \int_a^{m(n)} \frac{f(x-a) - f(x)}{\overline{F}(x)} dx, \quad b_n^2 = \int_a^{m(n)} \phi(x)\lambda(x)/\overline{F}(x)dx, \tag{1}$$

for n such that $m(n) \geq a$, where

$$\phi(t) = \int_{\max\{a,t\}}^{\infty} (f(x-a) - f(x)) \left(2\frac{\overline{F}(x-a)}{\overline{F}(x)} - 1\right) dx, \tag{2}$$

for $t \geq 0$. If $\int_0^\infty \lambda(x)^2 dx = \infty$ and any of the following hypotheses hold:

(H1) *There exists a constant $C > 0$ such that $\lambda(x) \leq C$, for all $x \geq 0$,*
(H2) $\lim_{x\to\infty} \lambda(x) = \infty$ *and λ is differentiable, with* $\lim_{x\to\infty} \lambda'(x) = \alpha$, *for some $\alpha \geq 0$.*

Then $(D_n - a_n)/b_n \xrightarrow{D} N(0,1)$.

Remark 2.1. It follows from the proof of Proposition 3.1 that, under (H2), all the moments of X_1 exist (see (6) below).

Example 2.1. Theorem 2.1 is valid for any distribution with ultimately decreasing density, such that $\int_0^\infty \lambda(x)^2 dx = \infty$. This example is based on a family of distributions which contains heavy-, medium-, and light-tailed distributions. For $\beta > 0$ and $r \in [-1/2, 1]$, let $\lambda(x) = \beta x^r$, for $x > 0$. Then F is heavy-tailed for $r < 0$, has an exponential tail for $r = 0$ and is light-tailed for $r > 0$. For these distributions, the hypotheses of Theorem 2.1 hold, so we obtain $(D_n - a_n)/b_n \xrightarrow{D} N(0,1)$. In most cases, we calculate explicit formulas for the normalizing sequences a_n, b_n, in terms of parameters β, r, as shown below. First note that $m(n) = \left(\frac{r+1}{\beta}\log n\right)^{1/(r+1)}$.

(a) $r = -1/2 \Rightarrow a_n = 2\beta^2 a \log\log n$, $b_n^2 = a_n$.
(b) $r \in (-1/2, 0) \Rightarrow b_n^2 = \frac{a\beta^2}{2r+1}\left(\frac{r+1}{\beta}\log n\right)^{\frac{2r+1}{r+1}}$.
(c) $r = 0 \Rightarrow a_n = (e^{a\beta} - 1)\log n$, $b_n^2 = (2e^{a\beta} - 1)a_n$.
(d) $r \in (0, 1) \Rightarrow b_n^2 = \frac{m(n)}{ar} e^{2\beta a m(n)^r}$.
(e) $r = 1 \Rightarrow a_n = \frac{1}{a} e^{-\beta a^2/2} e^{\beta a m(n)} \left(m(n) - (a + \frac{1}{\beta a})\right)$, $b_n^2 = \frac{1}{a} e^{-\beta a^2} m(n) e^{2\beta a m(n)}$.

Remark 2.2. Note that $r < -1/2$ is not included in the example above because, in this case, $\int_0^\infty \lambda(x)^2 dx < \infty$ and so D_n converges a.s. to a finite random variable; see Gouet et al. (in press) for details.

3. Intermediate Results and Proofs

The basis of our main result is a version of the well-known martingale central limit theorem (CLT), presented in the Appendix as Theorem A.1. We must therefore construct a martingale by finding an "appropriate" compensator for D_n, in the sense that conditions of Theorem A.1 can be verified and the random centering process can be replaced by a deterministic sequence. This requires more than just subtracting conditional expectations from D_n. On the other hand, it is interesting to see that both conditions for the martingale CLT can be stated in terms of sums of minima of iid random variables, which is quite advantageous because detailed results for such processes are available in the literature; see Theorem A.2, in the Appendix, and other results in Deheuvels (1974). This approach has already been successfully used in Gouet et al. (2007), to prove the asymptotic normality of record-like statistics in a discrete setting.

The martingale in the following Proposition is defined with respect to the natural filtration $\mathbb{F} = \{\mathcal{F}_n\}$, which is the increasing family of σ-algebras generated by the observations. In other words, $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$, $n \geq 1$ and $\mathcal{F}_0$ is the trivial σ-algebra.

Proposition 3.1. *Suppose* (H1) *or* (H2) *holds and let, for* $t \geq 0$,

$$\psi(t) = \int_0^t \frac{f(x-a) - f(x)}{\overline{F}(x)} dx. \tag{3}$$

Then $D_n - \psi(M_n)$, $n \geq 1$, *is a cubic integrable* $\mathbb{F}$*-martingale.*

Proof. Clearly, $D_n - \psi(M_n)$ is adapted to $\mathbb{F}$. For cubic integrability, it suffices to check that $E[|\psi(M_n)|^3] < \infty$, for all $n \geq 1$, which is implied by

$$E[|\psi(X_1)|^3] \leq \int_0^\infty \left(\int_0^t \frac{|f(x-a) - f(x)|}{\overline{F}(x)} dx \right)^3 f(t) dt < \infty.$$

The integral above is finite if and only if $\int_a^\infty \left(\int_a^t \frac{f(x-a)-f(x)}{\overline{F}(x)} dx \right)^3 f(t) dt < \infty$ and this is implied by $\mathcal{I}_1 < \infty$, with $\mathcal{I}_1 := \int_a^\infty \left(\int_a^t \frac{f(x-a)}{\overline{F}(x)} dx \right)^3 f(t) dt$.

Assume (H1). Then, from (22) in the Appendix we have

$$\int_a^t \frac{f(x-a)}{\overline{F}(x)} dx \leq D \int_a^{t+a} \lambda(x-a) dx = -D \log \overline{F}(t).$$

So $\mathcal{I}_1 \leq \int_a^\infty (-D \log \overline{F}(t))^3 f(t) dt < \infty$.

Assume now (H2). Note that, for every $x > 0$ and some constants $K, C, C' > 0$,

$$\begin{aligned}\int_0^x \frac{f(t-a)}{\overline{F}(t)} dt &= \int_0^x \lambda(t-a) e^{\int_{t-a}^t \lambda(y) dy} \leq K \int_0^x \lambda(t-a) e^{C\lambda(t-a)} dt \\ &\leq K \int_0^x e^{(C+1)\lambda(t-a)} dt \leq K \int_0^x e^{C' t} dt,\end{aligned}$$

where we have used that λ' is bounded, in the first and third inequalities, and that $t \le e^t$, in the second one. Thus,

$$\mathcal{I}_1 < E\left[\int_0^{X_1} \frac{f(t-a)}{\overline{F}(t)} dt\right]^3 \le K' E\left[e^{MX_1}\right], \tag{4}$$

with $K' = (K/C')^3$, $M = 3C'$. Now, since $\lambda(x) \to \infty$, we have $\Lambda(t) - \beta t \to \infty$ for all $\beta > 0$ so

$$\lim_{t\to\infty} \overline{F}(t)/e^{-\beta t} = \lim_{t\to\infty} e^{-(\Lambda(t)-\beta t)} = 0, \tag{5}$$

for all $\beta > 0$. The conclusion follows from

$$E\left[e^{MX_1}\right] = \int_0^\infty P\left[e^{MX_1} > y\right] dy = \int_0^\infty \overline{F}\left(\frac{\log y}{M}\right) dy \le 1 + K'' \int_1^\infty e^{-2\log y} dy < \infty, \tag{6}$$

where, in the last inequality, we have used (5), with $\beta = 2M$. The cubic integrability of $D_n - \psi(M_n)$ has been established under (H1) and (H2).

Finally, let Δ be defined as $\Delta a_n = a_n - a_{n-1}$. Then, $\Delta D_n = I_n$ and

$$\Delta\psi(M_n) = \int_{M_{n-1}}^{M_n} \frac{f(x-a) - f(x)}{\overline{F}(x)} dx. \tag{7}$$

Also, $E[I_n \mid \mathcal{F}_{n-1}] = P[X_n \in (M_{n-1} - a, M_{n-1}] \mid \mathcal{F}_{n-1}] = \int_{M_{n-1}-a}^{M_{n-1}} f(x)dx$ and

$$\begin{aligned} E[\Delta\psi(M_n) \mid \mathcal{F}_{n-1}] &= \int_{M_{n-1}}^\infty \left(\int_{M_{n-1}}^t \frac{f(x-a) - f(x)}{\overline{F}(x)} dx\right) f(t)dt \\ &= \int_{M_{n-1}}^\infty \left(\int_x^\infty f(t)dt\right) \frac{f(x-a) - f(x)}{\overline{F}(x)} dx \\ &= \int_{M_{n-1}}^\infty (f(x-a) - f(x))dx = \int_{M_{n-1}-a}^{M_{n-1}} f(x)dx. \end{aligned}$$

We have $E[\Delta(D_n - \psi(M_n)) \mid \mathcal{F}_{n-1}] = 0$ and so, $D_n - \psi(M_n)$ is a martingale. □

Proposition 3.2. *Assume either* (H1) *or* (H2). *Then, for* $n \ge 1$,

$$E[(I_n - \Delta\psi(M_n))^2 \mid \mathcal{F}_{n-1}] = \int_{M_{n-1}}^\infty (f(x-a) - f(x)) \left(2\frac{\overline{F}(x-a)}{\overline{F}(x)} - 1\right) dx. \tag{8}$$

Proof. The calculations below are justified because the martingale is square integrable, by Proposition 3.1. Note that $I_n \Delta\psi(M_n) = 0$ because, when X_n is a near-record ($I_n = 1$), then $M_n = M_{n-1}$ and so $\Delta\psi(M_n) = 0$. Therefore, $E[(I_n - \Delta\psi(M_n))^2 \mid \mathcal{F}_{n-1}] = E[I_n \mid \mathcal{F}_{n-1}] + E[(\Delta\psi(M_n))^2 \mid \mathcal{F}_{n-1}]$. Moreover,

$$\begin{aligned} E[(\Delta\psi(M_n))^2 \mid \mathcal{F}_{n-1}] &= \int_{M_{n-1}}^\infty \left(\int_{M_{n-1}}^t \frac{f(x-a) - f(x)}{\overline{F}(x)} dx\right)^2 f(t)dt \\ &= 2\int_{M_{n-1}}^\infty (f(t-a) - f(t)) \left(\int_{M_{n-1}}^t \frac{f(x-a) - f(x)}{\overline{F}(x)} dx\right) dt \end{aligned}$$

$$= 2\int_{M_{n-1}}^{\infty} \frac{f(x-a)-f(x)}{\overline{F}(x)} \left(\int_{x}^{\infty} (f(t-a)-f(t))dt\right) dx$$
$$= 2\int_{M_{n-1}}^{\infty} \frac{f(x-a)-f(x)}{\overline{F}(x)}\overline{F}(x-a)dx - 2E[I_n \mid \mathcal{F}_{n-1}],$$

where the second equality above follows from the integration-by-parts rule, noting that

$$\lim_{t\to\infty} \overline{F}(t)\left(\int_{M_{n-1}}^{t} \frac{f(x-a)-f(x)}{\overline{F}(x)}dx\right)^2 = 0,$$

by Lemma A.4(ii). Collecting terms above we obtain (8). □

For the Lyapunov-type condition of Theorem A.1 we calculate an upper bound of the third conditional moment of the martingale.

Proposition 3.3. *For n large enough, there exist $K > 0$ such that, if* (H1) *holds, then*

$$E[|I_n - \Delta\psi(M_n)|^3 \mid \mathcal{F}_{n-1}] \leq K(\overline{F}(M_{n-1}-a) - \overline{F}(M_{n-1})) \tag{9}$$

and, if (H2) *holds, then*

$$E[|I_n - \Delta\psi(M_n)|^3 \mid \mathcal{F}_{n-1}] \leq K(\overline{F}(M_{n-1}-3a) - \overline{F}(M_{n-1})). \tag{10}$$

Proof. Observe that

$$E[|I_n - \Delta\psi(M_n)|^3 \mid \mathcal{F}_{n-1}] \leq E[I_n \mid \mathcal{F}_{n-1}] + 3E[I_n|\Delta\psi(M_n)| \mid \mathcal{F}_{n-1}] + 3E[I_n(\Delta\psi(M_n))^2 \mid \mathcal{F}_{n-1}] + E[|\Delta\psi(M_n)|^3 \mid \mathcal{F}_{n-1}].$$

As in Proposition 3.2, we have $I_n\Delta\psi(M_n) = I_n(\Delta\psi(M_n))^2 = 0$ and so,

$$E[|I_n - \Delta\psi(M_n)|^3 \mid \mathcal{F}_{n-1}] \leq E[I_n \mid \mathcal{F}_{n-1}] + E[|\Delta\psi(M_n)|^3 \mid \mathcal{F}_{n-1}]. \tag{11}$$

Given that $M_n \uparrow \infty$ a.s., we assume that n is such that $M_{n-1} \geq a$. For the second term in (11) note $M_{n-1} \geq a$ implies $\Delta\psi(M_n) \geq 0$ because, in this case, the integrand in (7) is non negative. So,

$$E[|\Delta\psi(M_n)|^3 \mid \mathcal{F}_{n-1}] = \int_{M_{n-1}}^{\infty} \left(\int_{M_{n-1}}^{t} \frac{f(x-a)-f(x)}{\overline{F}(x)}dx\right)^3 f(t)dt$$
$$= 3\int_{M_{n-1}}^{\infty} (f(t-a)-f(t))\left(\int_{M_{n-1}}^{t} \frac{f(x-a)-f(x)}{\overline{F}(x)}dx\right)^2 dt$$
$$= 6\int_{M_{n-1}}^{\infty} \frac{f(t-a)-f(t)}{\overline{F}(t)}(\overline{F}(t-a)-\overline{F}(t)) \times \left(\int_{M_{n-1}}^{t} \frac{f(x-a)-f(x)}{\overline{F}(x)}dx\right) dt. \tag{12}$$

The second and third equations above follow from the integration-by-parts rule and Lemma A.4(ii), with $p = 3$ and $p = 2$, respectively.

Suppose (H1) holds. Then, from (22), (12) is bounded by

$$\begin{aligned}
&6D\int_{M_{n-1}}^{\infty}(f(t-a)-f(t))\left(\int_{M_{n-1}}^{t}\frac{f(x-a)-f(x)}{\overline{F}(x)}dx\right)dt\\
&\quad=6D\int_{M_{n-1}}^{\infty}\frac{f(x-a)-f(x)}{\overline{F}(x)}(\overline{F}(x-a)-\overline{F}(x))dx\\
&\quad<6D^2\int_{M_{n-1}}^{\infty}(f(x-a)-f(x))dx=6D^2(\overline{F}(M_{n-1}-a)-\overline{F}(M_{n-1})).
\end{aligned}$$

Finally, recalling that $E[I_n\,|\,\mathscr{F}_{n-1}]=\overline{F}(M_{n-1}-a)-\overline{F}(M_{n-1})$, the result follows from (11).

Suppose (H2) holds and let $h(y)=\int_y^\infty\frac{f(t-a)}{\overline{F}(t)}\overline{F}(t-a)\left(\int_y^t\frac{f(x-a)}{\overline{F}(x)}dx\right)dt$. Then the last member of (12) is bounded by $6h(M_{n-1})$ on $\{M_{n-1}\geq a\}$. Also, by Tonelli's theorem, by (27) and (24),

$$\begin{aligned}
h(y)&=\int_y^\infty\left(\int_x^\infty\frac{f(t-a)}{\overline{F}(t)}\overline{F}(t-a)dt\right)\frac{f(x-a)}{\overline{F}(x)}dx\\
&\sim K'\int_y^\infty\overline{F}(x-2a)\frac{f(x-a)}{\overline{F}(x)}dx\\
&\sim K''\overline{F}(y-3a)\sim K''(\overline{F}(y-3a)-\overline{F}(y)),
\end{aligned}$$

for some $K', K''>0$, as $y\to\infty$. So, for n large enough and some $K'''>0$,

$$E[|\Delta\psi(M_n)|^3\,|\,\mathscr{F}_{n-1}]\leq K'''(\overline{F}(M_{n-1}-3a)-\overline{F}(M_{n-1})).$$

Hence, from (11), for n large enough,

$$\begin{aligned}
E[|I_n-\Delta\psi(M_n)|^3\,|\,\mathscr{F}_{n-1}]&\leq\overline{F}(M_{n-1}-a)-\overline{F}(M_{n-1})+K'''(\overline{F}(M_{n-1}-3a)-\overline{F}(M_{n-1}))\\
&\leq(K'''+1)(\overline{F}(M_{n-1}-3a)-\overline{F}(M_{n-1})).
\end{aligned}$$
□

Proposition 3.4. *Assume* (H1) *or* (H2) *and* $\int_0^\infty\lambda(x)^2dx=\infty$. *Let* b_n^2 *be defined as in* (1). *Then,*

$$\frac{1}{b_n^2}\sum_{k=1}^n E[(I_k-\Delta\psi(M_k))^2\,|\,\mathscr{F}_{k-1}]\xrightarrow{P}1. \tag{13}$$

Proof. From (8) we have $E[(I_k-\Delta\psi(M_k))^2\,|\,\mathscr{F}_{k-1}]=\phi(M_{k-1})$ on $\{M_{k-1}\geq a\}$. Then, given that $M_k\uparrow\infty$ a.s.,

$$\sum_{k=1}^n E[(I_k-\Delta\psi(M_k))^2\,|\,\mathscr{F}_{k-1}]\sim\sum_{k=1}^n\phi(M_k). \tag{14}$$

Moreover, as ϕ is decreasing (see Lemma A.3), we have $\phi(M_k)=\min\{\phi(X_1),\ldots,\phi(X_k)\}$ and so, the right-hand side of (14) is a sum of partial minima of iid random variables, to which Theorem A.2 can be applied.

To that end, let $Z_n = \phi(X_n), n \geq 1$, whose common distribution $G(z) = P[\phi(X_1) \leq z] = \overline{F}(\phi^{-1}(z))$ for $z \in (0, \phi(a))$ and $G(\phi(a)) = 1$ satisfies the conditions of Theorem A.2, with $b = \phi(a)$, $\gamma = 1/P[Z_1 < \phi(a)] = 1/\overline{F}(a)$, and

$$H(\log t) = \int_\gamma^t \phi(m(z))dz = \int_a^{m(t)} \phi(x)\lambda(x)/\overline{F}(x)dx,$$

where the last equality follows from the change of variable $x = m(z)$.

Suppose (H1) holds and observe that, by (29), $\lim_{t\to\infty} H(t) \geq a\int_a^\infty \lambda(t)^2 dt = \infty$, so we can proceed to check conditions (34) and (35).

For (34) we must determine a sequence $v_n \uparrow \infty$ such that

$$\frac{\int_{m(n)}^{m(nv_n)} \phi(x)\lambda(x)/\overline{F}(x)dx}{\int_a^{m(n)} \phi(x)\lambda(x)/\overline{F}(x)dx} \to 0. \tag{15}$$

Observe that the numerator above can be bounded as follows:

$$\begin{aligned}\int_{m(n)}^{m(nv_n)} \phi(x)\lambda(x)/\overline{F}(x)dx &\leq K\int_{m(n)}^{m(nv_n)} \lambda(x-a)^2 dx \\ &\leq KC\int_{m(n)-a}^{m(nv_n)} \lambda(x)dx \\ &\leq KC\left(aC + \int_{m(n)}^{m(nv_n)} \lambda(x)dx\right) \\ &= K_1 + K_2 \log\frac{\overline{F}(m(n))}{\overline{F}(m(nv_n))} = K_1 + K_2\log v_n,\end{aligned}$$

where the first inequality is obtained from (29), and the others from the boundedness of λ, with K_1, K_2 positive constants. So (15) follows with $v_n = H(\log n)$.

For (35) note that, from (29),

$$kG^{-1}(1/k) = k\phi(m(k)) = \frac{\phi(m(k))}{\overline{F}(m(k))} \leq L\lambda(m(k)-a) \leq LC,$$

with L, C positive constants. Hence,

$$\frac{\sum_{k=\lceil\gamma\rceil}^n kG^{-1}(1/k)^2}{\left(\sum_{k=\lceil\gamma\rceil}^n G^{-1}(1/k)\right)^2} \leq \frac{LC}{\sum_{k=\lceil\gamma\rceil}^n G^{-1}(1/k)} \to 0$$

and (35) is proved. Having checked both conditions of Theorem A.2 under (H1), we conclude that $\sum_{k=1}^n \phi(M_k)/H(\log n) \xrightarrow{P} 1$, with $H(\log n) = \int_a^{m(n)} \phi(u)\lambda(u)/\overline{F}(u)du$, $n \geq 1$. Hence, the result follows from (14).

Assume (H2). Then it is easy to check, using, for instance, L'Hôpital's rule and (27), that $\phi(t) \sim 2\int_t^\infty \frac{f(x-a)}{\overline{F}(x)}\overline{F}(x-a)dx \sim 2e^{a^2\alpha}\overline{F}(t-2a)$, as $t \to \infty$. So, for t large enough

$$\int_a^t \phi(x)\lambda(x)/\overline{F}(x)dx \sim 2e^{a^2\alpha}\int_a^t \overline{F}(x-2a)\lambda(x)/\overline{F}(x)dx \geq 2e^{a^2\alpha}\int_a^t \lambda(x)dx$$

and therefore, $\lim_{t\to\infty} H(t) = \infty$.

Moreover, by (24) $\overline{F}(t-2a) \sim \overline{F}(t-2a) - \overline{F}(t)$ so, $\phi(t) \sim 2e^{a^2\alpha}(\overline{F}(t-2a) - \overline{F}(t))$, as $t \to \infty$, and so $\sum_{k=1}^{n} \phi(M_k) \sim 2e^{a^2\alpha} \sum_{k=1}^{n}(\overline{F}(M_{k-1}-2a) - \overline{F}(M_{k-1}))$, as $n \to \infty$.

Observe that $\overline{F}(M_{k-1}-2a) - \overline{F}(M_{k-1})$ is the conditional expectation of the indicator, say $I_k^{(2)}$, of near-record with parameter $2a$ instead of a. Moreover, the conditional Borel-Cantelli lemma (see p. 152 of Neveu, 1972) implies that

$$D_n^{(2)} := \sum_{k=1}^{n} I_k^{(2)} \sim \sum_{k=1}^{n} E[I_k^{(2)} \,|\, \mathcal{F}_{k-1}].$$

On the other hand, from the law of large numbers for D_n, proved in Theorem 2.2 (ii) of Gouet et al. (in press), we have $D_n^{(2)}/d_n^{(2)} \xrightarrow{P} 1$, with $d_n^{(2)} \sim \int_a^{m(n)} \overline{F}(x-2a)\lambda(x)/\overline{F}(x)dx \sim b_n^2/(2e^{a^2\alpha})$. □

Proposition 3.5. *Assume* (H1) *or* (H2) *and* $\int_0^\infty \lambda(x)^2 dx = \infty$. *Let* b_n *be defined as in* (1). *Then,*

$$\frac{1}{b_n^3} \sum_{k=1}^{n} E[|I_k - \Delta\psi(M_k)|^3 \,|\, \mathcal{F}_{k-1}] \xrightarrow{P} 0. \tag{16}$$

Proof. Assume (H1). Then, noting that $E[I_k \,|\, \mathcal{F}_{k-1}] = \overline{F}(M_{k-1}-a) - \overline{F}(M_{k-1})$, $k \geq 1$, convergence (16) follows from (9) if $\sum_{k=1}^{n} E[I_k \,|\, \mathcal{F}_{k-1}]/b_n^3 \xrightarrow{P} 0$.

Furthermore, as in the proof of Proposition 3.4, the conditional Borel-Cantelli lemma (see p. 152 of Neveu, 1972) implies that $D_n = \sum_{k=1}^{n} I_k \sim \sum_{k=1}^{n} E[I_k \,|\, \mathcal{F}_{k-1}]$. Again, from the law of large numbers for D_n, in Theorem 2.2(i) of Gouet et al. (in press), we have $D_n/d_n \xrightarrow{P} 1$, with $d_n = \int_a^{m(n)} g(x)\lambda(x)/\overline{F}(x)dx$ and g is defined in the proof of Lemma A.2. So, the conclusion is obtained if we prove that $d_n/b_n^3 \to 0$. Indeed, from (31) we have

$$\frac{d_n}{b_n^3} = \frac{\int_a^{m(n)} g(x)\lambda(x)/\overline{F}(x)dx}{\left(\int_a^{m(n)} \phi(x)\lambda(x)/\overline{F}(x)dx\right)^{3/2}} \leq \frac{1}{b_n} \to 0.$$

Assume (H2). Then convergence (16) follows from (10) if

$$\sum_{k=1}^{n}(\overline{F}(M_{k-1}-3a) - \overline{F}(M_{k-1}))/b_n^3 = \sum_{k=1}^{n} E[I_k^{(3)} \,|\, \mathcal{F}_{k-1}]/b_n^3 \xrightarrow{P} 0,$$

where $I_k^{(3)}$ represents the indicator of a near-record with parameter $3a$. We proceed as above by applying the conditional Borel-Cantelli lemma, which yields $D_n^{(3)} = \sum_{k=1}^{n} I_k^{(3)} \sim \sum_{k=1}^{n} E[I_k^{(3)} \,|\, \mathcal{F}_{k-1}]$.

As in Proposition 3.4, from the law of large numbers for D_n in Theorem 2.2(ii) of Gouet et al. (in press), we have $D_n^{(3)}/d_n^{(3)} \xrightarrow{P} 1$, with $d_n^{(3)} \sim \int_a^{m(n)} \overline{F}(x-3a)\lambda(x)/\overline{F}(x)dx$. So, the conclusion is obtained if we prove that $d_n^{(3)}/b_n^3 \to 0$. Indeed, recalling that under (H2) $b_n^2 \sim 2e^{a^2\alpha} \int_a^{m(n)} \overline{F}(x-2a)\lambda(x)/\overline{F}(x)dx$ and using (24), we have

$$b_n^2 \sim 2e^{-a^2\alpha} \int_a^{m(n)} \lambda(x)e^{2a\lambda(x)}dx \tag{17}$$

and $d_n^{(3)} \sim e^{-9a^2\alpha/2} \int_a^{m(n)} \lambda(x) e^{3a\lambda(x)} dx$. From L'Hôpital's rule we have

$$\lim_{t\to\infty} \frac{\int_a^t \lambda(x)e^{3a\lambda(x)}dx}{\left(\int_a^t \lambda(x)e^{2a\lambda(x)}dx\right)^{3/2}} = \frac{2}{3}\lim_{t\to\infty} \frac{\lambda(t)e^{3a\lambda(t)}}{\lambda(t)e^{2a\lambda(t)}\left(\int_a^t \lambda(x)e^{2a\lambda(x)}dx\right)^{1/2}}$$

$$= \frac{2}{3}\left(\lim_{t\to\infty} \frac{e^{2a\lambda(t)}}{\int_a^t \lambda(x)e^{2a\lambda(x)}dx}\right)^{1/2}$$

$$= \frac{2}{3}\left(\lim_{t\to\infty} \frac{2a\lambda'(t)e^{2a\lambda(t)}}{\lambda(t)e^{2a\lambda(t)}}\right)^{1/2} = 0$$

and the conclusion follows.

In the following Propositions we prove tightness results needed for changing the stochastic centering sequence of the martingale by a deterministic one.

Proposition 3.6. *Assume* (H1). *Then, for any sequence of positive numbers* $c_n \to \infty$,

$$\frac{\psi(M_n) - \psi(m(n))}{c_n} \xrightarrow{P} 0. \tag{18}$$

Proof. We establish first

$$(\Lambda(M_n) - \Lambda(m(n)))/c_n \xrightarrow{P} 0, \tag{19}$$

where $\Lambda(t) = \int_0^t \lambda(s)ds, t \geq 0$. To that end observe that $\Lambda(M_n) - \Lambda(m(n)) = -\log \overline{F}(M_n) + \log \overline{F}(m(n)) = -(\log \overline{F}(M_n) + \log n)$. Hence, (19) is equivalent to both

$$P[\log \overline{F}(M_n) + \log n > \epsilon c_n] \to 0 \quad \text{and} \quad P[\log \overline{F}(M_n) + \log n < -\epsilon c_n] \to 0,$$

for all $\epsilon > 0$. We have

$$P[\log \overline{F}(M_n) + \log n > \epsilon c_n] = P[M_n < \overline{F}^{-1}(e^{\epsilon c_n}/n)]$$
$$= P[X_1 < \overline{F}^{-1}(e^{\epsilon c_n}/n)]^n = (1 - e^{\epsilon c_n}/n)^n \mathbf{1}_{\{e^{\epsilon c_n} \leq n\}} \to 0$$

and

$$P[\log \overline{F}(M_n) + \log n < -\epsilon c_n] = P[M_n > \overline{F}^{-1}(e^{-\epsilon c_n}/n)]$$
$$= 1 - P[X_1 < \overline{F}^{-1}(e^{-\epsilon c_n}/n)]^n = 1 - (1 - e^{-\epsilon c_n}/n)^n \to 0.$$

Thus, (19) follows.

On the other hand, let u, v such that $a < u \leq v$. Then, by (22),

$$\psi(v) - \psi(u) \leq \int_u^v \frac{f(x-a)}{\overline{F}(x)}dx \leq D\int_{u-a}^{v-a} \lambda(x)dx \leq D\int_u^v \lambda(x)dx + K,$$

for some $K > 0$. Hence,

$$|\psi(M_n) - \psi(m(n))|/c_n \leq D|\Lambda(M_n) - \Lambda(m(n))|/c_n + K/c_n \xrightarrow{P} 0,$$

where the convergence follows from (19).

Proposition 3.7. *Assume* (H2) *and let* b_n *be defined as in* (1). *Then,*

$$\frac{\psi(M_n) - \psi(m(n))}{b_n} \xrightarrow{P} 0. \tag{20}$$

Proof. By (24) and (26), there exists $K > 0$ such that, for $a < u \leq v$,

$$\psi(v) - \psi(u) \leq \int_u^v \frac{f(x-a)}{\overline{F}(x)} dx \leq K \int_u^v \lambda(x) e^{a\lambda(x)} dx.$$

Let $\theta(t) = \int_0^t \lambda(x) e^{a\lambda(x)} dx$ and $\varphi(t) = \int_0^t \lambda(x) e^{2a\lambda(x)} dx$, for $t \geq 0$. Observe that $|\psi(M_n) - \psi(m(n))| \leq K|\theta(M_n) - \theta(m(n))|$ and $b_n^2 \sim 2e^{-a^2\alpha}\varphi(m(n))$ by (17). Then, (20) follows if we prove

$$\frac{\theta(M_n) - \theta(m(n))}{\varphi(m(n))^{1/2}} \xrightarrow{P} 0, \tag{21}$$

which is equivalent to both limits: $P[\theta(M_n) > \theta(m(n)) + \epsilon\varphi(m(n))^{1/2}] \to 0$ and $P[\theta(M_n) < \theta(m(n)) - \epsilon\varphi(m(n))^{1/2}] \to 0$, for all $\epsilon > 0$. Using a well-known result from extreme-value theory, we find that the limits above are, respectively, equivalent to

$$\frac{\overline{F}(\theta^{-1}[\theta(m(n)) + \epsilon\varphi(m(n))^{1/2}])}{\overline{F}(m(n))} \to 0, \quad \frac{\overline{F}(\theta^{-1}[\theta(m(n)) - \epsilon\varphi(m(n))^{1/2}])}{\overline{F}(m(n))} \to \infty.$$

Let us consider first the limit of $l(t) := \overline{F}(\theta^{-1}(\theta(t) + \epsilon\varphi(t)^{1/2}))/\overline{F}(t)$, as $t \to \infty$. We require the existence of a bounded positive function β such that, for sufficiently large t, $\theta^{-1}(\theta(t) + \epsilon\varphi(t)^{1/2}) > t + \beta(t)$ (a condition which is implied by $(\theta(t+\beta(t)) - \theta(t))/\varphi(t)^{1/2} \to 0$) and such that $\overline{F}(t+\beta(t))/\overline{F}(t) \to 0$. If such β exists, then $l(t) \leq \overline{F}(t+\beta(t))/\overline{F}(t) \to 0$.

Since β is bounded, then, from the mean-value theorem we have $\int_t^{t+\beta(t)} \lambda(x) e^{a\lambda(x)} dx = \beta(t)\lambda(\xi(t)) e^{a\lambda(\xi(t))}$, for $\xi(t) \in (t, t+\beta(t))$. Moreover, given that λ has a bounded derivative, there exists a constant $K > 0$ such that $\lambda(\xi(t)) e^{a\lambda(\xi(t))} \leq K\lambda(t) e^{a\lambda(t)}$ and so

$$\limsup_{t\to\infty} \frac{\left(\int_t^{t+\beta(t)} \lambda(x) e^{a\lambda(x)} dx\right)^2}{\int_0^t \lambda(x) e^{2a\lambda(x)} dx} \leq K^2 \lim_{t\to\infty} \frac{(\beta(t)\lambda(t) e^{a\lambda(t)})^2}{\int_0^t \lambda(x) e^{2a\lambda(x)} dx}.$$

We choose $\beta(t) = \lambda(t)^{-3/4}$ and show that the limit of the r.h.s. above is 0. Applying L'Hôpital's rule we obtain

$$\lim_{t\to\infty} \frac{\lambda(t)^{1/2} e^{2a\lambda(t)}}{\int_0^t \lambda(x) e^{2a\lambda(x)} dx} = \lim_{t\to\infty} (\lambda(t)^{-3/2}/2 + 2a\lambda(t)^{-1/2})\lambda'(t) = 0.$$

We have thus checked the first requirement for β. For the second we observe that $\overline{F}(t+\beta(t))/\overline{F}(t) = e^{-\int_t^{t+\beta(t)} \lambda(x)dx} = e^{-\lambda(\xi(t))\beta(t)}$, where $\xi(t) \in (t, t+\beta(t))$. Again, because β and λ' are bounded, the limit of $e^{-\lambda(\xi(t))\beta(t)}$ is equivalent to $\lim_{t\to\infty} e^{-\lambda(t)\beta(t)} = \lim_{t\to\infty} e^{-\lambda(t)^{1/4}} = 0$.

For the second limit let $l(t) = \overline{F}(\theta^{-1}(\theta(t) - \epsilon\varphi(t)^{1/2}))/\overline{F}(t)$. Then, we choose $\beta(t) > 0$ such that, for sufficiently large t, $\theta^{-1}(\theta(t) - \epsilon\varphi(t)^{1/2}) < t - \beta(t)$ (implied by $(\theta(t) - \theta(t-\beta(t)))/\varphi(t)^{1/2} \to 0$) and such that $\overline{F}(t-\beta(t))/\overline{F}(t) \to \infty$. Then $l(t) \geq \overline{F}(t-\beta(t))/\overline{F}(t) \to \infty$. Details are similar to the first case and are omitted.

Proof of main result. Conditions (32) and (33) of Theorem A.1 have been checked for the martingale $D_n - \psi(M_n)$, in Propositions 3.4 and 3.5, respectively, under conditions (H1) and (H2). Hence, we have $\frac{D_n - \psi(M_n)}{b_n} \xrightarrow{D} N(0,1)$, with b_n defined in Proposition 3.4. The final result follows from Propositions 3.6 and 3.7.

Appendix

We collect here some useful technical facts about functions related to distributions satisfying hypothesis (H1) or (H2) of Theorem 2.1. See Sec. 1 for notation. Recall that a is a fixed positive constant.

Lemma A.1. *Suppose* (H1) *holds. Then there exists a positive constant D such that, for* $t \geq a$,

$$\overline{F}(t-a) - \overline{F}(t) < \overline{F}(t-a) = \overline{F}(t)\exp\left(\int_{t-a}^{t} \lambda(s)ds\right) \leq D\overline{F}(t). \tag{22}$$

Proof. It follows from the well-known formula $\overline{F}(t) = e^{-\Lambda(t)}$ and (H1). □

Lemma A.2. *Suppose* (H2) *holds. Then, as* $t \to \infty$, *for any integer* $p \geq 1$ *and some* $K_p > 0$:

$$(i)\quad \frac{\overline{F}(t-a)}{\overline{F}(t)} \sim e^{a^2\alpha}\frac{\overline{F}(t-2a)}{\overline{F}(t-a)}; \tag{23}$$

$$(ii)\quad \frac{\overline{F}(t-a)}{\overline{F}(t)} \sim e^{-a^2\alpha/2}e^{a\lambda(t)}; \tag{24}$$

$$(iii)\quad \overline{F}(t)\left(\frac{\overline{F}(t-a)}{\overline{F}(t)}\right)^p \sim K_p\overline{F}(t-pa); \tag{25}$$

$$(iv)\quad \frac{\lambda(t+a)}{\lambda(t)} \to 1. \tag{26}$$

$$(v)\quad \frac{f(t-a)}{\overline{F}(t)}\overline{F}(t-pa) \sim e^{pa^2\alpha}f(t-(p+1)a). \tag{27}$$

Proof. (i) Note that

$$\frac{\overline{F}(t-a)^2}{\overline{F}(t)\overline{F}(t-2a)} = e^{v(t,a)}, \tag{28}$$

where $v(t,a) = -2\Lambda(t-a) + \Lambda(t) + \Lambda(t-2a)$.

We show that $v(t,a) \to \alpha^2 a$ as $t \to \infty$. To that end, consider the following expansions, for $t \geq 2a$:

$$\Lambda(t-a) = \Lambda(t-2a) + \Lambda'(t-2a)a + \Lambda''(\theta(t))a^2/2,$$
$$\Lambda(t) = \Lambda(t-2a) + \Lambda'(t-2a)2a + \Lambda''(\widetilde{\theta}(t))4a^2/2,$$

with $\theta(t) \in (t-2a, t-a)$ and $\widetilde{\theta}(t) \in (t-2a, t)$. Then, noting that $\Lambda'' = \lambda'$, we have $v(t,a) = 2a^2\Lambda''(\widetilde{\theta}(t)) - a^2\Lambda''(\theta(t)) = a^2(2\lambda'(\widetilde{\theta}(t)) - \lambda'(\theta(t)))$, which converges to $a^2\alpha$, as $t \to \infty$.

(ii) $\overline{F}(t-a)/\overline{F}(t) = e^{\Lambda(t)-\Lambda(t-a)} = e^{\Lambda(t)-(\Lambda(t)-a\Lambda'(t)+\Lambda''(\theta(t))a^2/2)} \sim e^{-a^2\alpha/2}e^{a\lambda(t)}$, where $\theta(t) \in [t-a, t]$.

(iii) We reason inductively as follows:

$$\overline{F}(t)\left(\frac{\overline{F}(t-a)}{\overline{F}(t)}\right)^{p+1} \sim K_p\overline{F}(t-pa)\frac{\overline{F}(t-a)}{\overline{F}(t)}$$
$$\sim K_p\overline{F}(t-pa)e^{pa^2\alpha}\frac{\overline{F}(t-(p+1)a)}{\overline{F}(t-pa)} = K_{p+1}\overline{F}(t-(p+1)a),$$

as $t \to \infty$, where the second equivalence follows from (23) applied p times.

(iv) The limit follows from the expansion $\lambda(t+a) = \lambda(t) + a\lambda'(\theta(t))$, with $\theta(t) \in (t, t+a)$. Then, (H2) implies $\lambda(t+a)/\lambda(t) = 1 + a\lambda'(\theta(t))/\lambda(t) \to 1$.

(v)

$$\lim_{t\to\infty}\frac{f(t-a)\overline{F}(t-pa)}{\overline{F}(t)f(t-(p+1)a)} = \lim_{t\to\infty}\frac{\lambda(t-a)\overline{F}(t-a)\overline{F}(t-pa)}{\overline{F}(t)\lambda(t-(p+1)a)\overline{F}(t-(p+1)a)}$$
$$= \lim_{t\to\infty}\frac{\overline{F}(t-a)\overline{F}(t-pa)}{\overline{F}(t)\overline{F}(t-(p+1)a)}.$$

The second equality above was obtained from (26) and the result follows from the application of (23) p times.

Lemma A.3. *Let $\phi(t)$ be defined as in* (2). *Then*:

(i) *If* (H1) *or* (H2) *hold, ϕ is well defined* (*finite*), *is decreasing on* $(0,\infty)$ *and is strictly decreasing on* $[a,\infty)$.
(ii) *If* (H1) *holds, there exist positive constants K_1, K_2 such that, for $t \geq a$,*

$$a\lambda(t)^2 \leq \frac{\phi(t)\lambda(t)}{\overline{F}(t)} \leq K_1\lambda(t-a)^2 \quad \text{and} \quad \phi(t)/\overline{F}(t) \leq K_2\lambda(t-a). \tag{29}$$

Proof. (i) Observe that $\phi(t) \leq 2\int_t^\infty \frac{f(x-a)}{\overline{F}(x)}\overline{F}(x-a)dx$, $t \geq a$. Then, if (H1) holds we use (22) to obtain $\phi(t) \leq 2D\int_t^\infty f(x-a)dx < \infty$. If (H2) holds, $\phi(t) < \infty$ follows from Lemma A.2(v). Finally, ϕ is constant on $[0,a)$ and, since f is continuous and strictly decreasing, the integrand in (2) is positive on $[a,\infty)$ and so, ϕ is strictly decreasing on $[a,\infty)$.

(ii) Observe that $g(t) := \overline{F}(t-a) - \overline{F}(t) = \int_{t-a}^{t} f(x)dx$, hence, since f is decreasing,

$$af(t) \leq g(t) \leq af(t-a), \tag{30}$$

for $t \geq a$. On the other hand, from (22) we have $1 \leq \overline{F}(t-a)/\overline{F}(t) \leq D$, for $t \geq a$, and therefore,

$$g(t) \leq \phi(t) \leq Kg(t), \tag{31}$$

for $t \geq a$, with $K = 2D - 1$. Now, from (30) and (31),

$$a\lambda(t)^2 = af(t)^2/\overline{F}(t)^2 \leq \phi(t)\lambda(t)/\overline{F}(t).$$

Similarly, for the upper bound we have

$$\phi(t)\lambda(t)/\overline{F}(t) \leq Kg(t)\lambda(t)/\overline{F}(t) \leq Kaf(t-a)^2/\overline{F}(t)^2 \leq K_1\lambda(t-a)^2,$$

with $K_1 = KaD^2$. The last inequality in (29) follows from

$$\phi(t)/\overline{F}(t) \leq Kg(t)/\overline{F}(t) \leq Kaf(t-a)/\overline{F}(t) \leq K_2\lambda(t-a),$$

with $K_2 = KaD$.

Lemma A.4. *Suppose that either* (H1) *or* (H2) *hold. Let* ψ *be defined as in* (3) *and* $w_p(t) = \overline{F}(t-a)\psi(t)^p$, *for* $t \geq 0$ *and* $p \geq 1$ *integer. Then*:

(i) $\psi(t) \to \infty$, *as* $t \to \infty$, *if* $\int_0^\infty \lambda(x)^2 dx = \infty$;
(ii) $\lim_{t\to\infty} w_p(t) = \lim_{t\to\infty} \overline{F}(t-a)\int_0^t f(x-a)/\overline{F}(x)dx = 0$.

Proof. (i) Note first that $\psi(t) = \int_0^t (\lambda(x-a)e^{\int_{x-a}^{x}\lambda(y)dy} - \lambda(x))dx$. From the mean value theorem, there exists $\theta(x) \in [x-a, x]$ such that $\int_{x-a}^{x}\lambda(y)dy = a\lambda(\theta(x))$. Assume (H1), then

$$\begin{aligned}\psi(t) &= \int_0^t (\lambda(x-a)e^{a\lambda(\theta(x))} - \lambda(x))dx \geq \int_0^t (\lambda(x-a)(1 + a\lambda(\theta(x))) - \lambda(x))dx \\ &\geq \int_0^t (\lambda(x-a) - \lambda(x))dx + C_1\int_0^t \lambda(x)\lambda(\theta(x))dx \\ &\geq \log\left(\frac{\overline{F}(t)}{\overline{F}(t-a)}\right) + C_1\int_0^t \lambda(x)^2\frac{\overline{F}(x)}{\overline{F}(\theta(x))}dx \geq C_2 + C_3\int_0^t \lambda(x)^2 dx \to \infty,\end{aligned}$$

where C_1, C_2, C_3 are positive constants. The second inequality above follows from (29) and the third because f is decreasing. Finally, for the last inequality observe that, by (22), $\overline{F}(x)/\overline{F}(\theta(x)) \geq \overline{F}(x)/\overline{F}(x-a) \geq D^{-1}$, for all $x \geq a$.

Suppose now (H2) holds. Then, it is easy to see that, as $t \to \infty$,

$$\psi(t) \sim \int_0^t f(x-a)/\overline{F}(x)dx \geq \int_0^t \lambda(x)dx \to \infty.$$

(ii) Observe that $-\Lambda(a) \leq \psi(t) \leq \int_0^\infty f(x-a)/\overline{F}(x)dx$. On the other hand, under (H1),

$$\int_0^t f(x-a)/\overline{F}(x)dx = \int_0^t \lambda(x-a)e^{\int_{x-a}^x \lambda(y)dy}dx \leq -C_1 \log \overline{F}(t-a),$$

for a positive constant C_1 and all $t \geq 0$. Therefore, $0 \leq \overline{\lim}_{t\to\infty} w_p^{1/p}(t) \leq -C_1 \lim_{t\to\infty} \overline{F}^{1/p}(t-a) \log \overline{F}(t-a) = 0$. Finally, if (H2) holds, from (25) we obtain

$$\overline{F}(t-a)\left(\int_0^t f(x-a)/\overline{F}(x)dx\right)^p \leq \overline{F}(t)\left(\frac{\overline{F}(t-a)}{\overline{F}(t)}\right)^{p+1} \sim K_{p+1}\overline{F}(t-(p+1)a) \to 0,$$

as $t \to \infty$, which yields $\lim_{t\to\infty} w_p(t) = 0$.

Theorem A.1. *Let* ξ_i, $i \geq 1$, *be a sequence of random variables, adapted to a filtration* $\mathcal{F}_i$, $i \geq 0$, *such that* $E[\xi_i \mid \mathcal{F}_{i-1}] = 0$ *and* $E[|\xi_i|^{2+\delta}] < \infty$, *for some* $\delta > 0$ *and all* $i \geq 1$. *Let* b_n, $n \geq 1$, *be an increasing sequence of positive real numbers, diverging to* ∞. *Then* $\sum_{i=1}^n \xi_i/b_n \xrightarrow{D} N(0,1)$ *if*

$$\textstyle\sum_{i=1}^n E[\xi_i^2 \mid \mathcal{F}_{i-1}]/b_n^2 \xrightarrow{P} 1 \quad \text{and} \tag{32}$$

$$\textstyle\sum_{i=1}^n E[|\xi_i|^{2+\delta} \mid \mathcal{F}_{i-1}]/b_n^{2+\delta} \xrightarrow{P} 0. \tag{33}$$

Proof. See p. 58 of Hall and Heyde (1980) for a version using Lindeberg's condition.

The following Theorem, which contains results for sums of partial minima, is adapted from Deheuvels (1974). We use this result in the proof of Proposition 3.4 about the rate of growth of the conditional variance process.

Theorem A.2. *Let* $Z_n, n \geq 1$, *be a sequence of nonnegative iid random variables, whose common distribution function* G *is continuous and strictly increasing on the interval* $(0, b)$, *for some* $b > 0$. *Let* G^{-1} *be the inverse of* G *on* $(0, G(b))$ *and* $H(t) = \int_\gamma^{e^t} G^{-1}(1/x)dx$, *for* $t \geq \log\gamma$, *where* $\gamma = 1/P[Z_1 < b]$. *Let* $\lceil\gamma\rceil$ *denote the least integer strictly greater than* γ.

If $\lim_{t\to\infty} H(t) = \infty$ *and there exists* $x_n \uparrow \infty$ *such that*

$$\lim_{n\to\infty} H(x_n + \log n)/H(\log n) = 1 \quad \text{and} \tag{34}$$

$$\lim_{n\to\infty} \textstyle\sum_{k=\lceil\gamma\rceil}^n kG^{-1}(1/k)^2 \Big/ \left(\sum_{k=\lceil\gamma\rceil}^n G^{-1}(1/k)\right)^2 = 0, \tag{35}$$

then $\sum_{k=1}^n \min\{Z_1, \ldots, Z_k\}/H(\log n) \xrightarrow{P} 1$.

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