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Approximate controllability and homogenization of a semilinear elliptic problem ☆

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Abstract

The L^2- and H^1-approximate controllability and homogenization of a semilinear elliptic boundary-value problem is studied in this paper. The principal term of the state equation has rapidly oscillating coefficients and the control region is locally distributed. The observation region is a subset of codimension 1 in the case of L^2-approximate controllability or is locally distributed in the case of H^1-approximate controllability. By using the classical Fenchel–Rockafellar's duality theory, the existence of an approximate control of minimal norm is established by means of a fixed point argument. We consider its asymptotic behavior as the rapidly oscillating coefficients H-converge. We prove its convergence to an approximate control of minimal norm for the homogenized problem.

Keywords: Approximate controllability; Homogenization; Semilinear elliptic equation

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1. Introduction

1.1. Setting of the problem

In this paper, we consider a nonperiodic, nonlinear homogenization problem where the control is distributed in a relatively compact subdomain. Our goal is to study the approximate controllability of this problem when the operators in the state equation (given by a second order elliptic boundary-value problem) and in the cost functional (involving a Dirichlet type integral of the state function) both have rapidly oscillating coefficients.

Let Ω be a connected bounded open set in $\mathbb{R}^N$, $N \geqslant 2$, with a smooth boundary $\partial\Omega$. We consider two nonempty subdomains of Ω which are the observable region ω and the region where the error between the obtained and the desired state is minimized, that we denote by S.

For given constants $0 < \alpha_m \leqslant \alpha_M$, we denote by $\mathcal{M}(\alpha_m, \alpha_M)$ the set of all $N \times N$ matrices $A = A(x)$ such that

$$A \in L^\infty(\Omega)^{N\times N}, \tag{1.1}$$

$$\alpha_m I \leqslant A(x) \quad \text{and} \quad \bigl|A(x)\xi\bigr| \leqslant \alpha_M|\xi| \quad \forall \xi \in \mathbb{R}^N, \text{ and for a.e. } x \in \Omega, \tag{1.2}$$

where I is the $N \times N$ identity matrix. (It is well-known that if A is symmetric, then the second condition in (1.2) is equivalent to $A(x) \leqslant \alpha_M I$.)

For each $\varepsilon > 0$, we consider a matrix $A_\varepsilon \in \mathcal{M}(\alpha_m, \alpha_M)$ and the state equation

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla y_\varepsilon(v)) + f(y_\varepsilon(v)) = \chi_\omega v & \text{in } \Omega, \\ y_\varepsilon(v) = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where χ_ω is the characteristic function of ω, v is the control, and $y_\varepsilon(v)$ the associated state. Here f is a real-valued continuous function for which we assume that

$$f(0) = 0 \quad \text{and} \quad \exists\gamma > 0, \quad 0 \leqslant \frac{f(s)}{s} \leqslant \gamma \quad \forall s \in \mathbb{R} \setminus \{0\}. \tag{1.4}$$

There are two possible locations of the observation zone S that allow different kinds of approximate controllability. One is the case where S is an open subset of Ω which is compactly contained in ω. In this case the H^1-approximate controllability is studied. The other case occurs when the observation zone S is a smooth subset of Ω of codimension 1 nonintersecting the control zone ω. In this case the L^2-approximate controllability can be considered.

The study of the H^1-approximate controllability involves a more general analysis and we will consider it in this paper. The analysis of the L^2-approximate controllability is simpler, and we have included a number of remarks at each step of the paper with the necessary changes to recover the L^2-case from the H^1-case.

Given $y_1 \in H^1(S)$, a constant $\alpha \geqslant 0$, and a *symmetric positive definite* matrix B, our aim is to find a control $v_\varepsilon \in L^2(\omega)$ such that

$$\bigl\| y_\varepsilon(v_\varepsilon) - y_1 \bigr\|_{B,S} \leqslant \alpha, \tag{1.5}$$

where, by definition,

$$\|y_\varepsilon(v_\varepsilon) - y_1\|_{B,S} \stackrel{\text{def}}{=} \left(\int_S B\nabla\big(y_\varepsilon(v_\varepsilon) - y_1\big) \cdot \nabla\big(y_\varepsilon(v_\varepsilon) - y_1\big)\,dx + \int_S \big|y_\varepsilon(v_\varepsilon) - y_1\big|^2\,dx \right)^{1/2}.$$

This means that the error between $y_\varepsilon(v_\varepsilon)$ and y_1 is bounded from above by α when using a norm equivalent to the H^1-norm and defined in terms of the matrix B.

Remark 1.1. In the case of the L^2-approximate controllability, the corresponding error condition is obtained by taking $y_1 \in L^2(S)$ and the L^2-norm $\|\cdot\|_{0,S}$ in (1.5), which formally corresponds to the case $B = 0$.

Notice that the case $\alpha = 0$ is the extreme situation of exact controllability. In this paper, we will just be concerned by *approximate* controllability, that is $\alpha > 0$.

Remark 1.2. If $S \subset\subset \omega$, then the case of L^2-approximate controllability can be treated as the H^1-case. Conversely, if $\overline{S} \cap \overline{\omega} = \emptyset$ and S is an nonempty open set, we can show by contradiction that the L^2-approximate controllability is not possible. Indeed, take σ a relatively compact open subset of S and define $y_1 = 0$ in $S \setminus \overline{\sigma}$ and $y_1 = 1$ in σ. The approximate controllability of the problem

$$\begin{cases} -\Delta y(v) = \chi_\omega v & \text{in } \Omega, \\ y(v) = 0 & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

implies that, for each $n \in \mathbb{N}$, there exists $v_n \in L^2(\Omega)$ such that

$$\|y(v_n) - 1\|_{0,\sigma} \leqslant 1/n \quad \text{and} \quad \|y(v_n)\|_{0,S\setminus\overline{\sigma}} \leqslant 1/n. \tag{1.7}$$

From this we derive that $y(v_n) \to y^*$ strongly in $L^2(S)$, where

$$y^* = 1 \quad \text{in } \sigma \quad \text{and} \quad y^* = 0 \quad \text{in } S \setminus \overline{\sigma}. \tag{1.8}$$

We write (1.6) for $v = v_n$ and we take the restriction to S. Passing to the limit, we derive

$$-\Delta y^* = 0 \quad \text{in } S \tag{1.9}$$

which contradicts (1.8).

1.2. *Presentation of the main results*

Our aim is to establish the approximate controllability for each $\varepsilon > 0$ and to study the H-convergence of minimal norm controls to an approximate control linked to homogenized problems.

We notice that problem (1.3), (1.5) does not generally have a unique solution. We are therefore interested in the optimal control v_ε^* which minimizes, over all $v \in L^2(\omega)$, the cost functional

$$I_\varepsilon(v) \stackrel{\text{def}}{=} \frac{1}{2}\|v\|_{0,\omega}^2 + \begin{cases} 0 & \text{if } \|y_\varepsilon(v)|_S - y_1\|_{B,S} \leqslant \alpha, \\ \infty & \text{otherwise.} \end{cases} \tag{1.10}$$

In order to do so, we first develop the fixed point strategy introduced by Fabre et al. [1]. Secondly, we pass to the limit as $\varepsilon \to 0$ using H-convergence methods (see Murat and Tartar [6] or Tartar [8]). Our main results are Theorems 2.5 and 3.2 below.

Remark 1.3. All the results of Sections 2 and 3 (including Lemma 2.3, Theorems 2.5 and 3.2) are also valid in the case of the L^2-approximate controllability and under the following hypothesis:

(i) Each point on S can be connected by an arc included in Ω to some point in ω without intersecting S.
(ii) The coefficients of A_ε are of class $C^1(\overline{\Omega})$ or $L^\infty(\Omega)$ under some geometrical restrictions that allow a certain unique continuation property (see Remark 2.4).

To adapt the results and proofs to this case, it suffices to take all the variables with subindex 1 (like y_1, φ_1) in $L^2(S)$, to replace $\|\cdot\|_{B,S}$ and $(\cdot\,,\cdot)_{B,S}$ by the usual norm $\|\cdot\|_{0,S}$ and inner product $(\cdot\,,\cdot)_{0,S}$ in $L^2(S)$ (that is with $B = 0$), and to replace χ_S by a Dirac mass on S.

2. Existence of an optimal control

2.1. The linearized problem

For technical reasons, and without loss of generality, we assume that $f \in \mathcal{C}^1(\mathbb{R})$. (Otherwise, we can argue by density, approximating f by a sequence of smooth functions.) This allows us to introduce the function

$$g(s) \stackrel{\text{def}}{=} \begin{cases} f(s)/s & \text{if } s \neq 0, \\ f'(0) & \text{if } s = 0. \end{cases} \tag{2.1}$$

The assumptions on f imply

$$g \in \mathcal{C}^0(\mathbb{R}) \quad \text{and} \quad 0 \leqslant g(s) \leqslant \gamma \quad \forall s \in \mathbb{R}. \tag{2.2}$$

We associate with g the linear problem

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla y_\varepsilon(z,v)) + g(z) y_\varepsilon(z,v) = \chi_\omega v & \text{in } \Omega, \\ y_\varepsilon(z,v) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where z is a given function in $L^2(\Omega)$. We consider the cost functional

$$I_\varepsilon^z(v) \stackrel{\text{def}}{=} \frac{1}{2}\|v\|_{0,\omega}^2 + \begin{cases} 0 & \text{if } \|y_\varepsilon(z,v)|_S - y_1\|_{B,S} \leqslant \alpha, \\ \infty & \text{otherwise.} \end{cases} \tag{2.4}$$

By classical linear control theory (see, e.g., Lions [4]), it is well-known that for any given $z \in L^2(\Omega)$ there exists a unique minimal norm control $v_\varepsilon^*(z)$ such that

$$I_\varepsilon^z\big(v_\varepsilon^*(z)\big) = \min_{v \in L^2(\omega)} I_\varepsilon^z(v) < +\infty. \tag{2.5}$$

We denote by $y_\varepsilon^* = y_\varepsilon(z, v_\varepsilon^*(z))$ the corresponding state.

With the help of the minimal norm control $v_\varepsilon^*(z)$, we introduce the operator

$$\mathcal{F}_\varepsilon : L^2(\Omega) \to L^2(\Omega), \quad z \mapsto y_\varepsilon^*\big(z, v_\varepsilon^*(z)\big). \tag{2.6}$$

Our goal is to find a fixed point of $\mathcal{F}_\varepsilon$, which will obviously solve problem (1.3).

2.2. *Adjoint problem and dual formulation*

It is useful to work with the adjoint problem in a dual formulation. To this end, we introduce the operator L defined by

$$L : L^2(\omega) \to H^1(S), \quad v \mapsto y_\varepsilon(z, v)|_S, \tag{2.7}$$

where $y_\varepsilon(z, v)$ is the solution of (2.3). Its adjoint L^* is given by

$$L^* : H^1(S) \to L^2(\omega), \quad \varphi_1 \mapsto \varphi_\varepsilon(z)|_\omega, \tag{2.8}$$

where φ_ε is the solution of the so-called adjoint problem, which is obtained by solving the following Dirichlet problem

$$\begin{cases} -\operatorname{div}({}^tA_\varepsilon \nabla\varphi_\varepsilon(z, \varphi_1)) + g(z)\varphi_\varepsilon(z, \varphi_1) = -\operatorname{div}(\chi_S B\nabla\varphi_1) + \chi_S\varphi_1 & \text{in } \Omega, \\ \varphi_\varepsilon(z, \varphi_1) = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

which defines $\varphi_\varepsilon(z, \varphi_1)$ uniquely.

Remark 2.1. In this paper, we are concerned with approximate controllability in the sense of inequality (1.5). There is an alternative approach to approximate controllability which consists in proving that the set $\{y_\varepsilon(z, v) \mid v \in L^2(\omega)\}$ is dense in $H^1(S)$. An equivalent condition to establish this density is to prove that $\operatorname{Ker}(L^*) = 0$. In our present case, this can be proved as follows. Given $h \in H^{-1}(\Omega)$, let us introduce $\varphi_1 \in H_0^1(\Omega)$, the unique solution of

$$\begin{cases} -\operatorname{div}(B\nabla\varphi_1) + \varphi_1 = h & \text{in } \Omega, \\ \varphi_1 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Clearly

$$\big\{\varphi_1|_S \;\big|\; h \in H^{-1}(\Omega)\big\} = H^1(S).$$

Therefore, if $L^*(\varphi_1) = 0$ in $L^2(\omega)$, using as test functions $\xi \in H_0^1(S)$ and $\xi \in H_0^1(\omega)$ successively in (2.9), we obtain

$$\begin{cases} -\operatorname{div}(B\nabla\varphi_1) + \varphi_1 = 0 & \text{in } S, \\ \int_{\partial S} B\nabla\varphi_1 \cdot n\xi \, ds = 0 \quad \forall \xi \in H_0^1(\omega), \end{cases} \tag{2.10}$$

since $S \subset\subset \omega$. Here n denotes the unit outward normal to both boundaries that of ω and that of S. It follows that $B\nabla\varphi_1 \cdot n = 0$ on ∂S, and hence $\varphi_1 = 0$ in S.

Remark 2.2. In the case of an L^2-approximate controllability, the corresponding definitions (2.7) and (2.8) of L and L^* respectively can be given with $H^1(S)$ replaced by $L^2(S)$.

The corresponding adjoint problem is the same as in (2.10) taking $B = 0$ and replacing χ_S by a Dirac mass concentrated on S. A direct proof of the approximate controllability can be done as in the previous remark under the geometrical hypothesis mentioned before in Remark 1.3 (see [7]).

The approximate controllability of the nonlinear problem (1.3) is obtained here by using a more constructive approach, which provides an explicit method to find a control of minimal norm. This method was introduced by Lions [5] (see also Osses and Puel [7]), and is based on the classical Fenchel–Rockafellar's duality theory.

We can write down the functional I_ε^z under the form

$$I_\varepsilon^z(v) = F(v) + G(Lv) \tag{2.11}$$

with

$$F(v) = \frac{1}{2}\|v\|_{0,\omega}^2 \quad \text{and} \quad G(Lv) = \begin{cases} 0 & \text{if } \|Lv - y_1\|_{B,S} \leqslant \alpha, \\ \infty & \text{otherwise.} \end{cases} \tag{2.12}$$

Denoting by F^* and G^* the conjugate functions of F and G respectively, the duality theory states that

$$\inf_{v\in L^2(\omega)} I_\varepsilon^z(v) = -\inf_{\varphi_1\in H^1(S)} J_\varepsilon^z(\varphi_1) = -\inf_{\substack{h\in H^{-1}(\Omega),\\ \varphi_1 \text{ solution of (2.10)}}} J_\varepsilon^z(\varphi_{1\varepsilon}), \tag{2.13}$$

where

$$\begin{cases} J_\varepsilon^z(\varphi_{1\varepsilon}) = F^*(L^*\varphi_{1\varepsilon}) + G^*(-\varphi_{1\varepsilon}), \\ F^*(L^*\varphi_{1\varepsilon}) = \frac{1}{2}\|\varphi_\varepsilon(z,\varphi_{1\varepsilon})\|_{0,\omega}^2, \\ G^*(L^*\varphi_{1\varepsilon}) = \alpha\|\varphi_{1\varepsilon}\|_{B,S} + (\varphi_{1\varepsilon}, y_1)_{B,S}, \end{cases} \tag{2.14}$$

that is

$$J_\varepsilon^z(\varphi_{1\varepsilon}) = \frac{1}{2}\big\|\varphi_\varepsilon(z,\varphi_{1\varepsilon})\big\|_{0,\omega}^2 + \alpha\|\varphi_{1\varepsilon}\|_{B,S} - (\varphi_{1\varepsilon}, y_1)_{B,S}. \tag{2.15}$$

The following lemma, whose proof is given below in Section 2.4, summarizes the main properties of J_ε^z.

Lemma 2.3 (Coercivity property of J_ε^z). *For each $\alpha > 0$ and $y_1 \in H^1(S)$, the functional J_ε^z defined in* (2.14) *is continuous, strictly convex, and satisfies*

$$\liminf_{\|\varphi_1\|_{B,S}\to+\infty} \frac{J_\varepsilon^z(\varphi_1)}{\|\varphi_1\|_{B,S}} \geqslant \alpha. \tag{2.16}$$

Let us denote by $\varphi_{1,\varepsilon}^*(z) \in H^1(S)$ the unique optimal element which minimizes $J_\varepsilon^z(\varphi_1)$ over $H^1(S)$ and let φ_ε^* be the corresponding element defined by (2.9). It is well-known that the duality theory provides extremal relations that the optimal controls satisfy, namely

$$\begin{cases} F(v_\varepsilon^*(z)) + F^*(L^*\varphi_{1,\varepsilon}^*(z)) - (L^*\varphi_{1,\varepsilon}^*(z), v_\varepsilon^*(z))_{0,\omega} = 0, \\ G(Lv_\varepsilon^*(z)) + G^*(-\varphi_{1,\varepsilon}^*(z)) + (\varphi_{1,\varepsilon}^*(z), Lv_\varepsilon^*(z))_{B,S} = 0. \end{cases} \tag{2.17}$$

From the first of these relations, we derive the following explicit formula for the minimal norm control:

$$v_\varepsilon^*(z) = \varphi_\varepsilon\bigl(z, \varphi_{1,\varepsilon}^*(z)\bigr)\big|_\omega. \tag{2.18}$$

Remark 2.4. The proof of Lemma 2.3 is based on the following unique continuation property: if the solution of problem (2.9) is zero in ω then it is zero in the whole of Ω. In the case of H^1-approximate controllability, this property is quite easy to prove under the regularity hypothesis (1.1) since $S \subset\subset \omega$. In the case of L^2-approximate controllability, S does not intersect ω and the result is a Holmgren's unique continuation property [2]. This requires more regularity in the coefficients of A_ε (at least C^1) and an additional geometrical hypothesis as mentioned in Remark 1.3. Nevertheless, if the coefficients of A_ε are only L^∞ but piecewise C^1, the unique continuation property remains valid because of transmission conditions on the discontinuity interfaces.

2.3. Fixed point strategy

Thanks to this dual formulation, we are now in a position to develop our fixed point strategy for $\mathcal{F}_\varepsilon$. It consists in three steps. First, we establish the continuity of $\mathcal{F}_\varepsilon$ from $L^2(\Omega)$ into itself. Next, we prove that it maps the whole of $L^2(\Omega)$ into a bounded subset of $L^2(\Omega)$. Last, we check that $\mathcal{F}_\varepsilon$ is compact, and using Schauder's fixed point theorem, we conclude the existence of a solution of problem (1.3). More precisely, we have

Theorem 2.5. *For a given $\varepsilon > 0$, let A_ε be a matrix in $\mathcal{M}(\alpha_m, \alpha_M)$. Assume that the real-valued function f satisfies condition* (1.4). *Then there exists at least an element $\bar z_\varepsilon \in L^2(\Omega)$ which is a fixed point of the operator $\mathcal{F}_\varepsilon$ defined by* (2.6). *This element satisfies the equation $\bar z_\varepsilon = y_\varepsilon^*(\bar z_\varepsilon, v_\varepsilon^*(\bar z_\varepsilon))$, where $y_\varepsilon^*(\bar z_\varepsilon, v_\varepsilon^*(\bar z_\varepsilon))$ is the state solution of problem* (1.3) *and $v = v_\varepsilon^*(\bar z_\varepsilon)$ is the optimal control of the functional I_ε (see* (1.10)).

The remaining part of Section 2 is entirely devoted to the proof of the above theorem.

Step 1. Continuity of $\mathcal{F}_\varepsilon$

Let z_n be any converging sequence in $L^2(\Omega)$, say

$$z_n \to z_0 \quad \text{strongly in } L^2(\Omega). \tag{2.19}$$

Denote $\varphi_{\varepsilon,n} = \varphi_\varepsilon(z_n, \varphi_1)$ the solution of (2.9) corresponding to $z = z_n$. Taking $\varphi_{\varepsilon,n}$ as a test function in the adjoint problem (2.9), we obtain (using (2.10))

$$\|\varphi_{\varepsilon,n}\|_{1,\Omega} \leqslant C\|\varphi_{1\varepsilon}\|_{B,S} \leqslant C,$$

where, here and in the following, C denotes different constants independent of z and n. Hence, up to a subsequence still denoted n, we have

$$\varphi_{\varepsilon,n} \rightharpoonup \varphi_{\varepsilon,0} \quad \text{weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega). \tag{2.20}$$

Let $\varphi \in L^2(\Omega)$. We have

$$
\begin{aligned}
&\left| \int_\Omega g(z_n)\varphi_{\varepsilon,n}\varphi\,dx - \int_\Omega g(z_0)\varphi_{\varepsilon,0}\varphi\,dx \right| \\
&\quad \leqslant \left| \int_\Omega g(z_n)(\varphi_{\varepsilon,n} - \varphi_{\varepsilon,0})\varphi\,dx \right| + \left| \int_\Omega \big(g(z_n) - g(z_0)\big)\varphi_{\varepsilon,0}\varphi\,dx \right| \\
&\quad \leqslant \|g\|_\infty \|\varphi_{\varepsilon,n} - \varphi_{\varepsilon,0}\|_{0,\Omega}\|\varphi\|_{0,\Omega} + \left| \int_\Omega \big(g(z_n) - g(z_0)\big)\varphi_{\varepsilon,0}\varphi\,dx \right|.
\end{aligned} \tag{2.21}
$$

The first term in the right-hand side tends to zero by (2.20). Besides, by (2.19), up to a subsequence, we have

$$z_n \to z_0 \quad \text{for a.e. } x \in \mathbb{R};$$

hence, by (2.2), up to another subsequence, we also have

$$g(z_n) \rightharpoonup g(z_0) \quad \text{weakly* in } L^\infty(\Omega).$$

Therefore, the second term in the right-hand side of (2.21) tends to zero by virtue of Lebesgue's dominated convergence theorem. Hence, up to a subsequence,

$$g(z_n)\varphi_{\varepsilon,n} \underset{n\to+\infty}{\rightharpoonup} g(z_0)\varphi_{\varepsilon,0} \quad \text{weakly in } L^2(\Omega) \text{ and strongly in } H^{-1}(\Omega). \tag{2.22}$$

Let us now pass to the limit in the adjoint problem (2.9) written for z_n and $\varphi_{\varepsilon,n}$. Using a test function $\varphi \in H_0^1(\Omega)$, integrating by parts in Ω and passing to the limit using convergence (2.20), (2.22), we deduce

$$\begin{cases} -\operatorname{div}({}^t A_\varepsilon \nabla\varphi_{\varepsilon,0}) + g(z_0)\varphi_{\varepsilon,0} = -\operatorname{div}(\chi_S B\nabla\varphi_1) + \chi_S\varphi_1 & \text{in } \Omega, \\ \varphi_{\varepsilon,0} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.23}$$

This means that

$$\varphi_{\varepsilon,0} = \varphi_\varepsilon(z_0, \varphi_1). \tag{2.24}$$

Let us now prove that the convergence in (2.20) is actually a strong one, that is

$$\varphi_\varepsilon(z_n, \varphi_1) \underset{n\to+\infty}{\longrightarrow} \varphi_\varepsilon(z_0, \varphi_1) \quad \text{strongly in } H^1(\Omega). \tag{2.25}$$

In fact, multiplying (2.9) (written for $\varphi_{\varepsilon,n}$) by $\varphi_{\varepsilon,n}$, integrating by parts in Ω, and passing to the limit, we obtain

$$
\begin{aligned}
&\lim_{n\to\infty} \int_\Omega {}^t A_\varepsilon \nabla\varphi_{\varepsilon,n} \cdot \nabla\varphi_{\varepsilon,n}\,dx \\
&\quad = -\int_\Omega g(z_0)\,|\varphi_{\varepsilon,0}|^2\,dx + \big\langle -\operatorname{div}(\chi_S B\nabla\varphi_1) + \chi_S\varphi_1, \varphi_{\varepsilon,0}\big\rangle,
\end{aligned} \tag{2.26}
$$

where the bracket is the classical duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. On the other hand, multiplying (2.23) by $\varphi_{\varepsilon,0}$, integrating by parts in Ω and comparing with (2.26), we deduce

$$\lim_{n\to\infty}\int_{\Omega} {}^tA_\varepsilon\nabla\varphi_{\varepsilon,n}\cdot\nabla\varphi_{\varepsilon,n}\,dx=\int_{\Omega} {}^tA_\varepsilon\nabla\varphi_{\varepsilon,0}\cdot\nabla\varphi_{\varepsilon,0}\,dx. \tag{2.27}$$

Since the mapping $\varphi\mapsto\int_\Omega {}^tA_\varepsilon\nabla\varphi\cdot\nabla\varphi\,dx$ defines a norm in $H_0^1(\Omega)$ which is equivalent to the one induced by the usual H^1-topology, we conclude the strong convergence (2.25) from (2.27).

We now prove that the sequence of optimal elements $\varphi_{1,\varepsilon}^*(z_n)$ remains bounded in $H^1(S)$ as $\varepsilon\to 0$ and $n\to\infty$. More precisely, we have

Lemma 2.6. *Assume that z_n satisfies* (2.19). *Then there exists a constant C_ε, independent of n, such that*

$$\|\varphi_{1,\varepsilon}^*(z_n)\|_{B,S}\leqslant C_\varepsilon\quad\forall n\in\mathbb{N}. \tag{2.28}$$

Proof. We argue by contradiction. Assume that there exists a subsequence, which we will still denote by n, such that

$$\|\varphi_{1,\varepsilon}^*(z_n)\|_{1,B_{\varepsilon_n},S}\to\infty\quad\text{as } n\to\infty. \tag{2.29}$$

Since $\varphi_{1,\varepsilon}^*(z_n)$ minimizes $J_\varepsilon^{z_n}$, we have

$$J_\varepsilon^{z_n}\big(\varphi_{1,\varepsilon}^*(z_n)\big)\leqslant J_\varepsilon^{z_n}(\varphi_1)\quad\forall\varphi_1\in H^1(S). \tag{2.30}$$

But

$$J_\varepsilon^{z_n}(\varphi_1)=\frac12\|\varphi_\varepsilon(z_n,\varphi_1)\|_{0,\omega}^2+\alpha\|\varphi_1\|_{B,S}-(\varphi_1,y_1)_{B,S}.$$

Thanks to (2.24), (2.25), $J_\varepsilon^{z_n}(\varphi_1)$ converges, when $n\to\infty$, to

$$J_\varepsilon^{z_0}(\varphi_1)=\frac12\|\varphi_0(z_0,\varphi_1)\|_{0,\omega}^2+\alpha\|\varphi_1\|_{B,S}-(\varphi_1,y_1)_{B,S}.$$

Then, combining this result with (2.30), for any $\delta>0$ and for n large enough, we have

$$J_\varepsilon^{z_n}\big(\varphi_{1,\varepsilon}^*(z_n)\big)\leqslant J_\varepsilon^{z_0}(\varphi_1)+\delta,$$

which obviously contradicts the coercivity property of Lemma 2.3. □

From (2.28), up to a subsequence, there exists a limiting function $\xi_\varepsilon\in H^1(S)$ such that

$$\varphi_{1,\varepsilon}^*(z_n)|_S\xrightarrow[n\to+\infty]{}\xi_\varepsilon\quad\text{weakly in } H^1(S)\text{ and strongly in } L^2(S). \tag{2.31}$$

Arguing as in the proof of (2.25), we deduce from (2.31)

$$\varphi_\varepsilon\big(z_n,\varphi_{1,\varepsilon}^*(z_n)\big)\xrightarrow[n\to+\infty]{}\varphi_\varepsilon(z_0,\xi_\varepsilon)\quad\text{strongly in } H_0^1(\Omega). \tag{2.32}$$

Our next step consists in proving that

$$\xi_\varepsilon=\varphi_{1,\varepsilon}^*(z_0), \tag{2.33}$$

which is the optimal element minimizing $J_\varepsilon^{z_0}$, that is

$$J_\varepsilon^{z_0}(\xi_\varepsilon) \leqslant J_\varepsilon^{z_0}(\varphi_1) \quad \forall \varphi_1 \in H^1(S). \tag{2.34}$$

Since $\varphi_{1,\varepsilon}^*(z_n)$ minimizes $J_\varepsilon^{z_n}$, we have

$$J_\varepsilon^{z_n}\big(\varphi_{1,\varepsilon}^*(z_n)\big) \leqslant J_\varepsilon^{z_n}(\varphi_1) \quad \forall \varphi_1 \in H^1(S),$$

which implies

$$\liminf_{n\to\infty} J_\varepsilon^{z_n}\big(\varphi_{1,\varepsilon}^*(z_n)\big) \leqslant \lim_{n\to\infty} J_\varepsilon^{z_n}(\varphi_1) = J_\varepsilon^{z_0}(\varphi_1). \tag{2.35}$$

Therefore, to prove (2.34), it suffices to show that

$$J_\varepsilon^{z_0}(\xi_\varepsilon) \leqslant \liminf_{n\to\infty} J_\varepsilon^{z_n}\big(\varphi_{1,\varepsilon}^*(z_n)\big). \tag{2.36}$$

Using convergence (2.31) and the definition of $J_\varepsilon^{z_n}$, we have

$$\liminf_{n\to\infty} J_\varepsilon^{z_n}\big(\varphi_{1,\varepsilon}^*(z_n)\big) \geqslant \liminf_{n\to\infty}\left(\frac{1}{2}\big\|\varphi_\varepsilon\big(z_n, \varphi_{1,\varepsilon}^*(z_n)\big)\big\|_{0,\omega}^2\right) + \alpha\|\xi_\varepsilon\|_{B,S} - (\xi_\varepsilon, y_1)_{B,S}.$$

Combining with (2.32), we conclude (2.36), which completes the proof of (2.33). Hence (2.32) becomes

$$\varphi_\varepsilon\big(z_n, \varphi_{1,\varepsilon}^*(z_n)\big) \to \varphi_\varepsilon\big(z_0, \varphi_{1,\varepsilon}^*(z_0)\big) \quad \text{strongly in } H_0^1(\Omega). \tag{2.37}$$

Using the explicit formula (2.18) for the optimal control v_ε^* of problem (2.3), we have

$$\begin{cases} v_\varepsilon^*(z_n) = \varphi_\varepsilon(z_n, \varphi_{1,\varepsilon}^*(z_n))|_\omega, \\ v_\varepsilon^*(z_0) = \varphi_\varepsilon(z_0, \varphi_{1,\varepsilon}^*(z_0))|_\omega. \end{cases}$$

Therefore, from (2.37), we derive

$$v_\varepsilon^*(z_n) \to v_\varepsilon^*(z_0) \quad \text{strongly in } H^1(\omega). \tag{2.38}$$

Finally, arguing as we did for the adjoint problem, we can pass to the limit in problem (2.3) using convergence (2.38), and we obtain

$$y_\varepsilon\big(z_n, v_\varepsilon^*(z_n)\big) \to y_\varepsilon\big(z_0, v_\varepsilon^*(z_0)\big) \quad \text{strongly in } H^1(\Omega). \tag{2.39}$$

This ends the proof of the continuity of $\mathcal{F}_\varepsilon$.

Remark 2.7. In the particular case when φ_1 is defined on the whole of Ω by (2.10), it is worthwhile to notice that it is merely the restriction $\varphi_1|_S$ of φ_1 to S which plays a role in the proof of Theorem 2.5.

Step 2. $\mathcal{F}_\varepsilon(L^2(\Omega))$ *is bounded in* $L^2(\Omega)$

Since for all $z \in L^2(\Omega)$ we have $\|g(z)\|_\infty \leqslant \gamma$, then

$$\big\|\varphi_\varepsilon(z, \varphi_1)\big\|_{1,\Omega} \leqslant C\|\varphi_1\|_{B,S},$$

with C independent of z and ε. This implies the existence of a constant $C = C(\varphi_1)$ such that

$$J_\varepsilon^z(\varphi_{1\varepsilon}) \leqslant C(\varphi_1) \quad \forall \varphi_1 \in H^1(S).$$

In particular, for the optimal element $\varphi^*_{1,\varepsilon}$, we have

$$J^z_\varepsilon\big(\varphi^*_{1,\varepsilon}(z)\big) \leqslant C(\varphi_1) \quad \forall \varphi_1 \in H^1(S).$$

This holds in particular for $\varphi_1 = 0$, thus

$$J^z_\varepsilon\big(\varphi^*_{1,\varepsilon}(z)\big) \leqslant C,$$

with C independent of z and ε.

Using again the coercivity of J^z_ε (see Lemma 2.3), we prove that $\|\varphi^*_{1,\varepsilon}(z)\|_{1,B_\varepsilon,S}$ is bounded independently of z and of ε. Thus we have

$$\big\|\varphi_\varepsilon\big(z, \varphi^*_{1,\varepsilon}(z)\big)\big\|_{1,\Omega} \leqslant C$$

with C independent of z and of ε. This clearly implies that both $v^*_\varepsilon(z)$ and $y_\varepsilon(z, v^*_\varepsilon(z))$ are bounded in their corresponding spaces, i.e., there exists C independent of z and ε such that

$$\big\|v^*_\varepsilon(z)\big\|_{0,\omega} \leqslant C \tag{2.40}$$

and

$$\big\|y_\varepsilon\big(z, v^*_\varepsilon(z)\big)\big\|_{0,\Omega} \leqslant \big\|y_\varepsilon\big(z, v^*_\varepsilon(z)\big)\big\|_{1,\Omega} \leqslant C, \tag{2.41}$$

which concludes the second step.

Step 3. *$\mathcal{F}_\varepsilon$ is compact*

In the second step, a stronger result than the one announced was proved. Indeed, from (2.41), we see that $\mathcal{F}_\varepsilon$ maps the whole of $L^2(\Omega)$ into a bounded subset of $H^1(\Omega)$, and hence into a relatively compact subset of $L^2(\Omega)$. This proves the compactness of $\mathcal{F}_\varepsilon$, and hence completes the proof of Theorem 2.5, provided the coercivity Lemma 2.3 is established. □

2.4. *Proof of Lemma 2.3*

To simplify matters, in this subsection we drop the index ε in the notation for φ_1. From (2.14), for $\varphi_1 \in H^1(S)$ and $\varphi_1 \neq 0$, we have

$$\frac{J^z_\varepsilon(\varphi_1)}{\|\varphi_1\|_{B,S}} = \frac{1}{2\|\varphi_1\|_{B,S}} \int_\omega \big|\varphi_\varepsilon(z,\varphi_1)\big|^2\,dx + \alpha - \int_S B\nabla\bigg(\frac{\varphi_1}{\|\varphi_1\|_{B,S}}\bigg)\cdot\nabla y_1\,dx.$$

Let $\varphi_{1,n} \in H^1(S)$ be a sequence such that

$$\|\varphi_{1,n}\|_{B,S} \xrightarrow[n\to+\infty]{} \infty$$

and

$$\lim_{n\to\infty} \frac{J^z_\varepsilon(\varphi_{1,n})}{\|\varphi_{1,n}\|_{B,S}} = \liminf_{\|\varphi_1\|_{B,S}\to\infty} \frac{J^z_\varepsilon(\varphi_1)}{\|\varphi_1\|_{B,S}}. \tag{2.42}$$

We introduce the following normalizations:

$$\hat{\varphi}_{1,n} = \frac{\varphi_{1,n}}{\|\varphi_{1,n}\|_{B,S}} \quad \text{and} \quad \hat{\varphi}_{\varepsilon,n} = \frac{\varphi_{\varepsilon,n}(z, \varphi_{1,n})}{\|\varphi_{1,n}\|_{1,B,S}}. \tag{2.43}$$

Then we have

$$\frac{J_\varepsilon^z(\varphi_{1,n})}{\|\varphi_{1,n}\|_{B,S}} = \alpha + \frac{1}{2}\|\varphi_{1,n}\|_{B,S} \int_\omega |\hat{\varphi}_{\varepsilon,n}|^2\,dx - \int_S B\nabla\hat{\varphi}_{1,n} \cdot \nabla y_1\,dx. \tag{2.44}$$

Also, since $\|\hat{\varphi}_{1,n}\|_{B,S} = 1$, using $\varphi_{\varepsilon,n}$ as a test function in the adjoint problem (2.9), we deduce

$$\|\hat{\varphi}_{\varepsilon,n}\|_{1,\Omega} \leqslant C \quad (C \text{ independent of } \varepsilon \text{ and } n). \tag{2.45}$$

Therefore, up to a subsequence, we have

$$\begin{cases} \hat{\varphi}_{1,n} \rightharpoonup \tilde{\varphi}_1 & \text{weakly in } L^2(S), \\ \hat{\varphi}_{\varepsilon,n} \rightharpoonup \tilde{\varphi}_\varepsilon & \text{weakly in } H^1(\Omega) \text{ and strongly in } L^2(\Omega). \end{cases} \tag{2.46}$$

Let us distinguish various cases.

Case (i). Assume that

$$\lim_{n\to\infty} \int_\omega |\hat{\varphi}_{\varepsilon,n}|^2\,dx \left(= \int_\omega |\tilde{\varphi}_\varepsilon|^2\,dx\right) > 0,$$

then the second term in the right-hand side of (2.44) tends to infinity while the third term has a limit. Hence (2.16) holds in this case.

Case (ii). If

$$\lim_{n\to\infty} \int_\omega |\hat{\varphi}_{\varepsilon,n}|^2\,dx \left(= \int_\omega |\tilde{\varphi}_\varepsilon|^2\,dx\right) = 0,$$

then $\tilde{\varphi}_\varepsilon = 0$ in ω. From the smoothness hypothesis on the coefficients of the matrix A_ε (see (1.1)), since $S \subset\subset \omega$, we have an homogeneous problem (2.9) in $\Omega \setminus \omega$ with homogeneous Dirichlet boundary conditions, and this implies that $\tilde{\varphi}_\varepsilon = 0$ in Ω. Therefore $\tilde{\varphi}_1 = 0$, and so

$$\lim_{n\to\infty} \frac{J_\varepsilon^z(\varphi_{1,n})}{\|\varphi_{1,n}\|_{B,S}} \geqslant \alpha + \liminf_{n\to\infty}\left(\|\varphi_{1,n}\|_{B,S} \int_\omega |\hat{\varphi}_{\varepsilon,n}|^2\,dx\right) \geqslant \alpha > 0,$$

which ends the proof of Lemma 2.3 and therefore that of Theorem 2.5. □

3. Homogenization of the approximate controllability problem

Our goal in this section is to pass to the limit in problem (1.3) when $v = v_\varepsilon^*$ is the optimal control constructed in Section 2.

To this general end, we begin by considering a sequence of matrices $A_\varepsilon \in \mathcal{M}(\alpha_m, \alpha_M)$ and the corresponding state equations

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla y_\varepsilon(v)) + f(y_\varepsilon(v)) = \chi_\omega v & \text{in } \Omega, \\ y_\varepsilon(v) = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $v \in L^2(\omega)$. The main additional assumption in this section is that

$$A_\varepsilon \ H\text{-converges to } A_0 \tag{3.2}$$

(see [6,8] for details about H-convergence). It is well-known that $A_0 \in \mathcal{M}(\alpha_m, \alpha_M^2/\alpha_m)$.

3.1. Homogenization of the state equation for a fixed control

In this section, we assume that the control v is a fixed element in $L^2(\omega)$. We prove the following homogenization result:

Proposition 3.1. *Assume that the hypotheses of Theorem* 2.5 *hold and that A_ε satisfies* (3.2). *Then, up to a subsequence, there exists $y_0(v)$ such that*

$$\begin{cases} y_\varepsilon(v) \rightharpoonup y_0(v) & \text{weakly in } H_0^1(\Omega), \\ A_\varepsilon \nabla y_\varepsilon(v) \rightharpoonup A_0 \nabla y_0(v) & \text{weakly in } L^2(\Omega)^N. \end{cases} \tag{3.3}$$

Moreover, $y_0(v)$ satisfies the homogenized state equation

$$\begin{cases} -\operatorname{div}(A_0 \nabla y_0(v)) + f(y_0(v)) = \chi_\omega v & \text{in } \Omega, \\ y_0(v) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Proof. Since f satisfies (1.4), the first convergence in (3.3) is straightforward. We now wish to establish a convergence result on $f(y_\varepsilon(v))$. For all $\varphi \in L^2(\Omega)$, we have

$$\left| \int_\Omega \big(f\big(y_\varepsilon(v)\big) - f\big(y_0(v)\big)\big)\varphi\, dx \right| \leqslant \left| \int_\Omega g\big(y_\varepsilon(v)\big)\big(y_\varepsilon(v) - y_0(v)\big)\varphi\, dx \right| + \left| \int_\Omega \big(g\big(y_\varepsilon(v)\big) - g\big(y_0(v)\big)\big) y_0(v)\varphi\, dx \right|.$$

Arguing as we did in Section 2.3 to establish (2.22), we prove that, up to a subsequence,

$$f\big(y_\varepsilon(v)\big) \rightharpoonup f\big(y_0(v)\big) \quad \text{weakly in } L^2(\Omega) \text{ and strongly in } H^{-1}(\Omega). \tag{3.5}$$

We are now in a position to pass to the limit in problem (3.1). Thanks to (3.5) and H-convergence properties, we end the proof of Proposition 3.1. □

3.2. Homogenization of the state equation for an optimal control

Denote $\bar{z}_\varepsilon$ the fixed point of $\mathcal{F}_\varepsilon$ constructed in Section 2 using Schauder's theorem. Since the constant in (2.40) is independent of z and ε, the sequence of optimal controls $v_\varepsilon^*(\bar{z}_\varepsilon)$ remains bounded in $L^2(\omega)$ as $\varepsilon \to 0$. Thus, up to a subsequence, there exists $v_0 \in L^2(\omega)$ such that

$$\begin{cases} v_\varepsilon^*(\bar{z}_\varepsilon) \rightharpoonup v_0 & \text{weakly in } L^2(\omega) \text{ and strongly in } H^{-1}(\omega), \\ \chi_\omega v_\varepsilon^*(\bar{z}_\varepsilon) \rightharpoonup \chi_\omega v_0 & \text{weakly in } L^2(\Omega) \text{ and strongly in } H^{-1}(\Omega). \end{cases} \tag{3.6}$$

As in Section 2, the estimate (2.40) implies that the solution $y_\varepsilon^* = y_\varepsilon(\bar{z}_\varepsilon, v_\varepsilon^*(\bar{z}_\varepsilon))$ of problem (1.3) satisfies

$$\|y_\varepsilon^*\|_{0,\Omega} \leqslant \|y_\varepsilon^*\|_{1,\Omega} \leqslant C,$$

where C is independent of ε. Hence there exists $y_0 \in H_0^1(\Omega)$ such that, up to a subsequence,

$$y_\varepsilon^* \rightharpoonup y_0(v_0) \quad \text{weakly in } H_0^1(\Omega). \tag{3.7}$$

Clearly, as in Section 3.1, we derive from (3.2), (3.6), (3.7), that $y_0(v_0)$ is solution of the homogenized problem

$$\begin{cases} -\operatorname{div}(A_0 \nabla y_0(v_0)) + f(y_0(v_0)) = \chi_\omega v_0 & \text{in } \Omega, \\ y_0(v_0) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Our aim is to prove that v_0 satisfies the following approximate controllability inequality:

$$\big\| y_0(v_0)|_S - y_1 \big\|_{B,S} \leqslant \alpha.$$

Furthermore, we will prove that v_0 is optimal in the sense that it minimizes, over all $v \in L^2(\omega)$, the cost functional

$$I_0(v) \stackrel{\text{def}}{=} \frac{1}{2}\|v\|_{0,\omega}^2 + \begin{cases} 0 & \text{if } \|y_0(v)|_S - y_1\|_{B,S} \leqslant \alpha, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.9}$$

where $y_0(v)$ is the solution of (3.8) corresponding to the control v.

To reach this aim, we begin by writing down the fixed point identity

$$\bar z_\varepsilon = y_\varepsilon\big(\bar z_\varepsilon, v_\varepsilon^*(\bar z_\varepsilon)\big) = y_\varepsilon^*.$$

Thus, from (3.7) there exists $z_0 \in H_0^1(\Omega)$ such that, up to a subsequence,

$$\bar z_\varepsilon \rightharpoonup z_0 \quad \text{weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega). \tag{3.10}$$

For any given control $v \in L^2(\omega)$, let $y_0(z_0, v)$ be the solution of the homogenized linearized problem

$$\begin{cases} -\operatorname{div}(A_0 \nabla y_0(z_0, v)) + g(z_0, v) = \chi_\omega v & \text{in } \Omega, \\ y_0(z_0, v) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

To this state equation, we associate the cost functional

$$I_0^{z_0}(v) = \frac{1}{2}\|v\|_{0,\omega}^2 + \begin{cases} 0 & \text{if } \|y_0(z_0, v)|_S - y_1\|_{B,S} \leqslant \alpha, \\ \infty & \text{otherwise.} \end{cases} \tag{3.12}$$

By classical linear control theory and Proposition 3.1 there exists a unique optimal control $v_0^*(z_0)$ such that

$$I_0^{z_0}\big(v_0^*(z_0)\big) = \min_{v \in L^2(\omega)} I_0^{z_0}(v) < +\infty. \tag{3.13}$$

We denote by $y_0^* = y_0(z_0, v_0^*(z_0))$ the corresponding state.

We are now in a position to prove our main result, namely

Theorem 3.2. *We make the hypotheses of Theorem* 2.5 *and we also assume the H-convergence* (3.2) *of A_ε to A_0. Let v_0 be the limit of the optimal controls defined in* (3.6). *Then*

$$v_0 = v_0^*(z_0), \tag{3.14}$$

where $v_0^*(z_0)$ is the optimal control of the linearized problem (3.11), (3.13).

Proof. We proceed in several steps.

Step 1. Existence of the optimal control $v_0^*(z_0)$

We use again the classical Fenchel–Rockafellar's duality theory which provides an explicit control of minimal norm. Given $\varphi_1 \in H^1(S)$, we introduce $\varphi_0(z_0, \varphi_1)$, the solution of

$$\begin{cases} -\operatorname{div}({}^t A_0 \nabla \varphi_0(z_0, \varphi_1)) + g(z_0)\varphi_0(z_0, \varphi_1) \\ \quad = -\operatorname{div}(\chi_S B \nabla \varphi_1) + \chi_S \varphi_1 \quad \text{in } \Omega, \\ \varphi_0(z_0, \varphi_1) = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.15}$$

By duality, as in Section 2, we have

$$\inf_{v \in L^2(\omega)} I_0^{z_0}(v) = -\inf_{\varphi_1 \in H^1(S)} J_0^{z_0}(\varphi_1), \tag{3.16}$$

where

$$J_0^{z_0}(\varphi_1) = \frac{1}{2} \left\| \varphi_0(z_0, \varphi_1) \right\|_{0,\omega}^2 + \alpha \|\varphi_1\|_{B,S} - (\varphi_1, y_1)_{B,S}. \tag{3.17}$$

It is also well-known, from the extremal relations for the above optimization problem, that

$$v_0^*(z_0) = \varphi_0\big(z_0, \varphi_1^*(z_0)\big)\big|_\omega, \tag{3.18}$$

where $\varphi_1^*(z_0) \in H^1(S)$ is the unique optimal element which minimizes $J_0^{z_0}$ over $H^1(S)$.

Step 2. Passage to the limit in the adjoint problem

From system (2.9) and convergence (3.10), we derive easily that there exists a function $\bar{\varphi}_0$ such that, up to a subsequence

$$\varphi_\varepsilon(\bar{z}_\varepsilon, \varphi_1) \rightharpoonup \bar{\varphi}_0 \quad \text{weakly in } H_0^1(\Omega). \tag{3.19}$$

By H-convergence results, we pass to the limit in (2.9) and we deduce that $\bar{\varphi}_0$ is the solution of

$$\begin{cases} -\operatorname{div}({}^t A_0 \nabla \bar{\varphi}_0) + g(z_0)\bar{\varphi}_0 = -\operatorname{div}(\chi_S B \nabla \varphi_1) + \chi_S \varphi_1 \quad \text{in } \Omega, \\ \bar{\varphi}_0 = 0 \quad \text{on } \partial\Omega, \end{cases}$$

that is (compare with (3.15))

$$\bar{\varphi}_0 = \varphi_0(z_0, \varphi_1). \tag{3.20}$$

We are now in a position to pass to the limit in $J_\varepsilon^{\bar{z}_\varepsilon}$ defined by (2.14). Recall that

$$J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_1) = \frac{1}{2}\left\|\varphi_\varepsilon(\bar{z}_\varepsilon, \varphi_1)\right\|_{0,\omega}^2 + \alpha \int_\Omega \left(\chi_S B \nabla\varphi_1 \cdot \nabla\varphi_1 + \chi_S|\varphi_1|^2\right) dx - \int_\Omega \chi_S B \nabla\varphi_1 \cdot \nabla y_1 \, dx.$$

To pass to the limit, we use convergence (3.19). We obtain

$$\lim_{\varepsilon\to 0} J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_1) = J_0^{z_0}(\varphi_1) \quad \forall \varphi_1 \in H^1(S). \tag{3.21}$$

From (3.21), we derive that between the optimal elements $\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)$ and $\varphi_1^*(z_0)$, we have the following relation:

$$J_\varepsilon^{\bar{z}_\varepsilon}\left(\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)\right) = \min_{\varphi_1} J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_1) \to \min_{\varphi_1} J_0^{z_0}(\varphi_1) = J_0^{z_0}\left(\varphi_1^*(z_0)\right). \tag{3.22}$$

Step 3. Convergence of the optimal controls for the state equation

Using the uniform coercivity property of the functionals $J_\varepsilon^{\bar{z}_\varepsilon}$ (see Lemma 2.3) and arguing as in the proof of Lemma 2.6, we deduce the existence of a constant C independent of ε such that

$$\left\|\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)\right\|_{1,B_\varepsilon,S} \leqslant C.$$

Since the matrices B_ε are equi-coercive, we derive the existence of an element $\xi^* \in H^1(S)$ such that, up to a subsequence,

$$\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon) \rightharpoonup \xi^* \quad \text{weakly in } H^1(S). \tag{3.23}$$

This implies that, up to another subsequence,

$$\varphi_\varepsilon\left(\bar{z}_\varepsilon, \varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)\right) \rightharpoonup \varphi_0(z_0, \xi^*) \quad \text{weakly in } H_0^1(\Omega). \tag{3.24}$$

Our next aim is to prove that ξ^* is equal to $\varphi_1^*(z_0)$, the unique minimizer of $J_0^{z_0}$, that is

$$J_0^{z_0}(\xi^*) \leqslant J_0^{z_0}(\varphi_1) \quad \forall \varphi_1 \in H^1(S). \tag{3.25}$$

Since $\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)$ minimizes $J_\varepsilon^{\bar{z}_\varepsilon}$, we have

$$J_\varepsilon^{\bar{z}_\varepsilon}\left(\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)\right) \leqslant J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_1) \quad \forall \varphi_1 \in H^1(S).$$

Thanks to (3.21), we deduce that

$$\liminf_{\varepsilon\to 0} J_\varepsilon^{\bar{z}_\varepsilon}\left(\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)\right) \leqslant \lim_{\varepsilon\to 0} J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_1) = J_0^{z_0}(\varphi_1).$$

Therefore, to prove (3.25), it suffices to show that

$$J_0^{z_0}(\xi^*) \leqslant \liminf_{\varepsilon\to 0} J_\varepsilon^{\bar{z}_\varepsilon}\left(\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)\right). \tag{3.26}$$

Using the definition of $J_\varepsilon^{\bar{z}_\varepsilon}$, we have

$$\liminf_{\varepsilon\to 0} J_\varepsilon^{\bar z_\varepsilon}\big(\varphi_{1,\varepsilon}^*(\bar z_\varepsilon)\big)$$
$$= \lim_{\varepsilon\to 0}\left(\frac{1}{2}\big\|\varphi_\varepsilon\big(\bar z_\varepsilon, \varphi_{1,\varepsilon}^*(\bar z_\varepsilon)\big)\big\|_{0,\omega}^2\right) + \alpha\|\xi^*\|_{B,S} - (\xi^*, y_1)_{B,S} = J_0^{z_0}(\xi^*),$$

which proves (3.26) and hence (3.25). Thus (3.23), (3.24) become

$$\begin{cases} \varphi_{1,\varepsilon}^*(\bar z_\varepsilon) \rightharpoonup \varphi_1^*(z_0) \quad \text{weakly in } H^1(S), \\ \varphi_\varepsilon(\bar z_\varepsilon, \varphi_{1,\varepsilon}^*(\bar z_\varepsilon)) \rightharpoonup \varphi_0(z_0, \varphi_1^*(z_0)) \quad \text{weakly in } H_0^1(\Omega). \end{cases} \tag{3.27}$$

To conclude, let us write the explicit formula (2.18) for $z = \bar z_\varepsilon$:

$$v_\varepsilon^*(\bar z_\varepsilon) = \varphi_\varepsilon\big(\bar z_\varepsilon, \varphi_{1,\varepsilon}^*(\bar z_\varepsilon)\big)\big|_\omega.$$

From (3.6), the left-hand side converges to v_0 and from (3.27), the right-hand side converges to $\varphi_0(z_0, \varphi_1^*(z_0))|_\omega$. Then, combining with (3.18), we deduce

$$v_0 = \varphi_0\big(z_0, \varphi_1^*(z_0)\big)\big|_\omega = v_0^*(z_0),$$

which completes the proof of Theorem 3.2. □

4. Homogenization of a cost functional with rapidly oscillating coefficients. Open questions

Our aim in this section is to study the same problems when the fixed symmetric matrix B is replaced by an ε-dependent symmetric matrix $B_\varepsilon \in \mathcal{M}(\alpha_m, \alpha_M)$ with rapidly oscillating coefficients.

The approximate controllability inequality (1.5) is now replaced by

$$\big\|y_\varepsilon(v) - y_1\big\|_{B_\varepsilon,S} \leqslant \alpha. \tag{4.1}$$

Since in Section 2 the parameter ε was fixed, all the results therein hold true in this new framework. Our goal is to pass to the limit as $\varepsilon \to 0$ and to see how the results of Section 3 are modified or can be generalized. We still assume that (3.2) holds as well as a H-convergence result for B_ε, namely

$$B_\varepsilon \ H\text{-converges to } B_0. \tag{4.2}$$

In what follows, we will need some kind of limiting matrix $(\chi_S B)_\#$ whose definition requires the introduction of three auxiliary functions, namely $X_k^\varepsilon, Y_k^\varepsilon, \psi_k^\varepsilon$, which are defined by

$$\begin{cases} X_k^\varepsilon \rightharpoonup 0 \quad \text{weakly in } H^1(\Omega), \\ \operatorname{div}(A_\varepsilon\nabla(-X_k^\varepsilon + x_k)) \to \operatorname{div}(A_0 e_k) \quad \text{strongly in } H^{-1}(\Omega), \end{cases} \tag{4.3}$$

$$\begin{cases} Y_k^\varepsilon \rightharpoonup 0 \quad \text{weakly in } H^1(\Omega), \\ \operatorname{div}(B_\varepsilon\nabla(-X_k^\varepsilon + x_k)) \to \operatorname{div}(B_0 e_k) \quad \text{strongly in } H^{-1}(\Omega), \end{cases} \tag{4.4}$$

$$\begin{cases} \psi_k^\varepsilon \rightharpoonup \psi_k^0 \quad \text{weakly in } H^1(\Omega), \\ \operatorname{div}({}^t A_\varepsilon\nabla\psi_k^\varepsilon + B_\varepsilon\nabla(-X_k^\varepsilon + x_k)) = 0 \quad \text{in } \Omega. \end{cases} \tag{4.5}$$

Here, $e_k \in \mathbb{R}^N$ is the kth standard basis vector and x_k denotes the function mapping $x \in \mathbb{R}^N$ to its kth coordinate.

The matrix $(\chi_S B)_\#$ is defined by means of the following formula:

$$(\chi_S B)_\# e_k = \chi_S B_0 + \lim_{\varepsilon\to 0}\bigl({}^tA_\varepsilon \nabla\psi_k^\varepsilon - {}^tA_0\nabla\psi_k^0\bigr) + \chi_S \lim_{\varepsilon\to 0}\bigl(B_\varepsilon\bigl(Y_k^\varepsilon - X_k^\varepsilon\bigr)\bigr). \tag{4.6}$$

The following proposition, whose proof can be found in Kesavan and Saint Jean Paulin [3], summarizes the main properties of $(\chi_S B)_\#$.

Proposition 4.1. *The matrix $(\chi_S B)_\#$ is symmetric and there exists $\tilde\alpha_M > 0$ such that $(\chi_S B)_\# \in \mathcal{M}(\alpha_m, \tilde\alpha_M)$.*

We use this matrix $(\chi_S B)_\#$ in order to pass to the limit in the adjoint problem (2.9), which we now rewrite in a slightly different form. Given $h \in H^{-1}(\Omega)$, let $\varphi_{1\varepsilon} \in H_0^1(\Omega)$ be the unique solution of

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla\varphi_{1\varepsilon}) = h \quad \text{in } \Omega, \\ \varphi_{1\varepsilon} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

The adjoint state $\varphi_\varepsilon = \varphi_\varepsilon(z, \varphi_{1\varepsilon})$ is defined as the unique solution of

$$\begin{cases} -\operatorname{div}({}^tA_\varepsilon \nabla\varphi_\varepsilon(z,\varphi_{1\varepsilon}) - (\chi_S B_\varepsilon)\nabla\varphi_{1\varepsilon}) \\ \quad = -g(z)\varphi_\varepsilon(z,\varphi_{1\varepsilon}) + \chi_S\varphi_{1\varepsilon} \quad \text{in } \Omega, \\ \varphi_\varepsilon(z,\varphi_{1\varepsilon}) = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.8}$$

Of course, Proposition 3.1 still holds true. Furthermore, if v_0 denotes the weak limit of the optimal controls $v_\varepsilon^*(\bar z_\varepsilon)$ (see (3.6)), then we still have (3.7), (3.8). This means that the state equation can be homogenized as in the easier case of a constant matrix B. The homogenization of the adjoint equation is not so easy and it requires the matrix $(\chi_S B)_\#$. Precisely, from systems (4.7), (4.8) and convergence (3.10), we derive easily that there exist functions φ_1 and $\bar\varphi_0$ such that, up to a subsequence,

$$\begin{cases} \varphi_{1\varepsilon} \rightharpoonup \varphi_1 \quad \text{weakly in } H_0^1(\Omega), \\ \varphi_\varepsilon(\bar z_\varepsilon, \varphi_{1\varepsilon}) \rightharpoonup \bar\varphi_0 \quad \text{weakly in } H_0^1(\Omega). \end{cases} \tag{4.9}$$

Of course, by H-convergence results, it is clear that φ_1 is the unique solution of

$$\begin{cases} -\operatorname{div}(A_0 \nabla\varphi_1) = h \quad \text{in } \Omega, \\ \varphi_1 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.10}$$

Besides, the right-hand side of (4.8) satisfies, up to a subsequence,

$$-g(\bar z_\varepsilon)\varphi_\varepsilon(\bar z_\varepsilon, \varphi_{1\varepsilon}) + \chi_S\varphi_{1\varepsilon} \rightharpoonup -g(z_0)\bar\varphi_0 + \chi_S\varphi_1$$
$$\text{weakly in } L^2(\Omega) \text{ and strongly in } H^{-1}(\Omega).$$

Therefore, a slight generalization of Theorem 3.1 in [3] allows us to pass to the limit in (4.8) and to deduce that $\bar\varphi_0$ is the solution of

$$\begin{cases} -\operatorname{div}({}^tA_0 \nabla\bar\varphi_0 - (\chi_S B)_\#\nabla\varphi_1) = -g(z_0)\bar\varphi_0 + \chi_S\varphi_1 \quad \text{in } \Omega, \\ \bar\varphi_0 = 0 \quad \text{on } \partial\Omega, \end{cases}$$

that is

$$\bar{\varphi}_0 = \varphi(z_0, \varphi_1), \tag{4.11}$$

where $\varphi(z_0, \varphi_1)$ is defined as the solution of a new homogenized adjoint problem analogous to (3.15) with $\chi_S B$ replaced by $(\chi_S B)_\#$.

Our next step would be to pass to the limit in the sequence $J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon))$ where $\varphi_{1,\varepsilon}^*$ is the minimizer of $J_\varepsilon^{\bar{z}_\varepsilon}$. This is the main open question of this section. Of course the desired result would be to prove that

$$\lim_{\varepsilon\to 0} J_\varepsilon^{\bar{z}_\varepsilon}\big(\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)\big) = J_0^{z_0}(\varphi_1^*), \tag{4.12}$$

where φ_1^* is the minimizer of the homogenized functional

$$J_0^{z_0}(\varphi_1) = \frac{1}{2}\left\|\varphi_0(z_0, \varphi_1)\right\|_{0,\omega}^2 + \alpha\|\varphi_1\|_{(\chi_S B)_\#, S} - (\varphi_1, y_1)_{(\chi_S B)_\#, S}.$$

This implies that Theorem 3.2 would also be true in the present case. However, we have a strong doubt about the validity of (4.12). Indeed, it is not difficult to check (using [3, Theorem 3.3]) that

$$\lim_{\varepsilon\to 0} J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_1) = \tilde{J}_0^{z_0}(\varphi_1) \quad \forall \varphi_1 \in H^1(S),$$

where

$$\tilde{J}_0^{z_0}(\varphi_1) = \frac{1}{2}\left\|\varphi_0(z_0, \varphi_1)\right\|_{0,\omega}^2 + \alpha\|\varphi_1\|_{(\chi_S B)_\#, S} - (\varphi_1, y_1)_{\tilde{B}, S}$$

(compare with (3.21)). Here, $\tilde{B}$ is another kind of limiting matrix, similar to $(\chi_S B)_\#$ which can be explicitly constructed using B_ε and the first correctors terms associated with the H-convergence sequence A_ε.

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