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THE STATIONARY INSTABILITY IN QUASI-REVERSIBLE SYSTEMS AND THE LORENZ PENDULUM

M. CLERC, P. COULLET* and E. TIRAPEGUI†
Institut Non Linéaire de Nice, UMR 6618 CNRS-UNSA,
1361 Route des lucioles, F-065600 Valbonne, France
†*Facultad de Ciencias Físicas y Mat., Depto. Física,*
Univ. de Chile, casilla 487-3, Santiago, Chile

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We study the resonance at zero frequency in presence of a neutral mode in quasi-reversible systems. The asymptotic normal form is derived and it is shown that in the presence of a reflection symmetry it is equivalent to the set of real Lorenz equations. Near the critical point an analytical condition for the persistence of an homoclinic curve is calculated and chaotic behavior is then predicted and its existence verified by direct numerical simulation. A simple mechanical pendulum is shown to be an example of the instability, and preliminary experimental results agree with the theoretical predictions.

1. Introduction

The stationary and Hopf bifurcation are the only two local bifurcations which occur generically in one parameter families of finite dimensional dissipative dynamical systems [Guckenheimer & Holmes, 1983]. In reversible systems, i.e. systems which are invariant under a time reversal transformation (see Appendix), linearization at a reversible equilibrium stable gives a matrix with purely imaginary eigenvalues whose number is equal to the dimension of the system. In this kind of systems the instabilities in one parameter families are: (a) The stationary instability denoted by (0^2) in Arnold's notation [Arnold, 1980], which we use from now on, corresponding to a resonance at zero frequency; and (b) the confusion of frequencies $(i\Omega^2)$ [Rocard, 1943] which corresponds to a resonance at a finite frequency.

We shall be interested here in quasi-reversible systems which are systems in which the terms that break the time reversal symmetry, i.e. irreversible effects, are small and can be considered as perturbative terms near instabilities. We find then in quasi-reversible systems the same instabilities which occur in reversible systems and irreversible terms will appear as unfolding terms in their normal forms and in some situations they will be responsible for asymptotic chaos as it has been remarked by Gibbon [Gibbon & McGuinness, 1982]. In this paper we focus on the stationary instability in the presence of a neutral mode which we denote $(0^2)(0)$, i.e. we have in the reversible system an eigenvalue zero of multiplicity three with linear part

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

while the other eigenvalues are in the imaginary axis and correspond to nonzero frequencies. If the system has an odd number of variables one necessarily has the zero frequency mode. We shall discuss in the Appendix the appearance and coupling of

*Professeur a l'Institut Universitaire de France.

this zero frequency mode when the reversible system is a Hamiltonian one and we will see that the mode will automatically be there if the Hamiltonian system has a conserved quantity. In Sec. 2 we shall derive the normal form of this instability without additional symmetries and when symmetries are present. In the first case genericity arguments show that we can have the scenario of Shilnikov chaos in the irreversible unfolding of the asymptotic normal form and in the second case reflection symmetry of the variables associated with the Jordan block lead to an asymptotic normal form equivalent to the well-known real Lorenz equations which will present the classic scenario in their region of validity. In Sec. 3 we study in detail this last situation, which we call Lorenz Bifurcation [Clerc *et al.*, 1999], and we show that we can predict chaos analytically since we can explicitly calculate the condition for persistence of an unstable homoclinic curve through a Melnikov condition. The calculation can be done because we are in a quasi-reversible situation and we know analytically the homoclinic solution of the reversible equations. In Sec. 4 we shall give a simple mechanical example of the Lorenz bifurcation which consists of a pendulum oscillating in a fixed vertical plane with respect to a rotating support submitted to a constant torque: the Lorenz pendulum. Finally in Sec. 5 we present an asymmetric physical pendulum oscillating in a fixed vertical plane with respect to a rotating support.

2. Normal Form of the $(0^2)(0)$ Instability

Let x and y be the variables corresponding to the Jordan block ($y = dx/dt$) and z the variable representing the neutral mode. It is simple to show that the other critical variables associated with the pure imaginary eigenvalues with finite frequencies can be eliminated when the dissipative irreversible unfolding terms are considered. The relevant variables will then be $\{x, y, z\}$ and from the global characterization of normal forms in [Elphick *et al.*, 1987] we have that the normal form can be written as

$$\frac{d^2x}{dt^2} = F(x, z) + \frac{dx}{dt}G(x, z)$$
$$\frac{dz}{dt} = K(x, z)\,.$$

When these equations are invariant under the time reversal transformation $t \to -t$, $x \to x$, $z \to z$, the function $K(x, z) \equiv 0$, z is constant, and the equations are integrable. This property is lost when one adds the terms breaking the time reversibility as it has been remarked by Gibbon [Gibbon & McGuinness, 1982]. The asymptotic reversible normal form which is obtained from the previous equations is

$$\frac{d^2x}{dt^2} = \varepsilon - x^2 + azx \pm z^2,$$
$$\frac{dz}{dt} = 0$$

where ε is the small parameter measuring the distance to the threshold of the instability, a is of order one, x and z will be of the order $\varepsilon^{1/2}$ and d/dt of order $\varepsilon^{1/4}$. If we add now the irreversible unfolding terms we obtain

$$\frac{d^2x}{dt^2} = \varepsilon - x^2 + azx \pm z^2 - \nu\frac{dx}{dt}\,,$$
$$\frac{dz}{dt} = -\mu z + \gamma x + \rho \tag{1}$$

where (ν, μ, γ) are of order $\varepsilon^{1/4}$ and ρ is order $\varepsilon^{3/4}$. These equations have a saddle node bifurcation and the stable branch can then lose stability through a Hopf bifurcation. When the limit cycle, created in the Hopf bifurcation, intersects the unstable initial fixed point an homoclinic orbit appears and we can have Shilnikov chaos. This scenario is easily observed numerically through simulation of the previous equations (see Fig. 1).

We consider now the case of reflection symmetry in the $(x, dx/dt)$ plane. If one has the invariance $x \to -x$, $dx/dt \to -dx/dt$, the reversible

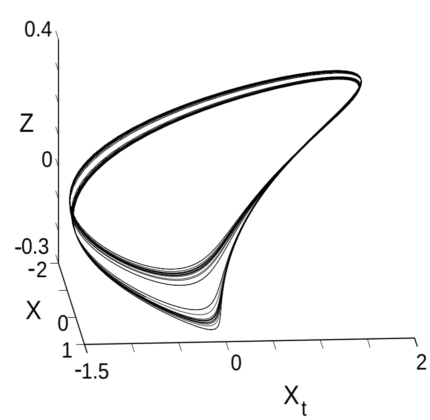


Fig. 1. Numerical simulation of Eq. (1) where $\varepsilon = 1$, $a = 1$, $\nu = 0.1$, $\rho = 0$, $\mu = 0.5$, $\delta = 0.205$.

asymptotic normal form will be

$$\frac{d^2x}{dt^2} = \epsilon x - x^3 - zx\,,$$
$$\frac{dz}{dt} = 0\,,$$

where ϵ is the small parameter measuring the distance to threshold. Now z is of order ϵ, x and d/dt are of order $\epsilon^{1/2}$. The sign of the cubic term has been chosen to have a supercritical instability and we remark that although one has $dz/dt = 0$ the role of the neutral mode is not trivial since it renormalizes the threshold parameter ε in the first equation and in fact we shall see that it will be responsible for the instability when it couples with the other modes through the small irreversible terms. Adding these terms the asymptotic normal form becomes

$$\frac{d^2x}{dt^2} = \epsilon x - x^3 - zx - \nu\frac{dx}{dt}\,,$$
$$\frac{dz}{dt} = -\mu z + \eta x^2, \tag{2}$$

where we choose $\{\nu, \mu\}$ positive (they represent then dissipative effects), η positive or negative, and the three parameters are of order $\epsilon^{1/2}$. When η is negative the system has a Lyapunov function [Abarbanel *et al.*, 1993]. These equations are equivalent to the Lorenz model through the change of variables

$$x = \frac{x'}{\tau_o}$$
$$y = (y' - x')\frac{\eta+\mu}{\tau_o}$$
$$z = z'\frac{\eta+\mu}{\tau_o} - \frac{x'^2}{\tau_o^2}$$

where $\tau_o = |(\eta+\mu)/(\nu - (\eta+\mu))|$, $\sigma = \eta + \mu$, $r = (\epsilon - (\eta+\mu)^2 + \nu(\eta+\mu))$, $b = \mu/\tau_o$, which put Eqs. (2) in the standard Lorenz form

$$\begin{aligned}\partial_t x' &= \sigma(y' - x')\\ \partial_t y' &= Rx' \mp y' - x'z'\\ \partial_t z' &= -bz' + x'y'\end{aligned} \tag{3}$$

where the sign "$\mp$" is determined by the sign of the expression $-\nu + (\eta+\mu)$. Equations (2) will be discussed in detail in the next section where we show that they present Lorenz type chaos. These equations can be obtained in a very simple way due to the symmetries of the problem: consider a dynamical system of three variables (x, y, z) where the origin loses stability through the $(0^2)(0)$ instability, which is invariant under the time reversal transformation $t \to -t$, $x \to x$, $y \to -y$, $z \to z$ and which has reflection symmetry in the (x, y) plane. Then the system has necessarily the form

$$\begin{aligned}\dot{x} &= y + yf(x^2, y^2, z)\,,\\ \dot{y} &= xg(x^2, y^2, z)\,,\\ \dot{z} &= xyh(x^2, y^2, z)\,,\end{aligned} \tag{4}$$

where f, g, and h are nonlinear functions. Since we are interested in the quasi-reversible situation we add small terms that break the time reversal symmetry and we expand in Taylor series

$$\begin{aligned}\dot{x} &= y + f_{2,1,0}yx^2 + f_{0,3,0}y^3 + f_{0,1,1}yz + f_{0,1,2}yz^2\\ &\quad + \mu f_{1,0,0}x + \mu f_{1,0,1}xz + \cdots\,,\\ \dot{y} &= \varepsilon x + g_{3,0,0}x^3 + g_{1,2,0}xy^2 + g_{1,0,1}xz\\ &\quad + g_{1,0,2}xz^2 + \mu g_{0,1,0}y + \mu g_{1,0,0}yz + \cdots\,,\\ \dot{z} &= h_{1,1,0}xy + h_{1,1,1}xyz + \mu h_{0,0,1}z + \mu h_{0,0,1}x^2\\ &\quad + \mu h_{0,2,0}y^2 + \cdots\,.\end{aligned}$$

Here $g_{1,0,0}$ is the small control parameter term measuring the distance to criticality and all the other constants $f_{a,b,c}$, $g_{d,e,i}$ and $h_{j,k,l}$ are of order one. The parameter μ is small since it appears in all terms breaking time reversal invariance (quasi-reversible system). If we call ϵ the small parameter $g_{1,0,0}$ then we can easily check in the previous equations that we must have $\partial_t \sim O(\sqrt{\epsilon})$, $x \sim O(\sqrt{\epsilon})$, $y \sim O(\epsilon)$, $z \sim O(\epsilon)$, $\mu \sim O(\sqrt{\epsilon})$. We can then keep in the equations the dominant terms in ϵ and obtain the asymptotic form [Arneodo *et al.*, 1985]

$$\begin{aligned}\dot{x} &= y + \mu f_{1,0,0}x\,,\\ \dot{y} &= g_{1,0,0}x + g_{3,0,0}x^3 + g_{1,0,1}xz + \mu g_{0,1,0}y\,,\\ \dot{z} &= h_{1,1,0}xy + \mu h_{0,0,1}z + \mu h_{0,0,1}x^2\,,\end{aligned} \tag{5}$$

where the first equation is $O(\epsilon)$, the two other ones of order $O(\epsilon\sqrt{\epsilon})$. The term $\mu f_{1,0,0}$ can be eliminated by the change of variables $y' = y + \mu f_{1,0,0}x$ which yields a renormalization of all the coefficients of the previous equations. In the same way the term $h_{1,1,0}$ can be eliminated putting $z' = z - h_{1,1,0}(x^2/2)$, and if we make a final rescaling we obtain the normal form as given before in Eqs. (2). The analysis we have done suggests that if in a quasi-reversible system a limit cycle becomes unstable when two Floquet multipliers $\exp(\pm i\omega)$ collide at the point (-1), a codimension one situation, the normal form should be the Lorenz equations if the phase variable associated with motion on the limit cycle decouples. In this case, the reflection invariance is imposed automatically by the time dependent transformation to the new variables which changes sign after one period. These considerations explain the appearance of the Lorenz attractor in the Poincaré section of the problem studied in [Moon, 1997]. We shall consider this problem elsewhere.

A prototype of reversible physical systems are the Hamiltonian systems where the number of variables is always even. We shall show in the Appendix that the presence of a cyclic variable in the Lagrangian of the system ensures the existence of a neutral mode which is a conserved quantity and reduces the number of first-order differential equations from $2n$ to $(2n-1)$.

3. Appearance of Chaos in the Asymptotic Normal Form of the $(0)^2 0$ Instability with Reflection Symmetry

Making appropriate scalings the asymptotic normal form (2) can be written for $\epsilon > 0$ as

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x - x^3 - xz - \frac{\nu}{\sqrt{\epsilon}} y \\ \frac{dz}{dt} &= -\frac{\mu}{\sqrt{\epsilon}} z + \frac{\eta}{\sqrt{\epsilon}} x^2 \end{aligned} \tag{6}$$

The reversible system, which corresponds to $\nu = \mu = \eta = 0$, has the homoclinic solution

$$\begin{aligned} x_0 &= \sec h\left(\frac{(t-t_0)}{\sqrt{2}}\right) \\ z_0 &= 0 \end{aligned} \tag{7}$$

which is an unstable two-dimensional curve in the (x, y) plane. We shall obtain now the condition of persistence of this homoclinic curve when we consider the irreversible terms, i.e. when ν, μ and η do not vanish. We first write the equation for z in integral form imposing the appropriate boundary conditions for the homoclinic $(x(t)$, $y(t)$ and $z(t)$ must tend to zero for $t \to \pm\infty$)

$$z(t) = \frac{\eta}{\sqrt{\varepsilon}} \int_{-\infty}^{t} e^{\mu^*(s-t)} x^2(s) ds \tag{8}$$

where $\mu^* = \mu/\sqrt{\epsilon}$. Replacing this expression in the equation for x we obtain an exact integro-differential equation

$$\ddot{x} = x - x^3 - \frac{\nu}{\sqrt{\varepsilon}}\dot{x} + \frac{\eta}{\sqrt{\varepsilon}} x \int_{-\infty}^{t} e^{\mu^*(s-t)} x^2(s) ds$$

Let us consider $\nu/\sqrt{\varepsilon}$ and $\eta/\sqrt{\varepsilon}$ as small quantities. We put $x = x_0 + w$ in the latter equation, with w of order $O(\nu/\sqrt{\varepsilon},\, \eta/\sqrt{\varepsilon})$. Then, keeping only linear terms in w, we obtain the linear equation

$$\begin{aligned} \mathcal{L}w &\equiv \left(\frac{d^2}{dt^2} + 3x_o^2 - 1\right) w = I\,, \\ I &\equiv -\frac{\nu}{\sqrt{\varepsilon}}\dot{x} + \frac{\eta}{\sqrt{\varepsilon}} x(t) \int_{-\infty}^{t} e^{\mu(s-t)} x^2(s) ds\,. \end{aligned} \tag{9}$$

The solvability condition for w in the previous equation gives a Melnikov type condition for the persistence of the homoclinic. The condition is $\langle I,\, dx_0/dt\rangle = 0$, where $\langle\cdot,\cdot\rangle$ is the usual scalar product. Explicitly one obtains the following relation

$$\nu = 2\eta \left(1 - \frac{\mu}{\sqrt{\varepsilon}} \frac{\displaystyle\int_{-\infty}^{\infty} dt x_o^2(t) \int_{-\infty}^{t} ds e^{\mu(s-t)} x_o^2(s)}{\displaystyle\int_{-\infty}^{\infty} dt (\partial_t x_o)^2}\right). \tag{10}$$

We remark that this result is valid for arbitrary μ since the homoclinic (7) is a solution of (6) for $\nu = \eta = 0$ and Eqs. (8) and (10) make no assumptions on μ.

Numerical calculation of the integral and its interpolation are shown in Fig. 2. The relation can be approximated by

$$\begin{aligned} \nu = 2\eta \Bigg\{ 1 - \frac{3}{2}\frac{\mu}{\sqrt{\varepsilon}} \Bigg[& 1.9996 - 1.92052 \frac{\mu}{\sqrt{\varepsilon}} \\ & + 1.2441 \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^2 - 0.387 \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3 \Bigg] \Bigg\}. \end{aligned} \tag{11}$$

It is obvious that the $(0^2)(0)$ instability has an homoclinic curve since it contains the (0^2) instability which has one. From the previous formula we see that when $\mu/\sqrt{\epsilon}$ is small ($\mu \ll \sqrt{\epsilon}$) the persistence condition reduces to

$$\nu = 2\eta \tag{12}$$

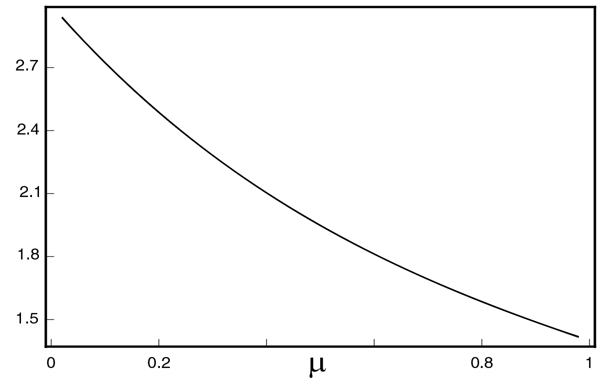


Fig. 2. Numerical calculation of integral of the right side of Eq. (10) and its interpolations.

The plane homoclinic (7) of the reversible system becomes slightly three-dimensional for the quasi-reversible system. Its explicit perturbed form at lowest order is

$$x_o = \sec\ h\left(\frac{\sqrt{t-t_o}}{\sqrt{2}}\right)\,,$$

$$z_o = \frac{\eta}{\sqrt{\epsilon}}\sqrt{2}e^{(1-\mu^*\sqrt{2})(t-t_o)}\ \sec\ h(t-t_o)\,. \qquad (13)$$

The linearization of Eq. (6) around the origin 0 ($x = 0$, $y = 0$, $z = 0$) tells us how the homoclinic behaves near the origin. We have one positive eigenvalue λ_1 and two negative eigenvalues (λ_2, λ_3). They are

$$\lambda_1 = \frac{1}{\sqrt{\varepsilon}}\left[\sqrt{\varepsilon + \frac{\nu^2}{4}} - \frac{\nu}{2}\right]$$

$$\lambda_2 = -\frac{1}{\sqrt{\varepsilon}}\left[\sqrt{\varepsilon + \frac{\nu^2}{4}} + \frac{\nu}{2}\right]$$

$$\lambda_3 = -\frac{\mu}{\sqrt{\varepsilon}} \qquad (14)$$

The eigenvectors corresponding to (λ_1, λ_2) determine two lines L_1 and L_2 in the (x, y) plane and the eigenvector associated to λ_3 has the direction of the z-axis. The linear stable manifold of the origin, i.e. the tangent plane at 0 to the stable manifold of the origin, is the plane determined by (L_2, z), and the linear unstable manifold is the line L_1. We have two situations:

(a) If $0 < \mu < \sqrt{\varepsilon + (\nu^2/4)} - \nu/2$ the homoclinic approaches the origin asymptotically by the z-axis because $|\lambda_3| < |\lambda_2|$ and it is unstable since $|\lambda_3| < |\lambda_1|$.

(b) If $0 < \sqrt{\varepsilon + (\nu^2/4)} - \nu/2 < \mu$ the homoclinic is always stable and we can distinguish two cases: if $|\lambda_3| < |\lambda_2|$ the stable homoclinic approaches the origin by the z-axis and if $|\lambda_3| > |\lambda_2|$ it approaches the origin by the line L_2.

We shall analyze case (a) which is easy to realize taking μ small enough and adjusting η to have $\nu = 2\eta$. In a sufficiently small neighborhood of the origin the unstable homoclinic will then be asymptotically in the (L_1, z) plane and a trajectory near to the homoclinic is shown schematically in Fig. 3 where we have drawn a small square OCFA in the plane (L_1, z) in order to construct a useful map to study the behavior of the system. If the trajectory goes inside the square crossing the line l_1 at point B its evolution will be essentially determined by the linear part of Eq. (12) till E where it leaves the square. We make the natural assumption that point E is reinjected isometrically in the square at point B', i.e. $|CE| = |AB'| = u'$. The part of the trajectory joining E to B' goes out of the plane (L_1, z) and since the homoclinic is unstable we have $u' = |AB'| > |AB| = |CD| = u$. With our assumptions the map $u \to u'$ is of the form

$$u' = \alpha u^\sigma\,, \quad \sigma = \frac{\mu}{\lambda_1} < 1\,, \qquad (15)$$

where α is a constant depending on the details of the system.

We recall here that due to the original reflection invariance one has, together with the curve of Fig. 3, the symmetric situation through reflection which corresponds to the equation $u' = -\alpha|u|^\sigma$. The latter equation is the map for the values of the parameters for which the unstable homoclinic

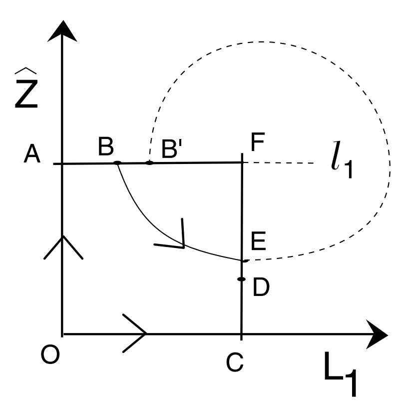


Fig. 3. Schematic representation of the dynamical evolution near the homoclinic solution.

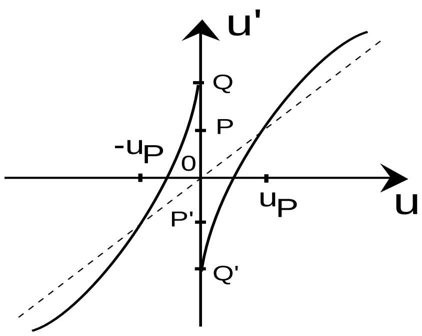
exists, i.e. the persistence condition (10) is satisfied. If we perturb slightly the system the map $u' \to u$ becomes

$$u' = \begin{cases} \alpha u^\sigma + \rho\,, & u > 0\,, \\ -\alpha |u|^\sigma - \rho\,, & u < 0\,, \end{cases} \tag{16}$$

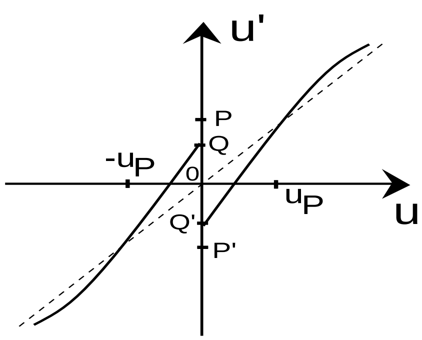
where ρ can be positive or negative. When ρ is negative complex behavior appears and we have drawn in Fig. 4 the two qualitatively different forms of the map which correspond to $|OQ| > |OP|$ and $|OQ| < |OP|$. The lower branch of the map obtained by reflection in the figures is due to the original reflection symmetry $(x \to -x, y \to -y)$ of Eqs. (6).

The asymptotic normal form (2) shows that for $\epsilon < 0$ the origin O with coordinates ($x = 0$, $y \equiv dx/dt = 0$, $z = 0$) is a stable fixed point which loses stability through a pitchfork bifurcation at $\epsilon = 0$. For $\epsilon > 0$ we have then the stable solutions

$$(x,\, y,\, z) = \left(x_\pm = \pm\sqrt{\frac{\epsilon\mu}{\mu + \eta}}\,,\ y = 0,\, z = \frac{\varepsilon\eta}{\mu + \eta} \right) \tag{17}$$



(a)



(b)

Fig. 4. Schematic representation of the mapping (16).

Linear stability analysis around these new solutions leads to the characteristic polynomial

$$\lambda^3 + \lambda^2(\mu + \nu) + \lambda\left(\mu\nu + 2\frac{\epsilon}{\mu + \nu}\right) + 2\mu\epsilon = 0 \tag{18}$$

whose roots will be the eigenvalues associated with the linear equation. Studying these roots we can check that the solutions (17) will lose stability to unstable limit cycles through a Hopf bifurcation when the following condition is satisfied

$$(\nu + \mu)\left(\nu + 2\frac{\epsilon}{\eta + \mu}\right) = 2\varepsilon \tag{19}$$

If $(\eta/\sqrt{\epsilon},\ \nu/\sqrt{\epsilon},\ \mu/\sqrt{\epsilon}) \ll 1$ the latter condition reduces to

$$\nu = \eta \tag{20}$$

and recalling condition $\nu = 2\eta$ for the persistence of the homoclinic (formula (12)) we shall see using the map (16) that between these two conditions we shall have, successively, "metastable" chaos, coexistence of a chaotic attractor with the two point attractors given by (17) and finally only a chaotic attractor after the Hopf bifurcation. This well-known scenario is represented schematically in Fig. 5 where A corresponds to the pitchfork bifurcation giving the solutions (17), B to the persistence condition $\nu = 2\eta$ (relation (10) in general), C to the change from case (a) to case (b) in Fig. 4, and D to the Hopf bifurcation at $\nu = \eta$ (relation (19) in general). In the region between B and C we have "metastable" chaos as it can be seen from the map [Fig. 4(a)] since for initial conditions in the interval $(-u_P, u_P)$

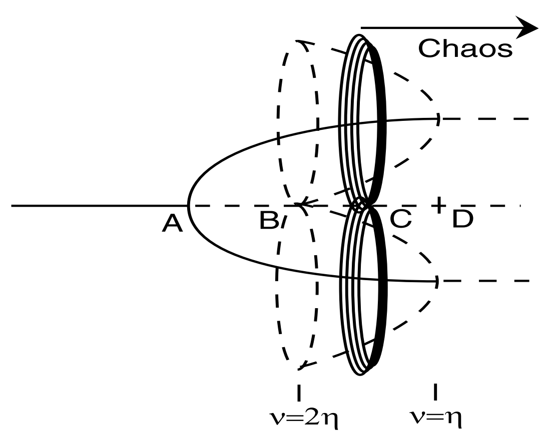


Fig. 5. Bifurcation diagram of the Lorenz model: (A) pitchfork bifurcation, (B) the homoclinic bifurcation, (C) apparition of chaos and (D) inverse Hopf bifurcation.

the system will remain for some time in that interval until it finally goes out of it and tend to one of the two point attractors given by expression (17) (it is this kind of transient erratic behavior that is called "metastable" chaos [Yorke & Yorke, 1979] and here it corresponds to the time the system spends in the interval $(-u_P, u_P)$). When we approach point C in Fig. 5 (which corresponds to $|OQ| = |OP|$ in Fig. 4) the time that the system remains in $(-u_P, u_P)$ for an initial condition in that interval increases and when we cross C we have $|OQ| < |OP|$ [Fig. 4(b)] and the system never leaves the interval. We have then between C and D a strange attractor coexisting with the two point attractors given by (17) and as we proceed from C to D the basin of attraction of the strange attractor grows until D becomes the only attractor

In case (b) the Hopf bifurcation will be supercritical to stable limit cycles leading to a stable homoclinic curve through a first homoclinic bifurcation. A cascade of homoclinic bifurcations ending in chaos will then occur in the gluing scenario described in [Arneodo *et al.*, 1981]. This scenario is only possible with the plus sign in the second equation of the Lorenz model as given by (3).

4. A Mechanical Example of the $(0^2)(0)$ Instability with Reflection: the Lorenz Pendulum

The system consists of a pendulum oscillating in a vertical plane fixed with respect to a support which rotates around a fixed vertical axis [Clerc *et al.*, 1999]. A constant external torque τ (see Fig. 6) is applied to the support. The kinetic energy of the system is

$$T = \frac{1}{2}(I + ML^2 \sin^2\theta)\dot{\phi}^2 + \frac{1}{2}ML^2\dot{\theta}^2 \qquad (21)$$

where M is the mass of the particle, L is the length of the pendulum, and I is the moment of inertia of the support with respect to its vertical axis of rotation. The potential energy due to gravity is

$$V = -Mgl\cos\theta \qquad (22)$$

and the Hamiltonian is

$$H = \frac{1}{2}\frac{P_1^2}{ML^2} + \frac{1}{2}\frac{P_2^2}{(I + ML^2\sin^2\theta)} - Mgl\cos\theta \qquad (23)$$

in terms of the generalized momentum

$$P_1 = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = ML^2\dot{\theta}\,,$$

$$P_2 = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = (I + ML^2\sin^2\theta)\dot{\phi} \qquad (24)$$

where $\mathcal{L} \equiv T - V$ is the Lagrangian. Hamilton's equations are

$$\dot{P}_1 = \frac{ML^2}{2}\sin(2\theta)\left(\frac{P_2}{I + ML^2\sin^2\theta}\right)^2 - MgL\sin(\theta)\,, \qquad (25)$$

$$\dot{\theta} = \frac{P_1}{ML^2}\,,$$

$$\dot{P}_2 = 0\,,$$

$$\dot{\phi} = \frac{P_2}{(I + ML^2\sin^2\theta)}\,, \qquad (26)$$

and we see that we are in the situation described in the Appendix since ϕ is a cyclic variable corresponding to the conservation of the total angular momentum with respect to the vertical axis which is here P_2. We can then consider the dynamical system formed by the first three equations of the latter set of equations which are invariant under the two time reversal transformations (see Eqs. (A.12) and (A.17))

$$t \to -t,\ \theta \to \theta,\ \phi \to \phi,\ P_1 \to -P_1,\ P_2 \to \pm P_2\,, \qquad (27)$$

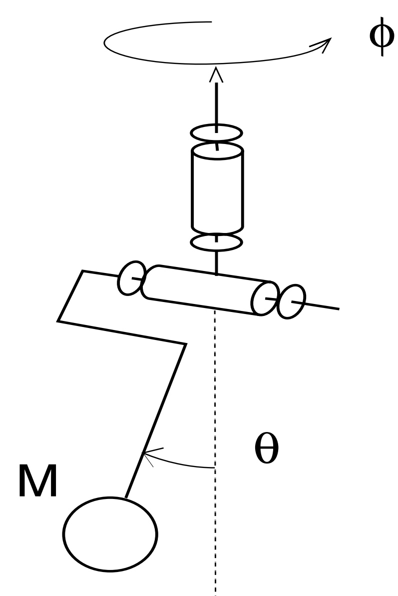


Fig. 6. Schematic representation of Lorenz pendulum.

and also have the reflection symmetry ($\theta \to -\theta$, $P_1 \to -P_1$). We can write the dynamical system as

$$ML^2\ddot{\theta} = \frac{ML^2}{2}\sin(2\theta)\left(\frac{P_2}{I + ML^2\sin^2\theta}\right)^2 - MgL\,\sin(\theta)\,, \tag{28}$$

$$\dot{P}_2 = 0\,, \tag{29}$$

These equations have a stationary solution where P_2 is an arbitrary constant $P_2^{(0)}$ and $\theta = \dot{\theta} = 0$. We put $P_2^{(0)} \equiv I\Omega$, where Ω, which is defined by the last equality, is the angular velocity with respect to the vertical axis. Displacing $P_2 = P_2^{(0)} + p_2$, scaling the time as $t' = \sqrt{g/L}t$, and defining $\tilde{\theta}(t') = \theta(t)$, $\varsigma(t') = p_2/P_2^{(0)}$, we obtain the dimensionless form

$$\frac{d^2\tilde{\theta}}{dt'^2} = \sin(2\tilde{\theta})\tilde{\Omega}^2\frac{1 + 2\varsigma + \varsigma^2}{(I + \alpha\,\sin^2\tilde{\theta})^2} - \sin(\tilde{\theta})\,,$$

$$\frac{d\varsigma}{dt'} = 0\,, \tag{30}$$

with $\tilde{\Omega} \equiv \Omega\sqrt{L/g}$, $\alpha \equiv ML^2/I$. The stationary solution is now ($\tilde{\theta} = d\tilde{\theta}/dt' = \varsigma = 0$) and expanding around it we obtain the truncated form

$$\frac{d^2\tilde{\theta}}{dt'^2} = \varepsilon\tilde{\theta} - \sigma\tilde{\theta}^3 + 2\tilde{\Omega}\varsigma\tilde{\theta}$$

$$\frac{d\varsigma}{dt'} = 0\,, \tag{31}$$

where $\varepsilon \equiv \tilde{\Omega}^2 - 1$ is the bifurcation parameter and σ is

$$\sigma \equiv \frac{4\tilde{\Omega}^2 - 1}{6} + 2\tilde{\Omega}^2\alpha\,. \tag{32}$$

We see immediately that at $\varepsilon = 0$ the latter equations present the $(0^2)(0)$ instability since linearly they are

$$\frac{d}{dt'}\begin{pmatrix}\tilde{\theta}\\ \frac{d\tilde{\theta}}{dt'}\\ \varsigma\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{\theta}\\ \frac{d\tilde{\theta}}{dt'}\\ \varsigma\end{pmatrix} \tag{33}$$

For ε positive the solution is unstable and σ is positive: the bifurcation is supercritical. We can now justify the truncated form (31) since for ε small one has $\partial_t \sim \sqrt{\varepsilon}$, $\tilde{\theta} \sim \sqrt{\varepsilon}$, $\varsigma \sim \varepsilon$, and then one can check that the other terms in the Taylor expansion of (30) are of higher order in ε. The scaling ($\tilde{\theta} = x/\sqrt{\sigma}$, $\varsigma = -z/(2\tilde{\Omega}^2)$) gives the final form

$$\frac{d^2x}{dt'^2} = \varepsilon x - x^3 - zx$$

$$\frac{dz}{dt'} = 0\,, \tag{34}$$

which is invariant under the time reversal transformation

$$t' \to -t',\; x \to x,\; z \longrightarrow z \tag{35}$$

in agreement with formula (A.24) of the Appendix.

We shall add now the small irreversible terms which will break the invariance under (35). These terms will be of two types: dissipative effects and injection of kinetic energy. We go back to Eqs. (28) and (29) where $P_2 = (I + ML^2\,\sin^2\theta)\dot{\phi}$ is the total angular momentum with respect to the vertical axis and since we inject energy through a constant external torque τ applied to the support Eq. (29) will get a term τ on the right-hand side. Dissipation will occur through two mechanisms:

- friction in the rotation of the support around its axis which will add a term $-\tilde{\mu}\dot{\phi}$ to Eq. (29).
- loss of energy due to the motion of the pendulum of mass M in the fluid surrounding it (for example the atmosphere).

This last effect is modelized by Stokes's law which says that the force on the sphere moving in the fluid is $\mathbf{F} = -\lambda\mathbf{u}$, where λ is a constant and $\mathbf{u}$ the velocity of the center of mass. This force will then add terms $-\tilde{\nu}\dot{\theta}$ in Eq. (28) and $-\tilde{\nu}\,\sin^2(\theta)\dot{\phi}$ in Eq. (29), with $\tilde{\nu} \equiv \lambda L^2$. We obtain the equations

$$ML^2\ddot{\theta} = \frac{ML^2}{2}\sin(2\theta)\dot{\phi}^2 - MgL\,\sin(\theta) - \tilde{\nu}\dot{\theta}$$

$$\frac{d}{dt}[(I + ML^2\,\sin^2\theta)\dot{\phi}] = \tau - \tilde{\mu}\dot{\phi} - \tilde{\nu}\,\sin^2(\theta)\dot{\phi} \tag{36}$$

we put $\tau = \tilde{\mu}\Omega$, with Ω the order one, since we want the injection of energy to be of the same order of the dissipations characterized by the small coefficient ($\tilde{\mu}$, $\tilde{\nu}$) which we take to be of the same order (notice that all the coefficients as written in the latter equations are positive). We rewrite finally our equations in terms of the angular momentum P_2 which we used in the reversible case, where it was

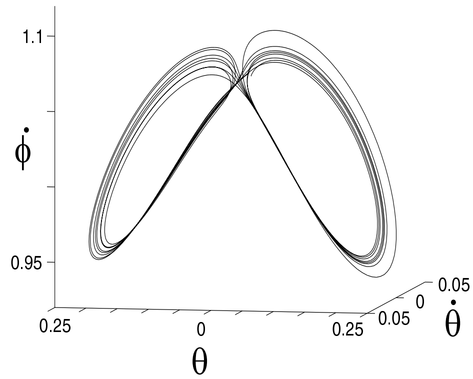


Fig. 7. Numerical simulation of Lorenz pendulum [Eq. (36)] with $\Omega = 1.4142$, $\nu = 0.1210$, $I = 0.3770$, $\mu = 0.037$.

a conserved quantity. Equations (28) and (29) are then replaced by

$$\ddot{\theta} = \frac{1}{2}\sin(2\theta)\left(\frac{P_2}{I + ML^2\sin^2\theta}\right)^2 - \frac{g}{L}\sin(\theta) - \frac{\tilde{\nu}}{ML^2}\dot{\theta}$$

$$\dot{P}_2 = \tilde{\mu}\left(\Omega - \frac{P_2}{I + ML^2\sin^2\theta}\right) - \tilde{\nu}\frac{\sin^2(\theta)P_2}{I + ML^2\sin^2\theta}$$

The stationary solutions is ($\theta = \dot{\theta} = 0$, $P_2 = I\Omega$), where Ω, which was a free parameter for the reversible system, is now fixed by the external torque τ. Doing the same scaling as before, i.e. all the steps leading to Eqs. (34) starting from Eqs. (28) and (29), we obtain

$$\frac{d^2x}{dt'^2} = \varepsilon x - x^3 - zx - \nu\frac{dx}{xt}$$

$$\frac{dz}{dt'} = -\mu z + \eta x^2, \tag{37}$$

where

$$\mu = \frac{\tilde{\mu}}{I}\sqrt{\frac{g}{L}}, \quad \nu = \frac{\tilde{\nu}}{\alpha I}\sqrt{\frac{g}{L}},$$

$$\eta = \frac{2\tilde{\Omega}^2(\nu - \mu)}{\frac{4\tilde{\Omega}^2 - 1}{6}\frac{I}{ML^2} + 2\tilde{\Omega}^2} \tag{38}$$

We have then exactly obtained the normal form of the $(0^2)(0)$ instability of Sec. 3 and we can have η positive which is necessary in order to have chaotic behavior. Numerical simulations of Eqs. (36) are shown in Fig. 7. It is simple to check here that one can satisfy the condition $\nu = 2\eta$ for the persistence of the homoclinic curve (formula (10)), but it is not possible to satisfy the condition of Hopf bifurcation ($\nu = \eta$) and consequently the whole Lorenz scenario cannot be observed. We shall see in the next section that the replacement of the ideal pendulum used here by an asymmetric physical pendulum introduces a new parameter which eliminates the previous problem.

5. The Physical Lorenz Pendulum

We shall make here some changes in the pendulum of the previous section. The difference is that we replace the ideal pendulum, i.e. the homogeneous sphere of mass M attached by a rod of negligible mass to the support, by a physical pendulum. The system at rest is represented schematically in Fig. 8, where O is the point of contact of the physical pendulum with the support, OZ is the vertical axis of rotation of the support, and P is the center of mass of the pendulum which is in the z-axis. We assume OP is one of the principal axis of inertia of the pendulum at the point O, then the two other principal axis can be chosen as (OX, OY), where OX is orthogonal to the plane (OY, OZ) of the drawing and gravity acts in the direction of the negative z-axis. We denote by (I_1^*, I_2^*, I_3^*) the moments of inertia

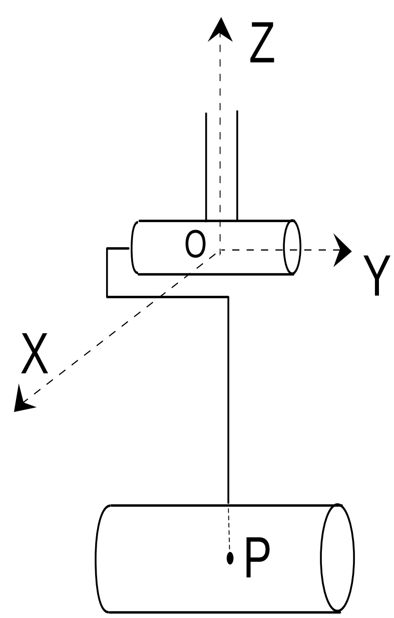


Fig. 8. Schematic representation of physical Lorenz pendulum.

with respect to (OX, OY, OZ), respectively. Let ϕ be the angle measuring the rotation of the support around the fixed vertical axis OZ measured as before and I the moment of inertia of the support with respect to OZ. We consider the frame of reference (OX, OY, OZ) as fixed with respect to the support. The pendulum oscillates in the plane (OX, OZ) and we call θ the angle between the line OP, which is contained in the plane, and the negative z-axis (see Fig. 6 where θ is indicated for the ideal pendulum). The kinetic energy of the support is then $T^{(S)} = (1/2)I(d\phi/dt)^2$. The kinetic energy $T^{(P)}$ of the pendulum will be

$$T^{(P)} = +\frac{1}{2}(I_3^* + (I_1^* - I_3^*)\sin^2(\theta))\left(\frac{d\phi}{dt}\right)^2$$
$$\frac{1}{2}I_2^*\left(\frac{d\theta}{dt}\right)^2 \tag{39}$$

and the total kinetic energy T of the system will be $T = T^{(S)} + T^{(P)}$. If $L = |OP|$ and M is the mass of the pendulum one has

$$I_1^* = ML^2 + I_1,\ I_2^* = ML^2 + I_2, I_3^* = I_3 \tag{40}$$

where (I_1, I_2, I_3) are the moments of inertia with respect to axis parallel to (OX, OY, OZ) passing through the center of mass P of the pendulum. For an anisotropic pendulum $I_1 \neq I_2$ and in that case Stokes's law will give two different coefficients $\tilde{\nu}$ and $\tilde{\chi}$ in the equations for $\ddot{\theta}$ and $\ddot{\phi}$ as a consequence of different contact sections with the fluid. The other forces and the torque will be the same as before and the new equations of motion for θ and ϕ will be

$$(ML^2 + I_2)\ddot{\theta} = \frac{(ML^2 + I_1 - I_3)}{2}\sin(2\theta)\dot{\phi}^2$$
$$- MgL\sin(\theta) - \tilde{\nu}\dot{\theta}$$
$$\frac{d}{dt}[(I + I_3 + (ML^2 + I_1 - I_3)\sin^2\theta)\dot{\phi}] =$$
$$\tau - \tilde{\mu}\dot{\phi} - \tilde{\chi}\sin^2(\theta)\dot{\phi} \tag{41}$$

The reversible system $(\tau = \tilde{\mu} = \tilde{\chi} = \tilde{\nu} = 0)$ has the Hamiltonian

$$H = \frac{1}{2}\frac{P_1^2}{I_2 + ML^2} - Mgl\cos\theta$$
$$+ \frac{1}{2}\frac{P_2^2}{(I + I_3 + (ML^2 + I_1 - I_3)\sin^2\theta)} \tag{42}$$

and the corresponding Hamilton's equation are

$$\dot{P}_1 = \frac{\sin(2\theta)(ML^2 + I_1 - I_3)}{2}$$
$$\times\left(\frac{P_2}{(I + I_3)(1 + \alpha\sin^2\theta)}\right)^2 - MgL\sin(\theta)\,,$$
$$\dot{\theta} = \frac{P_1}{I_2 + ML^2}\,,$$
$$\dot{P}_2 = 0\,,$$
$$\dot{\phi} = \frac{P_2}{(I + I_3)(1 + \alpha\sin^2\theta)}\,, \tag{43}$$

where $\alpha \equiv (ML^2 + I_1 - I_3)/(I + I_3)$. We are again in the case of the Appendix, and the three first equations above are a dynamical system which is invariant by the time reversal transformations $(t \to -t, \theta \to \theta, P_1 \to -P_1, P_2 \to \pm P_2)$. Putting $\tau = \tilde{\mu}\Omega$ we add now the irreversible terms to obtain Eqs. (41) which we write now in the form

$$\ddot{\theta} = \frac{\sin(2\theta)(ML^2 + I_1 - I_3)}{2(ML^2 + I_2)}$$
$$\times\left(\frac{P_2}{(I + I_3)(1 + \alpha\sin^2\theta)}\right)^2$$
$$- \frac{MgL\sin\theta}{I_2 + ML^2} - \frac{\tilde{\nu}}{(ML^2 + I_2)}\dot{\theta}\,,$$
$$\dot{P}_2 = \tilde{\mu}\left(\Omega - \frac{P_2}{(I + I_3)(1 + \alpha\sin^2\theta)}\right)$$
$$- \tilde{\chi}\frac{\sin^2(\theta)P_2}{(I + I_3)(1 + \alpha\sin^2\theta)} \tag{44}$$

These equations again have the fix point $(\theta = \dot{\theta} = 0, P_2 = P_2^{(o)} = (I + I_3)\Omega)$. We define the new variables

$$x(t') = \sqrt{\sigma}\theta(t),\ z(t') = -\frac{2\tilde{\Omega}}{P_2^{(o)}}(P_2 - P_2^{(o)})\,,$$
$$t' = t\sqrt{\frac{MgL}{ML^2 + I_1}} \tag{45}$$

where $\tilde{\Omega} \equiv \sqrt{(ML^2 + I_2 - I_3)/MLg}\,\Omega$, $\sigma = (4\tilde{\Omega}^2 - 1)/6 + 2\tilde{\Omega}^2\alpha$. The fixed point is now $(x = dx/dt = z = 0)$ and expanding around it we obtain as expected

$$\frac{d^2x}{dt'^2} = \epsilon x - x^3 - zx - \nu\frac{dx}{xt}$$

$$\frac{dz}{dt'} = -\mu z + \eta x^2, \tag{46}$$

where the bifurcation parameter $\epsilon \equiv \tilde{\Omega}^2 - 1$ and

$$\nu = \frac{\tilde{\nu}(ML^2 + I_1 - I_3)}{\alpha(I + I_3)(ML^2 + I_1)}\sqrt{\frac{MLg}{ML^2 + I_1}}$$

$$\mu = \frac{\tilde{\mu}}{I + I_3}\sqrt{\frac{MLg}{ML^2 + I_2}}\,, \eta = \frac{2\tilde{\Omega}^2(\chi - \mu)}{\sigma}$$

$$\chi = \frac{\tilde{\chi}}{I + I_3}\sqrt{\frac{MLg}{ML^2 + I_2}} \tag{47}$$

We have again the $(0^2)(0)$ instability at $\epsilon = 0$ but now we have a new parameter χ which will allow us to realize the condition of the Hopf bifurcation which could not be realized with the ideal pendulum of the previous section.

6. Conclusion

We have considered in this paper the stationary instability (resonance at zero frequency) in the presence of a neutral mode for quasi-reversible systems. We have shown that when one has reflection symmetry for the variables associated with the resonance the asymptotic normal form is the Lorenz model which has then a universal character. Due to this the Lorenz equations will describe numerous physical systems near the threshold of this instability. Some examples studied are (a) The Lorenz pendulum (this paper); (b) the onset of chaos in the one-dimensional reversible Ginzburg–Landau equation [Clerc *et al.*, 2000]; (c) the quasi-reversible limit cycle which loses its stability through period doubling [Moon, 1997]; (d) the interaction of two Bose condensates of attractive atoms [Coullet & Vandenberghe].

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Appendix

We consider a dynamical system

$$\frac{d\mathbf{q}}{dt} = \mathbf{A}(\mathbf{q}) , \quad \mathbf{q} \equiv (q^1, \ldots , q^n) \tag{A.1}$$

where $\mathbf{A} = (A^1(\mathbf{q}), A^2(\mathbf{q}), \ldots , A^n(\mathbf{q}))$ is an n-dimensional vector field. The system (A.1) will be time reversal if there exists a linear transformation S (an $n \times n$ matrix) such that

$$\mathbf{A}(\mathbf{q}) = -S\mathbf{A}(S\mathbf{q}), \quad S^2 = 1 \tag{A.2}$$

Then the transformation

$$t' = -t,\ \mathbf{q}'(t') = S\mathbf{q}(t) \tag{A.3}$$

leaves Eqs. (A.1) invariant. Any transformation of the form (A.3) with $S^2 = 1$ is called a time reversal transformation and we say that a dynamical system is reversible if it is invariant under some time reversal transformation (reversibility is then defined with respect to a given time reversal transformation). Let $\mathbf{q}^{(0)}$ be a fixed point of (A.1), i.e. $\mathbf{A}(\mathbf{q}) = 0$. Putting $\mathbf{q} = \mathbf{q}^{(0)} + \mathbf{q}'$ we can obtain from (A.1) the equation for $\mathbf{q}'$ which is

$$\frac{d\mathbf{q}'}{dt} = J\mathbf{q}' + O(\mathbf{q}'^2) \tag{A.4}$$

where J has matrix elements

$$J_{\mu\nu} = \left.\frac{\partial A^\mu}{\partial q^v}\right|_{\mathbf{q}'=\mathbf{q}'^{(0)}} \tag{A.5}$$

The eigenvalues of J determine the stability of the fixed point. If $S\mathbf{q}^{(0)} = \mathbf{q}^{(0)}$, i.e. if the fixed point is invariant under S, one has the property that if λ is an eigenvalue of J then $-\lambda$ is also one (this is a direct consequence of (A.2)). Furthermore since Eqs. (A.1) are real we have that λ and λ^* are eigenvalues simultaneously. Consequently a complex eigenvalue $\lambda = a + ib$ generates three others: $\lambda^* = a - ib$, $-\lambda = -(a + ib)$, and $-\lambda^* = -a + ib$. We see then that for reversible systems linear growth can only be avoided if all eigenvalues are in the imaginary axis and in that case we call the fixed point an equilibrium. If we have eigenvalues with real parts the system will be unstable. It follows from the previous properties that if a reversible system has an odd number of variables ($n = 2m + 1$ in (A.1) with m an integer) then it has necessarily a zero eigenvalue. An important example of reversible dynamical systems are the Hamiltonian systems when the Hamiltonian is quadratic in the momentums, i.e. if it has the form

$$H(\mathbf{p}, \mathbf{q}) = \frac{g^{\mu\nu}(\mathbf{q})}{2}\, p_\mu p_\nu + V(\mathbf{q}) \tag{A.6}$$

where we sum over repeated indices from 1 to n, $\mathbf{q} = (q^1, \ldots , q^n)$, $\mathbf{p} = (p_1, p_2, \ldots , p_n)$. The time reversal transformation which leaves Hamilton's equations invariant is

$$t \to -t,\ q^\mu \to q^\mu,\ p_\mu \to -p_\mu \quad \mu = 1, \ldots , n \tag{A.7}$$

We shall discuss now the appearance of a neutral mode at zero frequency in reversible Hamiltonian systems. If the variable q^n is cyclic, i.e. it does not appear in $H(\mathbf{p}, \mathbf{q})$, $g^{\mu\nu}(\mathbf{q}) = g^{\mu\nu}(q^1, \ldots , q^{n-1})$, $V(\mathbf{q}) = V(q^1, \ldots , q^{n-1})$, one has a conserved quantity which is the conjugate momentum p_n since $\dot{p}_n = 0$. The Hamilton's equations will be (Latin indices go from 1 to $(n - 1)$, Greek indices from 1 to n, $\partial_\mu = \partial/\partial q^\mu$)

$$\dot{q}^j = g^{j\mu}(q^1, \ldots , q^{n-1})p_\mu,\ j = 1, \ldots , n - 1 \tag{A.8}$$

$$\dot{p}_j = -\partial_j V(q^1, \ldots , q^{n-1}) - \frac{\partial_j g^{\mu\nu}(\mathbf{q})}{2} p_\mu p_\nu \tag{A.9}$$

$$\dot{p}_n = 0 \tag{A.10}$$

$$\dot{q}^n = g^{n\mu}(q^1, \ldots , q^{n-1})p_\mu \tag{A.11}$$

Equation (A.10) can be integrated giving $p_n = p_n^{(o)} =$ constant. We can then replace $p_n(t)$ by $p_n^{(o)}$ in Eqs. (A.8) and (A.9) and we obtain a set of $2(n - 1)$ autonomous first order differential equations for $(p_1, \ldots , p_{n-1};\ q^1, \ldots , q^{n-1})$ in which $p_n^{(o)}$ appears as a parameter. Once we know $(p_1, p_2, \ldots , p_{n-1}, q^1, q^2, \ldots , q^{n-1})$ as functions of t through integration of Eqs. (A.9) and (A.10) the variable $q^n(t)$ is determined by direct integration from (A.11). We see then that the system is reduced to the $(2n - 1)$ equations (A.8)–(A.10) which are a time reversible dynamical system invariant under the time reversal transformation

$$\begin{aligned} &t \to -t, \quad q^j \to q^j, \quad j = 1, \ldots , n - 1\,; \\ &p_\mu \to -p_\mu\,, \quad \mu = 1, \ldots , n \end{aligned} \tag{A.12}$$

We now make the further assumption that the symmetric matrix $g^{\mu\nu}$ is such that

$$\begin{aligned} &g^{nj} = 0\,, \quad j = 1, \ldots , n - 1 \\ &g^{nn} \neq 0 \end{aligned} \tag{A.13}$$

Equations (A.8)–(A.10) take now the form $(j,\, k,\, l = 1, \ldots,\, n-1)$

$$\dot{q}^j = g^{jk}(q^1, \ldots,\, q^{n-1})p_k \tag{A.14}$$

$$\dot{p}_j = -\partial_j V(q^1, \ldots,\, q^{n-1}) - \frac{\partial_j g^{kl}(\mathbf{q})}{2} p_k p_l - \frac{\partial_j g^{nn}(\mathbf{q})}{2} p_n^2 \tag{A.15}$$

$$\dot{p}_n = 0 \tag{A.16}$$

and we can check that they are invariant under the new time reversal transformation

$$\begin{gathered} t \to -t\,, \quad p_n \to p_n \\ (q^j,\, p_j) \to (q^j,\, -p_j)\,, \quad j = 1, \ldots, n-1\,; \end{gathered} \tag{A.17}$$

The set of equations (A.14)–(A.16) are then invariant under two time reversal transformations: (A.12) and (A.17). Fixed points of these equations are of the form

$$\begin{gathered} p_1 = p_2 = \ldots = p_{n-1} = 0\,, \\ p_n = p_n^{(o)};\ q = q^1_{(o)},\ q^2 = q^2_{(o)}, \ldots,\ q^{n-1} = q^{n-1}_{(o)} \end{gathered} \tag{A.18}$$

where $\{q^j_{(o)}\}$ satisfy the set of $(n-1)$ equations $(j = 1, \ldots, n-1)$

$$\frac{\partial_j g^{nn}(\mathbf{q}_{(o)})}{2} p_n^2 = -\partial_j V(q^1_{(o)}, \ldots,\, q^{n-1}_{(o)}) \tag{A.19}$$

We remark that for $p_n^{(o)} \neq 0$ these fixed points are time reversal invariant, i.e. invariant under S, only for the time reversal transformation (A.17). They will also be invariant under (A.12) if $p_n^{(o)} = 0$. Let us write now our equations in terms of the displaced variables $(p_1, \ldots,\, p_{n-1},\, P_n;\, Q^1, \ldots,\, Q^n)$ defined by

$$\begin{gathered} p_n = p_n^{(o)} + P_n \\ q^j = q^j_{(o)} + Q^j, \quad j = 1, \ldots, n-1\,; \end{gathered} \tag{A.20}$$

We put

$$\tilde{V}(Q^1, \ldots,\, Q^n) \equiv V(q^1_{(o)} + Q^1, \ldots,\, q^n_{(o)} + Q^n)\,,$$
$$\tilde{g}^{\mu\nu}(Q^1, \ldots,\, Q^n) \equiv g^{\mu\nu}(q^1_{(o)} + Q^1, \ldots,\, q^n_{(o)} + Q^n)\,.$$

The new equations are

$$\dot{Q}^j = \tilde{g}^{jk}(Q^1, \ldots,\, Q^n)p_k \tag{A.21}$$

$$\dot{p}_j = -\partial_j \tilde{V}(Q^1, \ldots,\, Q^n) - \frac{\partial_j \tilde{g}^{jk}(Q^1, \ldots,\, Q^n)}{2}\, p_k p_l - \frac{\partial_j \tilde{g}^{nn}(\mathbf{q})}{2}\, (p_n^{(o)} + P_n)^2 \tag{A.22}$$

$$\dot{P}_n = 0 \tag{A.23}$$

The time reversal transformation (A.17) implies now that Eqs. (A.21)–(A.23) will be invariant under the time reversal transformation

$$\begin{gathered} t \to -t\,, \quad P_n \to P_n \\ (q^j,\, p_j) \to (q^j,\, -p_j)\,, \quad j = 1, \ldots, n-1\,; \end{gathered} \tag{A.24}$$

This last case corresponds exactly to the example of the pendulum we have presented in Sec. 3 and we have remarked that in the final variables $(p_1, \ldots,\, p_{n-1},\, P_n;\ Q^1, \ldots,\, Q^n)$ obtained after the displacement (A.20) only one time reversal transformation leaving the new equations invariant survives, namely (A.24).