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BOUNDARY LAYERS IN THE HOMOGENIZATION OF A SPECTRAL PROBLEM IN FLUID–SOLID STRUCTURES*

GRÉGOIRE ALLAIRE† AND CARLOS CONCA‡

Abstract. This paper is devoted to the asymptotic analysis of the spectrum of a mathematical model that describes the vibrations of a coupled fluid–solid periodic structure. In a previous work [*Arch. Rational Mech. Anal.*, 135 (1996), pp. 197–257] we proved by means of a Bloch wave homogenization method that, in the limit as the period goes to zero, the spectrum is made of three parts: the macroscopic or homogenized spectrum, the microscopic or Bloch spectrum, and a third component, the so-called *boundary layer spectrum*. While the two first parts were completely described as the spectrum of some limit problem, the latter was merely defined as the set of limit eigenvalues corresponding to sequences of eigenvectors concentrating on the boundary. It is the purpose of this paper to characterize explicitly this boundary layer spectrum with the help of a family of limit problems revealing the intimate connection between the periodic microstructure and the boundary of the domain. We therefore obtain a "completeness" result, i.e., a precise description of all possible asymptotic behaviors of sequences of eigenvalues, at least for a special class of polygonal domains.

Key words. homogenization, Bloch waves, spectral analysis, boundary layers, fluid–solid structures

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1. Introduction.

1.1. Setting of the problem. This paper is devoted to the study of some boundary layer phenomena which arise in the asymptotic analysis of the spectrum of a mathematical model describing the vibrations of a coupled periodic system of solid tubes immersed in a perfect incompressible fluid. This simple model is due to Planchard, who studied it intensively (see [31], [32]). Since we introduced it at length in section 1.2 of our previous work [3] we content ourselves with briefly recalling the statement of this problem.

We consider a periodic bounded domain Ω_ϵ obtained from a fixed bounded open set Ω in $\mathbb{R}^N$ by removing a collection of identical, periodically distributed holes $(T_p^\epsilon)_{1\le p\le n(\epsilon)}$. The distance between adjacent holes as well as their size are both of the order of ϵ, the size of the period which is a small parameter going to zero. Correspondingly, the number of holes $n(\epsilon)$ is of the order of ϵ^{-N}, where N is the spatial dimension. More precisely, let us first define the standard unit cell $Y=(0;1)^N$ which, upon rescaling to size ϵ, becomes the period in Ω. Let T be a smooth, simply connected, closed subset of Y, assumed to be *strictly included* in Y (i.e., T does not touch the boundary of Y). The set T represents the reference tube (or rod) and the unit fluid cell is defined as

$$Y^* = Y \setminus T.$$

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†Commissariat à l'Energie Atomique, DRN/DMT/SERMA, C.E. Saclay, 91191 Gif sur Yvette, France and Laboratoire d'Analyse Numérique, Université Paris 6 (allaire@ann.jussieu.fr).

‡Departamento de Ingeniería Matemática, Universidad de Chile, Casilla 170/3, Correo 3, Santiago, Chile (cconca@dim.uchile.cl).

For each value of the small positive parameter ϵ, the fluid domain Ω_ϵ is obtained from the reference domain Ω by removing a periodic arrangement of tubes ϵT with period ϵY. Denoting by (T_p^ϵ) the family of all translates of ϵT by vectors ϵp (where p is a multi-index in $\mathbb{Z}^N$) and by (Y_p^ϵ) the corresponding family of cells, we define

$$\Omega_\epsilon = \Omega \setminus \bigcup_{p=1}^{n(\epsilon)} T_p^\epsilon. \tag{1}$$

Although p is a multi-index in $\mathbb{Z}^N$, for simplicity we denote its range by $1 \le p \le n(\epsilon)$. To obtain the fluid domain Ω_ϵ in (1), we remove from the original domain Ω only those tubes T_p^ϵ which belong to a cell Y_p^ϵ completely included in Ω. This has the effect that no tube meets the boundary $\partial\Omega$. Analogously, (Γ_p^ϵ) denotes the family of tubes boundaries (∂T_p^ϵ).

We are interested in the following spectral problem in Ω: find the eigenvalues λ_ϵ and the corresponding normalized eigenvectors u_ϵ, solutions of

$$\begin{cases} -\Delta u_\epsilon = 0 & \text{in } \Omega_\epsilon, \\ \lambda_\epsilon \frac{\partial u_\epsilon}{\partial n} = \epsilon^{-N} \vec{n} \cdot \displaystyle\int_{\Gamma_p^\epsilon} u_\epsilon \vec{n}\, ds & \text{on } \Gamma_p^\epsilon \text{ for } 1 \le p \le n(\epsilon), \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\vec{n}$ denotes the exterior unit normal to Ω_ϵ.

The homogenization of this model has already attracted the attention of several authors (see [1], [14], [16], [17]). Even though it is a spectral problem involving the Laplace operator, it is easily seen to admit only finitely many eigenvalues, exactly $Nn(\epsilon)$ (the number of tubes times the number of degrees of freedom in their displacements). To this end, a finite-dimensional operator S_ϵ is introduced, which acts on the family of tube displacements $\vec{s} = (\vec{s}_p)_{1\le p\le n(\epsilon)}$ with $\vec{s}_p \in \mathbb{R}^N$,

$$\begin{aligned} S_\epsilon :\ & \mathbb{R}^{Nn(\epsilon)} \longrightarrow \mathbb{R}^{Nn(\epsilon)}, \\ & (\vec{s}_p)_{1\le p\le n(\epsilon)} \ \mapsto\ \left(\frac{1}{\epsilon^N} \int_{\Gamma_p^\epsilon} u_\epsilon \vec{n}\, ds \right)_{1\le p\le n(\epsilon)}, \end{aligned} \tag{3}$$

where the fluid potential u_ϵ is now the unique solution in $H^1(\Omega_\epsilon)$ of

$$\begin{cases} -\Delta u_\epsilon = 0 & \text{in } \Omega_\epsilon, \\ \frac{\partial u_\epsilon}{\partial n} = \vec{s}_p \cdot \vec{n} & \text{on } \Gamma_p^\epsilon \text{ for } 1 \le p \le n(\epsilon), \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

According to [17], S_ϵ is self-adjoint, positive definite, and its spectrum, denoted by $\sigma(S_\epsilon)$, coincides with the set of eigenvalues of (2). Of course, since S_ϵ acts in a finite-dimensional space, $\sigma(S_\epsilon)$ is made up of $Nn(\epsilon)$ real numbers. It has been further proved that all eigenvalues of S_ϵ are uniformly bounded away from zero and from infinity (see, e.g., Proposition 1.2.1 and Lemma 1.2.2 in [3]). As the period ϵ goes to zero, $\sigma(S_\epsilon)$, considered as a subset of $\mathbb{R}^+$, converges to a limit set σ_∞ which, by definition, is the set of all cluster points of (sub)sequences of eigenvalues of S_ϵ

$$\sigma_\infty = \{\lambda \in \mathbb{R}^+ \mid \exists \text{ a subsequence } \lambda_{\epsilon'} \in \sigma(S_{\epsilon'}) \text{ such that } \lambda_{\epsilon'} \to \lambda\}.$$

Finding an adequate characterization of the limit set σ_∞ was the main goal of our previous paper [3]. A positive answer to this problem is given in the present article for a special class of polygonal domains.

1.2. Survey of the previous results. The characterization of σ_∞ amounts to studying the asymptotic behavior of the spectral problem (2), or, in other words, to homogenize (2) as the parameter ϵ goes to zero. To our knowledge, this can be done, at least, using two different approaches: the classical *homogenization process* for periodic structures (see, e.g., the reference books [7], [8], [24], [28], [35]) or the so-called *Bloch wave method* (also called the *nonstandard homogenization procedure* in [16]; see [8], [33], [34], [36] for an introduction to Bloch waves in spectral analysis). The former naturally yields the *homogenized or macroscopic spectrum* of (2), while the latter is associated with the so-called *Bloch or microscopic spectrum*.

Historically the second approach was the first applied to problem (2) by C. Conca, M. Vanninathan, and their coworkers [1], [15], [16], [17]. The key point in this method is to rescale the ϵ-network of tubes to size 1 and, therefore, as ϵ goes to zero, to obtain an infinite limit domain containing a periodic array of unit tubes. Then, the limit problem is amenable to the celebrated Bloch wave decomposition (also known as the Floquet decomposition; see the original work of F. Bloch [11] or the first mathematical papers [19], [30], [36] or the books [8], [33]). The spectrum of this limit problem is called the *Bloch spectrum*.

Although it seems the easiest to apply, the first approach (i.e., the classical homogenization) has only been recently applied to problem (2) in our previous article [3]. By homogenizing the operator S_ϵ with the help of the *two-scale convergence* (see [2], [29]), a homogenized equation is obtained in the domain Ω. Its spectrum is called the *homogenized spectrum*. It turns out that the homogenized spectrum is completely different from the Bloch spectrum, and therefore both approaches are complementary. This is possible since in neither case the underlying sequences of linear operators converge uniformly to their limit which are noncompact operators. In addition to this homogenization result, our paper [3] provides a unified theory for both approaches that we called the *Bloch wave homogenization* method. We refer to [3] for more details (see also [4], [5]), and we simply recall our main results.

The homogenization of model (2) amounts to analyzing the convergence of the sequence of operators S_ϵ. Since these operators are defined on a space which varies with ϵ, we extend them to the fixed space $[L^2(\Omega)^N]^{K^N}$, where K is an arbitrary positive integer. Denoting by S_ϵ^K this extension, it will be amenable to a standard asymptotic analysis, while keeping essentially the same spectrum as S_ϵ. Following the lead of Planchard [32], the reference cell of our homogenization procedure is KY instead of simply Y (this technique is referred to as homogenization by packets in [32]). To give a precise definition of S_ϵ^K we introduce two linear maps: a projection P_ϵ^K from $[L^2(\Omega)^N]^{K^N}$ into $\mathbb{R}^{Nn(\epsilon)}$ and an extension E_ϵ^K from $\mathbb{R}^{Nn(\epsilon)}$ into $[L^2(\Omega)^N]^{K^N}$ such that $S_\epsilon^K = E_\epsilon^K S_\epsilon P_\epsilon^K$. To do so, some notation is required concerning the two indices p (indexing constant vectors in $\mathbb{R}^{Nn(\epsilon)}$) and j (indexing vector functions in $[L^2(\Omega)^N]^{K^N}$).

Definition 1.1. *Let KY be the reference cell $(0,K)^N$ which is made of K^N subcells Y_j of the type $(0,1)^N$ containing a single tube T_j. The multi-integer $j = (j_1,\ldots,j_N)$ which enumerates all the tubes in KY takes its values in $\{0,1,\ldots,K-1\}^N$ (we use the notation $0 \le j \le K-1$). Let $p = (p_1,\ldots,p_N)$ be the multi-integer which enumerates all the tubes in Ω_ϵ (see (1)). We define a third multi-integer $\ell = (\ell_1,\ldots,\ell_N)$ which enumerates all the periodic reference cells $\epsilon(KY)$ in Ω_ϵ (its range is denoted by $1 \le \ell \le n_K(\epsilon)$). These three indices are assumed to be related by the* *following one-to-one map:*

$$\ell_m = E\left(\frac{p_m}{K}\right), \quad j_m = p_m - K\ell_m \quad \forall m = 1, ..., N, \tag{5}$$

where $E(\cdot)$ denotes the integer-part function.

Then, P_ϵ^K and E_ϵ^K are defined by

$$\begin{array}{rcl} P_\epsilon^K : [L^2(\Omega)^N]^{K^N} & \longrightarrow & \mathbb{R}^{Nn(\epsilon)}, \\ (\vec{s}_j(x))_{0\le j\le K-1} & \longrightarrow & \left(\vec{s}_p = \frac{1}{|\epsilon(KY)_\ell|}\int_{\epsilon(KY)_\ell} \vec{s}_j(x)dx\right)_{1\le p\le n(\epsilon)}, \end{array} \tag{6}$$

$$\begin{array}{rcl} E_\epsilon^K : \mathbb{R}^{Nn(\epsilon)} & \longrightarrow & [L^2(\Omega)^N]^{K^N}, \\ (\vec{s}_p)_{1\le p\le n(\epsilon)} & \longrightarrow & \left(\vec{s}_j(x) = \sum_\ell \chi_{\epsilon(KY)_\ell}(x)\vec{s}_p\right)_{0\le j\le K-1,} \end{array} \tag{7}$$

where p is related to (ℓ, j) by formula (5). One can easily check that the adjoint $(P_\epsilon^K)^*$ of P_ϵ^K is nothing but $(\epsilon K)^{-N}E_\epsilon^K$ and that $P_\epsilon^K E_\epsilon^K$ is equal to the identity in $\mathbb{R}^{Nn(\epsilon)}$. Therefore, S_ϵ^K is also self-adjoint compact and its spectrum is exactly that of S_ϵ, plus the new eigenvalue 0 which has infinite multiplicity.

The homogenization of the extended operator S_ϵ^K is now amenable to the two-scale convergence method [2], [29]. However, the limit operator S^K has a complicated form which can be simplified by using the following discrete Bloch wave decomposition (see [1]).

LEMMA 1.2. *For any family $(\vec{s}_j)_{0\le j\le K-1}$ of vectors in $\mathbb{C}^N$, let $\vec{s}(y)$ be the following KY-periodic function, piecewise constant in each subcell Y_j:*

$$\vec{s}(y) = \sum_{j=0}^{K-1} \vec{s}_j \chi_{Y_j}(y) \quad \forall y \in KY.$$

There exists a unique family of constant vectors $(\vec{t}_j)_{0\le j\le K-1}$ in $\mathbb{C}^N$ such that

$$\vec{s}(y) = \sum_{j=0}^{K-1} \vec{t}_j e^{2\pi\imath\frac{j}{K}\cdot E(y)} \quad \forall y \in KY, \tag{8}$$

where $E(\cdot)$ denotes the integer-part function. Moreover, the Bloch wave decomposition operator $\mathcal{B}$, defined by $\mathcal{B}(\vec{s}_j) = K^{N/2}(\vec{t}_j)$, is an isometry on $(\mathbb{C}^N)^{K^N}$.

The first main result in [3] (see Theorem 3.2.1) is the following theorem.

THEOREM 1.3. *The sequence $S_\epsilon^K = E_\epsilon^K S_\epsilon P_\epsilon^K$ converges strongly to a limit S^K; i.e., for any family $(\vec{s}_j(x))_{0\le j\le K-1}$, $S_\epsilon^K(\vec{s}_j)$ converges strongly to $S^K(\vec{s}_j)$ in $[L^2(\Omega)^N]^{K^N}$. Furthermore, the limit operator S^K is given by*

$$S^K = \mathcal{B}^* T^K \mathcal{B}, \text{ with } T^K = \operatorname{diag}\left[(T_j^K)_{0\le j\le K-1}\right], \tag{9}$$

where the entries T_j^K are self-adjoint continuous but noncompact operators in $L^2(\Omega)^N$, defined by

$$T_j^K \vec{t}_j = \begin{cases} (A(0) - I)\nabla u - (A(0) - |Y^*|I)\vec{t}_0 & \text{if } j = 0, \\ A(\frac{j}{K})\vec{t}_j & \text{if } j \ne 0, \end{cases} \tag{10}$$

where I is the identity matrix and u is the unique solution of the homogenized problem

$$\begin{cases} -\mathrm{div}(A(0)\nabla u) = \mathrm{div}((I - A(0))\vec{t_0}) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

and, for $\theta \in [0,1]^N$, $A(\theta)$ is the Bloch homogenized matrix with components $(A_{mm'}(\theta))_{1\leq m,m'\leq N}$ defined by

$$\bar{A}_{mm'}(\theta) = \int_{Y^*} \nabla w_m^\theta(y) \cdot \nabla \bar{w}_{m'}^\theta(y) dy, \tag{12}$$

where $(w_m^\theta)_{1\leq m\leq N}$ are solutions of the so-called cell problem at the Bloch frequency θ:

$$\begin{cases} -\Delta w_m^\theta = 0 & \text{in } Y^*, \\ (\nabla w_m^\theta - \vec{e}_m) \cdot \vec{n} = 0 & \text{on } \partial T, \\ y \to e^{-2\pi\imath\theta\cdot y} w_m^\theta(y) \ Y^*\text{-periodic.} \end{cases} \tag{13}$$

The first component T_0^K of the limit operator T^K is the same for all K and is denoted by S in what follows. It is called the macroscopic or homogenized limit of S_ϵ ((11) is also called the homogenized equation). The spectrum $\sigma(S)$ is essential and has been explicitly characterized in Theorems 2.1.4 and 2.1.5 of [3]. The other components of T^K are simple linear multiplication operators that represent the microscopic or Bloch limit behavior of the sequence S_ϵ^K.

According to Proposition 3.2.6 in [3], the matrix $A(\theta)$ is Hermitian and positive definite for any value of θ. Furthermore, it is a continuous function of θ, except at the origin $\theta = 0$. Nevertheless, it is continuous at the origin along rays of constant direction (see Proposition 3.4.4 in [3]). Denoting by $0 < \lambda_1(\theta) \leq \lambda_2(\theta) \leq \cdots \leq \lambda_N(\theta)$ its eigenvalues, we can define the so-called *Bloch spectrum* by

$$\sigma_{Bloch} = \bigcup_{m=1}^{N} \overline{\lambda_m(]0,1[^N)},$$

where $\overline{\lambda_m(]0,1[^N)}$ denotes the closure of the image of $]0,1[^N$ under the maps $\lambda_m(\cdot)$. We deduce our second main result.

THEOREM 1.4. *The strong convergence of S_ϵ^K to the limit operator S^K implies the lower semicontinuity of the spectrum*

$$\sigma(S^K) \subset \lim_{\epsilon\to 0} \sigma(S_\epsilon^K).$$

By letting K go to infinity, we obtain

$$\sigma(S) \cup \sigma_{Bloch} \subset \lim_{\epsilon\to 0} \sigma(S_\epsilon). \tag{14}$$

REMARK 1.5. *As a matter of fact, the Bloch spectrum σ_{Bloch} and the homogenized spectrum $\sigma(S)$ do not coincide. Therefore, both type of limit problems (macroscopic* (11) *and microscopic* (13)*) are complementary. As already mentioned, the Bloch spectrum has already been characterized by C. Conca and M. Vanninathan in* [17] *by means of a different method, the so-called nonstandard homogenization procedure (see also the book* [16]*).*

The question is now to see whether the inclusion in (14) is actually an equality, i.e., if our asymptotic analysis is complete. It turns out that the homogenized and the Bloch spectra are usually not enough to describe σ_∞ because the interaction between the boundary $\partial\Omega$ and the microstructure is not taken into account in our analysis. More precisely, there may well exist sequences of eigenvectors of (2) which concentrate near the boundary $\partial\Omega$ of Ω. They behave as *boundary layers* in the sense that they converge strongly to zero locally inside the domain. Clearly the oscillations of these eigenvectors cannot be captured by the usual homogenization method; neither are they filtered in the Bloch spectrum which is insensitive to the boundary.

Nevertheless, the third main result of our previous paper [3] shows that for any other type of sequences of eigenvectors (not concentrating on the boundary), the limits of the corresponding sequences of eigenvalues belong to $\sigma(S) \cup \sigma_{Bloch}$. More exactly, introducing the subset of σ_∞

$$\begin{aligned}\sigma_{boundary} = \quad & \{\lambda \in \mathbb{R} \mid \exists(\lambda_{\epsilon'}, \vec{s}^{\epsilon'}) \text{ such that } S^1_{\epsilon'}\vec{s}^{\epsilon'} = \lambda_{\epsilon'}\vec{s}^{\epsilon'},\ \lambda_{\epsilon'} \to \lambda, \\ & \|\vec{s}^{\epsilon'}\|_{L^2(\Omega)^N} = 1,\ \text{and } \forall\omega \text{ with } \overline{\omega} \subset \Omega,\ \|\vec{s}^{\epsilon'}\|_{L^2(\omega)^N} \to 0\},\end{aligned} \tag{15}$$

where ϵ' is a subsequence of ϵ and S^1_ϵ is the extension to $L^2(\Omega)^N$ of S_ϵ, we proved the following theorem (see Theorem 3.2.9 in [3]).

THEOREM 1.6. *The limit set of the spectrum of the operator S_ϵ is precisely made of three parts; the homogenized, the Bloch, and the boundary layer spectrum*

$$\lim_{\epsilon\to 0} \sigma(S_\epsilon) = \sigma_\infty = \sigma(S) \cup \sigma_{Bloch} \cup \sigma_{boundary}.$$

The proof of this *completeness result* is the focus of section 3.4 in [3]. It involves a new type of default measure for weakly converging sequences of eigenvectors of S_ϵ, the so-called *Bloch measures* which quantify its amplitude and direction of oscillations.

Of course the definition of $\sigma_{boundary}$ is not satisfactory, since it does not characterize that part of the limit set σ_∞ as the spectrum of some limit operator associated with the boundary $\partial\Omega$. In particular, it is not clear whether $\sigma_{boundary}$ is empty or included in $\sigma(S) \cup \sigma_{Bloch}$. It is the purpose of the present paper to characterize explicitly $\sigma_{boundary}$, at least for special rectangular domains Ω and associated sequences of parameters ϵ.

REMARK 1.7. *By their very definitions, the limit spectrum σ_∞ and the boundary layer spectrum $\sigma_{boundary}$ depend a priori on the choice of the sequence of small parameters ϵ. On the contrary, the homogenized spectrum $\sigma(S)$ and the Bloch spectrum σ_{Bloch} are independent of the sequence ϵ. We believe that $\sigma_{boundary}$ is actually strongly dependent on the sequence ϵ. In particular, we shall characterize it only for a specific sequence ϵ. We thank C. Castro and E. Zuazua for clarifying discussions on this topic* [12].

1.3. Presentation of the main new results. There are mainly two new results in this paper which correspond to the next two sections. First, in section 2 we introduce a new class of limit problems involving the interaction between the tubes array and the domain boundary. We assume that the domain Ω is cylindrical;

$$\Omega = \Sigma \times]0; L[, \tag{16}$$

where Σ is an open bounded set in $\mathbb{R}^{N-1}$ and $L > 0$ is a positive length. A generic point x in $\mathbb{R}^N$ is denoted by $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$ (x_N is the coordinate along the axis of Ω). Let us define a semi-infinite band

$$G = Y' \times]0; +\infty[,$$

where $Y' =]0,1[^{N-1}$ is the unit cell in $\mathbb{R}^{N-1}$. This new "boundary layer" limit problem takes place in the fluid part of G, denoted by G^* and defined by

$$G^* \; = \; G \setminus \bigcup_{q\geq 1} T_q,$$

where (T_q) is the infinite collection of tubes periodically disposed in G. With each tube T_q is associated a displacement $\vec{s}_q \in \mathbb{R}^N$. We denote by ℓ^2 the space of families $(\vec{s}_q)_{q\geq 1}$ such that $\sum_{q\geq 1} |\vec{s}_q|^2$ is finite. Introducing a Bloch parameter $\theta' \in [0,1]^{N-1}$, we define a "boundary layer" operator $d_{\theta'}$ by

$$\begin{aligned} d_{\theta'} \,:\, \ell^2 &\longrightarrow \ell^2, \\ (\vec{s}_q)_{q\geq 1} &\mapsto \left(\int_{\Gamma_q} u_{\theta'} \vec{n} ds \right)_{q\geq 1}, \end{aligned} \tag{17}$$

where $u_{\theta'}(y)$ is the unique solution of

$$\begin{cases} -\Delta u_{\theta'} = 0 & \text{in } G^*, \\ \frac{\partial u_{\theta'}}{\partial n} = \vec{s}_q \cdot \vec{n} & \text{on } \Gamma_q, \; q \geq 1, \\ u_{\theta'} = 0 & \text{if } y_N = 0, \\ y' \mapsto e^{-2\pi\imath\theta'\cdot y'} u_{\theta'}(y', y_N) & Y'\text{-periodic.} \end{cases}$$

Our first result (see Theorem 2.18) is concerned with the continuity of the spectrum of $d_{\theta'}$, considered as a subset of $\mathbb{R}$, with respect to the Bloch parameter θ'.

THEOREM 1.8. *For all $\theta' \in [0,1]^{N-1}$, $d_{\theta'}$ is a self-adjoint continuous but non-compact operator in ℓ^2. Its spectrum $\sigma(d_{\theta'})$ depends continuously on θ', except at $\theta' = 0$. Defining the* boundary layer spectrum *associated with the surface* Σ

$$\sigma_\Sigma \stackrel{\text{def}}{=} \bigcup_{\theta' \in]0,1[^{N-1}} \sigma(d_{\theta'}) \cup \sigma(d_0),$$

we have

$$\sigma_\Sigma \subset \lim_{\epsilon\to 0} \sigma(S_\epsilon).$$

In general, $\sigma(d_{\theta'})$ is not included in the previously found limit spectrum $\sigma(S) \cup \sigma_{Bloch}$ (see Proposition 2.17). Therefore, the new class of limit problems defined by (17) is not redundant with the homogenized or the Bloch limit problems. Our main tool for proving this theorem is a variant of the two-scale convergence adapted to boundary layers, using test functions which oscillate periodically in the directions parallel to the boundary Σ and decay asymptotically fast in the normal direction to Σ (see section 2.1). Remark that the above result holds for any cylindrical domain of the type (16) and for any sequence of periods ϵ going to zero.

Section 3 is devoted to our second main result which requires additional assumptions on the geometry of the domain and on the sequence of periods ϵ. More precisely, we now assume that Ω is a rectangle with integer dimensions

$$\Omega \; = \; \prod_{i=1}^{N}]0; L_i[\quad \text{and} \quad L_i \in \mathbb{N}^* \tag{18}$$

and that the sequence ϵ is exactly

$$\epsilon_n = \frac{1}{n},\ n \in \mathbb{N}^*.$$

These assumptions imply that, for any ϵ_n, the domain Ω is the union of a finite number of *entire* cells of size ϵ_n. Then, the above analysis of the boundary layer spectrum σ_Σ can be achieved for any face Σ of the rectangle Ω. Of course a completely similar analysis can be done for all the lower dimensional manifolds (edges, corners, etc.) of which the boundary of Ω is made up. For each type of manifold, a different family of limit problems arise which are straightforward generalizations of (17). For example, in two space dimensions, the corners of Ω give rise to a limit problem in the quarter of space $\mathbb{R}^+ \times \mathbb{R}^+$ filled with a periodic array of tubes (see section 3.3). Finally, we prove a completeness result (see Theorem 3.1).

THEOREM 1.9. *The limit set of the spectrum of the operator S_{ϵ_n} is precisely made of three parts; the homogenized, the Bloch, and the union of all boundary layer spectra, as defined in Theorem* 1.8*,*

$$\lim_{\epsilon_n \to 0} \sigma(S_{\epsilon_n}) = \sigma(S) \cup \sigma_{Bloch} \cup \sigma_{\partial\Omega},$$

with the notation

$$\sigma_{\partial\Omega} = \bigcup_{\Sigma \subset \partial\Omega} \sigma_\Sigma,$$

where the union is over all hypersurfaces and lower dimensional manifolds composing the boundary $\partial\Omega$.

REMARK 1.10. *The difference between the above completeness theorem and Theorem* 1.6 *is that, here, the boundary layer spectrum $\sigma_{\partial\Omega}$ is explicitly defined for the specific sequence of parameters ϵ_n as the spectrum of a family of limit operators, while, in our previous result, the boundary layer spectrum $\sigma_{boundary}$ was indirectly defined for any sequence ϵ but not explicitly characterized.*

We conclude this introduction by giving a few references to related works on boundary layers in homogenization and by a short discussion on numerical studies concerning problem (2). Apart from the classical books [7, Chapter 7] and [26], we refer mainly to the papers [6], [9], [10], and [27]. Planchard's model has already been studied numerically. The Bloch eigenvalues $\lambda_i(\theta)$ were computed by F. Aguirre in a two-dimensional example. A brief account of his work is given in [1]. On the other hand, direct numerical computations of the entire spectrum $\sigma(S_\epsilon)$ (for a fixed value of ϵ, and without using homogenization) have been reported in [23]. To our knowledge, these are the only available numerical results concerning a large tube array (see also [21], [22]). Of course, these results are consistent with Theorem 1.9 describing the asymptotic behavior of $\sigma(S_\epsilon)$. In particular, some vibration modes displayed in [23] are numerical evidence that $\sigma_{\partial\Omega}$ is not empty; i.e., there exist eigenvectors which are localized near the boundary or the corners of Ω.

2. Boundary layer homogenization. In this section we assume that Ω is a cylindrical bounded open set in $\mathbb{R}^N$ in the sense that it is defined by

$$\Omega = \Sigma \times]0; L[, \tag{19}$$

where Σ is an open bounded set in $\mathbb{R}^{N-1}$ and $L > 0$ is a positive length. With no loss of generality, we assume that the axis of the cylindrical domain Ω is parallel to the Nth canonical direction. Therefore, a generic point x in Ω is denoted by $x = (x', x_N)$ with $x' \in \Sigma$ and $x_N \in]0; L[$. The goal of this section is to analyze the asymptotic behavior of that part of the spectrum $\sigma(S_\epsilon)$ which corresponds to eigenvectors concentrating on the boundary $\Sigma \times \{0\}$, under the sole geometric assumption (19) (in particular, no restrictions are made on the sequence ϵ which goes to zero).

2.1. Two-scale convergence for boundary layers. We begin by adapting the classical two-scale convergence method of Allaire [2] and Nguetseng [29] to the case of boundary layers, that is, sequences of functions in Ω which concentrate near the boundary $\Sigma \times \{0\}$. This method of "two-scale convergence for boundary layers" will allow us to understand this phenomenon of concentration of oscillations near the boundary. The usual two-scale convergence relies on periodically oscillating test functions with a unit period $Y =]0, 1[^N$. Here, we use test functions which oscillate only in the directions parallel to the boundary Σ (with period $Y' =]0, 1[^{N-1}$) and which simply decay in the Nth direction orthogonal to Σ.

Let us define a semi-infinite band $G = Y' \times]0; +\infty[$, where $Y' =]0, 1[^{N-1}$ is the unit cell in $\mathbb{R}^{N-1}$. A generic point y is denoted by $y = (y', y_N)$ with $y' \in Y'$ and $y_N \in]0; +\infty[$. We introduce the space $L^2_\#(G)$ of square integrable functions in G which are periodic in the $(N-1)$ first variables, i.e.,

$$L^2_\#(G) = \{\phi(y) \in L^2(G) \mid y' \mapsto \phi(y', y_N) \text{ is } Y'\text{-periodic}\}.$$

We also denote by $C(\overline{\Sigma})$ the space of continuous functions on the closure of Σ, a compact set in $\mathbb{R}^{N-1}$.

Combining the concentration effect in y_N and the periodic oscillations in Y', the following convergence result is obtained for a sequence $\phi(\frac{x}{\epsilon})$ when ϕ belongs to $L^2_\#(G)$ (further modulated by $x' \in \Sigma$).

LEMMA 2.1. *Let $\varphi(x', y) \in L^2_\#\left(G; C(\overline{\Sigma})\right)$. Then*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_\Omega \left|\varphi\left(x', \frac{x}{\epsilon}\right)\right|^2 dx = \frac{1}{|Y'|} \int_\Sigma \int_G |\varphi(x', y)|^2 dx' dy.$$

REMARK 2.2. *Remark that, in the left-hand side of the above equation, the second argument of φ is x/ϵ and not only x'/ϵ. This implies that there is a concentration effect near 0 in the x_N variable since φ is not periodic in this direction. This, in turn, explains the $1/\epsilon$ scaling in front of the left-hand side, in order to get a nonzero limit.*

As usual in the context of two-scale convergence, the above result is not specific to the space $L^2_\#\left(G; C(\overline{\Sigma})\right)$, which could be replaced, for example, by $L^2\left(\Sigma; C_{c\#}(\bar{G})\right)$, where $C_{c\#}(\bar{G})$ is the space of continuous functions in G, periodic in y' of period Y', and with bounded support in y_N.

In view of Lemma 2.1, we define a notion of "two-scale convergence for boundary layers."

DEFINITION 2.3. *Let $(u_\epsilon)_{\epsilon > 0}$ be a sequence in $L^2(\Omega)$. It is said to* two-scale converge in the sense of boundary layers *on Σ if there exists $u_0(x', y) \in L^2(\Sigma \times G)$ such that*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_\Omega u_\epsilon(x) \varphi\left(x', \frac{x}{\epsilon}\right) dx = \frac{1}{|Y'|} \int_\Sigma \int_G u_0(x', y) \varphi(x', y) dx' dy$$

for all smooth functions $\varphi(x', y)$ defined in $\Sigma \times G$ such that $y' \mapsto \varphi(x', y', y_N)$ is Y'-periodic and φ has a bounded support in $\Sigma \times G$.

This definition makes sense because of the following compactness theorem which generalizes the usual two-scale convergence compactness theorem in [2], [29].

THEOREM 2.4. *Let $(u_\epsilon)_{\epsilon>0}$ be a sequence in $L^2(\Omega)$ such that there exists a constant C, independent of ϵ, for which*

$$\frac{1}{\sqrt{\epsilon}}\|u_\epsilon\|_{L^2(\Omega)} \le C.$$

There exists a subsequence, still denoted by ϵ, and a limit function $u_0(x',y) \in L^2(\Sigma \times G)$ such that

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_\Omega u_\epsilon(x)\varphi\left(x',\frac{x}{\epsilon}\right)dx = \frac{1}{|Y'|}\int_\Sigma\int_G u_0(x',y)\varphi(x',y)dx'dy \tag{20}$$

for all functions $\varphi(x',y) \in L^2_\#\left(G; C(\overline{\Sigma})\right)$.

Remark that Theorem 2.4 does not apply to sequences which are merely bounded in $L^2(\Omega)$ but also converge strongly to zero in $L^2(\Omega)$ as the square root of ϵ. Of course, this is the case for a sequence of the type $\varphi(x', \frac{x}{\epsilon})$, where $\varphi(x',y)$ is as in Lemma 2.1; then, the limit is nothing but $\varphi(x',y)$ itself.

It is not difficult to check that the L^2-norm is weakly lower semicontinuous with respect to the two-scale convergence (see Proposition 1.6 in [2]); i.e., in the present situation

$$\lim_{\epsilon\to 0}\frac{1}{\sqrt{\epsilon}}\|u_\epsilon\|_{L^2(\Omega)} \ge \frac{1}{|Y'|^{1/2}}\|u_0\|_{L^2(\Sigma\times G)}.$$

The next proposition asserts a corrector-type result when the above inequality is actually an equality.

PROPOSITION 2.5. *Let $(u_\epsilon)_{\epsilon>0}$ be a sequence in $L^2(\Omega)$ which two-scale converges in the sense of boundary layers to a limit $u_0(x',y) \in L^2(\Sigma \times G)$. Assume further that it two-scale converges* strongly*, that is,*

$$\lim_{\epsilon\to 0}\frac{1}{\sqrt{\epsilon}}\|u_\epsilon\|_{L^2(\Omega)} = \frac{1}{|Y'|^{1/2}}\|u_0\|_{L^2(\Sigma\times G)}.$$

Then,

(i) *for any sequence $(v_\epsilon)_{\epsilon>0}$ in $L^2(\Omega)$ which two-scale converges in the sense of boundary layers to a limit $v_0(x',y) \in L^2(\Sigma \times G)$, one has*

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_\Omega u_\epsilon v_\epsilon dx = \frac{1}{|Y'|}\int_\Sigma\int_G u_0(x',y)v_0(x',y)dx'dy;$$

(ii) *if $u_0(x',y)$ is smooth, say $u_0 \in L^2_\#\left(G; C(\overline{\Sigma})\right)$, then*

$$\lim_{\epsilon\to 0}\frac{1}{\sqrt{\epsilon}}\left\|u_\epsilon(x) - u_0\left(x',\frac{x}{\epsilon}\right)\right\|_{L^2(\Omega)} = 0.$$

In order to investigate the convergence of sequences of functions in $H^1_0(\Omega)$, we first have to define adequate functional spaces for the two-scale limit. Let $C^\infty_{c\#}(G)$ be the space of smooth functions in $\overline{G}$ which are Y'-periodic in y' and have a compact support in y_N (i.e., they vanish for sufficiently large and small y_N but not necessarily on the whole ∂G). Let $H^1_{0\#}(G)$ be the Sobolev space obtained by completion of $C^\infty_{c\#}(G)$ with respect to the $H^1(G)$-norm. We denote by $H^1_{0\#,loc}(G)$ the space of functions which are "locally" in $H^1_{0\#}(G)$, i.e., which coincide with a function of $H^1_{0\#}(G)$ in any compact set of $\overline{G}$. We define a Deny–Lions-type space (cf. [18]) $D^1_{0\#}(G)$ as the completion of $C^\infty_{c\#}(G)$ with respect to the $L^2(G)^N$-norm of the gradient

$$D^1_{0\#}(G) = \Big\{\psi(y) \in H^1_{0\#,loc}(G) \mid \exists \quad \psi_n \in C^\infty_{c\#}(G) \quad \text{such that} \quad \lim_{n\to+\infty} \|\nabla(\psi - \psi_n)\|_{L^2(G)^N} = 0\Big\}. \tag{21}$$

It is easily seen that a function in $D^1_{0\#}(G)$ vanishes when $y_N = 0$ but does not necessarily go to 0 when y_N goes to infinity since $D^1_{0\#}(G)$ contains functions which grow like y_N^α at infinity with $\alpha < 1/2$. We are now in a position to state our next result.

PROPOSITION 2.6. *Let $(u_\epsilon)_{\epsilon>0}$ be a sequence in $H^1_0(\Omega)$ such that there exists a constant C, independent of ϵ, for which*

$$\frac{1}{\sqrt{\epsilon}}\left(\|u_\epsilon\|_{L^2(\Omega)} + \|\nabla u_\epsilon\|_{L^2(\Omega)^N}\right) \le C.$$

Then, there exists a subsequence, still denoted by ϵ, and a limit $u_0(x',y) \in L^2(\Sigma; D^1_{0\#}(G))$ such that

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_\Omega u_\epsilon(x)\varphi\left(x', \frac{x}{\epsilon}\right) dx = 0,$$

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_\Omega \nabla u_\epsilon(x)\cdot\psi\left(x', \frac{x}{\epsilon}\right) dx = \frac{1}{|Y'|}\int_\Sigma\int_G \nabla_y u_0(x',y)\cdot\psi(x',y)dx'dy$$

for any functions $\varphi \in L^2_\#\left(G; C(\overline{\Sigma})\right)$ and $\psi \in L^2_\#\left(G; C(\overline{\Sigma})^N\right)$.

Remark that, in Proposition 2.6, the two-scale limit $u_0(x',y)$ does not belong to $L^2(\Sigma; H^1(G))$ as could be expected. The reason is that only $\nabla_y u_0 \in L^2(\Sigma\times G)$, while u_0 itself has no reason to belong to $L^2(\Sigma\times G)$. Since the proofs of the above results are very similar to those of the usual two-scale convergence theory, we simply sketch the proofs of Lemma 2.1, Theorem 2.4, and Proposition 2.6.

Proof of Lemma 2.1. Let us first assume that $\varphi(x',y) \in L^2_\#\left(G; C(\overline{\Sigma})\right)$ has bounded support in y_N; i.e., there exists $M > 0$ such that

$$\varphi(x',y) = 0 \text{ if } y_N \ge M.$$

Then, by the change of variables $y_N = x_N/\epsilon$ and for sufficiently small ϵ, we have

$$\begin{aligned} \tfrac{1}{\epsilon}\textstyle\int_\Omega |\varphi(x', \frac{x}{\epsilon})|^2 dx &= \tfrac{1}{\epsilon}\textstyle\int_0^L\int_\Sigma |\varphi(x', \frac{x'}{\epsilon}, \frac{x_N}{\epsilon})|^2 dx' dx_N \\ &= \textstyle\int_0^{L/\epsilon}\int_\Sigma |\varphi(x', \frac{x'}{\epsilon}, y_N)|^2 dx' dy_N \\ &= \textstyle\int_0^{M}\int_\Sigma |\varphi(x', \frac{x'}{\epsilon}, y_N)|^2 dx' dy_N. \end{aligned} \tag{22}$$

The usual convergence result for oscillating functions in $\mathbb{R}^{N-1}$ (see, e.g., [2] and references therein) yields that for almost everywhere $y_N \in (0; M)$

$$\lim_{\epsilon\to 0}\int_\Sigma \left|\varphi\left(x', \frac{x'}{\epsilon}, y_N\right)\right|^2 dx' = \frac{1}{|Y'|}\int_\Sigma\int_{Y'} |\varphi(x', y', y_N)|^2 dx'dy'$$

and that

$$\int_\Sigma \left|\varphi\left(x', \frac{x'}{\epsilon}, y_N\right)\right|^2 dx' \le |\Sigma| \int_{Y'} \max_{x'\in\overline{\Sigma}} |\varphi(x', y', y_N)|^2 dy'.$$

Therefore, applying the Lebesgue theorem, we deduce that

$$\lim_{\epsilon\to 0}\int_0^M \int_\Sigma \left|\varphi\left(x', \frac{x'}{\epsilon}, y_N\right)\right|^2 dx' dy_N = \frac{1}{|Y'|}\int_\Sigma\int_G |\varphi(x', y', y_N)|^2 dx' dy.$$

The density of such functions $\varphi(x', y)$ in $L^2_\#\left(G; C(\overline{\Sigma})\right)$ implies the desired result for any function in $L^2_\#\left(G; C(\overline{\Sigma})\right)$.

Proof of Theorem 2.4. Using the assumed uniform bound on u_ϵ, by the Schwarz inequality we obtain

$$\Big|\frac{1}{\epsilon}\int_\Omega u_\epsilon(x)\varphi\left(x', \frac{x}{\epsilon}\right) dx\Big| \le C\Big(\frac{1}{\epsilon}\int_\Omega \Big|\varphi\Big(x', \frac{x}{\epsilon}\Big)\Big|^2 dx\Big)^{\frac{1}{2}}.$$

Passing to the limit, up to a subsequence, which may depend on φ in the left-hand side and using Lemma 2.1 in the right-hand side, yield

$$\Big|\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_\Omega u_\epsilon(x)\varphi\left(x', \frac{x}{\epsilon}\right) dx\Big| \le C\Big(\int_\Sigma\int_G |\varphi(x', y)|^2 dx' dy\Big)^{\frac{1}{2}}. \tag{23}$$

Since $L^2_\#\left(G; C(\overline{\Sigma})\right)$ is separable, varying φ over a dense countable subset, by a standard diagonalization process, we can extract a subsequence of ϵ such that (23) is valid for all functions φ in this subset. By density, we conclude that the limit in the left side of (23), as a function of φ, defines a continuous linear form in $L^2(\Sigma\times G)$. Then, the classical Riesz representation theorem immediately implies the existence of a function $u_0(x, y) \in L^2(\Sigma\times G)$ which satisfies (20). This finishes the proof of Theorem 2.4.

Proof of Proposition 2.6. By application of Theorem 2.4, up to a subsequence, there exist two limits $u(x', y) \in L^2(\Sigma\times G)$ and $\xi^0(x', y) \in L^2(\Sigma\times G)^N$ such that u_ϵ and ∇u_ϵ two-scale converge in the sense of boundary layers to these respective limits; i.e.,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_\Omega u_\epsilon(x)\varphi\left(x', \frac{x}{\epsilon}\right) dx = \frac{1}{|Y'|}\int_\Sigma\int_G u(x', y)\varphi(x', y) dx' dy, \tag{24}$$

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_\Omega \nabla u_\epsilon(x)\cdot\psi\left(x', \frac{x}{\epsilon}\right) dx = \frac{1}{|Y'|}\int_\Sigma\int_G \xi_0(x', y)\cdot\psi(x', y) dx' dy \tag{25}$$

for any functions $\varphi \in L^2_\#\left(G; C(\overline{\Sigma})\right)$ and $\psi \in L^2_\#\left(G; C(\overline{\Sigma})^N\right)$. Integrating by parts in (25), we obtain

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_\Omega u_\epsilon(x)\mathrm{div}_y\psi\left(x', \frac{x}{\epsilon}\right) dx = 0.$$

In view of (24), this implies that

$$\frac{1}{|Y'|}\int_\Sigma\int_G u(x', y)\mathrm{div}_y\psi(x', y) dx' dy = 0.$$

Another integration by parts yields that $u(x', y)$ does not depend on y. On the other hand, it belongs to $L^2(\Sigma \times G)$ and G is unbounded. Since the only constant which belongs to $L^2(G)$ is zero, we deduce that $u = 0$. Now, specializing (25) to test functions ψ such that $\operatorname{div}_y \psi = 0$ and integrating by parts, we also obtain that

$$\frac{1}{|Y'|} \int_\Sigma \int_G \xi_0(x', y) \cdot \psi(x', y) dx' dy = 0.$$

As is well known, the orthogonal of divergence-free fields is exactly the set of gradients (see Proposition 1.14 in [2] for a precise statement and references). Therefore, there exists a function $u_0(x', y)$ in $L^2(\Sigma; D^1_{0\#}(G))$ such that $\xi_0 = \nabla_y u_0$ (we use the space $D^1_{0\#}(G)$ since u_0 has no reason to belong to $L^2(\Sigma \times G)$).

2.2. Convergence analysis. Recall that the original operator S_ϵ, defined by (3), acts in the space $\mathbb{R}^{Nn(\epsilon)}$ which depends on ϵ and that our strategy was to extend S_ϵ to a fixed space where a convergence analysis is possible. So far, the domain $\Omega = \Sigma \times]0, L[$ was considered periodic of period ϵY. Nevertheless, from now on, Ω is seen as a periodic domain with a new period G^K_ϵ defined by

$$G^K_\epsilon \stackrel{\text{def}}{=}]0; \epsilon K[^{N-1} \times]0; L[,$$

with K an integer larger than 1. We shall construct an extension of S_ϵ well suited for the previous two-scale convergence "in the sense of boundary layers" with such a period G^K_ϵ.

REMARK 2.7. *As already mentioned, we make no special hypothesis on the sequence of small parameters ϵ. However, the periodic arrangement of tubes in Ω is required to be aligned with Σ in such a way that the first row of periodic cells ϵY has a boundary which coincides with $\Sigma \times \{0\}$. In other words, the first layer of tubes close to Σ is at a fixed distance $\frac{\epsilon}{2}$ of $\Sigma \times \{0\}$ (see Figure 1).*

By a rescaling of ratio ϵ, this new period G^K_ϵ corresponds to a finite length truncation of the new reference cell

$$G^K \stackrel{\text{def}}{=} KG =]0; K[^{N-1} \times]0; +\infty[= KY' \times]0; +\infty[.$$

In the reference cell G^K (see Figure 2) we put infinitely many layers of tubes in the Nth direction, each layer being made of K^{N-1} tubes. The tubes in G^K are denoted by T_j, where $j = (j', j_N)$ is a multi-index such that $j_N \geq 1$ is an integer, which labels the corresponding layer in G^K, and j' is a multi-integer in $\{0, 1, \dots, K-1\}^{N-1}$, which locates the tube T_j in its layer j_N. The fluid part in G^K is denoted by G^{*K}, i.e.,

$$G^{*K} \ = \ G^K \setminus \bigcup_{\substack{0 \leq j' \leq K-1 \\ 1 \leq j_N}} T_j.$$

To each tube T_j in G^K we associate the subcell Y_j and the fluid subcell $Y^*_j = Y_j \setminus T_j$ analogous to Y and Y^*, respectively (see Figure 2). The main idea is to attach to each tube T_j in G^K a different displacement function $\vec{s}(x')$, depending only on the variable $x' \in \Sigma$, such that the family $(\vec{s}_j(x'))_{\substack{0 \leq j' \leq K-1 \\ 1 \leq j_N}}$ belongs to the space $L^2(\Sigma; \ell^2_K)$, where ℓ^2_K is the Hilbert space defined by

$$\ell^2_K = \left\{ (\vec{s}_j)_{\substack{0 \leq j' \leq K-1 \\ 1 \leq j_N}} \ \middle| \ \vec{s}_j \in \mathbb{C}^N, \quad \sum_{\substack{0 \leq j' \leq K-1 \\ 1 \leq j_N}} |\vec{s}_j|^2 < +\infty \right\}.$$

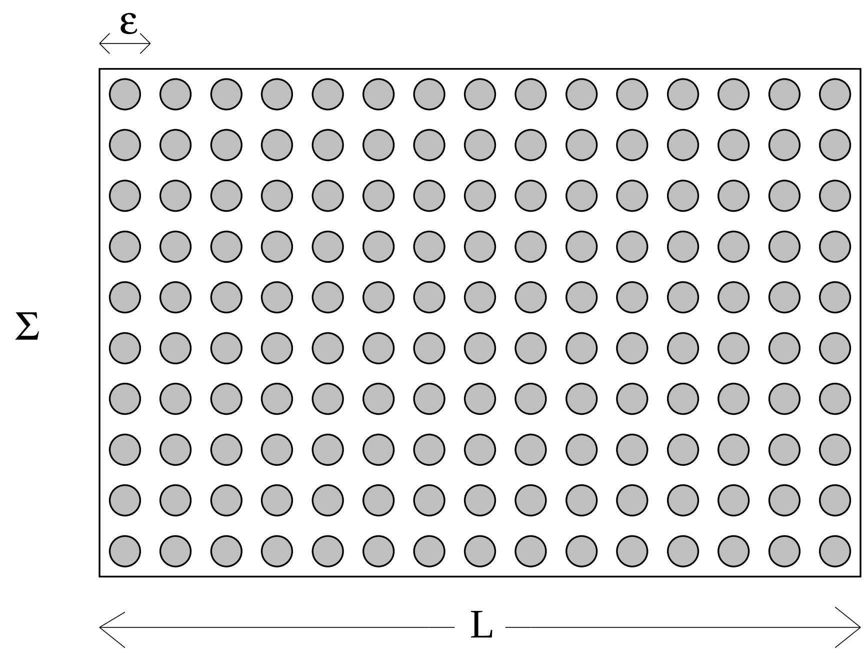

FIG. 1. *Cylindrical domain* $\Omega = \Sigma \times (0, L)$.

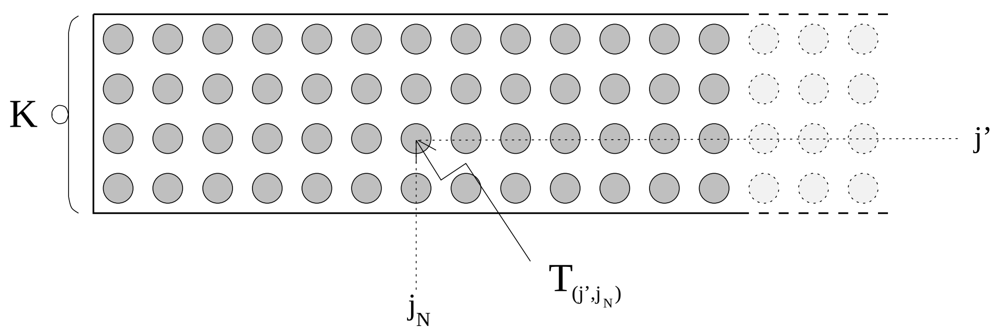

FIG. 2. *Reference cell* G^K.

Remark that this definition of ℓ^2_K implies a decay of the displacement function $\vec{s}_j$ as j_N goes to $+\infty$. Note also that each family $(\vec{s}_j(x')) \in L^2(\Sigma; \ell^2_K)$ can be identified with a function $\vec{s}(x', y) \in L^2(\Sigma \times G^K)$ which is constant in each subcell Y_j.

We now introduce the extended operator B^K_ϵ defined in $L^2(\Sigma; \ell^2_K)$ by

$$B^K_\epsilon = E^K_\epsilon S_\epsilon P^K_\epsilon,$$

where P^K_ϵ and E^K_ϵ are, respectively, projection and extension operators between $\mathbb{R}^{Nn(\epsilon)}$ and $L^2(\Sigma; \ell^2_K)$. To define precisely P^K_ϵ and E^K_ϵ we need the following notation.

DEFINITION 2.8. *Let $j = (j', j_N)$ denote the multi-index which enumerates all tubes in the periodic reference cell G^K. We use the notation $0 \le j' \le K - 1$ to indicate that j' varies in $\{0, 1, \ldots, K-1\}^{N-1}$ and $j_N \ge 1$ to indicate that j_N takes any positive integer value. Let $p = (p_1, \ldots, p_N)$ be the multi-integer which enumerates all the tubes in Ω (see Definition 1). The index p is such that the tube T^ϵ_p is located* *in the cell whose origin lies at the point $\epsilon p \in \Omega$. To describe its range we use the notation $1 \leq p \leq n(\epsilon)$, where $n(\epsilon)$ is the total numbers of tubes in Ω. We define a third multi-integer $\ell' = (\ell_1, \ldots, \ell_{N-1})$ which enumerates all the periodic reference cells $G^K_{\epsilon,\ell'}$ covering Ω (each being identical, up to a translation, to G^K_ϵ). For simplicity its range is denoted by $1 \leq \ell' \leq n_K(\epsilon)$. These three indices are assumed to be related by the following one-to-one relationship:*

$$\left\{ \begin{array}{l} \ell_m = E(\frac{p_m}{K}), \quad j_m = p_m - K\ell_m \quad \text{for} \quad 1 \leq m \leq N-1, \\ j_N = p_N, \end{array} \right. \tag{26}$$

where E denotes the integer-part function. This yields a one-to-one map between the tubes (T^ϵ_p) and their location in the cell $G^K_{\epsilon,\ell'}$ at the position j' in the layer j_N.

Then, we define a projection

$$\begin{array}{rl} P^K_\epsilon : L^2(\Sigma; \ell^2_K) \longrightarrow \mathbb{R}^{Nn(\epsilon)}, \\ (\vec{s}_j(x'))_{\substack{0\leq j'\leq K-1 \\ 1\leq j_N}} \mapsto (\vec{s}_p)_{1\leq p\leq n(\epsilon)} \end{array} \tag{27}$$

given by

$$\vec{s}_p = \frac{1}{|\epsilon KY'|} \int_{(\epsilon KY')_{\ell'}} \vec{s}_j(x')dx',$$

where (p, j, ℓ') are related by formula (26) and $(\epsilon KY')_{\ell'}$ is the cross section of the cell $G^K_{\epsilon,\ell'}$.

We also define an extension

$$\begin{array}{rl} E^K_\epsilon : \mathbb{R}^{Nn(\epsilon)} \longrightarrow L^2(\Sigma; \ell^2_K), \\ (\vec{s}_p)_{1\leq p\leq n(\epsilon)} \mapsto (\vec{s}_j(x'))_{\substack{0\leq j'\leq K-1 \\ 1\leq j_N}} \end{array} \tag{28}$$

given by

$$\vec{s}_j(x') = \sum_{\ell'} \chi_{(\epsilon KY')_{\ell'}}(x')\vec{s}_p,$$

where (p, j, ℓ') are related by formula (26) and $\chi_{(\epsilon KY')_{\ell'}}(x')$ is the characteristic function of $(\epsilon KY')_{\ell'}$. By convention, $\vec{s}_p$ is taken equal to 0 if the values of j and ℓ' correspond to a cell truncated by the boundary $\partial\Omega$ which therefore contains no tube.

One can easily check that P^K_ϵ and E^K_ϵ are adjoint operators (up to a multiplicative constant) and that the product $P^K_\epsilon E^K_\epsilon$ is nothing but the identity in $\mathbb{R}^{Nn(\epsilon)}$. Therefore, the spectrum of B^K_ϵ consists of that of S_ϵ and zero as an eigenvalue of infinite multiplicity. We summarize these results in the next lemma, the proof of which is safely left to the reader.

LEMMA 2.9. *The operators P^K_ϵ and E^K_ϵ satisfy the following properties;*

1. $(P^K_\epsilon)^\star = (\epsilon K)^{-(N-1)} E^K_\epsilon$,
2. $(E^K_\epsilon)^\star = (\epsilon K)^{(N-1)} P^K_\epsilon$,
3. $P^K_\epsilon E^K_\epsilon = Id_{\mathbb{R}^{Nn(\epsilon)}}$.

Therefore, the extended operator $B^K_\epsilon = E^K_\epsilon S_\epsilon P^K_\epsilon$ is self-adjoint and compact in $L^2(\Sigma; \ell^2_K)$. Its spectrum is

$$\sigma(B^K_\epsilon) = \sigma(S_\epsilon) \bigcup \{0\}.$$

The convergence analysis of this sequence of extended operators B^K_ϵ is amenable to the two-scale convergence method in the sense of boundary layers (as introduced in the previous section). It turns out that the corresponding limit operator B^K has a complicated form which can be considerably simplified by introducing the so-called Bloch wave decomposition. However, we emphasize that this decomposition will affect only the $(N-1)$ first variables and not the last one, orthogonal to the boundary Σ.

LEMMA 2.10. *Given a family* $(\vec{s}_j)_{\substack{0\le j'\le K-1\\ 1\le j_N}}$ *in* ℓ^2_K*, there exists a unique family* $(\vec{t}_j)_{\substack{0\le j'\le K-1\\ 1\le j_N}}$ *in* ℓ^2_K *such that, for any fixed* j_N*,*

$$\sum_{0\le j'\le K-1} \vec{s}_j \chi_{Y_{j'}}(y') = \sum_{0\le j'\le K-1} \vec{t}_j e^{2\pi\imath \frac{j'}{K}\cdot E(y')},$$

where $E(\cdot)$ *denotes the integer part function and* $(Y_{j'})_{0\le j'\le K-1}$ *is the family of subcells of* KY'*. Moreover, Parseval's identity holds true; i.e., for any fixed* j_N*,*

$$\sum_{0\le j'\le K-1} |\vec{s}_j|^2 \ = \ K^{N-1} \sum_{0\le j'\le K-1} |\vec{t}_j|^2.$$

The proof of Lemma 2.10 is standard (see, e.g., [1]). Remark that ℓ^2_K is isomorphic to $(\ell^2_1)^{K^{N-1}}$ by identifying an element $(\vec{s}_j)_{\substack{0\le j'\le K-1\\ 1\le j_N}}$ of ℓ^2_K as a collection of K^{N-1} elements $(\vec{s}_{(j',j_N)})_{j_N\ge 1}$ of ℓ^2_1. Therefore, in Lemma 2.10, one could replace ℓ^2_K by $(\ell^2_1)^{K^{N-1}}$. Let us define a linear map $\mathcal{B}'$

$$\begin{array}{rccl} \mathcal{B}' : & \ell^2_K & \longrightarrow & (\ell^2_1)^{K^{N-1}}, \\ & (\vec{s}_j) & \mapsto & (K^{\frac{N-1}{2}} \vec{t}_{(j',j_N)}), \end{array} \tag{29}$$

where the vectors $\vec{s}_j$ and $\vec{t}_j$ are related as in Lemma 2.10. This Bloch decomposition $\mathcal{B}'$ (the prime indicates that it concerns only the first $(N-1)$ variables) is easily seen to be an isometry from ℓ^2_K to $(\ell^2_1)^{K^{N-1}}$; namely, $(\mathcal{B}')^\star = (\mathcal{B}')^{-1}$.

We are now in a position to state the main result on the asymptotic behavior of B^K_ϵ.

THEOREM 2.11. *For each fixed* $K\ge 1$*, as* ϵ *goes to* 0*, the sequence* B^K_ϵ *converges strongly to a limit* B^K *into* $L^2(\Sigma;\ell^2_K)$*; i.e., for any function* $\vec{s}(x')\in L^2(\Sigma;\ell^2_K)$ *we have*

$$B^K_\epsilon \vec{s}(x') \longrightarrow B^K \vec{s}(x') \quad \text{in } L^2(\Sigma;\ell^2_K) \text{ strongly.}$$

By using the Bloch decomposition $\mathcal{B}'$ *defined in* (29)*, the operator* B^K *can be diagonalized*

$$B^K = (\mathcal{B}')^\star D^K \mathcal{B}' \quad \text{with} \quad D^K = \operatorname{diag}(D^K_{j'})_{0\le j'\le K-1},$$

where the entries $D^K_{j'}$ *are self-adjoint continuous (but not compact) operators in* $L^2(\Sigma;\ell^2_1)$ *defined, for any* $(\vec{s}_{j_N}(x'))_{j_N\ge 1}\in L^2(\Sigma;\ell^2_1)$*, by*

$$D^K_{j'}(\vec{s}_{j_N}(x')) = \left(\int_{\Gamma_{j_N}} u_{j'}\vec{n}ds\right)_{j_N\ge 1},$$

where $u_{j'}(y)$ is the unique solution of

$$\left\{\begin{array}{ll} -\Delta_y u_{j'} = 0 & \text{in } G^*, \\ \frac{\partial u_{j'}}{\partial n} = \vec{s}_{j_N}\cdot\vec{n} & \text{on } \Gamma_{j_N},\ j_N\geq 1, \\ u_{j'} = 0 & \text{on } y_N = 0, \\ y'\mapsto e^{-2\pi\imath\frac{j'}{K}\cdot y'}u_{j'}(y',y_N) & Y'-\text{periodic}, \end{array}\right. \tag{30}$$

where G^* is the fluid part of the semi-infinite band G (see Figure 2).

REMARK 2.12. *Of course, the solution $u_{j'}$ of (30) depends also on the variable $x'\in\Sigma$ since each displacement $\vec{s}_{j_N}(x')$ depends on x'. Nevertheless, x' plays the role of a parameter, since (30) is a partial differential equation in the variable y only. The limit problem (30) admits a unique solution $u_{j'}(x',y)$ in the space $L^2(\Sigma; D^1_{j',\#}(G^*))$, where $D^1_{j',\#}(G^*)$ is a Deny–Lions-type space. More precisely, it is defined as $D^1_{0\#}(G)$ in (21), the only difference being that functions in $D^1_{j',\#}(G^*)$ satisfy a $(e^{2\pi\imath\frac{j'}{K}}, Y')$ periodicity condition in y', instead of the usual Y' periodicity. Recall that a function $w(y)$ satisfying the periodicity condition of the limit problem (30) is said to be $(e^{2\pi\imath\frac{j'}{K}}, Y')$-periodic in y' because such a function also satisfies the following (generalized) periodicity condition:*

$$w(y+(k',0)) = e^{2\pi\imath\frac{j'\cdot k'}{K}}w(y) \quad \forall y=(y',y_N) \text{ and } \forall k'\in\mathbb{Z}^{N-1}.$$

For more details on this class of functions, we refer to [1], [16].

The key of the proof of Theorem 2.11 is the following homogenization result for the fluid potential when the displacements of the tubes are given in terms of the projection operator P^K_ϵ. Remark that, in view of definition (27) of P^K_ϵ, such a family of displacements concentrates near the boundary $\Sigma\times\{0\}$ as ϵ goes to 0.

PROPOSITION 2.13. *For any $\vec{s}(x')\in L^2(\Sigma;\ell^2_K)$ let us define $u_\epsilon = u_\epsilon(\vec{s})$ as the unique solution in $H^1(\Omega_\epsilon)$ of*

$$\left\{\begin{array}{ll} -\Delta u_\epsilon = 0 & \text{in } \Omega_\epsilon, \\ \frac{\partial u_\epsilon}{\partial n} = \left(P^K_\epsilon\vec{s}(x')\right)_p\cdot\vec{n} & \text{on } \Gamma^\epsilon_p,\ 1\leq p\leq n(\epsilon), \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{31}$$

Then, u_ϵ two scale converges in the sense of boundary layers to 0 and ∇u_ϵ two-scale converges in the sense of boundary layers to $\nabla_y u_0(x',y)$, where $u_0(x',y)$ is the unique solution in $L^2(\Sigma, D^1_{0\#}(G^{*K}))$ of

$$\left\{\begin{array}{ll} -\Delta_y u_0 = 0 & \text{in } G^{*K}, \\ \frac{\partial u_0}{\partial n} = \vec{s}_j\cdot\vec{n} & \text{on } \Gamma_j, \\ u_0 = 0 & \text{if } y_N = 0, \\ y'\mapsto u_0(x',y',y_N) & KY'\text{-periodic}, \end{array}\right. \tag{32}$$

and ∇u_ϵ two-scale converges strongly, i.e.,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\Omega_\epsilon}|\nabla u_\epsilon|^2dx = \frac{1}{|KY'|}\int_\Sigma\int_{G^K}|\nabla_y u_0|^2dx'dy. \tag{33}$$

Moreover, if $\vec{s}^\epsilon(x')$ is a sequence which converges weakly to a limit $\vec{s}(x')$ in $L^2(\Sigma;\ell^2_K)$, then the sequence of associated solutions $u_\epsilon(\vec{s}^\epsilon)$ two-scale converges in the sense of boundary layers to 0 and $\nabla u_\epsilon(\vec{s}^\epsilon)$ two-scale converges in the sense of boundary layers to $\nabla u_0(x',y)$, where u_0 is still the solution of (32).

REMARK 2.14. *A priori, the solution u_ϵ of* (31) *is defined only in the fluid domain Ω_ϵ which is a varying set as ϵ goes to* 0*. However, it is a standard matter (see* [13]*) to build an extension operator X_ϵ acting from $H^1(\Omega_\epsilon)$ into $H^1(\Omega)$ such that, for any $v \in H^1(\Omega_\epsilon)$,*

$$X_\epsilon v = v \text{ in } \Omega_\epsilon \text{ and } \|X_\epsilon v\|_{H^1(\Omega)} \leq C\|v\|_{H^1(\Omega_\epsilon)},$$

where C is a positive constant independent of ϵ. In what follows, we shall always identify functions in $H^1(\Omega_\epsilon)$ (as u_ϵ) with their extension in $H^1(\Omega)$ (as $X_\epsilon u_\epsilon$).

To prove Proposition 2.13 we need two technical lemmas.

LEMMA 2.15. *The extension and projection operators E_ϵ^K and P_ϵ^K satisfy the following estimates:*

(i) $\|P_\epsilon^K \vec{s}(x')\|_{\mathbb{R}^{Nn(\epsilon)}} \leq C\epsilon^{-\frac{N-1}{2}} \|\vec{s}(x')\|_{L^2(\Sigma;\ell_K^2)}$,

(ii) $\|E_\epsilon^K(\vec{s}_p)\|_{L^2(\Sigma;\ell_K^2)} \leq C\epsilon^{\frac{N-1}{2}} \|(\vec{s}_p)_{1\leq p\leq n(\epsilon)}\|_{\mathbb{R}^{Nn(\epsilon)}}$,

where C is a constant independent of ϵ and the norms are defined by

$$\|(\vec{s}_p)_{1\leq p\leq n(\epsilon)}\|^2_{\mathbb{R}^{Nn(\epsilon)}} = \sum_{1\leq p\leq n(\epsilon)} |\vec{s}_p|^2,$$

$$\|\vec{s}(x')\|^2_{L^2(\Sigma;\ell_K^2)} = \int_\Sigma \sum_{\substack{0\leq j'\leq K-1\\ 1\leq j_N}} |\vec{s}_j(x')|^2 dx'.$$

Proof. Let us prove (i) (the other inequality (ii) has a similar proof). By definition of P_ϵ^K,

$$\|P_\epsilon^K \vec{s}(x')\|^2_{\mathbb{R}^{Nn(\epsilon)}} = \sum_{1\leq p\leq n(\epsilon)} \Big(\frac{1}{|\epsilon KY'|}\int_{(\epsilon KY')_{\ell'}} \vec{s}_j(x')dx'\Big)^2,$$

where (p, j, ℓ') are related by formula (26). Applying the Cauchy–Schwarz inequality and summing over ℓ' yield

$$\begin{aligned}\|P_\epsilon^K \vec{s}(x')\|^2_{\mathbb{R}^{Nn(\epsilon)}} &\leq \sum_{1\leq p\leq n(\epsilon)} \tfrac{1}{|\epsilon KY'|}\int_{(\epsilon KY')_{\ell'}} |\vec{s}_j(x')|^2 dx' \\ &\leq \tfrac{1}{(K\epsilon)^{N-1}}\int_\Sigma \sum_j |\vec{s}_j(x')|^2 dx',\end{aligned} \tag{34}$$

which is the desired result.

LEMMA 2.16. *Let $\vec{s}^\epsilon(x')$ be a sequence of functions which converges weakly to $\vec{s}(x')$ in $L^2(\Sigma;\ell_K^2)$. Define a piecewise constant function*

$$\vec{a}^\epsilon(x) = \sum_{\ell'}\sum_j \Big(\frac{1}{|\epsilon KY'|}\int_{(\epsilon KY')_{\ell'}} \vec{s}_j^\epsilon(x')dx'\Big)\chi_{Y^\epsilon_{j\ell'}}(x),$$

where $\chi_{Y^\epsilon_{j\ell'}}(x)$ is the characteristic function of the jth subcell of the periodic cell $G^K_{\epsilon,\ell'}$. Then, $\vec{a}^\epsilon$ two-scale converges in the sense of boundary layers to a limit $\vec{a}^0(x,y) \in L^2(\Sigma\times G^K)$ defined by

$$\vec{a}^0(x,y) = \sum_j \vec{s}_j(x')\chi_{Y_j}(y),$$

where $\chi_{Y_j}(y)$ *is the characteristic function of the jth subcell of the reference cell* G^K. *Moreover, if* $\vec{s}^\epsilon(x')$ *converges* strongly *to* $\vec{s}(x')$ *in* $L^2(\Sigma;\ell^2_K)$, *then* $\vec{a}^\epsilon$ *two-scale converges* strongly *to* $\vec{a}^0$ *in the sense of boundary layers, i.e.,*

$$\lim_{\epsilon\to 0}\frac{1}{\sqrt{\epsilon}}\|\vec{a}^\epsilon(x)\|_{L^2(\Omega)} = \frac{1}{K^{\frac{N-1}{2}}}\|\vec{a}^0(x',y)\|_{L^2(\Sigma\times G^K)}.$$

Proof. The proof is very similar to that of Lemma 3.3.2 in our previous work [3], so we briefly sketch it. Let $\vec{\varphi}(x',y)$ be a suitable smooth test function defined on $\Sigma\times G^K$ with values in $\mathbb{R}^N$ such that $y'\to\vec{\varphi}(x',y',y_N)$ is KY'-periodic and $\vec{\varphi}$ vanishes for sufficiently large y_N. We check the definition of two-scale convergence:

$$\begin{aligned}&\tfrac{1}{\epsilon}\int_\Omega \vec{a}^\epsilon(x)\cdot\vec{\varphi}(x',\tfrac{x}{\epsilon})dx\\ &=\tfrac{1}{\epsilon}\sum_{\ell,j}\Big(\tfrac{1}{(\epsilon K)^{N-1}}\int_{\epsilon(KY')_{\ell'}}\vec{s}^\epsilon_j(x')dx'\Big)\cdot\int_{Y^\epsilon_{j\ell'}}\vec{\varphi}(x',\tfrac{x}{\epsilon})dx\\ &=\tfrac{1}{K^{N-1}}\sum_j\int_\Sigma \vec{s}^\epsilon_j(x')\cdot\Big[\sum_{\ell'}\Big(\tfrac{1}{\epsilon^N}\int_{Y^\epsilon_{j\ell'}}\vec{\varphi}(x',\tfrac{x}{\epsilon})dx\Big)\chi_{\epsilon(KY')_{\ell'}}(x')\Big]dx'.\end{aligned}$$

It is easily seen that for each fixed j the term between brackets converges strongly to $\int_{Y_j}\vec{\varphi}(x',y)dy$ in $L^2(\Sigma)^N$. Remark that the sum in j is finite since $\vec{\varphi}$ has a bounded support in G^K. Thus we can pass to the limit and obtain the desired result

$$\frac{1}{K^{N-1}}\sum_j\int_\Sigma \vec{s}_j(x')\cdot\left(\int_{Y_j}\vec{\varphi}(x',y)dy\right)dx'.$$

If $\vec{s}^\epsilon_j$ converges strongly to $\vec{s}_j$, the strong two-scale convergence of $\vec{a}^\epsilon(x)$ is obtained by a similar proof, replacing in the above computation the test function $\vec{\varphi}$ by $\vec{a}^\epsilon(x)$.

Proof of Proposition 2.13. Multiplying (31) by u_ϵ and integrating by parts, we get

$$\begin{aligned}\int_{\Omega_\epsilon}|\nabla u_\epsilon|^2dx &= \sum_{1\le p\le n(\epsilon)}\left(P^K_\epsilon\vec{s}\right)_p\cdot\int_{\Gamma^\epsilon_p}u_\epsilon\vec{n}ds\\ &\le \|\left(P^K_\epsilon\vec{s}\right)\|_{\mathbb{R}^{Nn(\epsilon)}}\left\|\left(\int_{\Gamma^\epsilon_p}u_\epsilon\vec{n}ds\right)\right\|_{\mathbb{R}^{Nn(\epsilon)}}.\end{aligned}\tag{35}$$

An easy calculation (see Lemma 2.2.3 in [3] if necessary) shows that

$$\left\|\left(\int_{\Gamma^\epsilon_p}u_\epsilon\vec{n}ds\right)\right\|^2_{\mathbb{R}^{Nn(\epsilon)}}\le C\epsilon^N\|\nabla u_\epsilon\|^2_{L^2(\Omega_\epsilon)^N},$$

and hence, using Lemma 2.15 we conclude that

$$\int_{\Omega_\epsilon}|\nabla u_\epsilon|^2dx\le C\epsilon\|\vec{s}(x')\|^2_{L^2(\Sigma;\ell^2_K)}.$$

A standard Poincaré inequality in Ω yields the same estimate for u_ϵ in $L^2(\Omega_\epsilon)$:

$$\int_{\Omega_\epsilon}|u_\epsilon|^2dx\le C\epsilon\|\vec{s}(x')\|^2_{L^2(\Sigma;\ell^2_K)}.$$

We now apply the method of two-scale convergence for the asymptotic analysis of the sequence u_ϵ, using test functions with G^K as the periodic cell (since we decided to consider G^K to be the reference cell and not G). By virtue of Proposition 2.6 there exists a subsequence of u_ϵ and a limit function $u_0(x', y)$ in $L^2(\Sigma; D^1_{0\#}(G^K))$ such that $(u_\epsilon, \nabla u_\epsilon)$ two-scale converge in the sense of boundary layers to $(0, \nabla_y u_0)$. Let $\varphi(x', y)$ be a smooth function in $L^2(\Sigma; D^1_{0\#}(G^K))$. Multiplying the equation (31) by $\varphi(x', \frac{x}{\epsilon})$ we obtain

$$\frac{1}{\epsilon}\int_\Omega \chi_{\Omega_\epsilon}(x)\nabla u_\epsilon \cdot \nabla_y \varphi\left(x', \frac{x}{\epsilon}\right) dx + \int_\Omega \chi_{\Omega_\epsilon}(x)\nabla u_\epsilon \nabla_{x'}\varphi\left(x', \frac{x}{\epsilon}\right) dx$$

$$= \sum_{1\le p\le n(\epsilon)} \left(P^K_\epsilon \vec{s}\right)_p \cdot \int_{\Gamma^\epsilon_p} \varphi\left(x', \frac{x}{\epsilon}\right) \vec{n} ds$$

$$= \frac{1}{\epsilon}\int_\Omega \left(\chi_{\Omega_\epsilon}(x) - 1\right)\vec{a}^\epsilon(x) \cdot \left(\nabla_y\varphi\left(x', \frac{x}{\epsilon}\right) + \epsilon\nabla_{x'}\varphi\left(x', \frac{x}{\epsilon}\right)\right) dx,$$

where $\chi_{\Omega_\epsilon}(x)$ is the periodic characteristic function of Ω_ϵ and $\vec{a}^\epsilon(x)$ is a piecewise constant function defined as in Lemma 2.16 by

$$\vec{a}^\epsilon = \sum_{\ell'}\sum_j \Big(\frac{1}{|\epsilon K Y'|}\int_{(\epsilon K Y')_{\ell'}} \vec{s}_j(x')dx'\Big)\chi_{Y^\epsilon_{j\ell'}}(x).$$

Remark that both terms involving $\nabla_{x'}\varphi$ go to zero with ϵ. Applying Lemma 2.16, we pass to the two-scale limit in the remaining terms to get

$$\frac{1}{|KY'|}\int_\Sigma\int_{G^{*K}} \nabla_y u_0(x', y) \cdot \nabla_y\varphi(x', y)dx'dy = \frac{-1}{|KY'|}\int_\Sigma\sum_j\int_{T_j} \vec{s}_j(x') \cdot \nabla_y\varphi(x', y)dx'dy$$

which is nothing but the variational formulation of the limit equation (32). A standard application of the Lax–Milgram lemma yields uniqueness of the solution u_0 in $L^2(\Sigma; D^1_{0\#}(G^K))$. Thus the entire sequence u_ϵ converges to the same limit u_0.

The proof of the energy convergence (33) is standard by passing to the two-scale limit in the right-hand side of (35) since $\vec{a}^\epsilon$ two-scale converges strongly in the sense of Proposition 2.5 (see Proposition 2.2.4 in [3]).

To prove the two-scale convergence of $u_\epsilon(\vec{s}^\epsilon)$ to u_0, when $\vec{s}^\epsilon$ converges weakly to $\vec{s}$ in $L^2(\Sigma; \ell^2_K)$, it suffices to repeat the same above arguments since Lemma 2.16 asserts that $\vec{a}^\epsilon$ two-scale converges to $\vec{a}^0$ even if $\vec{s}^\epsilon$ converges weakly. Note that in this case we do not have the energy convergence.

Proof of Theorem 2.11. Let $\vec{s}(x') \in L^2(\Sigma; \ell^2_K)$ and $\vec{t}^\epsilon$ be a sequence which converges weakly to $\vec{t}$ in $L^2(\Sigma; \ell^2_K)$. Our goal is to prove that

$$\lim_{\epsilon\to 0}\left\langle B^K_\epsilon \vec{s}(x'), \vec{t}^\epsilon(x')\right\rangle_{L^2(\Sigma;\ell^2_K)} = \left\langle B^K \vec{s}(x'), \vec{t}(x')\right\rangle_{L^2(\Sigma;\ell^2_K)}.$$

By definition of B_ϵ^K, we have

$$\begin{aligned}
\left\langle B_\epsilon^K \vec{s}(x'), \vec{t}^\epsilon(x')\right\rangle_{L^2(\Sigma;\ell_K^2)} &= \left\langle E_\epsilon^K S_\epsilon P_\epsilon^K \vec{s}(x'), \vec{t}^\epsilon(x')\right\rangle_{L^2(\Sigma;\ell_K^2)} \\
&= (\epsilon K)^{N-1} \left\langle S_\epsilon P_\epsilon^K \vec{s}(x'), P_\epsilon^K \vec{t}^\epsilon(x')\right\rangle_{\mathbb{R}^{Nn(\epsilon)}} \\
&= (\epsilon K)^{N-1} \sum_{1\le p\le n(\epsilon)} \tfrac{1}{\epsilon^N}\left(\int_{\Gamma_p^\epsilon} u_\epsilon(\vec{s})\vec{n}ds\right)\cdot (P_\epsilon^K \vec{t}^\epsilon)_p \\
&= \tfrac{K^{N-1}}{\epsilon}\int_{\Omega_\epsilon} \nabla u_\epsilon(\vec{s})\cdot \nabla u_\epsilon(\vec{t}^\epsilon)dx.
\end{aligned}$$

By Proposition 2.13 we know that $\nabla u_\epsilon(\vec{s})$ two-scale converges *strongly* in the sense of boundary layers to $\nabla_y u_0(\vec{s})$ while $\nabla u_\epsilon(\vec{t}^\epsilon)$ two-scale converges weakly to $\nabla_y u_0(\vec{t})$. By virtue of Proposition 2.5 we can pass to the limit in the product and we get

$$\lim_{\epsilon\to 0}\left\langle B_\epsilon^K \vec{s}(x'), \vec{t}^\epsilon(x')\right\rangle_{L^2(\Sigma;\ell_K^2)} = \int_\Sigma\int_{G^{*K}} \nabla_y u_0(\vec{s})\cdot\nabla_y u_0(\vec{t})dx'dy,$$

where $u_0(\vec{s})$ and $u_0(\vec{t})$ are solutions of the homogenized problem (32) with $\vec{s}$ and $\vec{t}$, respectively, as the right-hand side. A simple integration by parts shows that

$$\int_\Sigma\int_{G^{*K}} \nabla_y u_0(\vec{s})\cdot\nabla_y u_0(\vec{t})dx'dy \;=\; \left\langle B^K \vec{s}(x'), \vec{t}(x')\right\rangle_{L^2(\Sigma;\ell_K^2)},$$

where the limit operator B^K is defined by

$$B^K\vec{s}(x') = \left(\int_{\Gamma_j} u_0(\vec{s})\vec{n}ds\right)_{\substack{0\le j'\le K-1\\ 1\le j_N}}. \tag{36}$$

This proves the strong convergence of B_ϵ^K to B^K on $L^2(\Sigma;\ell_K^2)$. Obviously, B^K is self-adjoint and continuous but not compact since x' plays the role of a parameter in the definition of B^K.

It remains to diagonalize B^K with the help of the Bloch decomposition $\mathcal{B}'$. This diagonalization process has already been exposed in section 3.3 of our previous paper [3] in a slightly different context. For the sake of brevity, we do not repeat this standard argument here. Let us simply indicate the three main steps of this Bloch diagonalization. First, we apply the operator $\mathcal{B}'$ to $\vec{s}(x') = (\vec{s}_j(x'))_{\substack{0\le j'\le K-1\\ j_N\ge 1}}$ which gives the Bloch decomposition of $\vec{s}(x')$ with respect to the multi-index j' (not including j_N). Secondly, plugging this Bloch decomposition in the limit equation (32) (which holds in G^{*K}) and using a similar Bloch decomposition of $u_0(\vec{s})$, we decompose (32) in a family of K^{N-1} equations defined in a single reference cell G^*. In a third step, applying again the Bloch decomposition $\mathcal{B}'$ to formula (36) yields the desired diagonalization of B^K.

2.3. Analysis of the limit spectrum. In this section we analyze the spectrum of the limit operator B^K and, from the strong convergence of B_ϵ^K to B^K, we deduce the lower semicontinuous convergence of the spectrum $\sigma(S_\epsilon)$ to the limit spectrum $\sigma(B^K)$. Recall that for any $K\ge 1$, the extended operator B_ϵ^K has a spectrum given by

$$\sigma(B_\epsilon^K) = \sigma(S_\epsilon)\cup\{0\}.$$

Since B^K_ϵ converges strongly to B^K in $L^2(\Sigma;\ell^2_K)$, by virtue of Proposition 2.1.11 in [3], we have

$$\sigma(B^K) \subset \sigma_\infty = \lim_{\epsilon\to 0} \sigma(S_\epsilon).$$

From Rellich's theorem, the strong convergence of the spectral family associated with B^K_ϵ to that of B^K is also easily deduced (see Theorem 3.2.5 in [3]). This gives some (partial) information on the convergence of eigenvectors that we shall not use below.

In view of Theorem 2.11,

$$B^K = (\mathcal{B}')^{-1} D^K \mathcal{B}' \quad \text{with} \quad D^K = \operatorname{diag}(D^K_{j'})_{0\le j'\le K-1},$$

where each $D^K_{j'}$ is a self-adjoint continuous operator in $L^2(\Sigma;\ell^2_1)$. Since $\mathcal{B}'$ is an isometry, we have

$$\sigma(B^K) = \bigcup_{0\le j'\le K-1} \sigma(D^K_{j'}).$$

By the very definition of $D^K_{j'}$, the macroscopic variable $x' \in \Sigma$ plays the role of a parameter. Therefore, for any fixed value of x', $D^K_{j'}$ can be identified with an operator $d_{\frac{j'}{K}}$ acting in ℓ^2_1 which does not depend on x'. Introducing the Bloch parameter $\theta' = \frac{j'}{K} \in [0,1]^{N-1}$, this new operator $d_{\theta'}$ is defined by

$$\begin{aligned} & d_{\theta'} \,:\, \ell^2_1 \longrightarrow \ell^2_1, \\ & (\vec{s}_q)_{q\ge 1} \;\mapsto\; \left(\int_{\Gamma_q} u_{\theta'} \cdot \vec{n} ds \right)_{q\ge 1}, \end{aligned} \tag{37}$$

where $u_{\theta'}(y)$ is the unique solution of

$$\begin{cases} -\Delta u_{\theta'} = 0 & \text{in } G^*, \\ \frac{\partial u_{\theta'}}{\partial n} = \vec{s}_q \cdot \vec{n} & \text{on } \Gamma_q,\ q \ge 1, \\ u_{\theta'} = 0 & \text{if } y_N = 0, \\ y' \mapsto e^{-2\pi\imath\theta'\cdot y'} u_{\theta'}(y', y_N) & Y'\text{-periodic.} \end{cases}$$

In (37) the positive integer q is nothing but the index j_N introduced in Definition 2.8. Clearly, we have

$$\sigma(D^K_{j'}) = \sigma(d_{\frac{j'}{K}}).$$

As is well known, the spectrum of a self-adjoint operator can be decomposed in its *discrete* part, made of, at most, a countable number of isolated eigenvalues of finite multiplicities, and its *essential* part, for which the Weyl criterion applies (see, e.g., [25], [33], [34]). The next proposition characterizes the spectrum of $d_{\theta'}$.

PROPOSITION 2.17. *For all $\theta' \in [0,1]^{N-1}$, $d_{\theta'}$ is a self-adjoint continuous but noncompact operator in ℓ^2_1. Labeling the eigenvalues of the discrete spectrum $\sigma_{disc}(d_{\theta'})$ by decreasing order, each discrete eigenvalue is piecewise continuous in θ'. The essential spectrum is given by*

$$\sigma_{ess}(d_{\theta'}) = \bigcup_{\theta_N\in[0,1]} \sigma(A(\theta',\theta_N)),$$

where $A(\theta', \theta_N)$ *is the Bloch homogenized matrix, defined by* (12)*, which is continuous in* $\theta \in]0,1[^N$ *but discontinuous at* $\theta = 0$*. Moreover, the entire spectrum* $\sigma(d_{\theta'})$*, considered as a subset of* $\mathbb{R}^+$*, depends continuously on* θ'*, except at* $\theta' = 0$*.*

Because we use the usual convenient labeling of the discrete eigenvalues by decreasing order, we can merely prove that they are piecewise continuous. This is due to the fact that, when θ' varies, an analytical branch (if any) of discrete eigenvalues may merge into the essential spectrum: this yields a "jump" in the labeling of discrete eigenvalues. Therefore, one cannot hope to prove a global continuity of these eigenvalues with such an ordering.

Let us postpone for a moment the proof of Proposition 2.17 and define the so-called *boundary layer spectrum* associated with the surface Σ:

$$\sigma_\Sigma \stackrel{\text{def}}{=} \bigcup_{\theta' \in]0,1[^{N-1}} \sigma(d_{\theta'}) \cup \sigma(d_0). \tag{38}$$

By virtue of Proposition 2.17, we have

$$\sigma_{Bloch} \subset \sigma_\Sigma. \tag{39}$$

Therefore σ_Σ also has a band structure since it includes the Bloch spectrum, but it may include new bands of eigenvalues of $\sigma_{disc}(d_{\theta'})$. It also contains the isolated eigenvalues of $\sigma_{disc}(d_0)$. Therefore σ_Σ can contain elements which are not included in the previous limit spectrum $\sigma(S) \cup \sigma_{Bloch}$ (see section 1.2). The continuity of $\sigma(d_{\theta'})$ with respect to θ' ensures that σ_Σ is the closure of the union of all spectra $\sigma(d_{\theta'})$ with θ' rational.

$$\overline{\bigcup_{K \geq 1} \bigcup_{0 \leq j' \leq K-1} \sigma(d_{\frac{j'}{K}})} = \sigma_\Sigma.$$

We summarize our results in the following theorem.

THEOREM 2.18. *The boundary layer spectrum associated to* Σ *is included in the limit spectrum*

$$\sigma_\Sigma \subset \sigma_\infty.$$

REMARK 2.19. *Of course* σ_Σ *is not the complete boundary layer spectrum since it is concerned only with that part of the spectrum concentrating near* Σ*. A completely similar analysis has to be done for all the* $(N-1)$*-dimensional surfaces and all other lower dimensional manifolds (edges, corners, etc.) of which the boundary of* Ω *is made up. Then, we shall prove in the next section that the union of all these contributions, the so-called boundary layer spectrum, plus the usual homogenized spectrum and the Bloch spectrum, is equal to* σ_∞*, at least when* Ω *is made up only of entire cells* ϵY*.*

Proof of Proposition 2.17. Let us first prove that the essential spectrum of $d_{\theta'}$ is included in the Bloch spectrum, and, more precisely,

$$\sigma_{ess}(d_{\theta'}) = \bigcup_{0 \leq \theta_N \leq 1} \sigma(A(\theta', \theta_N)),$$

where $A(\theta)$ is the usual Bloch homogenized matrix defined in (12). In particular, this proves that $\sigma_{ess}(d_{\theta'}) \neq \{0\}$, so $d_{\theta'}$ is not compact.

Let $\lambda(\theta)$ be an eigenvalue of $A(\theta)$ and $u(\theta)$ be the associated potential solution of

$$\begin{cases} -\Delta_y u(\theta) = 0 & \text{in } Y^*, \\ \frac{\partial u(\theta)}{\partial n} = \lambda^{-1}(\theta) \int\limits_\Gamma u(\theta)\vec{n}ds & \text{on } \Gamma, \\ y \mapsto e^{-2\pi\imath\theta\cdot y} u(\theta, y) & Y\text{-periodic}. \end{cases}$$

We construct a Weyl sequence u_n associated with the spectral value $\lambda(\theta)$ by

$$u_n = \frac{u(\theta)\psi_n}{\|u(\theta)\psi_n\|_{L^2(G^*)}},$$

where $\psi_n(y_N)$ is a cut-off function defined by

$$\begin{cases} \psi_n(y_N) = & y_N & \text{when } 0 \le y_N \le 1, \\ \psi_n(y_N) = & 1 & \text{when } 1 \le y_N \le n, \\ \psi_n(y_N) = & n+1-y_N & \text{when } n \le y_N \le n+1, \\ \psi_n(y_N) = & 0 & \text{when } y_N \ge n+1. \end{cases}$$

By definition, $\|u_n\|_{L^2(G^*)} = 1$ and $\lim_{n\to+\infty} \|u(\theta)\psi_n\|_{L^2(G^*)} = +\infty$. Then, it is easily checked that, for any $\varphi \in D^1_{0\#}(G)$ (the Deny–Lions-type space defined in (21)),

$$\int_{G^*} \nabla u_n \cdot \nabla\varphi dy = \frac{1}{\lambda(\theta)} \sum_{q\ge1} \Big(\int_{\Gamma_q} u_n\vec{n}ds \Big) \cdot \Big(\int_{\Gamma_q} \varphi\vec{n}ds \Big) + \langle r_n, \varphi \rangle ,$$

where r_n is a negligible remainder term in the sense that

$$\lim_{n\to+\infty} \frac{\langle r_n, \varphi\rangle}{\|\nabla\varphi\|_{L^2(G^*)^N}} = 0.$$

Furthermore, $\vec{s}_n = (\int_{\Gamma_q} u_n\vec{n}ds)_{q\ge1}$ converges weakly to 0 in ℓ^2_1 since

$$\lim_{n\to+\infty} \|u(\theta)\psi_n\|_{L^2(G^*)} = +\infty.$$

Therefore, $\vec{s}_n$ is a Weyl sequence associated with $\lambda(\theta)$ for the operator $d_{\theta'}$. This proves that $\lambda(\theta) \in \sigma_{ess}(d_{\theta'})$. To prove the converse inclusion,

$$\sigma_{ess}(d_{\theta'}) \subset \bigcup_{0\le\theta_N\le1} \sigma(A(\theta', \theta_N)),$$

we consider a Weyl sequence $\vec{s}_n$ for a spectral value $\lambda \in \sigma_{ess}(d_{\theta'})$. Let u_n be the associated potential solution, i.e.,

$$\begin{cases} -\Delta u_n = 0 & \text{in } G^*, \\ \frac{\partial u_n}{\partial n} = (\vec{s}_n)_q \cdot \vec{n} & \text{on } \Gamma_q,\ q \ge 0, \\ u_n = 0 & \text{if } y_N = 0, \\ y' \mapsto e^{-2\pi\imath\theta'\cdot y'} u_n(y', y_N) & Y'\text{-periodic}. \end{cases} \tag{40}$$

Since $\|\vec{s}_n\|_{\ell^2_1} = 1$ and $\vec{s}_n \rightharpoonup 0$ in ℓ^2_1 weakly, it is easily seen that u_n converges to 0 weakly in $H^1(G^*)$. Furthermore, since the weak convergence to 0 of $\vec{s}_n$ implies that its components $(\vec{s}_n)_q$ go to 0 for fixed q, it is not difficult to check that, for any compact set $\mathcal{K}$ of G^*, u_n converges strongly to 0 in $H^1(\mathcal{K})$ (multiply equation (40) by ϕu_n where ϕ is equal to 1 in $\mathcal{K}$ and is compactly supported away from infinity). Introducing a sequence

$$v_n = \frac{\psi u_n}{\|\psi u_n\|_{L^2(G^*)}},$$

where $\psi(y_N)$ is a cut-off function defined by

$$\begin{cases} \psi(y_N) = & 0 \qquad \text{for } y_N \leq 0, \\ \psi(y_N) = & y_N \qquad \text{for } 0 \leq y_N \leq 1, \\ \psi(y_N) = & 1 \qquad \text{for } y_N \geq 1, \end{cases}$$

it is straightforward to prove that

$$\int_{B^*} \nabla v_n \cdot \nabla \varphi dx = \frac{1}{\lambda} \sum_{q \in \mathbb{Z}} \Big(\int_{\Gamma_q} v_n \vec{n} ds \Big) \cdot \Big(\int_{\Gamma_q} \varphi \vec{n} ds \Big) + \langle r_n, \varphi \rangle$$

for any $\varphi \in D^1_\#(B^*)$, where B^* is the infinite band $Y' \times]-\infty; +\infty[$ perforated by the periodic arrangement of tubes $(T_q)_{q\in\mathbb{Z}}$, and r_n is another negligible remainder term such that

$$\lim_{n\to+\infty} \frac{\langle r_n, \varphi\rangle}{\|\nabla\varphi\|_{L^2(B^*)^N}} = 0.$$

Therefore,

$$\vec{t_n} = \left(\int_{\Gamma_q} v_n \vec{n} ds \right)_{q\in\mathbb{Z}}$$

is a Weyl sequence for an operator similar to $d_{\theta'}$ but defined in the whole infinite band B^* instead of the semi-infinite band G^*. A standard Bloch decomposition with respect to the variable y_N yields that λ belongs to $\bigcup_{0\leq\theta_N\leq 1} \sigma(A(\theta', \theta_N))$.

To conclude the proof of Proposition 2.17, it remains to prove that the isolated eigenvalues of finite multiplicity $\lambda(\theta') \in \sigma_{disc}(d_{\theta'})$ are piecewise continuous with respect to θ'. Let θ'_n be a sequence converging to θ' in $]0, 1[^{N-1}$. Obviously, the sequence of continuous operators $d_{\theta'_n}$ uniformly converges to $d_{\theta'}$ in ℓ^2_1. Now, let us invoke a classical theorem (see, e.g., Theorem 3.1., Chapter I.3 in [20]) which states that for any closed curve γ in the complex plane, which encloses a finite number of eigenvalues of $\sigma_{disc}(d_{\theta'})$ and does not intersect $\sigma(d_{\theta'})$, there exists n_0 such that for any $n \geq n_0$, the curve γ contains the same number of eigenvalues (including multiplicities) of $\sigma_{disc}(d_{\theta'_n})$ and does not intersect $\sigma(d_{\theta'_n})$. This is nothing but the local continuity of the eigenvalues of $\sigma_{disc}(d_{\theta'})$ (enumerated, for example, in decreasing order). Remark that the continuity of the pth eigenvalue of $\sigma_{disc}(d_{\theta'})$ breaks down only when one of the previous eigenvalues (with label between 1 and $p-1$) meets the essential spectrum $\sigma_{ess}(d_{\theta'})$ as θ' varies. In any case, since $\sigma_{ess}(d_{\theta'})$ depends continuously on $\theta' \neq 0$, this proves that the entire spectrum $\sigma(d_{\theta'})$ depends also continuously on $\theta' \neq 0$. The lack of continuity for $\sigma(d_{\theta'})$ at $\theta' = 0$ is a phenomenon already explained in our previous work (see Proposition 3.3.4 in [3]).

REMARK 2.20. *When the tube T is symmetric in Y (in other words, by reflexion with respect to the hyperplane $[y_N = 0]$, G^* yields the infinite periodic array of tubes B^*), it can readily be checked that there is no isolated eigenvalue of finite multiplicity* *for $d_{\theta'}$; i.e., $\sigma_{disc}(d_{\theta'}) = \emptyset$ for all $\theta' \in [0,1]^{N-1}$. If this were not the case, by symmetry an eigenvalue of $\sigma_{disc}(d_{\theta'})$ would also be an eigenvalue of finite multiplicity for a similar operator in the infinite band B^*, which is impossible since by translation there exists an infinite number of eigenvectors.*

We conclude this section by proving that the eigenvectors corresponding to isolated eigenvalues of finite multiplicity of $d_{\theta'}$ are localized in the vicinity of the boundary $[y_N = 0]$ since they decay exponentially at infinity.

PROPOSITION 2.21. *Let λ be an eigenvalue in $\sigma_{disc}(d_{\theta'})$ and let $(\vec{s}_q)_{q\geq 1}$ be a corresponding eigenvector. There exists a positive constant $\alpha > 0$ such that $(e^{\alpha p}\vec{s}_q)_{q\geq 1}$ belongs to ℓ^2_1.*

Proof. The argument is by contradiction of the Weyl property for eigenvalues in the essential spectrum. For $\lambda \in \sigma_{disc}(d_{\theta'})$, let $\vec{s} = (\vec{s}_q)_{q\geq 1}$ be a corresponding normalized eigenvector and $u(y)$ the corresponding potential, solution of

$$\left\{ \begin{array}{ll} -\Delta u = 0 & \text{in } G^*, \\ \frac{\partial u}{\partial n} = \vec{s}_q \cdot \vec{n} & \text{on } \Gamma_q,\ q \geq 1, \\ u = 0 & \text{if } y_N = 0, \\ y' \mapsto e^{-2\pi\imath\theta'\cdot y'} u(y', y_N) & Y'\text{-periodic.} \end{array} \right. \tag{41}$$

By definition, for all $q \geq 1$, it satisfies

$$\int_{\Gamma_q} u \cdot \vec{n} ds \; = \; \lambda \vec{s}_q.$$

Let us define a sequence $(\vec{s^n})_{n\geq 0}$ in ℓ^2_1 by

$$\vec{s^n} = (\vec{s^n_q})_{q\geq 1} \text{ with } \vec{s^n_q} = \left\{ \begin{array}{ll} 0 & \text{if } q < n, \\ \frac{\vec{s}_q}{\sqrt{\sum_{p=n}^{\infty} |\vec{s}_p|^2}} & \text{if } q \geq n. \end{array} \right.$$

It is easily seen that $\vec{s^n}$ converges weakly to 0 in ℓ^2_1 with $\|\vec{s^n}\|_{\ell^2_1} = 1$. However, since λ does not belong to the essential spectrum of $d_{\theta'}$, any subsequence of $\vec{s^n}$ cannot be a Weyl sequence for λ. This implies the existence of a positive constant C and an integer n_0 such that, for any $n \geq n_0$,

$$\|d_{\theta'}\vec{s^n} - \lambda\vec{s^n}\|_{\ell^2_1} \geq C > 0. \tag{42}$$

As usual $u_n(y)$ is the potential associated with $\vec{s^n}$ through an equation similar to (41). We introduce a smooth cut-off function $\psi_n(y_N)$ such that $\psi_n = 0$ on all tubes T_q for $q < n$, and $\psi_n = 1$ on all tubes T_q for $q \geq n$. Let us denote by ω_n the bounded support of $\nabla\psi_n$ which lies between T_{n-1} and T_n. Introducing an approximation v_n of the potential u_n, defined by

$$v_n(y) = \frac{\psi_n(y_N)\,(u(y) - c_n)}{\sqrt{\sum_{p=n}^{\infty} |\vec{s}_p|^2}} \text{ with } c_n = \frac{1}{|\omega_n|}\int_{\omega_n} u(y) dy,$$

we write

$$d_{\theta'}\vec{s^n} \; = \; \lambda\vec{s^n} + \left(\int_{\Gamma_q} (u_n - v_n) \cdot \vec{n} ds \right)_{q\geq 1}.$$

From (42) we deduce

$$\|\nabla(u_n - v_n)\|_{L^2(G^*)^N} \geq C > 0 \text{ for } n \geq n_0.$$

Using the equations for u and u_n, a simple computation yields

$$\int_{G^*} |\nabla(u_n - v_n)|^2 dy = \int_{G^*} \nabla\psi_n \cdot \frac{(u_n - v_n)\nabla u - (u - c_n)\nabla(u_n - v_n)}{\sqrt{\sum_{p=n}^{\infty} |\vec{s}_p|^2}}. \tag{43}$$

Remark that the integral in the right-hand side reduces to ω_n since $\nabla\psi_n$ has bounded support in ω_n. Applying the Poincaré–Wirtinger inequality in ω_n to $(u-c_n)$ and $(u_n - v_n)$ (this last term has not zero average in ω_n, but (43) is invariant by substraction of a constant to $(u_n - v_n)$), we obtain from (43)

$$\|\nabla(u_n - v_n)\|_{L^2(G^*)^N} \leq C \frac{\|\nabla u\|_{L^2(\omega_n)^N}}{\sqrt{\sum_{p=n}^{\infty} |\vec{s}_p|^2}},$$

which implies

$$\sum_{p=n}^{\infty} |\vec{s}_p|^2 \leq C\|\nabla u\|^2_{L^2(\omega_n)^N}. \tag{44}$$

On the other hand, multiplying equation (41) by $\psi_n(u - c_n)$ and integrating by parts gives

$$\int_{G^*} \psi_n |\nabla u|^2 dy + \int_{G^*} (u - c_n)\nabla u \cdot \nabla\psi_n dy = \lambda \sum_{p=n}^{\infty} |\vec{s}_p|^2.$$

Applying again the Poincaré–Wirtinger inequality in ω_n to $(u - c_n)$ yields

$$\int_{G^*} \psi_n |\nabla u|^2 dy \leq \lambda \sum_{p=n}^{\infty} |\vec{s}_p|^2 + C\|\nabla u\|^2_{L^2(\omega_n)^N}. \tag{45}$$

Let us denote by G_n the subset of G^* defined by $G_n = \{y \in G^* | y_N > n\}$. From (44) and (45) we deduce

$$\|\nabla u\|^2_{L^2(G_{n+1})^N} \leq C\|\nabla u\|^2_{L^2(\omega_n)^N} \leq C \left(\|\nabla u\|^2_{L^2(G_n)^N} - \|\nabla u\|^2_{L^2(G_{n+1})^N} \right),$$

which implies, for $n \geq n_0$,

$$\|\nabla u\|^2_{L^2(G_n)^N} \leq \left(\frac{C}{1 + C} \right)^{n-n_0} \|\nabla u\|^2_{L^2(G_{n_0})^N}. \tag{46}$$

It is easily seen that (46) implies the desired result.

3. Completeness of the boundary layer spectrum. In this section we assume that Ω is a rectangle with integer dimensions, i.e.,

$$\Omega \;=\; \prod_{i=1}^{N}]0; L_i[\quad \text{ and } \quad L_i \in \mathbb{N}^*. \tag{47}$$

The sequence of small parameters ϵ is also assumed to be

$$\epsilon_n = \frac{1}{n}, \ n \in \mathbb{N}^*. \tag{48}$$

Remark that all the previous results in this paper hold for any type of sequence ϵ going to zero. From now on, we restrict ourselves to the sequence ϵ_n since, for any $n \geq 1$, the domain Ω is the union of a finite number of *entire periodic cells* $Y_p^{\epsilon_n}$. However, to simplify the notation, we shall not indicate the dependence on n and simply denote by ϵ the particular sequence defined in (48).

Remark that the assumption on the geometry of Ω can be slightly relaxed. Any polygonal domain with faces parallel to the axis (i.e., the normal is everywhere one of the basis vectors) and having vertex with integer coordinates could equally be considered.

3.1. Presentation of the main result. This section is devoted to the so-called completeness of the limit spectrum. Recall that in our previous work [3] we proved that

$$\sigma_\infty = \sigma(S) \cup \sigma_{Bloch} \cup \sigma_{boundary}, \tag{49}$$

where $\sigma_{boundary}$ is defined in (15). In section 2, we proved that

$$\sigma_\infty \supset \sigma_\Sigma,$$

where σ_Σ is the boundary layer spectrum associated with the surface Σ, defined by (38). Remark that, *due to our hypotheses on the domain* Ω *and on the sequence* ϵ, the surface Σ can be any of the faces of Ω defined by

$$\prod_{\substack{j=1 \\ j\neq i}}^{N}]0; L_j[\times \{0\} \quad \text{or} \quad \prod_{\substack{j=1 \\ j\neq i}}^{N}]0; L_j[\times \{L_i\} \quad \text{for } 1 \leq i \leq N.$$

Of course, the analysis of section 2 can be repeated for any other lower dimensional manifolds (edges, corners, etc.) which compose the boundary of Ω. For $0 \leq m \leq N-1$, let us define the m-dimensional parts of $\partial\Omega$ as

$$\Sigma_{m,\tau} = \prod_{j=1}^{m}]0; L_{\tau(j)}[\times \prod_{j=m+1}^{N} \{x_{\tau(j)} = 0 \text{ or } L_{\tau(j)}\},$$

where τ is any permutation of the numbers $\{1, 2, \ldots, N\}$. There are $2^{N-m} C_N^{N-m}$ m-dimensional manifolds of the type $\Sigma_{m,\tau}$. A simple adaptation of the two-scale convergence in the sense of boundary layers for such manifolds allows us to prove that, for any m and τ,

$$\sigma_\infty \supset \sigma_{\Sigma_{m,\tau}},$$

where $\sigma_{\Sigma_{m,\tau}}$ is the spectrum of a family of limit problems posed, not in a semi-infinite band as in section 2, but rather in a periodic domain bounded in the variables $x_{\tau(1)}, \ldots, x_{\tau(m)}$ and unbounded with respect to the other variables (see section 3.3 for the case of corners in two space dimension). Eventually, defining the union of all these spectra

$$\sigma_{\partial\Omega} = \bigcup_{m,\tau} \sigma_{\Sigma_{m,\tau}}, \tag{50}$$

we deduce from Theorem 2.18 and from the geometric assumptions (47), (48) that

$$\sigma_\infty \supset \sigma_{\partial\Omega}. \tag{51}$$

Comparing our results (49) and (51), a completeness result amounts to link the two definitions of the boundary layer spectrum $\sigma_{\partial\Omega}$ and $\sigma_{boundary}$.

THEOREM 3.1. *For the sequence ϵ_n defined by* (48)*, the boundary layer spectrum satisfies*

$$\sigma_{boundary} \subset \sigma_{\partial\Omega}.$$

Therefore, the limit spectrum of the sequence S_{ϵ_n} is precisely made of three parts; the homogenized, the Bloch, and the boundary layer spectrum

$$\lim_{\epsilon_n \to 0} \sigma(S_{\epsilon_n}) = \sigma(S) \cup \sigma_{Bloch} \cup \sigma_{\partial\Omega},$$

where the boundary layer spectrum $\sigma_{\partial\Omega}$ is explicitly defined by (50)*.*

REMARK 3.2. *Remark that Theorem* 3.1 *does not state that $\sigma_{boundary}$, defined by* (15)*, and $\sigma_{\partial\Omega}$ coincide. Indeed, we have shown in* (39) *that $\sigma_{\partial\Omega}$ contains the Bloch spectrum. It is not clear whether $\sigma_{boundary}$ contains the Bloch spectrum too. The comparison of $\sigma_{\partial\Omega}$ and $\sigma_{boundary}$ is definitely a very difficult question. We suspect that if the definition of $\sigma_{boundary}$ is modified in such a way that it contains only limit eigenvalues corresponding to sequences of eigenvectors which decay exponentially fast away from the boundary, then it may coincide with that part of $\sigma_{\partial\Omega}$ made of discrete eigenvalues (which also have exponentially decreasing corresponding eigenvectors).*

To prove this completeness result, we need an intermediate result in the spirit of section 2.

THEOREM 3.3. *As in section* 2*, let Ω be a domain defined by*

$$\Omega = \Sigma \times]0; L[,$$

with Σ a bounded open set in $\mathbb{R}^{N-1}$ and $L > 0$. Recall that S^1_ϵ is the extension of S_ϵ to $L^2(\Omega)^N$. Consider a sequence of eigenvalues λ_ϵ and eigenvectors $\vec{s}^\epsilon$ such that

$$S^1_\epsilon \vec{s}^\epsilon = \lambda_\epsilon \vec{s}^\epsilon \quad \text{with} \quad \|\vec{s}^\epsilon\|_{L^2(\Omega)^N} = 1 \quad \text{and} \quad \lim_{\epsilon\to 0} \lambda_\epsilon = \lambda.$$

Assume that for all subset ω such that $\overline{\omega} \subset \Omega$, we have

$$\lim_{\epsilon\to 0} \|\vec{s}^\epsilon\|_{L^2(\omega)^N} = 0. \tag{52}$$

Assume further that there exists an $(N-1)$-dimensional open set σ, with $\overline{\sigma} \subset \Sigma$, a positive number l, with $0 < l < L$, and a positive constant c such that

$$\lim_{\epsilon\to 0} \|\vec{s}^\epsilon\|_{L^2(\sigma\times]0,l[)^N} \geq c > 0. \tag{53}$$

Then, λ belongs to the boundary layer spectrum associated with the surface Σ

$$\lambda \in \sigma_\Sigma,$$

where σ_Σ is defined by (38)*.*

The proof of Theorem 3.3 is the focus of the next section. If we admit it for the moment, as well as its generalizations concerning all other manifolds $\Sigma_{m,\tau}$ making up the boundary $\partial\Omega$, we are in a position to complete the following proof.

Proof of Theorem 3.1. Let $\lambda \in \sigma_{boundary}$. By definition there exists a subsequence (still denoted by ϵ), eigenvalues λ_ϵ, and eigenvectors $\vec{s}^\epsilon$ of S^1_ϵ such that

$$S^1_\epsilon \vec{s}^\epsilon = \lambda_\epsilon \vec{s}^\epsilon \quad \text{with} \quad \|\vec{s}^\epsilon\|_{L^2(\Omega)^N} = 1 \quad \text{and} \quad \lim_{\epsilon\to 0} \lambda_\epsilon = \lambda,$$

and, for all subset ω satisfying $\overline{\omega} \subset \Omega$,

$$\lim_{\epsilon\to 0} \|\vec{s}^\epsilon\|_{L^2(\omega)^N} = 0.$$

If there exists an $(N-1)$-dimensional open subset σ_i, compactly embedded in $\prod_{\substack{j=1\\ j\neq i}}^N]0; L_j[$, a positive length $0 < l_i < L_i$, a positive constant c, and another subsequence (still denoted by ϵ) such that

$$\lim_{\epsilon\to 0} \|\vec{s}^\epsilon\|_{L^2(\sigma_i\times]0,l_i[)^N} \geq c > 0 \quad \text{or} \quad \lim_{\epsilon\to 0} \|\vec{s}^\epsilon\|_{L^2(\sigma_i\times]l_i,L_i[)^N} \geq c > 0, \tag{54}$$

then, by application of Theorem 3.3, the limit eigenvalue belongs to $\sigma_{\partial\Omega}$ as desired.

If (54) does not hold true for any such σ_i, l_i, c, and subsequence ϵ, it implies that the L^2-norm of $\vec{s}^\epsilon$ concentrates near the lower dimensional edges of the rectangle Ω. In this case, we repeat the above argument with an $(N-2)$-dimensional open set included in one of the set $\Sigma_{N-2,\tau}$, and so on up to the 0-dimensional set made of one of the vertices of Ω. A tedious but simple induction argument on the dimension m shows that there exists at least a dimension $0 \leq m \leq N-1$, a permutation τ, positive lengths $(l_{\tau(j)})_{m+1\leq j\leq N}$, a positive constant c, and a subsequence ϵ such that

$$\lim_{\epsilon\to 0} \|\vec{s}^\epsilon\|_{L^2(\omega)^N} \geq c > 0,$$

with $\omega \subset \Omega$ of the type

$$\omega = \sigma \times \prod_{j=m+1}^{N} \left( ]0, l_{\tau(j)}[\text{ or }]l_{\tau(j)}, L_{\tau(j)}[\right) \quad \text{and} \quad \overline{\sigma} \subset \prod_{j=1}^{m}]0; L_{\tau(j)}[.$$

Then, applying an adequate generalization of Theorem 3.3, this proves that the limit eigenvalue belongs to $\sigma_{\partial\Omega}$.

3.2. Proof of the completeness. This section is devoted to the proof of Theorem 3.3 which is divided in several lemmas and propositions. Let us begin by recalling the definition of the associated potential u_ϵ, solution of

$$\begin{cases} -\Delta u_\epsilon = 0 & \text{in } \Omega_\epsilon, \\ \frac{\partial u_\epsilon}{\partial n} = \vec{s}^\epsilon_p \cdot \vec{n} & \text{on } \Gamma^\epsilon_p, \quad 1 \leq p \leq n(\epsilon), \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{55}$$

The spectral equation $\widetilde{S}_\epsilon \vec{s}^\epsilon = \lambda_\epsilon \vec{s}^\epsilon$ implies that

$$\left(\int_{\Gamma^\epsilon_p} u_\epsilon \vec{n} \right)_{1\leq p\leq n(\epsilon)} = \lambda_\epsilon \vec{s}^\epsilon_p. \tag{56}$$

By assumption (52), for all subsets ω such that $\overline{\omega} \subset \Omega$, we have

$$\lim_{\epsilon\to 0} \|\vec{s}^\epsilon\|_{L^2(\omega)^N} = 0.$$

In other words, all the energy of the eigenvectors $\vec{s}^\epsilon$ concentrates near the boundary $\partial\Omega$. This concentration effect has important consequences on the associated potential u_ϵ.

LEMMA 3.4. *The sequence u_ϵ defined in* (55) *converges to* 0 *in $H^1_0(\Omega)$ weakly and strongly in $L^2(\Omega)$. Furthermore, u_ϵ converges strongly to* 0 *in $H^1_{loc}(\Omega)$.*

Proof. Multiplying equation (55) by a test function $v \in H^1_0(\Omega)$ yields

$$\int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla v dx = \sum_{p=1}^{n(\epsilon)} \vec{s}^{\,p}_\epsilon \cdot \Big(\int_{\Gamma^\epsilon_p} v\vec{n} ds \Big) = \int_\Omega \vec{s}^\epsilon(x) \cdot \vec{z}^\epsilon(x) dx,$$

where

$$\vec{z}^\epsilon(x) = -\sum_{p=1}^{n(\epsilon)} \frac{1}{\epsilon^N} \Big(\int_{T^\epsilon_p} \nabla v(x) dx \Big) \chi_{Y^\epsilon_p}(x).$$

It is easily seen that $\vec{z}^\epsilon$ converges strongly to $-\frac{|T|}{|Y|}\nabla v(x)$ in $L^2(\Omega)^N$. Since $\vec{s}^\epsilon$ converges weakly to 0 in $L^2(\Omega)^N$ by virtue of (52), we deduce that u_ϵ converges to 0 weakly in $H^1_0(\Omega)$ and, by the Rellich theorem, strongly in $L^2(\Omega)$. Finally, for any open set ω such that $\overline{\omega} \subset \Omega$, let φ be a smooth function with compact support in Ω and equal to 1 on ω. Multiplying (55) by $\varphi^2 u_\epsilon$ and integrating by parts leads to

$$\int_{\Omega_\epsilon} \varphi^2 |\nabla u_\epsilon|^2 dx = -2 \int_{\Omega_\epsilon} \varphi u_\epsilon \nabla\varphi \cdot \nabla u_\epsilon dx + \sum_{p=1}^{n(\epsilon)} \vec{s}^{\,\epsilon}_p \cdot \Big(\int_{\Gamma^\epsilon_p} \varphi^2 u_\epsilon \vec{n} ds \Big). \tag{57}$$

Since u_ϵ converges weakly to 0 in $H^1_0(\Omega)$, the first term in the right-hand side of (57) goes to 0 with ϵ. In view of (56), the second term is bounded by

$$\|\varphi\|^2_{L^\infty(\Omega)} \|\vec{s}^\epsilon\|^2_{L^2(\mathrm{supp}(\varphi))},$$

which goes to 0 by virtue of the assumption (52). Therefore, we deduce from (57) that ∇u_ϵ converges strongly to 0 in $L^2(\omega)^N$. This concludes the proof of Lemma 3.4.

By assumption (53), there exists an $(N-1)$-dimensional open set σ, with $\overline{\sigma} \subset \Sigma$, such that the sequence of eigenvectors concentrates partly near σ. By translation, one can always assume that the origin lies inside σ. The strategy of the proof is to rescale the domain Ω by the change of variables $y = \frac{x}{\epsilon}$ and then to transform the sequence of eigenvectors $\vec{s}^\epsilon$ in a Weyl sequence for a limit operator. The limit domain will be $\mathbb{R}^N_+ = \{y \in \mathbb{R}^N | y_N > 0\}$ since we have carefully choose the origin to belong to σ. The limit fluid domain is denoted by $G^{*\infty}$, which is defined by

$$G^{*\infty} = \mathbb{R}^N_+ \setminus \bigcup_{j \in \mathbb{Z}^N_+} T_j,$$

where T_j denotes the tube j placed in the subcell Y_j (centered at the point $j = (j', j_N)$ with $j' \in \mathbb{Z}^{N-1}$ and $j_N \in \mathbb{Z}_+$). In this limit domain we define a limit operator B^∞, which acts from ℓ^∞_2 in itself, by

$$B^\infty \vec{s} = \left(\int_{\Gamma_j} u\vec{n} ds \right)_{j \in \mathbb{Z}^N_+} \qquad \forall \vec{s} \in \ell^\infty_2,$$

where $u(y)$ is the unique solution in $D_0^1(G^{*\infty})$ of

$$\left\{\begin{array}{ll} -\Delta u = 0 & \text{in } G^{*\infty}, \\ \frac{\partial u}{\partial n} = \vec{s}_j \cdot \vec{n} & \text{on } \Gamma_j,\ j \in \mathbb{Z}_+^N, \\ u = 0 & \text{on } \mathbb{R}^{N-1} \times \{0\}. \end{array}\right. \tag{58}$$

Recall that elements in $D_0^1(G^{*\infty})$ are restrictions to $G^{*\infty}$ of functions wnin $D_0^1(\mathbb{R}_+^N)$ which, in its turn, is the closure, with respect to the L^2-norm of the gradient, of smooth functions with compact support in $\mathbb{R}_+^N$.

REMARK 3.5. *The limit domain $G^{*\infty}$ is nothing but the limit as K goes to infinity of the domain G^{*K} defined in section 2.2. By the same token, the Hilbert space ℓ_2^∞ is the limit of ℓ_2^K (it is also equal to $\ell_2(\mathbb{Z}_+^N;\mathbb{C}^N)$). In some sense the limit operator B^∞ is also the limit of the operator B^K defined in Theorem 2.11.*

Let φ be a smooth function, equal to 1 in $\omega = \sigma\times]0,L[$, with compact support in $\Sigma\times]-L;L[$ (i.e., φ vanishes on all faces of Ω except on that defined by $x_N = 0$). We use φ to localize the sequence of eigenvectors $\vec{s}^\epsilon$ in a vicinity of ω. Let us define a sequence $\vec{t}^\epsilon$ by

$$\vec{t}^\epsilon = E_\epsilon^1 P_\epsilon^1(\varphi(x)\vec{s}^\epsilon(x)),$$

where $E_\epsilon^1 P_\epsilon^1$ is the projection operator in $L^2(\Omega)^N$ on piecewise constant functions (cf. their definitions (27) and (28)).

Remark that, by assumption (53), the sequence $\vec{t}^\epsilon$ satisfies

$$\lim_{\epsilon\to 0} \|\vec{t}^\epsilon\|_{L^2(\omega)^N} \geq c > 0.$$

Let us define $G_\epsilon^{*\infty}$ as $G^{*\infty}$ rescaled to size ϵ. Let v_ϵ be the potential in $G_\epsilon^{*\infty}$ associated with $\vec{t}^\epsilon$, defined by

$$\left\{\begin{array}{ll} -\Delta v_\epsilon = 0 & \text{in } G_\epsilon^{*\infty}, \\ \frac{\partial v_\epsilon}{\partial n} = \vec{t}_p^\epsilon \cdot \vec{n} & \text{on } \Gamma_p^\epsilon,\ p \in \mathbb{Z}_+^N, \\ v_\epsilon = 0 & \text{on } \mathbb{R}^{N-1} \times \{0\}. \end{array}\right. \tag{59}$$

LEMMA 3.6. *The sequence v_ϵ defined by* (59) *converges to zero in $D_0^1(\mathbb{R}_+^N)$ weakly and in $H_{loc}^1(\mathbb{R}_+^N)$ strongly.*

Proof. The argument is similar to that of Lemma 3.4, except that the Rellich theorem applies only for compact sets in $\overline{\mathbb{R}_+^N}$.

LEMMA 3.7. *The difference $w_\epsilon = v_\epsilon - \varphi u_\epsilon$ converges strongly to zero in $D_0^1(\mathbb{R}_+^N)$.*

Proof. A simple calculation provides the following key identity:

$$\int_{\mathbb{R}_+^N} |\nabla w_\epsilon|^2 = \int_{\mathbb{R}_+^N} \nabla v_\epsilon \cdot \nabla w_\epsilon - \int_{\mathbb{R}_+^N} \nabla u_\epsilon \cdot \nabla(\varphi w_\epsilon) - \int_{\mathbb{R}_+^N} \nabla\varphi \cdot (u_\epsilon \nabla w_\epsilon - w_\epsilon \nabla u_\epsilon). \tag{60}$$

By virtue of Lemmas 3.4 and 3.6, u_ϵ and w_ϵ converge to zero strongly in L^2 of the support of φ. Therefore, the last term in (60) goes to zero with ϵ. On the other hand, an integration by parts yields

$$\int_{G_{\epsilon\infty}^*} \nabla v_\epsilon \cdot \nabla w_\epsilon - \int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla(\varphi w_\epsilon) = \sum_{p=1}^{n(\epsilon)} \left[\vec{t}_p^\epsilon \cdot \Big(\int_{\Gamma_p^\epsilon} w_\epsilon \vec{n} \Big) - \vec{s}_p^\epsilon \cdot \Big(\int_{\Gamma_p^\epsilon} \varphi w_\epsilon \vec{n} \Big) \right].$$

Since $\vec{t}^\epsilon_p = \frac{1}{\epsilon^N}\int_{Y^\epsilon_p}\varphi\vec{s}^\epsilon_p dx$ and $|\varphi(x)-\varphi(x^\epsilon_p)| \le \epsilon\|\nabla\varphi\|_{L^\infty}$, where x^ϵ_p is the center of the cube Y^ϵ_p which contains x, we obtain

$$\left|\int_{G^*_{\epsilon\infty}}\nabla v_\epsilon\cdot\nabla w_\epsilon - \int_{\Omega_\epsilon}\nabla u_\epsilon\cdot\nabla(\varphi w_\epsilon)\right| \le \epsilon\|\nabla\varphi\|_{L^\infty}\|\vec{s}^\epsilon\|_{L^2(\Omega)^N}\|\nabla w_\epsilon\|_{L^2(\mathbb{R}^+_N)^N},$$

which gives the desired result.

Lemma 3.8. *From Lemma* 3.7 *we deduce the following approximation result for the displacement vector* $\vec{t}^\epsilon$*:*

$$\lim_{\epsilon\to 0}\sum_{p\in\mathbb{Z}^N_+}\epsilon^N\left|\frac{1}{\epsilon^N}\int_{\Gamma^\epsilon_p} v_\epsilon\vec{n}ds - \lambda_\epsilon\vec{t}^\epsilon_p\right|^2 = 0.$$

Proof. We have

$$\begin{aligned}\epsilon^N\sum_{p\in\mathbb{Z}^N_+}\left|\tfrac{1}{\epsilon^N}\textstyle\int_{\Gamma^\epsilon_p}(v_\epsilon-\varphi u_\epsilon)\vec{n}ds\right|^2 &\le \sum_{p\in\mathbb{Z}^N_+}\|\nabla(v_\epsilon-\varphi u_\epsilon)\|^2_{L^2(T^\epsilon_p)^N}\\ &\le \|\nabla(v_\epsilon-\varphi u_\epsilon)\|^2_{L^2(\mathbb{R}^N_+)^N},\end{aligned}\tag{61}$$

which goes to zero as $\epsilon\to 0$ by virtue of Lemma 3.7. Furthermore,

$$\epsilon^N\sum_{p\in\mathbb{Z}^N_+}\left|\frac{1}{\epsilon^N}\int_{\Gamma^\epsilon_p}\varphi u_\epsilon\vec{n}ds - \lambda_\epsilon\vec{t}^\epsilon_p\right|^2 \le \epsilon\|\nabla\varphi\|_{L^\infty}\|\nabla u_\epsilon\|_{L^2(\Omega)^N}$$

since, $\vec{s}^\epsilon$ being constant in each cell Y^ϵ_p,

$$\frac{1}{\epsilon^N}\int_{\Gamma^\varepsilon_p}\varphi u_\epsilon\vec{n}ds = \frac{1}{\varepsilon^N}\int_{\Gamma^\epsilon_p}\Big(\varphi(s)-\frac{1}{\varepsilon^N}\int_{Y^\epsilon_p}\varphi(t)dt\Big)u_\epsilon\vec{n}ds + \lambda_\epsilon(P^1_\epsilon\varphi\vec{s}^\epsilon)_p.$$

Summing these two estimates yields the desired result.

Now, let us define a sequence $\vec{\tau}^\epsilon$ in ℓ^∞_2 by

$$\vec{\tau}^\epsilon = \epsilon^{N/2}(\vec{t}^\epsilon_p)_{p\in\mathbb{Z}^N_+},$$

which plays the role of a Weyl sequence for the limit operator B^∞.

Proposition 3.9. *The sequence* $\vec{\tau}^\epsilon$ *satisfies*

$$\lim_{\epsilon\to 0}\|\vec{\tau}^\epsilon\|_{\ell^\infty_2} \ge c > 0,$$

and

$$B^\infty\vec{\tau}^\epsilon = \lambda\vec{\tau}^\epsilon + \vec{r}^\epsilon,\tag{62}$$

where $\vec{r}^\epsilon$ *is a remainder term which goes to zero strongly in* ℓ^∞_2.

Proof. A simple rescaling in (59) shows that $\tilde{v}_\epsilon(y) = \epsilon^{\frac{N}{2}-1}v_\epsilon(\epsilon y)$ is the unique solution in $D^1_0(G^{*\infty})$ of

$$\begin{cases} -\Delta\widetilde{v}_\epsilon = 0 & \text{in } G^{*\infty},\\ \frac{\partial\widetilde{v}_\epsilon}{\partial n} = \vec{\tau}^\epsilon_p\cdot\vec{n} & \text{on } \Gamma_p,\ p\in\mathbb{Z}^N_+,\\ \widetilde{v}_\epsilon = 0 & \text{on } \mathbb{R}^{N-1}\times\{0\}.\end{cases}\tag{63}$$

Furthermore, $\|\nabla_y \widetilde{v}_\epsilon\|_{L^2(G^{*\infty})^N} = \|\nabla_x v_\epsilon\|_{L^2(G_\epsilon^{*\infty})^N}$. By definition,

$$B^\infty \vec{\tau}^\epsilon = \left(\int_{\Gamma_p} \widetilde{v}_\epsilon \vec{n} ds \right)_{p\in\mathbb{Z}_+^N} = \epsilon^{-\frac{N}{2}} \left(\int_{\Gamma_p^\epsilon} v_\epsilon \vec{n} ds \right)_{p\in\mathbb{Z}_+^N}.$$

Defining $\vec{r}^\epsilon = (\vec{r}_p^\epsilon)_{p\in\mathbb{Z}_+^N}$ by

$$\vec{r}_p^\epsilon = \epsilon^{\frac{N}{2}} \left(\frac{1}{\epsilon^N} \int_{\Gamma_p^\epsilon} v_\epsilon \vec{n} ds - \lambda_\epsilon \vec{t}_p^\epsilon \right),$$

we get

$$B^\infty \vec{\tau}^\epsilon = \lambda_\epsilon \vec{\tau}^\epsilon + \vec{r}^\epsilon,$$

which, by virtue of Lemma 3.8, is the desired result.

To conclude the proof of Theorem 3.3, we remark that either $\vec{\tau}^\epsilon$ converges weakly in ℓ_2^∞ to a nonzero limit $\vec{\tau}$ (up to a subsequence) or $\vec{\tau}^\epsilon$ converges weakly to $\vec{0}$ in ℓ_2^∞. In the first case, passing to the limit as ϵ goes to 0, we obtain that $\vec{\tau} \neq \vec{0}$ is an eigenvector of B^∞ for λ (the limit of the sequence λ_ϵ). In the latter case, this proves that $\vec{\tau}^\epsilon$ is a Weyl sequence for the spectral value λ which belongs to the essential spectrum of B^∞. Now, it is a standard matter (see, e.g., [15], [16]) to show, by a Bloch wave decomposition analogous to that of section 2.3, that the spectrum of B^∞ is nothing but $\lim_{K\to+\infty} \sigma(B^K)$, i.e., the boundary layer spectrum associated with the face Σ of Ω.

REMARK 3.10. *Let us remark that Theorem* 3.3 *is valid for any choice of the sequence* ϵ *and not only for the particular sequence* ϵ_n *defined in* (48). *The interested reader will not fail to notice that the present proof of the completeness result is different from that of our previous work* [3]. *In this paper, we used the concept of Bloch measures in order to prove a similar completeness result by means of an energetic method. Here, we propose a new proof (in a slightly different context), based on a rescaling argument, which is simpler, although less precise, and which could equally be applied in* [3].

3.3. Analysis of the corner spectrum. In section 3.1 the boundary layer spectrum $\sigma_{\partial\Omega}$ was defined as the union of all spectra of the type σ_Σ, where Σ is any lower dimensional manifold composing the boundary $\partial\Omega$. When Σ is an $(N-1)$-dimensional hyperplane, a complete derivation of σ_Σ has been given in section 2. However, for lower dimensional manifold we have been a little cavalier in saying that the analysis of section 2 can be easily generalized to the case of edges, corners, and so on. The purpose of this section is to briefly indicate some details of this generalization when analyzing the *corner spectrum*. Since the physical problem of interest is truly two-dimensional, we restrict ourselves to the case of corners of the plane square domain Ω (this has the advantage of simplifying the exposition).

Therefore, our domain Ω is now a rectangle with integer dimensions, i.e.,

$$\Omega =]0; L_1[\times]0; L_2[.$$

We describe the limit spectrum associated with the corner located at the origin. We introduce the space ℓ_+^2 of displacements defined by

$$\ell_+^2 = \left\{ (\vec{s}_j)_{j=(j_1,j_2)\ j_1\geq 1, j_2\geq 1} \ \middle|\ \vec{s}_j \in \mathbb{R}^2, \quad \sum_{j_1,j_2=1}^{+\infty} |\vec{s}_{j_1,j_2}|^2 < +\infty \right\}.$$

Remark that this definition of ℓ^2_+ implies a decay of the displacement $\vec{s}_j$ as j_1 or j_2 goes to $+\infty$.

We extend the operator S_ϵ to the larger space ℓ^2_+ by the following formula:

$$C_\epsilon = E_\epsilon S_\epsilon P_\epsilon,$$

where P_ϵ and E_ϵ are, respectively, projection and extension operators between $\mathbb{R}^{Nn(\epsilon)}$ and ℓ^2_+. Their definition is very simple. Recall that a tube T^ϵ_j in Ω is located in a cell Y^ϵ_j whose origin is ϵj. We denote the range of all indices j such that T^ϵ_j is included in Ω by $1 \leq j \leq n(\epsilon)$. The projection is defined by

$$\begin{array}{lcl} P_\epsilon : \ell^2_+ & \longrightarrow & \mathbb{R}^{Nn(\epsilon)}, \\ (\vec{s}_j)_{j=(j_1,j_2)\ j_1\geq 1, j_2 \geq 1} & \mapsto & (\vec{s}_j)_{1\leq j\leq n(\epsilon)}, \end{array}$$

and the extension by

$$\begin{array}{lcl} E_\epsilon : \mathbb{R}^{Nn(\epsilon)} & \longrightarrow & \ell^2_+, \\ (\vec{s}_j)_{1\leq j\leq n(\epsilon)} & \mapsto & (\vec{t}_j)_{j=(j_1,j_2)\ j_1\geq 1, j_2\geq 1}, \end{array}$$

with $\vec{t}_j = \vec{s}_j$ if $1 \leq j \leq n(\epsilon)$ and $\vec{t}_j = 0$ otherwise.

One can easily check that P_ϵ and E_ϵ are adjoint operators and that the product $P_\epsilon E_\epsilon$ is equal to the identity in $\mathbb{R}^{Nn(\epsilon)}$. Therefore, the spectrum of C_ϵ consists of that of S_ϵ and zero as an eigenvalue of infinite multiplicity.

The convergence analysis of C_ϵ is much simpler than that in section 2 because ℓ^2_+ is not a space of periodically oscillating displacements. There is no need to introduce any notion of two-scale convergence for corner boundary layers. A simple rescaling argument is enough. More precisely, denoting by Q_+ the first quadrant in the plane

$$Q_+ \ = \ ]0;+\infty[\times]0;+\infty[,$$

we replace the two-scale convergence by the weak convergence in $L^2(Q_+)$: with each bounded sequence $u_\epsilon(x)$ in $L^2(\Omega)$, we associate the rescaled sequence $v_\epsilon(y)$ defined by

$$v_\epsilon(y) = \begin{cases} \epsilon^2 u_\epsilon(\epsilon y) & \text{if } \epsilon y \in \Omega, \\ 0 & \text{otherwise,} \end{cases}$$

which is also bounded in $L^2(Q_+)$.

Then, a similar analysis to that of section 2 shows that the sequence of operators C_ϵ converges strongly in $\mathcal{L}(\ell^2_+)$ to a limit operator C_∞ defined by

$$C_\infty : \ell^2_+ \longrightarrow \ell^2_+ \tag{64}$$

$$(\vec{s}_j)_{j=(j_1,j_2)\ j_1\geq 1, j_2\geq 1} \mapsto \left(\int_{\Gamma_j} u\vec{n}ds \right)_{j=(j_1,j_2)\ j_1\geq 1, j_2 \geq 1}, \tag{65}$$

where $u(y)$ is the unique solution of

$$\begin{cases} -\Delta u = 0 & \text{in } Q^*_+ = Q_+ \setminus \bigcup\limits_j T_j, \\ \frac{\partial u}{\partial n} = \vec{s}_j \cdot \vec{n} & \text{on } \Gamma_j,\ j_1 \geq 1, j_2 \geq 1, \\ u = 0 & \text{on } \partial Q_+, \\ \lim\limits_{|y|\to+\infty} u(y) = 0. \end{cases}$$

Clearly, C_∞ is a self-adjoint noncompact operator acting in ℓ^2_+. As in Proposition 2.17, one can prove that the essential spectrum of C_∞ is precisely the Bloch spectrum. However, the discrete spectrum of C_∞ may contain new eigenvalues which correspond to eigenvectors localized in the corner of Q_+.

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