Identification of immersed obstacles via boundary measurements

2005 Inverse Problems 21 1531

(http://iopscience.iop.org/0266-5611/21/5/003)

INSTITUTE OF PHYSICS PUBLISHING INVERSE PROBLEMS

Inverse Problems **21** (2005) 1531–1552 doi:10.1088/0266-5611/21/5/003

Identification of immersed obstacles via boundary measurements

C Alvarez[1], **C Conca**[1,2], **L Friz**[1,3], **O Kavian**[4] **and J H Ortega**[1,3]

[1] Centro de Modelamiento Matemático, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile UMI 2807 CNRS-UChile, Casilla 170/3-Correo 3, Santiago, Chile
[2] Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Casilla 170/3-Correo 3, Santiago, Chile
[3] Departamento de Ciencias Básicas, Facultad de Ciencias, Universidad del Bío-Bío, Casilla 447, Campus Fernando May, Chillán, Chile
[4] Laboratoire de Mathématiques Appliquées (UMR 7641), Université de Versailles-Saint-Quentin, Bâtiment Fermat, 45 Avenue des Etats-Unis, 78035 Versailles Cedex, France

E-mail: 251564256@iol.cz, cconca@dim.uchile.cl, lfriz@roble.fdo-may.ubiobio.cl, kavian@math.uvsq.fr and jortega@dim.uchile.cl

Received 23 November 2004, in final form 4 July 2005
Published 23 August 2005
Online at stacks.iop.org/IP/21/1531

Abstract
We study the following inverse problem: an inaccessible rigid body D is immersed in a viscous fluid, in such a way that D plays the role of an obstacle around which the fluid is flowing in a greater bounded domain Ω, and we wish to determine D (i.e., its form and location) via boundary measurement on the boundary $\partial\Omega$. Both for the stationary and the evolution problem, we show that under reasonable smoothness assumptions on Ω and D, one can identify D via the measurement of the velocity of the fluid and the Cauchy forces on some part of the boundary $\partial\Omega$. We also show that the dependence of the Cauchy forces on deformations of D is analytic, and give some stability results for the inverse problem.

1. Introduction

Let Ω be a smooth bounded open set in $\mathbb{R}^N$, filled with an incompressible fluid, and let $D \subset\subset \Omega$ be an unknown rigid body immersed in it. Let $\varphi \in C^1([0, T]; H^{3/2}(\partial\Omega)^N)$ be non-homogeneous Dirichlet boundary data satisfying the standard flux compatibility condition

$$\int_{\partial\Omega} \varphi \cdot \mathbf{n} \, \mathrm{d}s = 0, \tag{1.1}$$

and let $(v, p) \in L^2(0, T; H^1(\Omega\backslash\overline{D})^N) \times L^2(0, T; L^2(\Omega\backslash\overline{D}))$ be the unique solution of the Stokes (when $\varepsilon_* = 0$) or Navier–Stokes (when $\varepsilon_* = 1$) system of equations,

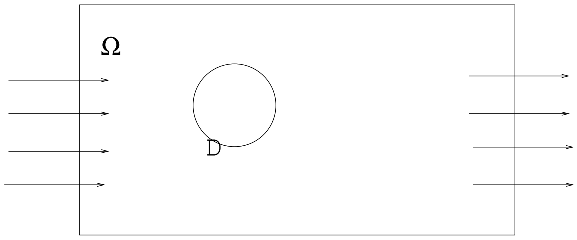

Figure 1.

$$\begin{cases} \dfrac{\partial v}{\partial t} - \operatorname{div}(\sigma(v, p)) + \varepsilon_* \operatorname{div}(v \otimes v) = 0 & \text{in} \quad (\Omega \backslash \overline{D}) \times (0, T) \\ \operatorname{div}(v) = 0 & \text{in} \quad (\Omega \backslash \overline{D}) \times (0, T) \\ v(0) = 0 & \text{in} \quad \Omega \backslash \overline{D} \\ v = \varphi & \text{on} \quad \partial\Omega \times (0, T) \\ v = 0 & \text{on} \quad \partial D \times (0, T), \end{cases} \tag{1.2}$$

with σ being the stress tensor defined as follows,

$$\sigma(v, p) = -pI + 2\nu e(v), \tag{1.3}$$

where I is the identity matrix, $\nu > 0$ is a given constant representing the *kinematic viscosity* of the liquid and $e(v)$ is the linear strain tensor defined by

$$e(v) = \tfrac{1}{2}(\nabla v + {}^t\nabla v). \tag{1.4}$$

The typical situation is illustrated in figure 1.

The problem we study is to obtain some information on the domain D (shape and location) through the observation of the Cauchy force $\sigma(v, p)\mathbf{n}$ on some part of the boundary (here $\mathbf{n}$ is the external unit normal to $\partial\Omega$). Indeed the stationary version of the problem, that is,

$$\begin{cases} -\operatorname{div}(\sigma(v, p)) + \varepsilon_* \operatorname{div}(v \otimes v) = 0 & \text{in} \quad \Omega \backslash \overline{D} \\ \operatorname{div}(v) = 0 & \text{in} \quad \Omega \backslash \overline{D} \\ v = \varphi & \text{on} \quad \partial\Omega \\ v = 0 & \text{on} \quad \partial D, \end{cases} \tag{1.5}$$

can also be considered and treated in the same way.

Such problems have been studied for a long time since the publication of the paper by Calderón [4] in 1980, in particular for the identification of the scalar coefficient a in operators of the form $u \mapsto -\operatorname{div}(a\nabla u)$. In this case u represents an electric potential and one assumes that the Poincaré–Steklov (also called the Dirichlet-to-Neumann) operator $\Lambda_a : H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega)$ is known (Λ_a is defined by $\Lambda_a(\varphi) := a\partial u/\partial\mathbf{n}$ where $\operatorname{div}(a\nabla u) = 0$ in Ω and $u = \varphi$ on $\partial\Omega$). The study of such problems can also be adapted in order to identify particular subdomains of Ω, or cracks and inhomogeneities included in the domain. The interested reader is referred to the review by Uhlmann [15] for key historical remarks on this matter and to the pioneering works by Kohn and Vogelius [10] and Sylvester and Uhlmann [12] for early results on this theory.

In the problem we investigate, we will look for the unknown domain D in the following class of admissible geometries

$$\mathcal{D}_{\text{ad}} := \{D \subset\subset \Omega : D \text{ is open, Lipschitz and } \Omega \backslash \overline{D} \text{ is connected}\}. \tag{1.6}$$

The corresponding Poincaré–Steklov operator Λ_D is defined by

$$\Lambda_D(\varphi) := \sigma(v, p)\mathbf{n} \quad \text{on} \quad \Gamma \times (0, T), \tag{1.7}$$

which maps $\varphi \in C^1([0,T]; H^{3/2}(\partial\Omega)^N)$ (with $\int_{\partial\Omega} \varphi \cdot \mathbf{n}\,\mathrm{d}s = 0$) into the Cauchy forces $\sigma(v,p)\mathbf{n} \in L^2(0,T; H^{-1/2}(\Gamma)^N)$, where (v,p) is the unique solution of the Stokes system (1.2) and Γ is a relatively open subset of the boundary $\partial\Omega$. In the case of the stationary version of the problem, the operator Λ_D has to be considered as acting between the spaces $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\Gamma)$, and in this case (v,p) is a solution of (1.5).

Our first result concerns the identifiability of D: it states that given fixed non-homogeneous Dirichlet boundary data φ, two different admissible geometries $D_0 \neq D_1 \in \mathcal{D}_{\mathrm{ad}}$, yield two different Steklov–Poincaré operators $\Lambda_{D_0} \neq \Lambda_{D_1}$.

Theorem 1.1. *Let $T > 0$ and $\Omega \subseteq \mathbb{R}^N$, $N = 2$ or $N = 3$, be a bounded $C^{1,1}$ domain, and Γ be a non-empty open subset of $\partial\Omega$. Let $D_0, D_1 \in \mathcal{D}_{\mathrm{ad}}$ and $\varphi \in C^1([0,T]; H^{3/2}(\partial\Omega)^N)$ with $\varphi \not\equiv 0$, satisfying the flux condition (1.1). For $\varepsilon_* = 0$ or $\varepsilon_* = 1$, let (v_j, p_j) for $j = 0, 1$, be a solution of*

$$\begin{cases} \dfrac{\partial v_j}{\partial t} - \mathrm{div}(\sigma(v_j,p_j)) + \varepsilon_* \,\mathrm{div}(v_j \otimes v_j) = 0 & \text{in} \quad (\Omega\backslash\overline{D}_j) \times (0,T) \\ \mathrm{div}(v_j) = 0 & \text{in} \quad (\Omega\backslash\overline{D}_j) \times (0,T) \\ v_j(x,0) = 0 & \text{for} \quad x \in \Omega\backslash\overline{D}_j \\ v_j(s,t) = \varphi(s,t) & \text{for} \quad (s,t) \in \partial\Omega \times (0,T) \\ v_j(s,t) = 0 & \text{for} \quad (s,t) \in \partial D_j \times (0,T). \end{cases} \tag{1.8}$$

Assume that (v_j, p_j) are such that

$$\sigma(v_0,p_0)\mathbf{n} = \sigma(v_1,p_1)\mathbf{n} \quad \text{on} \quad \Gamma \times (0,T).$$

Then $D_0 \equiv D_1$.

The same identification result holds for the stationary problem:

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^N$, $N = 2$ or $N = 3$, be a bounded Lipschitz domain, and Γ a non-empty open subset of $\partial\Omega$. Let $D_0, D_1 \in \mathcal{D}_{\mathrm{ad}}$ and $\varphi \in H^{3/2}(\partial\Omega)^N$ with $\varphi \not\equiv 0$, satisfying the flux condition (1.1). For $\varepsilon_* = 0$ or $\varepsilon_* = 1$, let (v_j, p_j) for $j = 0, 1$, be a solution of*

$$\begin{cases} -\mathrm{div}(\sigma(v_j,p_j)) + \varepsilon_* \,\mathrm{div}(v \otimes v) = 0 & \text{in} \quad \Omega\backslash\overline{D}_j \\ \mathrm{div}(v_j) = 0 & \text{in} \quad \Omega\backslash\overline{D}_j \\ v_j(s) = \varphi(s) & \text{for} \quad s \in \partial\Omega \\ v_j(s) = 0 & \text{for} \quad s \in \partial D_j. \end{cases} \tag{1.9}$$

Assume that (v_j, p_j) are such that

$$\sigma(v_0,p_0)\mathbf{n} = \sigma(v_1,p_1)\mathbf{n} \quad \text{on} \quad \Gamma.$$

Then $D_0 \equiv D_1$.

The identifiability of a sufficiently smooth inclusion within a given domain $\Omega \subset \mathbb{R}^N$ has been well studied for scalar elliptic and parabolic equations. The interested reader is referred to the papers by Beretta and Vessella [3], Canuto and Kavian [7], Canuto and Vessella [6], Canuto [5] for more details. We refer also to the paper by Alessandrini *et al* [1, 2] for estimates on the volume of the subdomain D immersed in a perfect fluid filling Ω; note that in this case, it is assumed that the dynamics of the fluid is governed by a velocity potential which satisfies a scalar Laplace equation. The proof by Beretta and Vessella [3] of injectivity and stability of the corresponding boundary map is mainly based on some structural properties of the Laplace equation like the maximum principle and Harnack's inequality. Since these properties are not valid anymore in our case, we were led to a different proof, namely we obtain our identifiability result by a suitable application of the unique continuation property for the Stokes equations due to Fabre and Lebeau [8].

Studying the stability of the identifiability is reduced to studying the continuity properties of the inverse of the boundary map. Given a background admissible configuration $\Omega_0 := \Omega\backslash\overline{D}_0$, let us consider a smooth perturbation of D_0, called D_1, and a bijective Lipschitz mapping $\Psi : \overline{\Omega} \to \overline{\Omega}$, such that Ψ^{-1} is also Lipschitz and $\Psi = I$ in a neighbourhood of the boundary $\partial\Omega$, with

$$\Psi(D_0) = D_1 \qquad \text{and} \qquad \Psi(\Omega_0) = \Omega_1 = \Omega\backslash\overline{D}_1.$$

Clearly we need Ω_1 to be also admissible, that is, $D_1 \in \mathcal{D}_{\mathrm{ad}}$ such that the new rigid body D_1, as well as D_0, is both Lipschitz and included in a fixed open set $\mathcal{O}$ satisfying

$$D_0 \cup D_1 \subset\subset \mathcal{O} \subset\subset \Omega.$$

In this setting, studying the stability of the inverse problem can be viewed as studying continuity properties for the mapping $\Psi \mapsto \Lambda^{-1}_{\Psi(D_0)}$. In practice, it suffices to prove continuity at $\Psi = I$, where I is the identity map, that is to say, for any given $\varphi \in C^1(0, T; H^{3/2}(\Omega)^N)$ (respectively $\varphi \in H^{3/2}(\partial\Omega)$ for the stationary problem), when $\left\|\Lambda_{D_0}(\varphi) - \Lambda_{\Psi(D_0)}(\varphi)\right\|_{L^2(0,T,H^{-1/2}(\Gamma)^N)}$ (respectively $\left\|\Lambda_{D_0}(\varphi) - \Lambda_{\Psi(D_0)}(\varphi)\right\|_{H^{-1/2}(\Gamma)^N}$) is small, then the norm $\|I - \Psi\|_{W^{1,\infty}(\Omega_0;\mathbb{R}^N)}$ is small.

Actually, our result concerning stability is weaker than the above desired continuity property: we are able to prove only a sort of *directional continuity*, or *hemicontinuity* of the mapping $\Psi \mapsto \Lambda^{-1}_{\Psi(D_0)}$. More precisely, we will consider admissible deformations of the type $\Psi_\tau = I + \tau\Psi_1$, where τ is a small real parameter, and for some integer $m \geqslant 1$ and a positive constant $C > 0$, both depending on $\Psi_1 \not\equiv 0$, provided $\Psi_\tau(D_0) \neq D_0$, we prove that

$$\left\|\Lambda_{D_0}(\varphi) - \Lambda_{\Psi_\tau(D_0)}(\varphi)\right\| \geqslant C|\tau|^m. \tag{1.10}$$

To be more specific, let us consider the linear stationary case (whose understanding is in fact the main point for all other cases). For each change of variables $\Psi : \Omega_0 \longrightarrow \Omega_\tau := \Psi_\tau(\Omega_0)$, let$(v_\tau, p_\tau) \in H^1(\Omega_\tau)^N \times L^2(\Omega_\tau)$ denote the unique solution of the Stokes system in the deformed domain, that is,

$$\begin{cases} -\mathrm{div}(\sigma(v_\tau, p_\tau)) + \varepsilon_* \,\mathrm{div}(v \otimes v) = 0 & \text{in} \quad \Omega_\tau \\ \mathrm{div}(v_\tau) = 0 & \text{in} \quad \Omega_\tau \\ v_\tau(s) = \varphi(s) & \text{for} \quad s \in \partial\Omega \\ v_\tau(s) = 0 & \text{for} \quad s \in \partial D_\tau. \end{cases} \tag{1.11}$$

(For $\tau = 0$ note that $\Psi_0 = I$ and $\Omega_\tau = \Omega_0$.) We will show that, under suitable assumptions on the function Ψ_1, the mapping

$$\Psi_\tau \mapsto \sigma(v_\tau, p_\tau)\mathbf{n} =: \Lambda_{\Psi_\tau(\Omega_0)}$$

is analytic in the open set of $W^{1,\infty}$-'diffeomorphisms' (by an abuse of language we shall say that Ψ is a $W^{1,\infty}$-diffeomorphism when both Ψ and Ψ^{-1} are in $W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$, and restricted respectively to the domains $\Omega\backslash\overline{D}_0$ and $\Psi(\Omega\backslash\overline{D}_0)$). In particular, it can be differentiated with respect to Ψ; as a matter of fact, the corresponding derivative is the so-called *shape differentiation* of the solution of (1.11) with respect to the geometry.

The following stability (directional continuity) result is proven in section 4.

Theorem 1.3. *Let* $D_0 \in \mathcal{D}_{\mathrm{ad}}, \Psi_0 := I$ *and* $\Psi_1 \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ *such that* $\Psi_1 \equiv 0$ *in a neighbourhood of the boundary* $\partial\Omega$ *and* $\Psi_1 \not\equiv 0$ *on* D_0. *Denote by* $\Psi_\tau = I + \tau\Psi_1$ *and* (v_τ, p_τ) *the solution of (1.11) (including for* $\tau = 0$*), and by* $\tau_1 > 0$ *a positive number such* *that* $\tau \mapsto \sigma(v_\tau, p_\tau)\mathbf{n}$ *is analytic on* $(-\tau_1, \tau_1)$*. Assume that for some* $\tau_* \in (-\tau_1, \tau_1)$ *one has* $\Psi_{\tau_*}(D_0) \neq D_0$*. Then there exist a strictly positive constant* $C = C(\Psi_1, \Omega, D_0, \varphi)$ *and a positive integer* $m = m(\Psi_1, \Omega, D_0, \varphi)$ *such that for some* $\tau_0 > 0$ *and all* $\tau \in [-\tau_0, \tau_0]$*, we have*

$$\left\| \Lambda_{D_0}(\varphi) - \Lambda_{\Psi_\tau(D_0)}(\varphi) \right\|_{H^{-1/2}(\Gamma)^N} \geqslant C|\tau|^m.$$

where $\Lambda_{\Psi_\tau(D_0)}(\varphi) := \sigma(v_\tau, p_\tau)\mathbf{n}$ *on* Γ.

The main ingredient of the proof is to write the equation satisfied by (v_τ, p_τ) in the fixed domain, using a change of variables, and then to observe that the operators involved and their inverses depend in a smooth manner on Ψ.

The same stability result can be established for the linear or nonlinear evolution problem.

To conclude this section, here is how this paper is organized. In section 2 we recall a few results on unique continuation properties for the Stokes or Navier–Stokes systems which are crucial in the proof of the identifiability result. In section 3 we prove the identifiability results for the evolution problem as well as the stationary one. Section 4 is devoted to establishing the analyticity of the mapping $\Psi \mapsto \Lambda_{\Psi(D_0)}$, for the linear stationary Stokes system, and there we prove the stability result mentioned above. We determine also the first derivative of this mapping, which is necessary in order to apply an optimization algorithm studied in an optimal control approach. In section 5 we present analyticity results for the linear or nonlinear evolution problems and the corresponding stability results.

2. Preliminary results

In this section, we gather some preliminary results about unique continuation properties, which are essential in the proof of the identifiability result. First we mention a unique continuation result for the Stokes equation due to Fabre and Lebeau [8, p 576].

Proposition 2.1. *Let* $\Omega_0 \subseteq \mathbb{R}^N$, $N \geqslant 2$*, be a bounded domain and* ω_0 *an open subset of* Ω_0*. If* $a \in L^\infty_{\text{loc}}(\Omega_0)^N$ *and* $(v, p) \in H^1_{\text{loc}}(\Omega_0)^N \times L^2_{\text{loc}}(\Omega_0)$ *is a solution of*

$$\begin{cases} -\Delta v + (a\cdot\nabla)v + \nabla p = 0, & \text{in} \quad \Omega_0 \\ \nabla\cdot v = 0, & \text{in} \quad \Omega_0, \end{cases} \tag{2.1}$$

with $v = 0$ *in* ω_0*, then* $v = 0$ *in* Ω_0 *and p is constant in* Ω_0.

A direct consequence of the above result is the following unique continuation property.

Corollary 2.2. *Let* $\Omega_0 \subseteq \mathbb{R}^N$, $N \geqslant 2$*, be a Lipschitz domain. If* $(v, p) \in H^1(\Omega_0)^N \times L^2(\Omega_0)$ *is a solution of*

$$\begin{cases} -\operatorname{div}\sigma(v, p) = 0 & \text{in} \quad \Omega_0 \\ \operatorname{div} v = 0 & \text{in} \quad \Omega_0 \\ v = 0 & \text{on} \quad \partial\Omega_0 \end{cases} \tag{2.2}$$

satisfying $\sigma(v, p)\mathbf{n} = 0$ *on* Γ_0*, where* $\Gamma_0 \subset \partial\Omega_0$ *is a relatively open non-empty subset, then* $v = 0$ *and p is constant in* Ω_0.

Analogously to the stationary case we recall the following unique continuation result due to Fabre and Lebeau [8, p 574], for the non-steady Stokes problem. Let $\Omega_0 \subset \mathbb{R}^N$ be an open connected set, $N \geqslant 2$ and $T > 0$. We consider an open non-empty subset O of $\Omega_0 \times (0, T)$. Let us define the horizontal component of O as

$$C(O) = \{(x, t) \in \Omega_0 \times (0, T) : \exists x_0 \in \Omega_0, (x_0, t) \in O\}.$$

Theorem 2.3. *Let $\Omega_0 \subset \mathbb{R}^N$ be a connected open set, $N \geqslant 2$ and $T > 0$. Let $a \in L^\infty_{\mathrm{loc}}(\Omega_0 \times (0,T))^N$ and $c \in C([0,T]; L^r_{\mathrm{loc}}(\Omega_0, \mathcal{M}_{N\times N}))$ be a matrix-valued function with $r > N$. If $(v,p) \in L^2(0,T; H^1_{\mathrm{loc}}(\Omega_0))^N \times L^2_{\mathrm{loc}}(\Omega_0 \times (0,T))$ is a solution of*

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + (a\cdot\nabla)v + cv + \nabla p = 0 & \text{in} \quad \Omega_0 \times (0,T) \\ \operatorname{div}(v) = 0 & \text{in} \quad \Omega_0 \times (0,T), \end{cases} \tag{2.3}$$

with $v = 0$ in O, then $v \equiv 0$ in $C(O)$.

The following result is also a consequence of the above theorem:

Corollary 2.4. *Let $\Omega_0 \subset \mathbb{R}^N$ be a Lipschitz domain, $N \geqslant 2$. For $\varepsilon_* = 0$ or $\varepsilon_* = 1$, if $(v,p) \in L^2(T_1,T_2; H^1(\Omega_0)^N) \times L^2(\Omega_0 \times (T_1,T_2))$ (with $0 \leqslant T_1 < T_2$) is a solution of*

$$\begin{cases} \dfrac{\partial v}{\partial t} - \operatorname{div}(\sigma(v,p)) + \varepsilon_* \operatorname{div}(v\otimes v) = 0 & \text{in} \quad \Omega_0 \times (T_1,T_2) \\ \operatorname{div}(v) = 0 & \text{in} \quad \Omega_0 \times (T_1,T_2) \\ v = 0 & \text{on} \quad \Gamma_0 \times (0,T), \end{cases} \tag{2.4}$$

satisfying $\sigma(v,p)\mathbf{n} = 0$ on $\Gamma_0 \times (0,T)$, where $\Gamma_0 \subset \partial\Omega_0$ is a relatively open non-empty subset, then $v = 0$ in $\Omega_0 \times (0,T)$.

Remark. Fabre and Lebeau prove the above results in the case $c \equiv 0$, using appropriate local Carleman inequalities. As a matter of fact, a careful examination of their arguments shows that if $c \in L^r_{\mathrm{loc}}(\Omega_0, \mathcal{M}_{N\times N})$ is a matrix-valued function with $r > N$, then assuming that $(v,p) \in H^1_{\mathrm{loc}}(\Omega_0) \times L^2_{\mathrm{loc}}(\Omega_0)$ satisfies

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + (a\cdot\nabla)v + cv + \nabla p = 0 & \text{in} \quad \Omega_0 \\ \nabla \cdot v = 0 & \text{in} \quad \Omega_0, \end{cases} \tag{2.5}$$

and $v = 0$ in ω_0 (an open subdomain of Ω_0) then $v \equiv 0$ in Ω_0 and p is constant. In particular, if Ω_0 is smooth enough and if (v,p) satisfies (2.5) and $v = 0$ on $\partial\Omega_0$ and if $\sigma(v,p)\mathbf{n} = 0$ on an open subset $\Gamma \subset \partial\Omega_0$, then $v = 0$ in Ω_0 and p is constant (see also Fernández-Cara [9]).

3. The proof of identifiability results

We begin with the proof of the identifiability result for the non-steady case, that is, the proof of theorem 1.1.

Consider first the linear case, that is, $\varepsilon_* = 0$. Let v_0, v_1 be solutions of system (1.8) for $j = 0, 1$, verifying

$$\sigma(v_0,p_0)\mathbf{n} = \sigma(v_1,p_1)\mathbf{n} \qquad \text{on} \quad \Gamma \times (0,T),$$

and define

$$v = v_0 - v_1, \qquad p = p_0 - p_1 \qquad \text{and} \qquad D = D_0 \cup D_1$$

(see figure 2).

It is straightforward to see that (v,p) is a solution of

$$\begin{cases} \dfrac{\partial v}{\partial t} - \operatorname{div}\sigma(v,p) = 0 & \text{in} \quad (\Omega\backslash\overline{D}) \times (0,T) \\ \operatorname{div}(v) = 0 & \text{in} \quad (\Omega\backslash\overline{D}) \times (0,T) \\ v(0) = 0 & \text{in} \quad \Omega\backslash\overline{D} \\ v = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ v = 0 & \text{on} \quad \partial D \times (0,T). \end{cases} \tag{3.1}$$

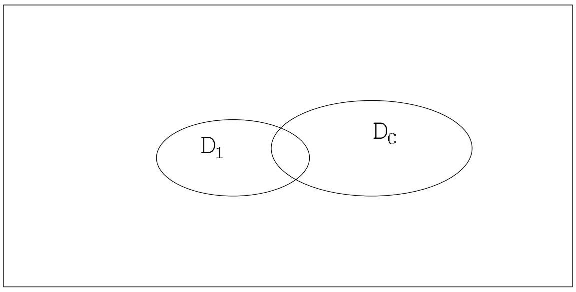

Figure 2. Proof of identifiability results.

Therefore, from the unique continuation property for the Stokes equation (corollary 2.4), we have that $v = 0$ in $\Omega\backslash\overline{D} \times (0, T)$ and then $v_0 = v_1$ in $\Omega\backslash\overline{D} \times (0, T)$.

Now, let us suppose that $D_0\backslash\overline{D}_1$ is an open non-empty subset of Ω, then we have that

$$\frac{\partial v_1}{\partial t} - \mathrm{div}(\sigma(v_1, p_1)) = 0 \qquad \text{in} \quad D_0\backslash\overline{D}_1 \times (0, T).$$

Multiplying this equation by v_1 and integrating by parts in $D_0\backslash\overline{D}_1$, noting that on ∂D_0 we have $v_1 = v_0 = 0$, while on ∂D_1 by assumption we know that $v_1 = 0$, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{D_0\backslash\overline{D}_1} |v_1(x, t)|^2\,\mathrm{d}x = -\int_{D_0\backslash\overline{D}_1} |e(v_1)(x, t)|^2\,\mathrm{d}x, \tag{3.2}$$

which implies that the function

$$t \mapsto E(t) = \int_{D_0\backslash\overline{D}_1} |v_1(x, t)|^2\,\mathrm{d}x$$

is a decreasing non-negative function but, since the initial data $v_1(0) = 0$, we conclude that $v_1|_{D_0\backslash\overline{D}_1} = 0$ for all $t \in (0, T)$. Thus, from theorem 2.3 we get that

$$v_1 = 0 \qquad \text{in} \quad (\Omega\backslash\overline{D}_1) \times (0, T), \tag{3.3}$$

which is impossible because $v_1 = \varphi$ and $\varphi \neq 0$ on $\Gamma \times (0, T)$. Therefore we have that $D_0\backslash\overline{D}_1 = \emptyset$. Analogously one proves that $D_1\backslash\overline{D}_0 = \emptyset$. Thus we have that $D_0 = D_1$. This completes the proof in the simpler linear case $\varepsilon_* = 0$.

Consider now the nonlinear case $\varepsilon_* = 1$. Then $(v, p) := (v_0 - v_1, p_0 - p_1)$ solves

$$\begin{cases} \dfrac{\partial v}{\partial t} - \mathrm{div}\,\sigma(v, p) + (v_0 \cdot \nabla)v + (v \cdot \nabla)v_1 = 0 & \text{in} \quad (\Omega\backslash\overline{D}) \times (0, T) \\ \mathrm{div}(v) = 0 & \text{in} \quad (\Omega\backslash\overline{D}) \times (0, T) \\ v(0) = 0 & \text{in} \quad \Omega\backslash\overline{D} \\ v = 0 & \text{on} \quad \partial\Omega \times (0, T) \\ v = 0 & \text{on} \quad \partial D \times (0, T). \end{cases} \tag{3.4}$$

Since $\varphi \in C^1([0, T]; H^{3/2}(\partial\Omega))$, classical existence results for the Navier–Stokes equation (see, for instance, [13] or [14]) show that there exists $T_* \in (0, T)$ such that a unique solution (v_j, p_j) to equation (1.8) exists on the interval $[0, T_*]$ in such a way that $v_j \in C([0, T_*], H^2(\Omega))$. Therefore, upon setting $c_{k\ell} := \partial_\ell v_{1,k}$, for $1 \leqslant k, \ell, N$, we have $c_{k\ell} \in C([0, T_*]; H^1(\Omega))$ and by Sobolev imbedding theorems the matrix-valued function $c := (c_{k\ell})_{1\leqslant k,\ell\leqslant N}$ satisfies the assumption of corollary 2.4, and since $\sigma(v, p)\mathbf{n} = 0$ on $\Gamma \times [0, T_*]$ we conclude that $v \equiv 0$ in $\Omega\backslash\overline{D} \times (0, T_*)$, that is, $v_0 = v_1$ in $\Omega\backslash\overline{D} \times (0, T_*)$.

If $D_0\backslash\overline{D}_1$ were non-empty, then we would have

$$\frac{\partial v_1}{\partial t} - \operatorname{div}(\sigma(v_1, p_1)) + (v_1 \cdot \nabla)v_1 = 0 \qquad \text{in} \quad D_0\backslash\overline{D}_1 \times (0, T_*).$$

Proceeding as above, multiplying this equation by v_1 and integrating by parts in $D_0\backslash\overline{D}_1$, noting that on ∂D_0 we have $v_1 = v_0 = 0$, while on ∂D_1 by assumption we know that $v_1 = 0$, and noting that

$$\int_{D_0\backslash\overline{D}_1} [(v_1(x,t)\cdot\nabla)v_1(x,t)]\cdot v_1(x,t)\,\mathrm{d}x = 0$$

we obtain that for $0 < t < T_*$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{D_0\backslash\overline{D}_1} |v_1(x,t)|^2\,\mathrm{d}x = -\int_{D_0\backslash\overline{D}_1} |e(v_1)(x,t)|^2\,\mathrm{d}x, \tag{3.5}$$

which implies that the function

$$t \mapsto E(t) = \int_{D_0\backslash\overline{D}_1} |v_1(x,t)|^2\,\mathrm{d}x$$

is a decreasing non-negative function on $[0, T_*]$. At this point the reader is easily convinced that the remainder of the argument follows exactly the lines of the proof of the linear case seen above, and that finally this implies $D_0\backslash\overline{D}_1 = \emptyset$. In the same manner one shows that $D_1\backslash\overline{D}_0 = \emptyset$ and thus $D_0 = D_1$.

Next we present the identifiability result for the stationary case, that is,

Theorem 3.1. *Let $\Omega \subseteq \mathbb{R}^N$, with $N \geqslant 2$, be a bounded Lipschitz domain and Γ be a non-empty open subset of $\partial\Omega$. Let $\varphi \in H^{1/2}(\partial\Omega)^N$ be a given non-homogeneous Dirichlet boundary condition. Assume that for $j = 0, 1$, $D_j \in \mathcal{D}_{\mathrm{ad}}$ are open sets in Ω and (v_j, p_j) are the solutions of*

$$\begin{cases} -\operatorname{div}(\sigma(v_j, p_j)) = 0 & \textit{in} \quad (\Omega\backslash\overline{D}_j) \\ \operatorname{div}(v_j) = 0 & \textit{in} \quad (\Omega\backslash\overline{D}_j) \\ v_j = \varphi & \textit{on} \quad \partial\Omega \\ v_j = 0 & \textit{on} \quad \partial D_0 \end{cases} \tag{3.6}$$

such that

$$\sigma(v_0, p_0)\mathbf{n} = \sigma(v_1, p_1)\mathbf{n} \qquad \textit{on} \quad \Gamma.$$

Then $D_0 \equiv D_1$.

Proof of theorem 3.1. Let us define

$$v := v_0 - v_1, \qquad p = p_0 - p_1 \qquad \text{and} \qquad D = D_0 \cup D_1.$$

One sees that (v, p) satisfies

$$\begin{cases} -\operatorname{div}\sigma(v,p) = 0 & \text{in} \quad \Omega\backslash\overline{D} \\ \operatorname{div} v = 0 & \text{in} \quad \Omega\backslash\overline{D} \\ v = 0 & \text{on} \quad \Gamma \\ \sigma(v,p)\mathbf{n} = 0 & \text{on} \quad \Gamma, \end{cases} \tag{3.7}$$

thus, in view of corollary 2.2, we have $v = 0$ in $\Omega\backslash\overline{D}$, and therefore $v_0 = v_1$ in $\Omega\backslash\overline{D}$.

Let us suppose that $D_0\backslash\overline{D}_1$ is an open non-empty subset of Ω. We know that

$$-\operatorname{div}(\sigma(v_1, p_1)) = 0 \qquad \text{in} \quad D_0\backslash\overline{D}_1,$$

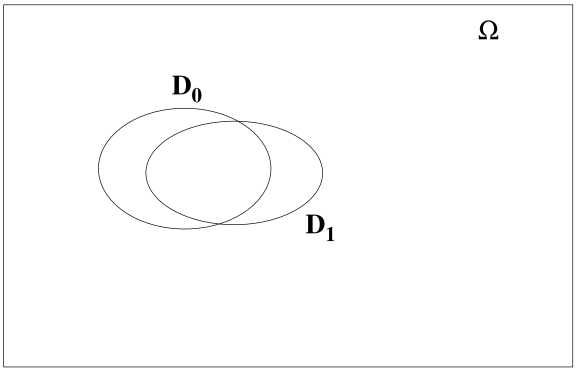

Figure 3. Admissible geometries.

hence, multiplying this equation by v_1 and integrating by parts in $D_0\backslash\overline{D}_1$, we obtain

$$\int_{D_0\backslash\overline{D}_1} |e(v_1)|^2\,\mathrm{d}x + \int_{\partial(D_0\backslash\overline{D}_1)} (\sigma(v_1, p_1)\mathbf{n})\cdot v_1\,\mathrm{d}s = 0.$$

Now, since $v_0 = v_1 = 0$ on $\partial D_0\backslash(\partial D_1 \cap \overline{D}_0)$, we have that

$$\int_{\partial(D_0\backslash\overline{D}_1)} (\sigma(v_1, p_1)\mathbf{n})\cdot v_1\,\mathrm{d}s = 0,$$

therefore we get

$$\int_{D_0\backslash\overline{D}_1} |e(v_1)|^2\,\mathrm{d}x = 0.$$

It follows immediately from Korn's inequality (see, for instance, [13] or [11, p 50]) that $v_1 \equiv 0$ in $D_0\backslash\overline{D}_1$, then, by proposition 2.1, $v_1 \equiv 0$ in $\Omega\backslash\overline{D}_1$, which is impossible because $v_1 = \varphi$ and $\varphi \not\equiv 0$ on $\partial\Omega$. This proves that $D_0\backslash\overline{D}_1 = \emptyset$. Analogously we prove that $D_1\backslash\overline{D}_0 = \emptyset$. Therefore $D_0 = D_1$. This completes the proof. □

4. Smooth dependence of Cauchy forces with respect to the deformation of the domain: the linear case

In this section, we consider the linear stationary Stokes equations and we prove that the velocity and the pressure depend smoothly on the deformations of the domain. For the sake of simplicity we set the viscosity to be $\nu = 1$. The proof of the result concerning the non-stationary case is essentially based on the results of this section and will be presented later on.

Let Ω and $\mathcal{O}$ be open connected sets in $\mathbb{R}^N$ and $D_0, D_1 \in \mathcal{D}_{\mathrm{ad}}$ (see figure 3) such that

$$D_0 \cup D_1 \subset\subset \mathcal{O} \subset\subset \Omega,$$

and set

$$\Omega_0 = \Omega\backslash\overline{D}_0 \qquad \text{and} \qquad \Omega_1 = \Omega\backslash\overline{D}_1.$$

For the sake of clarity, we introduce some basic notation and definitions. Let $\Psi = (\Psi^1, \ldots, \Psi^N) \in W^{1,\infty}(\Omega)^N$, then we write

$$\Psi' := \left(\frac{\partial \Psi^i}{\partial x_j}\right)_{i,j=1}^{N}, \qquad \mathrm{Jac}(\Psi) := |\det(\Psi')|,$$

the Jacobian matrix and the Jacobian determinant of Ψ (actually Ψ is going to be a smooth bijective map $\Omega_0 \to \Omega_1$). If $A \in W^{1,\infty}(\Omega; \mathcal{M}_{N\times N})$ is a matrix-valued function, then we define the vector $\mathrm{div}(A)$ as being

$$\mathrm{div}(A) := \left(\sum_{j=1}^{N} \frac{\partial A_{ij}}{\partial x_j} \right)_{i=1}^{N}. \tag{4.1}$$

For each function $f \in H^1(\Omega_1)$ and $\Psi \in W^{1,\infty}(\Omega_0; \Omega_1)$ we may set $g(x) := f(\Psi(x))$, in such a way that we have $g \in H^1(\Omega_0)$. Furthermore, when Ψ is bijective and $\Psi', (\Psi')^{-1} \in L^\infty(\Omega_0, \mathcal{M}_{N\times N})$, the mapping $f \mapsto g$ induces an isomorphism between the spaces $H_0^1(\Omega_1)^N$ and $H_0^1(\Omega_0)^N$ on the one hand, and between $L^2(\Omega_1)$ and $L^2(\Omega_0)$ on the other hand. We shall denote this isomorphism by $\widetilde{\Psi}$:

$$\widetilde{\Psi} : H_0^1(\Omega_1)^N \longrightarrow H_0^1(\Omega_0)^N, \qquad \widetilde{\Psi}(f) := g \qquad \text{with} \quad g(x) := f(\Psi(x)).$$

If y denotes the variable in Ω_1 and that in Ω_0 is denoted by x, using the change of variables $y := \Psi(x)$ and setting $g(x) := f(\Psi(x))$ we have

$$\frac{\partial f}{\partial y_i}(y) = \frac{\partial f}{\partial y_i}(\Psi(x)) = \sum_{j=1}^{N} \frac{\partial g}{\partial x_j}(x) \frac{\partial x_j}{\partial y_i}.$$

Let $\Psi \in W^{1,\infty}(\Omega_0, \Omega_1)$ be a diffeomorphism. We shall denote by $M \in L^\infty(\Omega_0, \mathcal{M}_{n\times n})$ the matrix

$$M := \left(\frac{\partial x_j}{\partial y_i} \right)_{i,j=1,N} = \left(\left(\frac{\partial \Psi_i}{\partial x_j} \right)^{-1}_{i,j=1,N} \right)^{*} = ((\Psi')^*)^{-1},$$

where A^* denotes the transpose matrix of the matrix A. Note that if Ψ is a small perturbation of the identity in the norm of $W^{1,\infty}$, then the matrix M is invertible, and D_1 is in some sense a 'small perturbation' of D_0. Note that with the above notation $y := \Psi(x)$ and $g(x) := f(\Psi(x))$, we have $\nabla f(y) = M(x)\nabla g(x)$.

Also if A and B are $N \times N$ matrices, we denote their *scalar product* by

$$A : B := \sum_{i,j=1}^{N} A_{ij} B_{ij},$$

and $|A|^2 := \sum_{i,j=1}^{N} |A_{ij}|^2$.

Instead of writing equations (3.6) using the stress tensor (1.3), we use the following equivalent form (4.2) (since for $\nu = 1$ and v such that $\mathrm{div}(v) = 0$ one has $\mathrm{div}(\sigma(v, p)) = \Delta v - \nabla p$); here for a matrix A we denote by $\mathrm{Tr}(A)$ its trace):

$$\begin{cases} -\Delta v_j + \nabla p_j = 0 & \text{in} \quad \Omega_j \\ \mathrm{div}(v_j) = \mathrm{Tr}(\nabla v_j) = 0 & \text{in} \quad \Omega_j \\ v_j = \varphi & \text{on} \quad \partial\Omega \\ v_j = 0 & \text{on} \quad \partial D_j. \end{cases} \tag{4.2}$$

Our aim is to show that $\sigma(v_1, p_1)\mathbf{n}$ and $\sigma(v_0, p_0)\mathbf{n}$ are close when Ψ is close to the identity map, but since the equations are set in different domains we will change (v_1, p_1) to a pair of functions (u_0, q_0) defined in Ω_0 which satisfy certain elliptic equations with variable L^∞ coefficients, and then we show that the dependence of (u_0, q_0), and thus that of $\sigma(u_0, q_0)\mathbf{n}$, on these coefficients is analytic.

Since both subdomains D_0 and D_1 have to be admissible, we will consider only those deformations Ψ which respect this requirement (see figure 4):

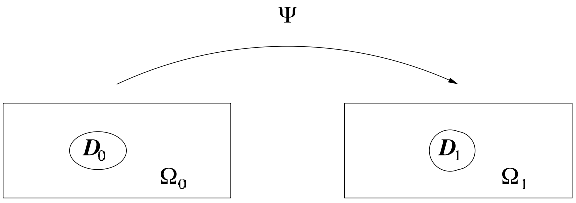

Figure 4. Admissible deformation.

Definition 4.1. *Let Ω be a bounded domain of $\mathbb{R}^N$. A mapping $\Psi \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ is called an admissible deformation of $D_0 \subset \Omega$, if Ψ is bijective, $\Psi^{-1} \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$, $\Psi(\Omega) = \Omega$ and there exists a Lipschitz domain $\mathcal{O} \subset\subset \Omega$ such that*

$$D_0 \cup \Psi(D_0) \subset\subset \mathcal{O}$$

and for some $\delta > 0$ one has $\Psi = I$ on $\Omega \backslash \overline{\mathcal{O}}_\delta$, where

$$\mathcal{O}_\delta := \{x \in \mathcal{O}; \operatorname{dist}(x, \mathcal{O}^c) > \delta\}.$$

Put in other terms, the class of admissible deformations Ψ we consider can be written in the form $\Psi = I + \Psi_1$ with Ψ_1 having a compact support contained in $\mathcal{O}$, or $\Psi_1 \in W_0^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$ (when $\mathcal{O}$ is Lipschitz, we may set $\widetilde{\Psi}_1 := \Psi$ in $\mathcal{O}$ and $\widetilde{\Psi}_1 = 0$ in $\mathbb{R}^N \backslash \overline{\mathcal{O}}$: this yields a function $\widetilde{\Psi}_1$ with compact support contained in Ω).

When φ satisfies the compatibility condition

$$\int_{\partial\Omega} \varphi(s) \cdot \mathbf{n}(s) \, \mathrm{d}s = 0, \tag{4.3}$$

one can find $\widetilde{\varphi}_1 \in H^1(\Omega)^N$ such that

$$\operatorname{div}(\widetilde{\varphi}_1) = 0 \quad \text{in} \quad \Omega, \qquad \widetilde{\varphi}_1 = 0 \quad \text{in} \quad \mathcal{O}, \qquad \widetilde{\varphi}_1 = \varphi \quad \text{on} \quad \partial\Omega, \tag{4.4}$$

and then, upon writing

$$v_1 =: \widetilde{\varphi}_1 + \widetilde{v}_1, \qquad \text{with} \quad \widetilde{v}_1 \in H_0^1(\Omega_1)^N,$$

solving equation (4.2) with $j = 1$ is equivalent to solving

$$\begin{cases} -\Delta\widetilde{v}_1 + \nabla p_1 = \Delta\widetilde{\varphi}_1, & \text{in} \quad \Omega_1 \\ \operatorname{div}(\widetilde{v}_1) = \operatorname{Tr}(\nabla\widetilde{v}_1) = 0, & \text{in} \quad \Omega_1 \\ \widetilde{v}_1 = 0 & \text{on} \quad \partial\Omega_1. \end{cases} \tag{4.5}$$

In the following, we shall denote by $L_0^2(\Omega_1)$ the space $L^2(\Omega_1)/\mathbb{R}$, that is,

$$L_0^2(\Omega_1) := \left\{ q \in L^2(\Omega_1); m(q) := \frac{1}{|\Omega_1|} \int_{\Omega_1} q(y) \, \mathrm{d}y = 0 \right\}.$$

Then equation (4.5) is equivalent to stating that $(\widetilde{v}_1, p_1) \in H_0^1(\Omega_1)^N \times L_0^2(\Omega_1)$ and for all $(w, p) \in H_0^1(\Omega_1)^N \times L_0^2(\Omega_1)$

$$\int_{\Omega_1} \nabla\widetilde{v}_1 : \nabla w \, \mathrm{d}y - \int_{\Omega_1} p_1 \operatorname{Tr}(\nabla w) \, \mathrm{d}y + \int_{\Omega_1} \operatorname{Tr}(\nabla\widetilde{v}_1) p \, \mathrm{d}y = -\int_{\Omega_1} \nabla\widetilde{\varphi}_1 : \nabla w \, \mathrm{d}y. \tag{4.6}$$

Next, Ψ being admissible as in definition 4.1, we define (u_0, q_0) by setting

$$\begin{cases} y := \Psi(x), \\ u_0(x) := \widetilde{v}_1(\Psi(x)), \\ q_0(x) := p_1(\Psi(x)), \\ \varphi_0(x) := \widetilde{\varphi}_1(\Psi(x)). \end{cases} \tag{4.7}$$

Note that since $\widetilde{\varphi}_1 = 0$ in $\mathcal{O}$ and $\Psi = I$ in $\Omega \backslash \overline{\mathcal{O}}_\delta$, we have $\varphi_0(x) = \widetilde{\varphi}_1(x)$ for all $x \in \Omega$, meaning that $\widetilde{\varphi}_1$ is invariant under the isomorphism $\widetilde{\Psi}$.

With the above notation (in particular $M(x) := (\Psi'(x)^{-1})^*$), one checks easily that solving the variational problem (4.6) is equivalent to finding $(u_0, q_0) \in H^1(\Omega_0) \times L^2(\Omega_0)$ satisfying for all $(z, q) \in H_0^1(\Omega_0)^N \times L^2(\Omega_0)$

$$\int_{\Omega_0} M\nabla u_0 : M\nabla z \,\mathrm{Jac}(\Psi)\,\mathrm{d}x - \int_{\Omega_0} q_0 \,\mathrm{Tr}(M\nabla z)\,\mathrm{Jac}(\Psi)\,\mathrm{d}x + \int_{\Omega_0} \mathrm{Tr}(M\nabla u_0) q \,\mathrm{Jac}(\Psi)\,\mathrm{d}x = -\int_{\Omega_0} M\nabla\varphi_0 : M\nabla z \,\mathrm{Jac}(\Psi)\,\mathrm{d}x. \tag{4.8}$$

This variational problem means that (u_0, q_0) satisfies the following equation in Ω_0:

$$\begin{cases} -\mathrm{div}(\mathrm{Jac}(\Psi)M^*M\nabla u_0) + \mathrm{div}(q_0\,\mathrm{Jac}(\Psi)M^*) = \mathrm{div}(\mathrm{Jac}(\Psi)M^*M\nabla\varphi_0) & \text{in } \Omega_0 \\ \mathrm{Tr}(M\nabla u_0) = 0 & \text{in } \Omega_0 \\ u_0 = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{4.9}$$

Now our aim is to prove that the solution (u_0, q_0) depends smoothly on Ψ, in the natural $W^{1,\infty}$ norm.

Define the space $L^2_\Psi(\Omega_0)$ as being the space $L^2(\Omega_0)$ endowed with the scalar product

$$(p|q)_\Psi := \int_{\Omega_0} p(x)q(x)\,\mathrm{Jac}(\Psi)(x)\,\mathrm{d}x,$$

(this can be viewed as the image of the usual scalar product of $L^2(\Omega_1)$ under the isomorphism $\widetilde{\Psi}$) and the elliptic operator

$$Au := -\frac{1}{\mathrm{Jac}(\Psi)}\mathrm{div}(\mathrm{Jac}(\Psi)M^*M\nabla u) \tag{4.10}$$

on $H_0^1(\Omega_0)^N$, which is self-adjoint with respect to the scalar product of $L^2_\Psi(\Omega_0)$, i.e., it is such that

$$(Au|v)_\Psi = (u|Av)_\Psi.$$

Define also the operator B by

$$Bz := -\mathrm{Tr}(M\nabla z), \qquad B : H_0^1(\Omega_0)^N \longrightarrow L^2_{0,\Psi}(\Omega_0), \tag{4.11}$$

where $L^2_{0,\Psi}(\Omega_0)$ is the subspace of functions p in $L^2_\Psi(\Omega_0)$ such that

$$\int_{\Omega_0} p(x)\,\mathrm{Jac}(\Psi)(x)\,\mathrm{d}x = 0.$$

(Note that the fact that for $z \in H_0^1(\Omega_0)^N$ one has $Bz \in L^2_{0,\Psi}(\Omega_0)$ is a consequence of the fact that for $\in H_0^1(\Omega_1)^N$ one has $\mathrm{Tr}(\nabla w) \in L_0^2(\Omega_1)$). We prove first the following lemma:

Lemma 4.2. *The range $R(B)$ of B is closed and the adjoint B^* of B is given by*

$$B^*q = \frac{1}{\mathrm{Jac}(\Psi)}\mathrm{div}(q\,\mathrm{Jac}(\Psi)M^*).$$

Moreover, the kernel $N(B^*)$ is precisely given by the functions which are constant on Ω_0, while $R(B) = N(B^*)^\perp = L^2_{0,\Psi}(\Omega_0)$.

Proof. We prove first that $R(B)$, the range of B, is closed. Indeed, it is a classical result that the range of the mapping $w \mapsto \mathrm{div}(w) = \mathrm{Tr}(\nabla w)$ from $H_0^1(\Omega_1)^N$ into $L^2(\Omega_1)$ is closed (see, for instance, [13] or [14]). Let $\widetilde{\Psi}$ be the mapping $w \mapsto z := w \circ \Psi := \widetilde{\Psi}(w)$, which is an isomorphism between the Hilbert spaces $H_0^1(\Omega_1)^N$ and $H_0^1(\Omega_0)^N$ on the one hand, and $L^2(\Omega_1)$

and $L^2_\Psi(\Omega_0)$ on the other hand, as recalled above. Now, since $Bz = -\widetilde{\Psi}(\mathrm{Tr}(\nabla\widetilde{\Psi}^{-1}(z)))$, one sees that B has a closed range.

Next we determine the adjoint of B with respect to the scalar product $(\cdot|\cdot)_\Psi$. If $q \in C_c^\infty(\Omega_0)$ and $z \in C_c^\infty(\Omega_0)^N$, we have (using, in the last step of the following, the definition of the divergence, div, of a matrix, see (4.1)

$$\begin{aligned}\langle B^*q, z\rangle = \langle q, Bz\rangle &= -\int_{\Omega_0} q(x)\,\mathrm{Tr}(M(x)\nabla z(x))\,\mathrm{Jac}(\Psi)(x)\,\mathrm{d}x \\ &= -\sum_{i=1}^N\sum_{j=1}^N \int_{\Omega_0} M_{ij}(x)q(x)\frac{\partial z_j(x)}{\partial x_i}\,\mathrm{Jac}(\Psi)(x)\,\mathrm{d}x \\ &= \sum_{i=1}^N\sum_{j=1}^N \int_{\Omega_0} \frac{\partial}{\partial x_i}[\mathrm{Jac}(\Psi)(x)M_{ij}(x)q(x)]z_j(x)\,\mathrm{d}x \\ &= \int_{\Omega_0} \mathrm{div}(q(x)\,\mathrm{Jac}(\Psi)(x)M^*(x))\cdot z(x)\,\mathrm{d}x. \end{aligned} \tag{4.12}$$

From this, and the density of smooth functions in $L^2(\Omega_0)$ and $H_0^1(\Omega_0)^N$, we conclude that

$$B^*q = \frac{1}{\mathrm{Jac}(\Psi)}\,\mathrm{div}(q\,\mathrm{Jac}(\Psi)(x)M^*).$$

To finish the proof of the lemma, assume that $q \in L^2(\Omega_0)$ is such that $B^*q = 0$. Then $q \in R(B)^\perp$, that is, for all $z \in H_0^1(\Omega_0)^N$ we have (upon setting $p(y) := q(\Psi^{-1}(y))$ and $w(y) := z(\Psi^{-1}(y)))$

$$0 = \langle B^*q, z\rangle = -\int_{\Omega_0} q(x)\,\mathrm{Tr}(M(x)\nabla z(x)), \quad \mathrm{Jac}(\Psi)(x)\,\mathrm{d}x$$

$$0 = -\int_{\Omega_1} p(y)\,\mathrm{Tr}(\nabla w(y))\,\mathrm{d}y,$$

for all $w \in H_0^1(\Omega_1)^N$. This implies that p is constant in Ω_1, and thus q is constant in Ω_0. □

Now one can see that equation (4.9) can be written in the form

$$\begin{cases} Au_0 + B^*q_0 = f & \text{in} \quad \Omega_0 \\ Bu_0 = 0 & \text{in} \quad \Omega_0 \\ u_0 = 0 & \text{on} \quad \partial\Omega_0 \end{cases} \tag{4.13}$$

in which we denote

$$f := \frac{1}{\mathrm{Jac}(\Psi)}\,\mathrm{div}(\mathrm{Jac}(\Psi)M^*M\nabla\varphi_0),$$

and where q_0 is determined uniquely up to the addition of an element in $N(B^*)$, that is, up to the addition of a constant.

We are finally in a position to state and show the analytic dependence of (u_0, q_0) on Ψ.

Theorem 4.3. *Let $\mathcal{O} \subset\subset \Omega$ be a Lipschitz domain, and $\Psi_1 \in W^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$ be such that $\Psi := I + \Psi_1$ is an admissible deformation according to definition 4.1. The mapping $\Psi_1 \mapsto (u_0, q_0)$ from $W_0^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$ into $H_0^1(\Omega_0)^N \times L^2(\Omega_0)$ is analytic in a neighbourhood of the origin.*

Proof. We begin by pointing out that, due to the assumptions on Ψ, the function f defined above is actually independent of Ψ, since where $\Psi \neq I$ we have $\varphi_0 \equiv 0$. It is elementary to verify that the operator : $H_0^1(\Omega_0)^N \longrightarrow H^{-1}(\Omega_0)^N$ is an isomorphism. Then the first equation of (4.13) implies that

$$u_0 + A^{-1}B^*q_0 = A^{-1}f, \tag{4.14}$$

and upon applying B to both sides of this, and using the fact that $Bu_0 = 0$, we find that q_0 is given by the equation

$$BA^{-1}B^*q_0 = BA^{-1}f. \tag{4.15}$$

However, due to the fact that A^{-1} is coercive, that is, for some $\alpha_0 > 0$ one has

$$\langle g, A^{-1}g\rangle \geqslant \alpha_0\|g\|^2_{H^{-1}},$$

one checks that the operator $BA^{-1}B^*$ is continuous and one-to-one in $N(B^*)^\perp$. Therefore, up to the addition of a constant, q_0 is uniquely determined by

$$q_0 = (BA^{-1}B^*)^{-1}BA^{-1}f. \tag{4.16}$$

Note that $q_0 \in L^2_{0,\Psi}(\Omega_0)$, that is,

$$\int_{\Omega_0} q_0(x)\,\mathrm{Jac}(\Psi)(x)\,\mathrm{d}x = 0.$$

However, since $N(B^*)$ is the one-dimensional subset of constant functions on Ω_0, and thus independent of Ψ, we may normalize q_0 by adding a constant so that

$$\int_{\Omega_0} q_0(x)\,\mathrm{d}x = 0.$$

From (4.16) and (4.14) one concludes that u_0 is given by

$$u_0 = (I - A^{-1}B^*(BA^{-1}B^*)^{-1}B)A^{-1}f. \tag{4.17}$$

It is clear that the mappings $\Psi_1 \mapsto A$ and $\Psi_1 \mapsto B^*$, in a neighbourhood of the origin, are analytic from $W_0^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$ into $\mathcal{L}\big(H_0^1(\Omega_0)^N; H^{-1}(\Omega_0)^N\big)$ and $\mathcal{L}(L^2(\Omega_0), H^{-1}(\Omega_0)^N)$ respectively, and since so is the inversion of continuous operators, the formulae (4.16) and (4.17) show that the mapping $\Psi_1 \mapsto (u_0, q_0)$ is analytic in a neighbourhood of the origin in $W_0^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$. □

Remark. As a matter of fact, denoting by A_0 and B_0 the corresponding operators when $\Psi_1 = 0$, that is, $A_0u := -\Delta u$ and $B_0u := -\mathrm{div}(u)$, using (4.17) one can write u_0 as a series. Indeed $A = A_0 + A_1$ and $B = B_0 + B_1$ where $\|A_1\|$ and $\|B_1\|$ are small. So we may write

$$BA^{-1}B^* =: B_0A_0^{-1}B_0^*(I - L), \qquad A^{-1} = \left(\sum_{k\geqslant 0}\left(-A_0^{-1}A_1\right)^k\right)A_0^{-1}$$

where $\|L\|$ is small. Therefore

$$(BA^{-1}B^*)^{-1} = \left(\sum_{k\geqslant 0} L^k\right)\left(B_0A_0^{-1}B_0^*\right)^{-1}.$$

From these expressions one sees that for a sequence of operators, say $(L_k)_k$, continuous from $H^{-1}(\Omega_0)^N$ into $H_0^1(\Omega_0)^N$, one has

$$u_0 = \sum_{k\geqslant 0} L_k f,$$

but, since we will not use this expression, we do not insist on determining the sequence $(L_k)_k$ in terms of Ψ_1.

Remark. We should point out that even though the above analysis shows that the mapping $\Psi_1 \mapsto (u_1, q_1)$ is analytic we cannot say anything about the analyticity of the mapping $\Psi_1 \mapsto (v_1, p_1)$: indeed the only thing we can infer is that the mapping

$$\Psi_1 \mapsto \sigma(v_1, p_1)\mathbf{n} = \sigma(u_1, q_1)\mathbf{n}$$

is analytic.

The following corollary is of interest in the next section, where we show that the smooth dependence of $\sigma(v_1, p_1)\mathbf{n}$ on Ψ extends to the linear or nonlinear evolution equations.

Corollary 4.4. *Let $\lambda > 0$ be a fixed parameter, let the assumptions of theorem 4.3 be satisfied, and for a given $f \in H^{-1}(\Omega)^N$, let $(u_{0,\lambda}, q_{0,\lambda})$ be the solution to the equation*

$$\begin{cases} u_{0,\lambda} + \lambda A u_{0,\lambda} + B^* q_{0,\lambda} = f & \text{in} \quad \Omega_0 \\ B u_{0,\lambda} = 0 & \text{in} \quad \Omega_0 \\ u_{0,\lambda} = 0 & \text{on} \quad \partial\Omega_0. \end{cases} \tag{4.18}$$

Then we can write $u_{0,\lambda} = R_\lambda f$ where the resolvent R_λ is defined by

$$R_\lambda := (I + \lambda A)^{-1} - (I + \lambda A)^{-1} B^* (B(I + \lambda A)^{-1} B^*)^{-1} B (I + \lambda A)^{-1}.$$

One has $\|R_\lambda f\|_{L^2(\Omega_0)} \leqslant \|f\|_{L^2(\Omega_0)}$, and the mapping $\Psi_1 \mapsto (u_{0,\lambda}, q_{0,\lambda})$ from $W_0^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$ into $H_0^1(\Omega_0)^N \times L^2(\Omega_0)$ is analytic in a neighbourhood of the origin.

Proof. Proceeding as in the proof of theorem 4.3, one sees that up to a constant $q_{0,\lambda}$ is given by

$$q_{0,\lambda} = (B(I + \lambda A)^{-1} B^*)^{-1} B (I + \lambda A)^{-1} f,$$

while u_0 is given by

$$u_{0,\lambda} = (I + \lambda A)^{-1} f - (I + \lambda A)^{-1} B^* (B(I + \lambda A)^{-1} B^*)^{-1} B (I + \lambda A)^{-1} f.$$

From this it is easily seen that the analytic dependence of $(u_{0,\lambda}, q_{0,\lambda})$ on Ψ_1 holds. □

Remark. As a matter of fact, the mapping $(\lambda, \Psi_1) \mapsto R_\lambda$ can be extended into an analytic function defined on $[\mathrm{Re}(z) > 0] \times B(0, \rho) \longrightarrow \mathcal{L}(H^{-1}(\Omega)^N, H_0^1(\Omega)^N)$, where $B(0, \rho)$ is the ball of $W_0^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$ centred at the origin with a radius ρ sufficiently small.

Our next result, a corollary of what we have proved in theorem 4.3 and the identifiability result of section 3, is the fact if $\Psi := \Psi_\tau = I + \tau\Psi_1$ for a fixed $\Psi_1 \not\equiv 0$, and if for $|\tau|$ small enough we have $D_0 \in \mathcal{D}_{\mathrm{ad}}$ and $D_\tau := \Psi_\tau(D_0) \in \mathcal{D}_{\mathrm{ad}}$ then we have a lower bound for $\left\|\Lambda_{D_0}(\varphi) - \Lambda_{D_\tau}(\varphi)\right\|_{H^{-1}}$ in terms of τ, that is, we have a certain directional stability:

Corollary 4.5. *Let $\Psi_1 \in W_0^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$ such that $\Psi_1 \not\equiv 0$ on D_0. Denote $\Psi_\tau := I + \tau\Psi_1$ for $\tau \in \mathbb{R}$, and $D_\tau := \Psi_\tau(D_0)$ and $\Omega_\tau := \Psi_\tau(\Omega_0)$. Let $\tau_1 > 0$ be a positive number such that $\tau \mapsto (v_\tau, p_\tau)$ is analytic on $(-\tau_1, \tau_1)$ and assume that for some $\tau_* \in (-\tau_1, \tau_1)$ one has $\Psi_{\tau_*}(D_0) \neq D_0$. Then there exist $\tau_0 > 0$, an integer $m \geqslant 1$ and a positive constant C, all depending on $\Psi_1, \varphi, \Gamma, \Omega_0$, such that for all $\tau \in [-\tau_0, \tau_0]$ one has*

$$\left\|\Lambda_{D_0}(\varphi) - \Lambda_{D_\tau}(\varphi)\right\|_{H^{-1/2}(\Gamma)^N} \geqslant C|\tau|^m \tag{4.19}$$

Proof. It is clear that for $|\tau|$ smaller than some $\tau_1 > 0$ the set D_τ belongs to the class of admissible subdomains $\mathcal{D}_{\mathrm{ad}}$. Now for such τ, let (v_τ, p_τ) be a solution of the Stokes system

$$\begin{cases} -\mathrm{div}(\sigma(v_\tau, p_\tau)) = 0 & \text{in} \quad \Omega\backslash\overline{D}_\tau \\ \mathrm{div}(v_\tau) = 0 & \text{in} \quad \Omega\backslash\overline{D}_\tau \\ v_\tau = 0 & \text{on} \quad \partial D_\tau \\ v_\tau = \varphi & \text{on} \quad \partial\Omega. \end{cases} \tag{4.20}$$

According to the analyticity result proved above, $\tau \mapsto \sigma(v_\tau, p_\tau)\mathbf{n}$ is analytic in a neighbourhood of the origin, and therefore so is the mapping

$$\tau \mapsto \Lambda_{D_\tau}(\varphi) - \Lambda_{D_0}(\varphi)$$

from $[-\tau_1, \tau_1]$ into $H^{-1/2}(\Gamma)^N$. Since for a sequence of $(F_k)_k$ in $H^{-1/2}(\Gamma)^N$ we have

$$\Lambda_{D_\tau}(\varphi) - \Lambda_{D_0}(\varphi) = \sum_{k=1}^{\infty} \tau^k F_k,$$

and since, for some $\tau_* \neq 0$, thanks to the identifiability result of section 3 we know that $\Lambda_{D_{\tau_*}}(\varphi) - \Lambda_{D_0}(\varphi) \neq 0$, because $D_0 \neq D_{\tau_*}$, all the F_k cannot be zero, and so there exists a least integer $m \geqslant 1$ such that $F_m \neq 0$. Upon choosing $0 < \tau_0 < \tau_1$ so that

$$\left\| \sum_{k=m+1}^{\infty} \tau_0^{k-m} F_k \right\|_{H^{-1/2}(\Gamma)^N} \leqslant \frac{1}{2} \|F_m\|_{H^{-1/2}(\Gamma)^N},$$

it follows that for $|\tau| \leqslant \tau_0$ we have

$$\left\|\Lambda_{D_\tau}(\varphi) - \Lambda_{D_0}(\varphi)\right\|_{H^{-1/2}(\Gamma)^N} \geqslant \tfrac{1}{2} \|F_m\|_{H^{-1/2}(\Gamma)^N} \tau^m,$$

and thus the result is proved. □

In the remainder of this section, we show that the Gâteaux derivative of the mapping $\Psi \mapsto \sigma(u_0, q_0)\mathbf{n}$ can be obtained quite easily. To be more specific, for $\tau \in \mathbb{R}$ and $\Psi_1 \in W_0^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$ consider a perturbation of I of the form

$$\Psi := \Psi_\tau := I + \tau \Psi_1.$$

Let us denote, for $|\tau|$ small enough, by M_τ the corresponding matrices M defined at the beginning of this section, namely

$$M_0 := I, \qquad M_\tau := ((\Psi_\tau')^*)^{-1}$$

and by A_τ and B_τ the corresponding operators A and B defined in (4.10) and (4.11). Also we shall denote by (u_0, q_0) and (u_τ, q_τ) the solutions of the corresponding equations (4.13) where A and B are replaced with A_0, B_0 and A_τ, B_τ respectively. (Note that as a matter of fact, $A_0 = -\Delta$ and $B_0 = -\mathrm{div}$, that is, the corresponding equations for u_0, q_0 are just the classical Stokes system).

Since we know that $\Psi_\tau \mapsto (u_\tau, q_\tau)$ is analytic for $|\tau|$ small enough, this means that the mapping $\tau \mapsto (u_\tau, q_\tau)$ is analytic and that we may write

$$u_\tau = u_0 + \tau u_1^* + O(\tau^2), \qquad q_\tau = q_0 + \tau q_1^* + O(\tau^2) \tag{4.21}$$

where the notation $O(\tau^2)$ refers to the norm of $H_0^1(\Omega_0)$ in the case of u_τ and that of $L^2(\Omega_0)$ for q_τ. We may now state the proposition concerning the first derivative of the mapping $\Psi_\tau \mapsto (u_\tau, q_\tau)$.

Proposition 4.6. *With the above notation, the first derivative of the mapping which maps Ψ into the solution of equation (4.13) is given by*

$$\frac{\partial}{\partial \tau}(u_\tau, q_\tau)_{|\tau=0} = (u_1^*, q_1^*)$$

where (u_τ, q_τ) satisfies the equation

$$\begin{cases} -\Delta u_1^* + \nabla q_1^* = F_0 & \text{in} \quad \Omega_0 := \Omega \backslash \overline{D}_0 \\ \mathrm{div}(u_1^*) = G_0 & \text{in} \quad \Omega_0 \\ u_1^* = 0 & \text{on} \quad \partial\Omega_0. \end{cases} \tag{4.22}$$

where

$$F_0 := -\mathrm{div}([\Psi_1' + (\Psi_1')^* - \mathrm{Tr}(\Psi_1')I]\nabla u_0) + \nabla(q_0 \,\mathrm{Tr}(\Psi_1')) - \mathrm{div}(q_0\Psi_1')$$

and

$$G_0 := \mathrm{Tr}((\Psi_1')^*\nabla u_0).$$

Proof. Indeed on the one hand we have $M_\tau = ((\Psi_\tau')^*)^{-1} = (I + \tau(\Psi_1')^*)^{-1}$, and therefore

$$M_\tau = I - \tau(\Psi_1')^* + O(\tau^2). \tag{4.23}$$

On the other hand $\mathrm{Jac}(\Psi_\tau) = \det(I + \tau\Psi_1')$, and so one has

$$\mathrm{Jac}(\Psi_\tau) = 1 + \tau\,\mathrm{Tr}(\Psi_1') + O(\tau^2). \tag{4.24}$$

Now recall that (u_τ, q_τ) is characterized by the fact that $(u_\tau, q_\tau) \in H^1(\Omega_0) \times L^2(\Omega_0)$ and for all $(z, q) \in H_0^1(\Omega_0)^N \times L^2(\Omega_0)$ one has

$$\int_{\Omega_0} M_\tau\nabla u_\tau : M_\tau\nabla z\,\mathrm{Jac}(\Psi_\tau)\,\mathrm{d}x - \int_{\Omega_0} q_\tau\,\mathrm{Tr}(M_\tau\nabla z)\,\mathrm{Jac}(\Psi_\tau)\,\mathrm{d}x + \int_{\Omega_0}\mathrm{Tr}(M_\tau\nabla u_\tau)q\,\mathrm{Jac}(\Psi_\tau)\,\mathrm{d}x$$
$$= -\int_{\Omega_0} M_\tau\nabla\varphi_0 : M_\tau\nabla z\,\mathrm{Jac}(\Psi_\tau)\,\mathrm{d}x. \tag{4.25}$$

Using expansions (4.23) and (4.24) on the one hand and the fact that

$$(u_\tau, q_\tau) = (u_0, q_0) + \tau(u_1^*, q_1^*) + O(\tau^2),$$

on the other hand, one finds that the three integrals on the left-hand side of (4.25) have the following expansions: the first one is

$$\int_{\Omega_0} M_\tau\nabla u_\tau : M_\tau\nabla z\,\mathrm{Jac}(\Psi_\tau)\,\mathrm{d}x = \int_{\Omega_0}\nabla u_0 : \nabla z\,\mathrm{d}x + \tau\int_{\Omega_0}\nabla u_1^* : \nabla z\,\mathrm{d}x$$
$$+\tau\int_{\Omega_0}[\mathrm{Tr}(\Psi_1')I - \Psi_1' - (\Psi_1')^*]\nabla u_0 : \nabla z\,\mathrm{d}x + O(\tau^2), \tag{4.26}$$

while the second one is

$$\int_{\Omega_0} q_\tau\,\mathrm{Tr}(M_\tau\nabla z)\,\mathrm{Jac}(\Psi_\tau)\,\mathrm{d}x = \int_{\Omega_0} q_0\,\mathrm{Tr}(\nabla z)\,\mathrm{d}x + \tau\int_{\Omega_0} q_1^*\,\mathrm{Tr}(\nabla z)\,\mathrm{d}x$$
$$+\tau\int_{\Omega_0}[\mathrm{Tr}(\Psi_1')q_0\,\mathrm{Tr}(\nabla z) - q_0\,\mathrm{Tr}((\Psi_1')^*\nabla z]\,\mathrm{d}x + O(\tau^2), \tag{4.27}$$

and finally the third one is

$$\int_{\Omega_0}\mathrm{Tr}(M_\tau\nabla u_\tau)q\,\mathrm{Jac}(\Psi_\tau)\,\mathrm{d}x = \int_{\Omega_0}\mathrm{Tr}(\nabla u_0)q\,\mathrm{d}x + \tau\int_{\Omega_0}\mathrm{Tr}(\nabla u_1^*)q\,\mathrm{d}x$$
$$-\tau\int_{\Omega_0}\mathrm{Tr}((\Psi_1')^*\nabla u_0)q\,\mathrm{d}x + O(\tau^2). \tag{4.28}$$

Analogously, the integral on the right-hand side of (4.25) can be expanded into

$$\int_{\Omega_0} M_\tau\nabla\varphi_0 : M_\tau\nabla z\,\mathrm{Jac}(\Psi_\tau)\,\mathrm{d}x = \int_{\Omega_0}\nabla\varphi_0 : \nabla z\,\mathrm{d}x$$
$$+\tau\int_{\Omega_0}[\mathrm{Tr}(\Psi_1')I - \Psi_1' - (\Psi_1')^*]\nabla\varphi_0 : \nabla z\,\mathrm{d}x + O(\tau^2)$$
$$= \int_{\Omega_0}\nabla\varphi_0 : \nabla z\,\mathrm{d}x + O(\tau^2), \tag{4.29}$$

since we have $[\mathrm{Tr}(\Psi_1')I - \Psi_1' - (\Psi_1')^*]\nabla\varphi_0 \equiv 0$ in Ω_0.

At this point recall that (u_0, q_0) is the solution of a variational problem corresponding to

$$\int_{\Omega_0} \nabla u_0 : \nabla z \, \mathrm{d}x - \int_{\Omega_0} q_0 \, \mathrm{Tr}(\nabla z) \, \mathrm{d}x + \int_{\Omega_0} \mathrm{Tr}(\nabla u_0) q \, \mathrm{d}x = \int_{\Omega_0} \nabla \varphi_0 : \nabla z \, \mathrm{d}x,$$

that is, $-\Delta u_0 + \nabla q_0 = \Delta \varphi_0$ and $\mathrm{div}(u_0) = \mathrm{Tr}(\nabla u_0) = 0$. Therefore, after reporting the expressions in (4.26)–(4.29) into the variational equation (4.25), using the equation satisfied by (u_0, q_0), dividing by τ and letting $\tau \to 0$, one finds that $(u_1^*, q_1^*) \in H_0^1(\Omega_0)^N \times L^2(\Omega_0)$ is characterized by the fact that for all $(z, q) \in H_0^1(\Omega_0)^N \times L^2(\Omega_0)$

$$\int_{\Omega_0} \nabla u_1^* : \nabla z \, \mathrm{d}x - \int_{\Omega_0} q_1^* \, \mathrm{Tr}(\nabla z) \, \mathrm{d}x + \int_{\Omega_0} \mathrm{Tr}(\nabla u_1^*) q \, \mathrm{d}x = \langle F_0, z \rangle + \int_{\Omega_0} \mathrm{Tr}((\Psi_1')^* \nabla u_0) q \, \mathrm{d}x,$$

where $F_0 \in H^{-1}(\Omega_0)^N$ is given by

$$F_0 := -\mathrm{div}([\Psi_1' + (\Psi_1')^* - \mathrm{Tr}(\Psi_1') I] \nabla u_0) + \nabla(q_0 \, \mathrm{Tr}(\Psi_1')) - \mathrm{div}(q_0 \Psi_1'). \tag{4.30}$$

This implies that $(u_1^*, q_1^*) \in H_0^1(\Omega_0) \times L^2(\Omega_0)$ satisfies the equation $-\Delta u_1^* + \nabla q_1^* = F_0$ in Ω_0, and that $\mathrm{div}(u_1^*) = \mathrm{Tr}(\Psi_1' \nabla u_0)$ in Ω_0, and the proof is over. □

5. Smooth dependence of Cauchy forces with respect to the deformation of the domain: the evolution case

In this section, we shall consider the nonlinear Navier–Stokes equation and prove that the Cauchy forces depend smoothly on the deformation of the obstacle D_0. We begin by considering the linear evolution equations

$$\begin{cases} \dfrac{\partial v_j}{\partial t} - \Delta v_j + \nabla p_j = 0 & \text{in} \quad \Omega_j \times (0, T) \\ \mathrm{div}(v_j) = \mathrm{Tr}(\nabla v_j) = 0 & \text{in} \quad \Omega_j \times (0, T) \\ v_j(s, t) = \varphi(s, t) & \text{on} \quad \partial\Omega \times (0, T) \\ v_j(s, t) = 0 & \text{on} \quad \partial D_j \times (0, T) \\ v_j(x, 0) = 0 & \text{in} \quad \Omega_j \end{cases} \tag{5.1}$$

and we show that when D_0 and D_1 are close, then the Cauchy forces $\sigma(v_0, p_0)\mathbf{n}$ and $\sigma(v_1, p_1)\mathbf{n}$ are close, and that there is a smooth dependence in the sense explained in the previous section, from which we use the notation.

First, observe that we may find $\widetilde{\varphi}_1 \in C^1([0, T], (H^1(\Omega))^N)$ such that

$$\begin{cases} \widetilde{\varphi}_1(x, 0) = 0 & \text{in} \quad \Omega \\ \mathrm{div}(\widetilde{\varphi}_1) = 0 & \text{in} \quad \Omega \times (0, T) \\ \widetilde{\varphi}_1 = 0 & \text{in} \quad \mathcal{O} \times (0, T) \\ \widetilde{\varphi}_1 = \varphi & \text{on} \quad \partial\Omega \times (0, T). \end{cases} \tag{5.2}$$

Then we define $\widetilde{v}_1$ by setting

$$\widetilde{v}_1 := v_1 - \widetilde{\varphi}_1$$

so that $\widetilde{v}_1$ is a solution to

$$\begin{cases} \dfrac{\partial \widetilde{v}_1}{\partial t} - \Delta \widetilde{v}_1 + \nabla p_1 = \dfrac{\partial \widetilde{\varphi}_1}{\partial t} - \Delta \widetilde{\varphi}_1 & \text{in} \quad \Omega_1 \times (0, T) \\ \mathrm{div}(\widetilde{v}_1) = \mathrm{Tr}(\nabla \widetilde{v}_1) = 0 & \text{in} \quad \Omega_1 \times (0, T) \\ \widetilde{v}_1(s, t) = 0 & \text{on} \quad \partial\Omega_1 \times (0, T) \\ \widetilde{v}_1(y, 0) = 0 & \text{in} \quad \Omega_1. \end{cases} \tag{5.3}$$

If Ψ is an admissible deformation as in definition 4.1, the operators A and B as in (4.10) and (4.11), and proceeding as we did in the previous section, after writing the variational formulation of equation (5.3) (and some lengthy but straightforward elementary calculations) one checks that

$$u_0(x,t) := \widetilde{v}_1(\Psi(x),t), \qquad q_0(x,t) := p_1(\Psi(x),t) \tag{5.4}$$

satisfy the equation

$$\begin{cases} \dfrac{\partial u_0}{\partial t} + Au_0 + B^* q_0 = f & \text{in} \quad \Omega_0 \times (0,T) \\ Bu_0 = 0 & \text{in} \quad \Omega_0 \times (0,T) \\ u_0 = 0 & \text{on} \quad \partial\Omega_0 \times (0,T) \\ u_0(x,0) = 0 & \text{in} \quad \Omega_0, \end{cases} \tag{5.5}$$

where

$$f(x,t) := -\frac{1}{\mathrm{Jac}(\Psi)} \left[\frac{\partial \varphi_0}{\partial t} - \mathrm{div}(\mathrm{Jac}(\Psi) M^* M \nabla \varphi_0) \right], \tag{5.6}$$

and

$$\varphi_0(x,t) := \widetilde{\varphi}_1(\Psi(x),t).$$

We can now establish the analytic dependence of (u_0, q_0) on Ψ.

Theorem 5.1. *Let $\mathcal{O} \subset\subset \Omega$ be a Lipschitz domain, and $\Psi_1 \in W^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$ be such that $\Psi := I + \Psi_1$ is an admissible deformation according to definition 4.1. If $\varphi \in C^1([0,T], H^{1/2}(\partial\Omega)^N)$ and $\int_{\partial\Omega} \varphi(s,t) \cdot \mathbf{n}(s)\, \mathrm{d}s = 0$ for all $t \in [0,T]$, then by mapping there exists a neighbourhood $B(0,\rho)$ of the origin in $W_0^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$ such that the mapping $\Psi_1 \mapsto (u_0, q_0)$ is analytic from $B(0,\rho)$ into $C([0,T], H_0^1(\Omega_0)^N \times L^2(\Omega_0))$.*

Proof. Note that since where $\Psi \neq I$ we have $\varphi_0 \equiv 0$, the function f defined above is actually independent of Ψ (note also that $f \in C([0,T], H^{-1}(\Omega))^N)$.

Now, for a given $u_{\mathrm{init}} \in H_0^1(\Omega_0)^N$ with $Bu_{\mathrm{init}} = 0$, it is known that there exists a unique $u \in C((0,\infty); H_0^1(\Omega_0)^N)$ solution to the evolution equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au + B^* q = 0 \\ Bu = 0 \\ u(x,0) = u_{\mathrm{init}}(x), \end{cases} \tag{5.7}$$

and that this solution may be defined via the semi-group $S(t)$ generated by A on $N(B)$. More precisely, with the notation of corollary 4.4, it is a classical result of the theory of semi-groups (see, for instance, [16], chapter IX) that for any $t > 0$ we have

$$S(t) u_{\mathrm{init}} := u(t) = \lim_{n\to\infty} (R_{t/n})^n u_{\mathrm{init}},$$

where the convergence takes place actually in the topology of $C([0,T], L^2(\Omega_0)^N)$. As a matter of fact $S(t)$ is a holomorphic semi-group and one can represent $S(t)$ via the resolvent R_λ as a path integral in the following way (see, for instance, [16], chapter IX, section 10). For $\theta_0 \in (\pi/2, \pi)$ fixed, let γ be the path in the complex plane $\mathbb{C}$ defined as

$$\gamma := \{s\,\mathrm{e}^{-\mathrm{i}\theta_0}; s \geqslant 1\} \cup \{\mathrm{e}^{\mathrm{i}\theta}; -\theta_0 \leqslant \theta \leqslant \theta_0\} \cup \{s\,\mathrm{e}^{\mathrm{i}\theta_0}; s \geqslant 1\},$$

in which we assume that γ is oriented as a path coming from the direction $s\,\mathrm{e}^{\mathrm{i}\theta_0}$ with s ranging from $s = -\infty$ to $s = 1$. Then we have

$$S(t) u_{\mathrm{init}} = \frac{1}{2\mathrm{i}\pi} \int_\gamma \mathrm{e}^{\lambda t} R_{1/\lambda} u_{\mathrm{init}} \frac{\mathrm{d}\lambda}{\lambda} = \frac{1}{2\mathrm{i}\pi} \int_\gamma \mathrm{e}^{\lambda t} (\lambda I + A)^{-1} u_{\mathrm{init}}\, \mathrm{d}\lambda,$$

where the integral converges uniformly for $t \in [t_0, T]$ for any $0 < t_0 < T$, and actually one may define $S(t)$ for $t \in \mathbb{C}$ with $|\arg(t)| < (2\theta_0 - \pi)/2$. According to corollary 4.4, in a fixed neighbourhood $B(0, \rho)$ of the origin in $W^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$, for any $f \in H^{-1}(\Omega_0)^N$ the mapping $\Psi_1 \mapsto R_\lambda f$ is analytic from $B(0, \rho)$ into $H_0^1(\Omega_0)^N$. In particular, if u_{init} has its support in $\mathcal{O}_\delta$, then $Bu_{\text{init}} = 0$ means that $\operatorname{div}(u_{\text{init}}) = 0$, and thus for any $\Psi_1 \in B(0, \rho)$ given, $S(t)u_{\text{init}}$ is well defined and one can see from the above integral representation of $S(t)$ that $\Psi_1 \mapsto S(t)u_{\text{init}}$ is also analytic from $B(0, \rho)$ into $L^2(\Omega_0)^N$. However, since the semi-group $S(t)$ is analytic, for any $t > 0$ and $u_{\text{init}} \in L^2(\Omega_0)^N$ with $\operatorname{div}(u_{\text{init}}) = 0$ we have actually $S(t)u_{\text{init}} \in H_0^1(\Omega_0)^N$ (with an estimate of the form $\|S(t)u_{\text{init}}\|_{H_0^1} \leqslant ct^{-1/2}\|u_{\text{init}}\|_{L^2}$), we can infer that $\Psi_1 \mapsto S(t)u_{\text{init}}$ is analytic from $B(0, \rho)$ into $C([t_0, T], H_0^1(\Omega_0)^N)$. In fact when $u_{\text{init}} \in H_0^1(\Omega_0)^N \cap D(A)$, one has (with $u(t) := S(t)u_{\text{init}}$)

$$u \in C([0, T], H_0^1(\Omega_0)^N) \cap C^1([0, T], L^2(\Omega_0)^N)$$

and one can conclude that the mapping $\Psi_1 \mapsto S(t)u_{\text{init}}$ from

$$B(0, \rho) \longrightarrow C([0, T], H_0^1(\Omega_0)^N) \cap C^1([0, T], L^2(\Omega_0)^N)$$

is analytic.

Now f being as in (5.6), if we denote by f_0 the projection of f in the space of divergence-free functions, then for any $\Psi_1 \in B(0, \rho)$, we also have $f_0 \in N(B)$, and the solution u_0 of (5.5) can be written as

$$u_0(t) = \int_0^t S(t - \tau) f_0(\tau)\, \mathrm{d}\tau. \tag{5.8}$$

This shows that $\Psi_1 \mapsto u_0$ is analytic from $B(0, \rho)$ into $C([0, T], H_0^1(\Omega_0)^N) \cap C^1([0, T], L^2(\Omega_0)^N)$. Using the equation satisfied by (u_0, q_0) one sees that

$$u_0 + A^{-1}B^* q_0 = A^{-1} f - A^{-1}\frac{\partial u_0}{\partial t}$$

and hence (because $Bu_0 = 0$)

$$BA^{-1}B^* q_0 = BA^{-1} f - BA^{-1}\frac{\partial u_0}{\partial t}.$$

From this it is straightforward to see that q_0 is determined up to a constant (in the spatial variable) which may be chosen so that $\int_{\Omega_0} q_0(x, t)\, \mathrm{d}x = 0$, and that $\Psi_1 \mapsto q_0$, as a mapping from $B(0, \rho)$ into $C([0, T], L^2(\Omega_0))$, is also analytic. □

Using the same procedures, the nonlinear Navier–Stokes equation written in Ω_1 can be transformed into a nonlinear equation written in Ω_0. Namely, if (v_1, p_1) is the solution of equation (1.8) with $\varepsilon_* = 1$ and $j = 1$, one defines first $(\widetilde{v}_1, p_1)$ with $\widetilde{v}_1 := v_1 - \widetilde{\varphi}_1$ and then one applies the change of variables Ψ by setting

$$u_0(x, t) := \widetilde{v}_1(\Psi(x), t), \qquad q_0(x, t) := p_1(\Psi(x), t).$$

After some cumbersome calculations, which we may omit, one checks that (u_0, q_0) satisfies the following equation,

$$\begin{cases} \dfrac{\partial u_0}{\partial t} + Au_0 + B^* q_0 + F(u_0) = f & \text{in} \quad \Omega_0 \times (0, T) \\ Bu_0 = 0 & \text{in} \quad \Omega_0 \times (0, T) \\ u_0 = 0 & \text{on} \quad \partial\Omega_0 \times (0, T) \\ u_0(x, 0) = 0 & \text{in} \quad \Omega_0, \end{cases} \tag{5.9}$$

where the nonlinearity F is defined by (for $1 \leqslant i \leqslant N$ and $u \in H_0^1(\Omega_0)^N$)

$$(F(u))_i := \sum_{j=1}^{N} u_j M_{jk} \partial_k u_i + \sum_{j=1}^{N} \varphi_{0,j} M_{jk} \partial_k u_i + \sum_{j=1}^{N} u_j M_{jk} \partial_k \varphi_{0,i}$$

and the right-hand side

$$(f(x,t))_i := -\frac{1}{\mathrm{Jac}(\Psi)} \left[\frac{\partial \varphi_0}{\partial t} - \mathrm{div}(\mathrm{Jac}(\Psi) M^* M \nabla \varphi_0) \right]_i - \sum_{j=1}^{N} \varphi_{0,j} M_{jk} \partial_k \varphi_{0,i}, \tag{5.10}$$

and

$$\varphi_0(x,t) := \widetilde{\varphi}_1(\Psi(x), t).$$

At this point, we know that the semi-group $S(t)$ depends in a smooth manner on $\Psi_1 \in B(0,\rho)$, and that the solution of equation (5.9) can be obtained as a fixed point for the mild version of that equation, that is,

$$u_0(t) = \int_0^t S(t-\tau) f_0(\tau)\,\mathrm{d}\tau - \int_0^t S(t-\tau) F_0(u_0(\tau))\,\mathrm{d}\tau,$$

where by f_0 and $F_0(u)$ we denote the projection of f and $F(u)$ on $N(B)$. It is known that, for some $T_* > 0$ small enough, the above equation admits a unique solution in $C\big([0,T_*], H_0^1(\Omega_0)^N\big)$. As a matter of fact, this solution is obtained as a fixed point of the mapping

$$\widetilde{\Phi}(u)(t) := \int_0^t S(t-\tau) f_0(\tau)\,\mathrm{d}\tau - \int_0^t S(t-\tau) F_0(u(\tau))\,\mathrm{d}\tau$$

in the space $C\big([0,T_*], H_0^1(\Omega_0)^N\big)$, and the fact that $\Psi_1 \mapsto \widetilde{\Phi}$ is analytic implies that the mapping $\Psi_1 \mapsto u_0$ is also analytic as a mapping from $B(0,\rho)$ into $C\big([0,T_*], H_0^1(\Omega_0)^N\big)$. In turn, this implies that $\Psi_1 \mapsto q_0$ is analytic, and finally we can state these observations in the following:

Corollary 5.2. *Let $\mathcal{O} \subset\subset \Omega$ be a Lipschitz domain, and $\Psi_1 \in W^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$ be such that $\Psi := I + \Psi_1$ is an admissible deformation according to definition 4.1. Using the above notation, if $\varphi \in C^1([0,T], H^{1/2}(\partial\Omega)^N)$ and $\int_{\partial\Omega} \varphi(s,t)\cdot \mathbf{n}(s)\,\mathrm{d}s = 0$ for all $t \in [0,T]$, then there exist $T_* > 0$ and $\rho > 0$ such that for all $\Psi_1 \in B(0,\rho)$ equation (5.9) has a unique solution $(u_0, q_0) \in C([0,T_*], H_0^1(\Omega_0)^N \times L^2(\Omega_0))$ (with $\int_{\Omega_0} q_0(x,t)\,\mathrm{d}x = 0$). Moreover, the mapping $\Psi_1 \mapsto (u_0, q_0)$ from $B(0,\rho) \subset W_0^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$ into $C\big([0,T_*], H_0^1(\Omega_0)^N \times L^2(\Omega_0)\big)$ is analytic.*

As we observed in corollary 4.5, the analyticity result together with the identification result established above, allows us to state the following stability result concerning the linear or nonlinear evolution problem. Let (v_τ, p_τ) be the solution of

$$\begin{cases} \dfrac{\partial v_\tau}{\partial t} - \mathrm{div}(\sigma(v_\tau, p_\tau)) + \varepsilon_* \,\mathrm{div}(v_\tau \otimes v_\tau) = 0 & \text{in} \quad (\Omega \backslash \overline{D}_\tau) \times (0, T_*) \\ \mathrm{div}(v_\tau) = 0 & \text{in} \quad (\Omega \backslash \overline{D}_\tau) \times (0, T) \\ v_\tau(x, 0) = 0 & \text{for} \quad x \in \Omega \backslash \overline{D}_\tau \\ v_\tau(s,t) = \varphi(s,t) & \text{for} \quad (s,t) \in \partial\Omega \times (0, T_*) \\ v_\tau(s,t) = 0 & \text{for} \quad (s,t) \in \partial D_\tau \times (0, T_*), \end{cases} \tag{5.11}$$

where for $\tau = 0$ the domain D_0 is supposed to be the *unperturbed* obstacle, and D_τ denotes the perturbation of D_0 according to the convention we have used in the above studies, and T_* is the minimum time of existence for the solutions (v_τ, p_τ).

Corollary 5.3. *Let $\Psi_1 \in W_0^{1,\infty}(\mathcal{O}, \mathbb{R}^N)$ be such that $\Psi_1 \not\equiv 0$ on D_0. Denote $\Psi_\tau := I + \tau\Psi_1$ for $\tau \in \mathbb{R}$, and $D_\tau := \Psi_\tau(D_0)$ and $\Omega_\tau := \Psi_\tau(\Omega_0)$, and let (v_τ, p_τ) be the solution of equation (5.11), for some $\varphi \in C^1([0, T], H^{1/2}(\partial\Omega))$ satisfying $\int_{\partial\Omega} \varphi(s, t) \cdot \mathbf{n}(s)\, \mathrm{d}s = 0$ for $t \in [0, T]$. Let $\tau_1 > 0$ be a positive number such that $\tau \mapsto (v_\tau, p_\tau)$ is analytic on $(-\tau_1, \tau_1)$ and assume that for some $\tau_* \in (-\tau_1, \tau_1)$ one has $\Psi_{\tau_*}(D_0) \neq D_0$. Then there exist $\tau_0 > 0$, an integer $m \geqslant 1$ and a positive constant C, all depending on $\Psi_1, \varphi, \Gamma, \Omega_0$, such that for all $\tau \in [-\tau_0, \tau_0]$ one has*

$$\left\| \Lambda_{D_0}(\varphi) - \Lambda_{D_\tau}(\varphi) \right\|_{C([0,T_*],H^{-1/2}(\Gamma)^N)} \geqslant C|\tau|^m. \tag{5.12}$$

Acknowledgments

The authors gratefully acknowledge the Chilean and French Governments through the Scientific Committee ECOS-CONICYT. They are also indebted to the anonymous referees for their careful reading and criticism of an earlier version of the paper, their comments allowed us not only to improve the presentation of our results, but also to strengthen and generalize some of them. This paper is partially supported by Grant CONICYT-FONDECYT 1030943 and Grant Dir. Investigación, U. del Bío-Bío 033607 1/R.

References

[1] Alessandrini G, Morassi A and Rosset E 2002 Detecting cavities by electrostatic boundary measurements, *Inverse Problems* **18** 1333–53
[2] Alessandrini G, Morassi A and Rosset E 2002 Detecting an inclusion in an elastic body by boundary measurements *SIAM J. Math. Anal.* **33** 1247–68
[3] Beretta E and Vesella S 1998 Stable determination of boundaries from Cauchy data *SIAM J. Math. Anal.* **30** 220–32
[4] Calderón A P 1980 On an inverse boundary value problem *Seminars on Numerical Analysis and its Application to Continuum Physics, Sociedade Brasileira de Matemática (Rio de Janeiro)* pp 65–73
[5] Canuto B 2002 Unique localization of unknown boundaries in a conducting medium from boundary measurements *ESAIM Control Optim. Calc. Var.* **7** 1–22 (electronic)
[6] Canuto B, Rosset E and Vessella S 2002 A stability result in the localization of cavities in a thermic conducting medium *ESAIM Control Optim. Calc. Var.* **7** 521–65 (electronic)
[7] Canuto B and Kavian O 2001 Determining coefficients in a class of heat equations via boundary measurements *SIAM J. Math. Anal.* **32** 963–86
[8] Fabre C and Lebeau G 1996 Prolongement unique des solutions de l'équation de Stokes *Commun. Part. Diff. Eqns* **21** 573–96
[9] Fernández-Cara E 1996 Some controllability problems for the Stokes and Navier–Stokes systems *Control of Partial Differential Equations and Applications (Laredo, 1994) (Lecture Notes in Pure and Applied Mathematics* vol 174*)* (New York: Dekker) pp 77–88
[10] Kohn R and Vogelius M 1984 Determining conductivity by boundary measurements *Commun. Pure Appl. Math.* **37** 289–98
[11] Raviart P A and Thomas J M 1983 *Introduction à l'Analyse Numérique des Equations aux Dérivées Partielles* (Paris: Masson)
[12] Sylvester J and Uhlmann G 1986 A uniqueness theorem for an inverse boundary problem in electrical prospection *Commun. Pure Appl. Math.* **39** 91–112
[13] Tartar L 1975 Nonlinear partial differential equations using compactness method *Report* 1584, Mathematics Research Center, University of Wisconsin, Madison
[14] Tartar L 2005 *Introduction to Navier–Stokes Equations and Oceanography* (Berlin: Springer) at press
[15] Uhlmann G 1998 Inverse boundary value problems for partial differential equations *Doc. Math. J.* Extra volume ICM 1–10
[16] Yosida K 1974 *Functional Analysis (Die Grundlehren der Mathematischen Wissenschaften* vol 123*)* (New York: Springer)