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ASYMPTOTIC ANALYSIS RELATING SPECTRAL MODELS IN FLUID–SOLID VIBRATIONS*

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Abstract. An asymptotic study of two spectral models which appear in fluid–solid vibrations is presented in this paper. These two models are derived under the assumption that the fluid is slightly compressible or viscous, respectively. In the first case, min-max estimations and a limit process in the variational formulation of the corresponding model are used to show that the spectrum of the compressible case tends to be a continuous set as the fluid becomes incompressible. In the second case, we use a suitable family of unbounded non-self-adjoint operators to prove that the spectrum of the viscous model tends to be continuous as the fluid becomes inviscid. At the limit, in both cases, the spectrum of a perfect incompressible fluid model is found. We also prove that the set of generalized eigenfunctions associated with the viscous model is dense for the L^2-norm in the space of divergence-free vector functions. Finally, a numerical example to illustrate the convergence of the viscous model is presented.

Key words. asymptotic distribution of eigenvalues, fluid–solid vibrations, completeness of generalized eigenfunctions

AMS subject classifications. 35P20, 35P10, 47A75, 73K70

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1. Introduction and main results.

1.1. Introduction. In this paper, we study the asymptotic behavior of some spectral models which represent the vibrations of a bundle of tubes surrounded by a fluid. These types of models have considerable importance in engineering, as they are used in the design and simulation of various sorts of industrial equipment. In recent decades, much effort has been devoted to experimental and theoretical research in this subject. For a more detailed treatment of these investigations, see, for example, the articles of R. Blevins [2], [3], S. Chen [4], [5], [6], H. Connors [14], D. Gorman [19], M. Paidoussis [27], [28], M. Pettigrew [29], and J. Planchard [30], [31], [32], [33], [34], [35]; see also [13].

To introduce the physical problem, let us imagine a mobile structure composed of K parallel tubes of constant section R_i with rigidity k and mass m, immersed in a fluid which occupies a three-dimensional region with a constant bounded section Ω (the region Ω is assumed to be connected). Let Γ_i be the boundary of each section R_i for $i = 1, \dots, K$ and let Γ_0 be the exterior boundary of Ω. We assume that all the boundaries are locally Lipschitz continuous, and we denote by $\mathbf{n}$ the unit normal oriented as in Figure 1.

Eigenfluctuations of the type $\mathbf{u}(x)e^{wt}$ (velocity) and $p(x)e^{wt}$ (pressure) are sought, where $w \in \mathbb{C}$ is called an eigenfrequency of the model, and $\mathbf{u}$ or p, the associated eigen-

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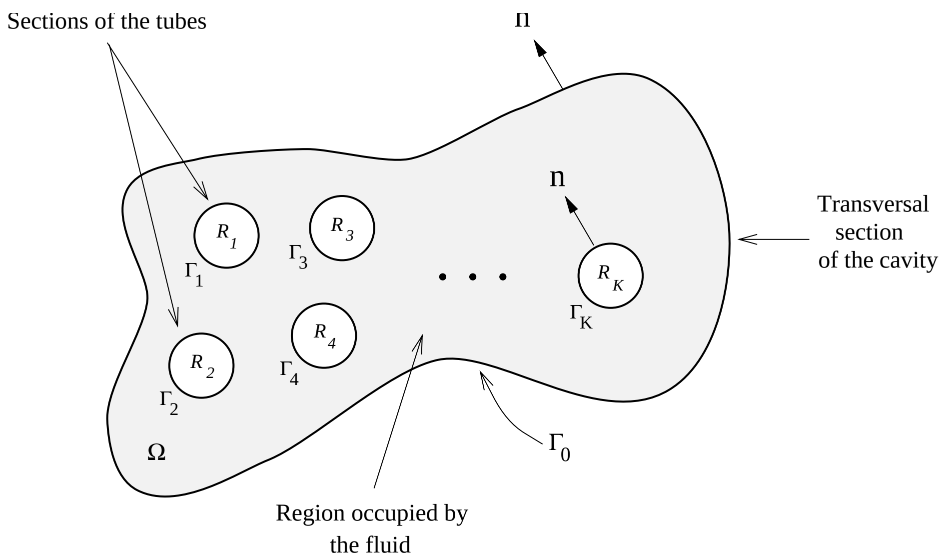

FIG. 1. *Section of the problem and principal notations.*

functions. In engineering terms, the imaginary and real parts of an eigenfrequency ω represent, respectively, the physical frequency and damping. Under certain simplifying assumptions, it is shown that the spectral modeling of this problem only depends on the physical properties of the fluid, basically on its compressibility and its viscosity (see [13, Chapters 2 and 6]).

We summarize as follows the principal models related to this article and some properties of their spectra which we will use later.

(a) *Case of a perfect incompressible fluid* (*Laplace model*). Find $w \in \mathbb{C}$ and $\psi \not\equiv 0$ (pressure) such that

$$\Delta\psi = 0 \quad \text{in } \Omega, \tag{1.1a}$$

$$\frac{\partial\psi}{\partial n} = 0 \quad \text{on } \Gamma_0, \qquad \int_\Omega \psi \; dx = 0, \tag{1.1b}$$

$$\frac{\partial\psi}{\partial n} = -\frac{w^2}{k+mw^2}\left(\int_{\Gamma_i} \psi \mathbf{n} ds\right)\cdot \mathbf{n} \qquad \text{on } \Gamma_i \;\forall i = 1,\ldots,K. \tag{1.1c}$$

In this model, we know (see [20], [21], [30], [31], [32], [33]) that there exist exactly $2K$ conjugate pairs of pure imaginary eigenfrequencies which we denote by $iw_{j,L}$, $-iw_{j,L}$ (where i is the imaginary unit in $\mathbb{C}$) for $j = 1,\ldots,2K$, such that

$$0 < w_{1,L}^2 \le w_{2,L}^2 \le \cdots \le w_{2K,L}^2 < \frac{k}{m}. \tag{1.2}$$

(b) *Case of a perfect slightly compressible fluid* (*Helmholtz model*). Find $w \in \mathbb{C}$ and $\phi \not\equiv 0$ (pressure) such that

$$c^2\Delta\phi - w^2\phi = 0 \quad \text{in } \Omega, \tag{1.3a}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \Gamma_0, \qquad \int_\Omega \phi \; dx = 0, \tag{1.3b}$$

$$\frac{\partial\phi}{\partial n} = -\frac{w^2}{k+mw^2}\left(\int_{\Gamma_i} \phi \mathbf{n} ds\right)\cdot \mathbf{n} \qquad \text{on } \Gamma_i \;\forall i = 1,\ldots,K. \tag{1.3c}$$

In this model, it is known (see [10], [11], or [12]) that there exists a sequence of conjugate pairs of pure imaginary eigenfrequencies which we denote by $i\omega_j(c)$, $-i\omega_j(c)$, $j \geq 1$. They are such that

$$0 < w_1^2(c) \leq w_2^2(c) \leq \cdots \leq w_{2K}^2(c) \leq w_{2K+1}^2(c) \cdots \to +\infty, \tag{1.4}$$

$$w_j^2(c) < k/m, \qquad 1 \leq j \leq 2K, \tag{1.5}$$

and we make explicit their dependency on c (the speed of sound in the fluid).

(c) *Case of a viscous incompressible fluid* (*Stokes model*). Find $w \in \mathbb{C}$, $\mathbf{u}$ (velocity), $\mathbf{u} \not\equiv \mathbf{0}$, and p (pressure) nonconstant such that

$$-\nu\Delta\mathbf{u} + \nabla p + w\mathbf{u} = \mathbf{0} \quad \text{ in } \Omega, \tag{1.6a}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.6b}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_0, \tag{1.6c}$$

$$\mathbf{u} = -\frac{w}{k + mw^2}\left(\int_{\Gamma_i} \sigma(\mathbf{u}, p)\mathbf{n}ds\right) \quad \text{ on } \Gamma_i \ \forall i = 1, \ldots, K\ . \tag{1.6d}$$

Finally, in this model (see [13], [8], [9], or [7]) the spectrum is composed of an unbounded sequence of strictly negative real eigenfrequencies and of at most $2K$ conjugate pairs of nonreal eigenfrequencies, which lie in the following region of the complex plane:

$$\{z \in \mathbb{C} \ : \ |z| < \sqrt{k/m}\ , \text{Re } z < 0\}, \tag{1.7}$$

where Re z denotes the real part of the complex number z.

Our main purpose is to study the limit spectral properties of the Helmholtz model (b) (a perfect compressible fluid) and Stokes model (c) (a viscous incompressible fluid), when the physical properties of the fluid are close to the ideal case (a perfect incompressible fluid) represented by the Laplace model (a).

We have divided our analysis into two cases: the limit spectral behavior of the Helmholtz model as the fluid tends to be incompressible, and the same problem for the Stokes model as the fluid becomes inviscid. Our task is to develop a suitable mathematical treatment of each problem and to obtain precise convergence results. From a numerical point of view, we are interested in carrying out computational experiments in the second case to verify our theoretical predictions.

The functional framework introduced to study the Stokes model leads us to state another interesting property: the denseness of their generalized eigenfunctions in the L^2-space of divergence-free vector functions with suitable boundary conditions.

1.2. Main results. Our work and main results are detailed as follows.

In section 2 we analyze the case when the fluid becomes incompressible, that is, when the speed of sound in the fluid tends to infinity ($c \to \infty$). We prove that a part of the spectrum of the Helmholtz model converges to the spectrum of the Laplace model, and the other part diverges. This is stated in the following theorem.

Let us define the constant δ_1 as the smallest eigenvalue of the following problem:

find $\delta \in \mathbb{R}$ and $\phi : \Omega \to \mathbb{R}$, $\phi \not\equiv 0$, such that

$$\Delta\phi + \delta^2\phi = 0 \quad \text{ in } \Omega, \tag{1.8a}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{ on } \Gamma_0, \tag{1.8b}$$

$$\frac{\partial \phi}{\partial n} = \mathbf{s}_i \cdot \mathbf{n} \quad \text{on } \Gamma_i \quad \text{for any} \quad \mathbf{s}_i \in \mathbb{R}^2,\ i = 1, \dots, K, \tag{1.8c}$$

$$\int_{\Gamma_i} \phi \mathbf{n}\, ds = 0 \quad \forall i = 1, \dots, K, \qquad \int_\Omega \phi\, dx = 0. \tag{1.8d}$$

It is easy to show that this constant is strictly positive and depends only on Ω.

THEOREM 1.1. *If $c > \frac{1}{\delta_1}\sqrt{k/m}$, then there exist exactly $2K$ conjugate pairs of eigenfrequencies of the Helmholtz model with absolute value in the interval $]0, \sqrt{k/m}[$, which converge as $c \to \infty$ to the eigenfrequencies of the Laplace model; that is,*

$$w_j^2(c) \longrightarrow w_{j,L}^2 \quad \text{for each} \quad j = 1, \dots, 2K \quad \text{as} \quad c \to \infty.$$

For the other eigenfrequencies we have, uniformly on j,

$$w_j^2(c) \longrightarrow +\infty \quad \text{for} \quad j \geq 2K + 1 \quad \text{as} \quad c \to \infty.$$

The proof of this result is based on sharp min-max estimates of the eigenfrequencies of the Helmholtz model and on a limit process in the variational formulation of this model. In [26] we can find analogous results for the case of an elastic structure in a fluid which occupies an unbounded region. The mathematical approach of [26] is different in the sense that the authors use scattering techniques instead of variational methods, which we use in this paper.

For the convergence of the eigenfunctions as $c \to \infty$, see the remark at the end of section 2.2.

In section 3 we perform the analysis as the fluid becomes inviscid, that is, when the viscosity parameter ν of the fluid converges to zero ($\nu \to 0$). We prove the following result of convergence.

THEOREM 1.2. *For a sufficiently small viscosity, the Stokes model has exactly $2K$ conjugate pairs of nonreal eigenfrequencies which converge to the eigenfrequencies of the Laplace model as the viscosity tends to zero. More precisely, if V is a neighborhood of an eigenfrequency of the Laplace model with multiplicity m, which does not contain any other eigenfrequency of the Laplace model, then for a sufficiently small viscosity, there are a number of eigenfrequencies of the Stokes model in V with total multiplicity m.*

To prove this theorem, we identify the nonreal eigenfrequencies of the Stokes model with the eigenvalues of a suitable family of unbounded non-self-adjoint operators which depend on ν. Then we study the resolvent convergence of this family as $\nu \to 0$, and we identify its limit with an operator whose spectrum is identical to that of the Laplace model. We arrive at the result using techniques developed in T. Kato [22] and the fact that the nonreal eigenfrequencies of the Stokes model are at most $2K$ conjugate pairs.

The behavior of the eigenfrequencies of a bounded cavity containing a slightly compressible viscous fluid (without tubes) as the viscosity converges to zero, and the relation with the eigenfrequencies of a cavity containing a slightly compressible perfect fluid has been treated in [16] (see also [24] for expansion series). The methods are also based on Kato's techniques [22].

For the convergence of the eigenfunctions in the case $\nu \to 0$, see the remark at the end of section 3.5.

In section 4, we prove a complementary result for the Stokes model using the framework developed in section 3. The result is the denseness of the generalized eigenfunctions of the Stokes model in the space of all free-divergence functions satisfying suitable boundary conditions.

THEOREM 1.3. *The generalized eigenfunctions of the Stokes model are dense in the space of all free-divergence $L^2(\Omega)^2$ functions* $\mathbf{v}$ *which satisfy the following boundary conditions:*

$$\mathbf{v}\cdot\mathbf{n}|_{\Gamma_0} = \mathbf{0}, \quad \mathbf{v}\cdot\mathbf{n}|_{\Gamma_i} = \mathbf{c}_i\cdot\mathbf{n} \quad \textit{for any} \quad \mathbf{c}_i \in \mathbb{C}^2,\ i = 1,\dots,K.$$

In section 5, we solve the Stokes model using a finite element method for a test problem, and we study the numerical convergence of the spectrum as $\nu \to 0$. The computed values validate the convergence result announced in Theorem 1.2.

2. Asymptotic analysis as $c \to \infty$ for the Helmholtz model. In section 2.1 we introduce some general spaces and notations. In section 2.2 we prove Theorem 1.1. The proof is basically based on [10].

2.1. Principal spaces and notations. We define the following space with components in $\mathbb{C}^2$:

$$\mathbb{C}^{2K} = \{(\mathbf{s}_1,\dots,\mathbf{s}_K) \mid \mathbf{s}_i \in \mathbb{C}^2,\ i = 1,\dots,K\}, \tag{2.1}$$

endowed with the inner product $(\mathbf{s},\mathbf{t})_{2K} = \sum_{i=1}^K \mathbf{s}_i\cdot\bar{\mathbf{t}}_i$ and the corresponding induced norm $\|\mathbf{s}\|_{2K} = (\mathbf{s},\mathbf{s})_{2K}^{1/2}$.

We also introduce the usual Sobolev space (see, e.g., [25]):

$$H^1(\Omega) = \{\mathbf{v} \in L^2(\Omega) \mid \nabla\mathbf{v} \in L^2(\Omega)^2\}$$

with its usual inner product and norm, and the zero mean functions set in $H^1(\Omega)$:

$$V_0 = \left\{q \in H^1(\Omega) \mid \int_\Omega q\ dx = 0\right\},$$

which is a Hilbert space with inner product $\int_\Omega \nabla p\cdot\nabla\overline{q}\ dx,\ \ p,q \in V_0$, and the norm $|p|_{1,\Omega} = (\int_\Omega |\nabla p|^2\,dx)^{1/2}$ by virtue of the generalized Poincaré inequality.

Finally, we define the following linear continuous finite rank operator:

$$\begin{aligned} &\mathbf{T} : H^1(\Omega)^N \longrightarrow \mathbb{C}^{2K}, \\ &\mathbf{T}(q) = \left(\int_{\Gamma_1} q\mathbf{n}\ ds\ ,\dots,\int_{\Gamma_K} q\mathbf{n}\ ds\right), \end{aligned} \tag{2.2}$$

where $N = 1$ or $N = 4$ (in this case q is a 2×2 matrix), and we note by $\mathbf{T}_i$ its components in $\mathbb{C}^2,\ i = 1,\dots,K$.

2.2. Proof of Theorem 1.1. Let $iw_{j,L}$, $-iw_{j,L}$, for $j = 1,\dots,2K$ be the eigenfrequencies of the Laplace model (1.1) ordered as in (1.2). Let us introduce the following positive numbers:

$$\lambda_j \equiv \frac{w_{j,L}^2}{k - mw_{j,L}^2}. \tag{2.3}$$

From the variational formulation of the Laplace model, it is easy to show that $\{\lambda_j\}_{j=1}^{2K}$ are all the characteristic values of the self-adjoint operator

$$R : \mathbf{a} \in \mathbb{C}^{2K} \to \mathbf{T}(\psi) \in \mathbb{C}^{2K}, \tag{2.4a}$$

where $\psi \in V_0$ is the (unique) solution of

$$\int_\Omega \nabla\psi \cdot \nabla\overline{\varphi}\, dx = (\mathbf{a}, \mathbf{T}(\varphi))_{2K} \quad \forall\varphi \in V_0. \tag{2.4b}$$

We recall the definition of δ_1 given in section 1 as the smallest eigenvalue of the problem (1.8). It is proved in [10] that under the following geometric condition:

$$k/m < (c\delta_1)^2, \quad \text{or equivalently} \quad c > \frac{1}{\delta_1}\sqrt{k/m},$$

the $2(K+1)$th eigenfrequency of the Helmholtz model (1.3) ordered as in (1.4) satisfies $w_{2K+1}^2(c) > k/m$. Therefore, thanks to (1.5), we have for $c > \frac{1}{\delta_1}\sqrt{k/m}$ that

$$0 < w_1^2(c) \leq \cdots \leq w_{2K}^2(c) < \frac{k}{m} < w_{2K+1}^2 \leq \cdots. \tag{2.5}$$

Furthermore, applying the min-max principle to a suitable self-adjoint operator associated to the Helmholtz model (1.3) (see [10, Theorem 4.8]), we obtain for c sufficiently large:

$$0 < a_0 \leq w_1^2(c) \leq \cdots \leq w_{2K}^2(c) \leq b_0 < \frac{k}{m} \tag{2.6}$$

and

$$w_{2K+1}^2(c) \geq \frac{c^4}{P^*}, \tag{2.7}$$

where $a_0 = \max\{m/k, 2\lambda_{2K}/k\}$, $b_0 = k/(m+\lambda_1)$, and

$$P^* = \inf\{P \mid \|\psi\|_{0,\Omega}^2 \leq P|\psi|_{1,\Omega}^2 \ \forall\psi \in V_0\}$$

are constants independent from c. Clearly, (2.7) implies that $w_j^2(c) \longrightarrow +\infty$ for $j \geq 2K+1$ as $c \to \infty$.

Now, let us focus our analysis on the case $1 \leq j \leq 2K$. We consider solutions of the Helmholtz model $\{(iw_j(c), \phi_j^c)\}_{j=1}^{2K}$ satisfying (2.6) and the following orthogonalization condition:

$$\int_\Omega \phi_j^c \overline{\phi_l^c}\, dx + (\mathbf{s}_j^c, \mathbf{s}_l^c)_{2K} = \delta_{jl}\ , \quad j, l = 1, \ldots, 2K. \tag{2.8}$$

In particular, taking $j = l$, we have

$$\left\|\phi_j^c\right\|_{0,\Omega}^2 + \left\|\mathbf{s}_j^c\right\|_{2K}^2 = 1, \tag{2.9}$$

where, for $j = 1, \ldots, 2K$, we set

$$\eta_j(c) \equiv \frac{w_j^2(c)}{k - m w_j^2(c)} \tag{2.10}$$

and

$$\mathbf{s}_j^c \equiv \eta_j(c)\, \mathbf{T}(\phi_j^c). \tag{2.11}$$

In a first part, to show the convergence of the solutions $(iw_j(c), \phi_j^c)$ for a fixed j as $c \to \infty$, we prove that there exists a pair (iw_j, ψ_j) such that, up to a subsequence,

(2.12a) $\quad w_j^2(c) \to w_j^2,$

(2.12b) $\quad \phi_j^c \rightharpoonup \psi_j$ in V_0 weakly and

(2.12c) $\quad (iw_j, \psi_j)$ is a solution of the Laplace model (1.1).

In a second part, we conclude the proof by showing that $\{iw_j, -iw_j\}_{j=1}^{2K}$ are all the eigenfrequencies of the Laplace model. This fact also implies that the whole sequence converges in (2.12a).

In the first part, we fix $j \in \{1, \dots, 2K\}$, and for the sake of simplicity we suppress the index in the following notations. As $c \to \infty$, from (2.6), we can choose an accumulation point w, such that (2.12a) holds up to a subsequence. Thus (2.10) and (2.11) also converge and we denote their limits, respectively, as

$$\eta(c) \to \eta \equiv \frac{w^2}{k - mw^2} \tag{2.13}$$

and

$$\mathbf{s}^c \to \mathbf{s} \equiv \eta\, \mathbf{T}(\psi). \tag{2.14}$$

Multiplying (1.3a) by $\varphi \in V_0$ and integrating by parts, we obtain, for all $\varphi \in V_0$,

$$\int_\Omega \nabla\phi^c \cdot \nabla\overline{\varphi}\; dx = \frac{w^2(c)}{c^2} \int_\Omega \phi^c\, \overline{\varphi}\; dx + \eta(c)(\mathbf{T}(\phi^c), \mathbf{T}(\varphi))_{2K}, \tag{2.15}$$

and in particular, if $\varphi = \phi^c$, we have

$$|\phi^c|_{1,\Omega}^2 = \frac{w^2(c)}{c^2}\, \|\phi^c\|_{0,\Omega}^2 + \eta(c)\, \|\mathbf{T}(\phi^c)\|_{2K}^2 . \tag{2.16}$$

By (2.9), (2.12a), (2.13), and (2.14), the right side of (2.16) remains bounded as $c \to \infty$, hence, $|\phi^c|_{1,\Omega}$ is also bounded. Therefore, up to a new subsequence, there exists $\psi \in V_0$, which satisfies (2.12b).

We are now able to prove that (2.12c) is verified. Taking the limit as $c \to \infty$ in (2.15) we obtain

$$\int_\Omega \nabla\psi \cdot \nabla\overline{\varphi}\; dx = \eta(\mathbf{T}(\psi), \mathbf{T}(\varphi))_{2K} \quad \forall \varphi \in V_0, \tag{2.17}$$

whose variational formulation is the Laplace model (1.1). Then (2.12c) follows if $\psi \not\equiv 0$. Effectively, taking the limit in the normalization condition (2.9) yields

$$\|\psi\|_{0,\Omega}^2 + \|\mathbf{s}\|_{2K}^2 = 1. \tag{2.18}$$

But, from (2.17) with $\varphi = \psi$, we have

$$|\psi|_{1,\Omega}^2 = \eta\, \|\mathbf{T}(\psi)\|_{2K}^2 . \tag{2.19}$$

Hence, from (2.14) we deduce that $\|\mathbf{s}\|_{2K}^2 = \eta^2\, \|\mathbf{T}(\psi)\|_{2K}^2 = \eta|\psi|_{1,\Omega}^2$, and the identity (2.18) becomes $\|\psi\|_{0,\Omega}^2 + \eta|\psi|_{1,\Omega}^2 = 1$, which implies $\psi \not\equiv 0$.

In the second part, we prove that the accumulation points $\{w_j^2\}_{j=1}^{2K}$ in (2.12a) are necessarily

$$w_j^2 = w_{j,L}^2 \quad \text{for} \quad j = 1, \dots, 2K, \tag{2.20}$$

which implies that the whole sequence converges in (2.12a) and completes the proof of Theorem 1.1. For this goal, we go to the limit in (2.8) and we infer that the set $\{(\psi_j, \mathbf{s}_j)\}_{j=1}^{2K}$ is also orthonormal

$$\int_\Omega \psi_j \overline{\psi}_l \, dx + (\mathbf{s}_j, \mathbf{s}_l)_{2K} = \delta_{jl}, \quad j, l = 1, \dots, 2K. \tag{2.21}$$

But, in this particular case, both $\{\psi_j\}_{j=1}^{2K}$ and $\{\mathbf{s}_j\}_{j=1}^{2K}$ are linearly independent sets. Indeed, from (2.14) and (2.17) we have for $j = 1, \dots, 2K$ that

$$\int_\Omega \nabla\psi_j \cdot \nabla\overline{\varphi} \, dx = \eta_j(\mathbf{T}(\psi_j), \mathbf{T}(\varphi))_{2K} = (\mathbf{s}_j, \mathbf{T}(\varphi))_{2K} \quad \forall \varphi \in V_0. \tag{2.22}$$

Then, if we take scalars $\alpha_j \in \mathbb{C}$ with $\varphi = \sum_{k=1}^{2K} \alpha_k \psi_k$ and $\varphi = \sum_{k=1}^{2K} \eta_k \alpha_k \psi_k$ in (2.22), we deduce

$$\sum_{k=1}^{2K} \alpha_k \psi_k = 0 \quad \text{if and only if} \quad \sum_{k=1}^{2K} \alpha_k \mathbf{s}_k = 0$$

and the linear independence of $\{\psi_j\}_{j=1}^{2K}$ and $\{\mathbf{s}_j\}_{j=1}^{2K}$ follows from (2.21).

It is clear from the definition (2.4) of R and (2.22) that $\mathbf{s}_j = \eta_j \, R(\mathbf{s}_j)$. Thus, in fact, the set $\{\mathbf{s}_j\}_{j=1}^{2K}$ is a basis of eigenvectors of the operator R associated to the characteristic values $\{\eta_j\}_{j=1}^{2K}$. Then $\{\eta_j\}_{j=1}^{2K} = \{\lambda_j\}_{j=1}^{2K}$, and from (2.3) and (2.13), we get

$$\{w_j^2\}_{j=1}^{2K} = \{w_{j,L}^2\}_{j=1}^{2K}. \tag{2.23}$$

Now, we observe by going to the limit in (2.6) that

$$0 < w_1^2 \leq \dots \leq w_{2K}^2 < k/m. \tag{2.24}$$

Finally, comparing the ordering in (2.6) and (2.24), the identity (2.23) implies the announced result (2.20). □

Remark (convergence of the eigenfunctions). The convergence in (2.12b) is strong. Effectively, from (2.9), (2.13), (2.16), and (2.19) we deduce

$$|\phi_j^c|_{1,\Omega} \to |\psi_j|_{1,\Omega} \quad \text{as} \quad c \to \infty.$$

Due to possible multiplicities of w_j, we cannot ensure the strong convergence of the whole sequence in (2.12b). However, since $\{\psi_j\}_{j=1}^{2K}$ are linearly independent, we obtain at the limit a basis of eigenfunctions of the Laplace model.

3. Asymptotic analysis as $\nu \to 0$ for the Stokes model. In section 3.1 we present the principal spaces involved in our analysis. In sections 3.2 and 3.3 we introduce the operators $\mathcal{A}_\nu$ and $\mathcal{A}_0$ whose spectra characterize the eigenfrequencies of the Stokes and Laplace models, respectively, and we establish their main properties. In section 3.4 we prove the strong resolvent convergence of $\mathcal{A}_\nu$ to $\mathcal{A}_0$. In section 3.5 we analyze the convergence of the nonreal eigenvalues of A_ν and we conclude the proof of Theorem 1.2.

3.1. Principal functional spaces. In addition to the definitions introduced in section 2.1, we will also use some additional functional spaces. First we present the following classical Hilbert spaces:

$$\mathcal{V} = \{\mathbf{u} \in \mathcal{D}(\Omega)^2 \mid \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega\},$$

$$V = \{\mathbf{u} \in H_0^1(\Omega)^2 \mid \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega\}$$

with the inner product $(\mathbf{u}, \mathbf{v})_V = \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v}\, dx$ (we note $A{:}B = \sum_i \sum_j a_{ij} b_{ij}$). We will denote by $\gamma(\mathbf{u})$ or $\mathbf{u}|_\Gamma$ the trace of $\mathbf{u}$ on Γ when it has a sense. We define

$$H = \{\mathbf{u} \in L^2(\Omega)^2 \mid \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u} \cdot \mathbf{n}|_\Gamma = 0\}$$

with the usual inner product of $L^2(\Omega)^2$. It is well known (see [37], [17]) that $\mathcal{V}$ is dense in V and that V is dense in H.

For $i = 0, \dots, K$, we will denote by $\gamma_i(\mathbf{u})$ or $\mathbf{u}|_{\Gamma_i}$ the trace of $\mathbf{u}$ on Γ_i. Let us define the following spaces, which are especially well adapted to our problem:

$$S_V = \{(\mathbf{v}, \mathbf{c}) \in H^1(\Omega)^2 \times \mathbb{C}^{2K} \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v}|_{\Gamma_0} = \mathbf{0},$$
$$\mathbf{v}|_{\Gamma_i} = \mathbf{c}_i \text{ (constant)},\ i = 1, \dots, K\},$$

$$S_H = \{(\mathbf{v}, \mathbf{c}) \in L^2(\Omega)^2 \times \mathbb{C}^{2K} \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v} \cdot \mathbf{n}|_{\Gamma_0} = \mathbf{0},$$
$$\mathbf{v} \cdot \mathbf{n}|_{\Gamma_i} = \mathbf{c}_i \cdot \mathbf{n},\ i = 1, \dots, K\}$$

with the inner products induced by $H^1(\Omega) \times \mathbb{C}^{2K}$ and $L^2(\Omega)^2 \times \mathbb{C}^{2K}$, respectively.

LEMMA 3.1. $S_H = \overline{S_V}^{L^2(\Omega)^2 \times\ \mathbb{C}^{2K}}$.

Proof. Let $(\mathbf{v}_0, \mathbf{c}^0) \in S_H$ be fixed and let B_0 be defined by

$$B_0 = \{\mathbf{v} \in H^1(\Omega)^2 \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v}|_{\Gamma_0} = \mathbf{0},$$
$$\mathbf{v}|_{\Gamma_i} = \mathbf{c}_i^0,\ i = 1, \dots, K\},$$

which is not empty since Ω is connected (see [17, Lemma 2.2]). If $\mathbf{v} \in B_0$, then $(\mathbf{v}, \mathbf{c}^0) \in S_V$, and, therefore, it suffices to find a sequence in B_0 converging to $\mathbf{v}_0$. Indeed, if $\psi \in B_0$ then

$$\mathbf{v}_0 - \psi \in H,$$

and by denseness there exists $\{\mathbf{w}_j\} \subset V$ such that $\mathbf{w}_j \to \mathbf{v}_0 - \psi$ in $L^2(\Omega)^2$. Then, if we define $\mathbf{v}_j = \mathbf{w}_j + \psi\ \in B_0$, we have $\mathbf{v}_j \to \mathbf{v}_0$ in $L^2(\Omega)^2$. ∎

Finally, we introduce the following Hilbert space:

$$U_H = \{(\mathbf{v}, \mathbf{s}, \mathbf{c}) \in L^2(\Omega)^2 \times \mathbb{C}^{2K} \times \mathbb{C}^{2K} \mid (\mathbf{v}, \mathbf{c}) \in S_H\}$$

with the inner product

$$((\mathbf{u}^1, \mathbf{s}^1, \mathbf{c}^1), (\mathbf{u}^2, \mathbf{s}^2, \mathbf{c}^2))_{U_H} = \frac{1}{m} \int_\Omega \mathbf{u}^1 \cdot \overline{\mathbf{u}}^2\, dx + \frac{k}{m} (\mathbf{s}^1, \mathbf{s}^2)_{2K} + (\mathbf{c}^1, \mathbf{c}^2)_{2K},$$

and we denote the associated norm by $\|\cdot\|_{U_H}$.

3.2. The operator $\mathcal{A}_\nu$ and its properties. For each $\nu > 0$ the idea is to relate the Stokes model to the spectral problem of an unbounded operator $\mathcal{A}_\nu$ in U_H. For this we define $D(\mathcal{A}_\nu) \subset U_H$ as follows:

$$(\mathbf{u}, \mathbf{s}, \mathbf{c}) \in D(\mathcal{A}_\nu) \text{ if and only if}$$
$$(\mathbf{u}, \mathbf{c}) \in S_V \text{ and}$$
$$\exists (\psi, \mathbf{t}) \in S_H,\ \exists p \in L^2(\Omega) \text{ such that}$$
$$\nu \Delta \mathbf{u} - \nabla p = \psi \qquad \text{in } \Omega, \tag{3.1a}$$
$$-\frac{1}{m}\left(\int_{\Gamma_i} \sigma(\mathbf{u}, p)\mathbf{n}\, ds + k\mathbf{s}_i\right) = \mathbf{t}_i \qquad \text{on } \Gamma_i,\ i = 1, \ldots, K, \tag{3.1b}$$

where $\sigma(\mathbf{u}, p) = -pI + 2\nu e(\mathbf{u})$ and $e(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^t)$.

Next, we define $\mathcal{A}_\nu$ as

$$\mathcal{A}_\nu : D(\mathcal{A}_\nu) \subset U_H \longrightarrow U_H,$$
$$\mathcal{A}_\nu(\mathbf{u}, \mathbf{s}, \mathbf{c}) = (\psi, \mathbf{c}, \mathbf{t}). \tag{3.2}$$

LEMMA 3.2. *For each $\nu > 0$ the operator $\mathcal{A}_\nu$ is well defined and linear.*

Proof. First we see that if $(\mathbf{u}, \mathbf{s}, \mathbf{c}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$ then $\mathcal{A}_\nu(\mathbf{u}, \mathbf{s}, \mathbf{c}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$. If $(\mathbf{u}, \mathbf{s}, \mathbf{c}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$, then by the definition of $\mathcal{A}_\nu$

$$(\psi, \mathbf{t}) \in S_H,$$
$$-\nabla p = \psi,$$
$$\frac{1}{m}\int_{\Gamma_i} p\mathbf{n}\, ds = \mathbf{t}_i.$$

Thus, in this case, $p \in H^1(\Omega)$ and it is a solution of

$$\Delta p = 0 \quad \text{in } \Omega,$$
$$\frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma_0,\ \int_\Omega p\, dx = 0,$$
$$\frac{\partial p}{\partial n} = -\mathbf{t}_i \cdot \mathbf{n} \quad \text{on } \Gamma_i\ \forall i = 1, \ldots, K,$$

where we have added a zero mean condition since p is uniquely defined up to an additive constant. It is easy to verify that p is the unique solution of

$$\int_\Omega \nabla p \cdot \overline{\nabla \tilde{p}}\, dx = -\frac{1}{m}(\mathbf{T}(p), \mathbf{T}(\tilde{p}))_{2K} \qquad \forall \tilde{p} \in V_0,$$

which obviously implies $p = 0$.

Now, if we take $(\psi_1, \mathbf{t}_1)$, p_1 and $(\psi_2, \mathbf{t}_2)$, p_2 as solutions of (3.1) for the same $(\mathbf{u}, \mathbf{s}, \mathbf{c})$ in $D(\mathcal{A}_\nu)$, subtracting the corresponding equations and reasoning as before, we easily verify that $p_1 = p_2$ and $(\psi_1, \mathbf{t}_1) = (\psi_2, \mathbf{t}_2)$. This proves that the operator is well defined and its linearity is verified without difficulty. ∎

Let us introduce a useful characterization of $D(\mathcal{A}_\nu)$.

LEMMA 3.3. $(\mathbf{u}, \mathbf{s}, \mathbf{c}) \in D(\mathcal{A}_\nu)$ *if and only if*

$$(\mathbf{u}, \mathbf{c}) \in S_V \textit{ and}$$
$$\exists (\psi, \mathbf{t}) \in S_H \textit{ such that } \forall (\phi, \mathbf{d}) \in S_V$$
$$2\nu \int_\Omega e(\mathbf{u}) : e(\bar{\phi})\, dx + k(\mathbf{s}, \mathbf{d})_{2K} = -m(\mathbf{t}, \mathbf{d})_{2K} - \int_\Omega \psi \cdot \bar{\phi}\, dx. \tag{3.3}$$

And if these conditions are satisfied, then $\mathcal{A}_\nu(\mathbf{u}, \mathbf{s}, \mathbf{c}) = (\psi, \mathbf{c}, \mathbf{t})$.

Proof. *First part:* We show that if $(\mathbf{u},\mathbf{s},\mathbf{c})$ satisfies (3.1) then it also satisfies (3.3). Multiplying (3.1a) by $\phi \in H^1(\Omega)^2$ such that $\operatorname{div}\phi = 0$, $\gamma_0(\phi) = \mathbf{0}$, $\gamma_i(\phi) = \mathbf{d}_i$, $i = 1,\ldots,K$, for any $\mathbf{d} \in \mathbb{C}^{2K}$, and integrating by parts, we obtain

$$2\nu \int_\Omega e(\mathbf{u}) : e(\bar{\phi})\, dx - \sum_{i=1}^K \int_{\Gamma_i} \sigma(\mathbf{u},p)\mathbf{n}\, ds \cdot \gamma_i(\bar{\phi}) = -\int_\Omega \psi \cdot \bar{\phi}\, dx. \tag{3.4}$$

But, by (3.1b), we have

$$\int_{\Gamma_i} \sigma(\mathbf{u},p)\mathbf{n}\, ds = -(k\mathbf{s}_i + m\mathbf{t}_i), \quad i = 1,\ldots,K, \tag{3.5}$$

and replacing in (3.4) we obtain (3.3) with $\mathbf{d}_i = \gamma_i(\phi)$, $i = 1,\ldots,K$.

Second part: We prove that if $(\mathbf{u},\mathbf{s},\mathbf{c})$ satisfies (3.3) then (3.1) holds. In particular, if we take $\phi \in \mathcal{V}$ in (3.3) we have

$$2\nu \int_\Omega e(\mathbf{u}) : e(\bar{\phi})\, dx = -\int_\Omega \psi \cdot \bar{\phi}\, dx \qquad \forall \phi \in \mathcal{V}, \tag{3.6}$$

from where, in the distribution sense:

$$\langle -2\nu \operatorname{div} e(\mathbf{u}) + \psi, \phi \rangle = 0 \qquad \forall \phi \in \mathcal{V}.$$

Therefore, by virtue of De Rham's lemma there exists a function $q \in L^2(\Omega)$ (unique up to an additive constant) such that

$$-2\nu \operatorname{div} e(\mathbf{u}) + \psi = \nabla q \qquad \text{in } \Omega, \tag{3.7}$$

and (3.1a) holds. Now, multiplying (3.7) by $\phi \in H^1(\Omega)^2$ such that $\operatorname{div}\phi = \mathbf{0}$, $\gamma_0(\phi) = \mathbf{0}$, $\gamma_i(\phi) = \mathbf{d}_i$, $i = 1,\ldots,K$, for any $\mathbf{d} \in \mathbb{C}^{2K}$, and integrating by parts, we obtain

$$2\nu \int_\Omega e(\mathbf{u}) : e(\bar{\phi})\, dx - \sum_{i=1}^K \int_{\Gamma_i} \sigma(\mathbf{u},q)\mathbf{n}\, ds \cdot \gamma_i(\bar{\phi}) = -\int_\Omega \psi \cdot \bar{\phi}\, dx,$$

and subtracting this expression with (3.3), we conclude that, $\forall \mathbf{d} \in \mathbb{C}^{2K}$,

$$\sum_{i=1}^K \left(\int_{\Gamma_i} \sigma(\mathbf{u},q)\mathbf{n}\, ds + k\mathbf{s}_i + m\mathbf{t}_i \right) \cdot \bar{\mathbf{d}}_i = 0, \quad i = 1,\ldots,K.$$

Thus (3.1b) holds. ☐

LEMMA 3.4. *For each $\nu > 0$, $\mathcal{A}_\nu$ is a closed, densely defined operator and its eigenvalues and eigenvectors are exactly the eigenfrequencies and eigenfunctions of the Stokes model.*

Proof. *First part:* To prove that $\mathcal{A}_\nu$ is densely defined in U_H, we will see that if $(\mathbf{v},\mathbf{z},\mathbf{r}) \in U_H$ satisfies

$$\frac{1}{m}\int_\Omega \mathbf{u} \cdot \bar{\mathbf{v}}\, dx + \frac{k}{m}(\mathbf{s},\mathbf{z})_{2K} + (\mathbf{c},\mathbf{r})_{2K} = 0 \qquad \forall (\mathbf{u},\mathbf{s},\mathbf{c}) \in D(\mathcal{A}_\nu), \tag{3.8}$$

then $(\mathbf{v},\mathbf{z},\mathbf{r}) = (\mathbf{0},\mathbf{0},\mathbf{0})$. We first remark that $(\mathbf{0},\mathbf{s},\mathbf{0}) \in D(\mathcal{A}_\nu)\ \forall \mathbf{s} \in \mathbb{C}^{2K}$. Effectively, from (3.1) it is sufficient to prove that for each $\mathbf{s} \in \mathbb{C}^{2K}$ the following problem is well posed:

$$\text{find } (\psi, \mathbf{t}) \in S_H, \quad q \in L^2(\Omega) \text{ such that}$$

(3.9a) $$\nabla q = -\psi,$$

(3.9b) $$-\frac{1}{m}(-\mathbf{T}_i(q) + k\mathbf{s}_i) = \mathbf{t}_i, \qquad i = 1, \dots, K;$$

that is,

$$\begin{aligned}
&\text{find } q \in V_0 \text{ such that}\\
&\Delta q = 0 \qquad \text{in } \Omega,\\
&\frac{\partial q}{\partial n} = 0 \qquad \text{on } \Gamma_0,\\
&\frac{\partial q}{\partial n} = \frac{1}{m}(-\mathbf{T}_i(q) + k\mathbf{s}_i) \qquad \text{on } \Gamma_i, \quad i = 1, \dots, K,
\end{aligned}$$

which admits a unique solution since its variational formulation is

$$\int_\Omega \nabla q \cdot \nabla \overline{\phi}\, dx + \frac{1}{m}(\mathbf{T}(q), \mathbf{T}(\phi))_{2K} = \frac{k}{m}(\mathbf{s}, \mathbf{T}(\phi))_{2K} \qquad \forall \phi \in V_0,$$

and the pair $(\psi, \mathbf{t})$ is obtained from (3.9). Hence, taking $(\mathbf{0}, \mathbf{s}, \mathbf{0}) \in D(\mathcal{A}_\nu)$ in (3.8), we obtain

(3.10) $$\mathbf{z} = 0.$$

Now it is clear from (3.1) that $(\mathbf{u}, \mathbf{0}, \mathbf{0}) \in D(\mathcal{A}_\nu)$ $\forall \mathbf{u} \in \mathcal{V}$, and by an obvious denseness argument,

$$\int_\Omega \mathbf{u} \cdot \overline{\mathbf{v}}\ dx = 0 \quad \forall \mathbf{u} \in H.$$

That is (see [37, Theorem 1.5]), there exists $p \in H^1(\Omega)$ such that $\mathbf{v} = \nabla p$ and

$$\Delta p = 0\ ,\ \frac{\partial p}{\partial n}\bigg|_{\Gamma_0} = 0\ ,\ \frac{\partial p}{\partial n}\bigg|_{\Gamma_i} = \mathbf{r}_i \cdot \mathbf{n}.$$

From (3.8), integrating by parts the first term, we obtain

(3.11) $$\frac{1}{m}(\mathbf{c}, \mathbf{T}(p))_{2K} + (\mathbf{c}, \mathbf{r})_{2K} = 0 \qquad \forall (\mathbf{u}, \mathbf{s}, \mathbf{c}) \in D(\mathcal{A}_\nu).$$

Then it is easy to verify that $(\mathbf{c}, \frac{1}{m}\mathbf{T}(p) + \mathbf{r})_{2K} = 0$ $\forall \mathbf{c} \in \mathbb{C}^{2K}$, and consequently

(3.12) $$\int_{\Gamma_i} p\mathbf{n}\ ds = -m\mathbf{r}_i.$$

Thus, p satisfies

$$\begin{aligned}
&\Delta p = 0 \qquad \text{in } \Omega,\\
&\frac{\partial p}{\partial n} = 0 \qquad \text{on } \Gamma_0,\\
&\frac{\partial p}{\partial n} = -\frac{1}{m}\int_{\Gamma_i} p\mathbf{n}\ ds \cdot \mathbf{n} \qquad \text{on } \Gamma_i, \qquad i = 1, \dots, K,
\end{aligned}$$

whose variational formulation is:

$$\text{find } p \in V_0 \text{ such that}$$
$$\int_\Omega \nabla p \cdot \nabla \overline{q}\, dx = -\frac{1}{m}(\mathbf{T}(p), \mathbf{T}(q))_{2K} \qquad \forall q \in V_0,$$

and if $q = p$ then $p = 0$ in V_0, so $\mathbf{v} = 0$ and from (3.12), $\mathbf{r} = 0$.

Second part: To prove that $\mathcal{A}_\nu$ is closed we consider a sequence $\{(\mathbf{u}^j, \mathbf{s}^j, \mathbf{c}^j)\}_{j\leq 1}$ in $D(\mathcal{A}_\nu)$ such that

$$(\mathbf{u}^j, \mathbf{s}^j, \mathbf{c}^j) \longrightarrow (\mathbf{u}^\infty, \mathbf{s}^\infty, \mathbf{c}^\infty) \qquad \text{in } U_H \tag{3.13}$$

and also

$$\mathcal{A}_\nu(\mathbf{u}^j, \mathbf{s}^j, \mathbf{c}^j) \equiv (\psi^j, \mathbf{c}^j, \mathbf{t}^j) \longrightarrow (\psi^\infty, \mathbf{c}^\infty, \mathbf{t}^\infty) \qquad \text{in } U_H. \tag{3.14}$$

Taking $\phi = \mathbf{u}^j$ and $\mathbf{d} = \mathbf{c}^j$ in (3.3) we obtain

$$2\nu \left\| e(\mathbf{u}^j) \right\|_{0,\Omega}^2 = -(m\mathbf{t}^j + k\mathbf{s}^j, \mathbf{c}^j)_{2K} - \int_\Omega \psi^j \cdot \overline{\mathbf{u}}^j\, dx.$$

Now, it is clear that if $j \to \infty$, then

$$\left\| e(\mathbf{u}^j) \right\|_{0,\Omega} \longrightarrow \alpha \in \mathbb{R}, \tag{3.15}$$

where α satisfies

$$2\nu\alpha^2 = -(m\mathbf{t}^\infty + k\mathbf{s}^\infty, \mathbf{c}^\infty)_{2K} - \int_\Omega \psi^\infty \cdot \overline{\mathbf{u}}^\infty\, dx.$$

From (3.13) and (3.15) it is obvious that $\{\mathbf{u}^j\}$ is bounded in $H^1(\Omega)^2$ so, up to a subsequence,

$$\mathbf{u}^j \rightharpoonup \mathbf{u}^* \qquad \text{in } H^1(\Omega)^2 \text{ weakly}, \tag{3.16}$$

and by the uniqueness of the limit $\mathbf{u}^* = \mathbf{u}^\infty \in H^1(\Omega)^2$, the whole sequence converges.

Using (3.16) we have

$$e(\mathbf{u}^j) \rightharpoonup e(\mathbf{u}^\infty) \qquad \text{in } L^2(\Omega)^2 \text{ weakly}.$$

Since $(\mathbf{u}^j, \mathbf{s}^j, \mathbf{c}^j)$ and $(\psi^j, \mathbf{c}^j, \mathbf{t}^j)$ satisfy (3.3), we can pass to the limit in this equation and obtain

$$\begin{aligned} 2\nu \int_\Omega e(\mathbf{u}^\infty) : \overline{e(\phi)}\, dx = &- (m\mathbf{t}^\infty + k\mathbf{s}^\infty, \mathbf{d})_{2K} \\ &- \int_\Omega \psi^\infty \cdot \overline{\phi}\, dx \quad \forall (\phi, \mathbf{d}) \in S_V. \end{aligned} \tag{3.17}$$

From (3.16)

$$\gamma_i(\mathbf{u}^j) \rightharpoonup \gamma_i(\mathbf{u}^\infty) \quad \text{in } L^2(\Gamma_i) \text{ weakly}, \quad i = 0, \ldots, K,$$

but $\gamma_i(\mathbf{u}^j) = \mathbf{c}_i^j$, $i = 1, \ldots, K$, $\gamma_0(\mathbf{u}^j) = \mathbf{0}$, and hence

$$\gamma_i(\mathbf{u}^j) \longrightarrow \gamma_i(\mathbf{u}^\infty) \text{ in } \mathbb{C}^2, \quad i = 0, \ldots, K.$$

On the other hand, from (3.13), we have

$$\gamma_i(\mathbf{u}^\infty) = \mathbf{c}_i^\infty, \qquad i = 1, \ldots, K, \tag{3.18}$$

$$\gamma_0(\mathbf{u}^\infty) = \mathbf{0}, \tag{3.19}$$

and from (3.16),

$$\operatorname{div} \mathbf{u}^j \rightharpoonup \operatorname{div} \mathbf{u}^\infty \qquad \text{in } L^2(\Omega) \text{ weakly.}$$

Since $\operatorname{div} \mathbf{u}^j = 0$, we conclude that

$$\operatorname{div} \mathbf{u}^\infty = 0 \qquad \text{in } \Omega. \tag{3.20}$$

It is clear from (3.14) and (3.17)–(3.20) that $(\mathbf{u}^\infty, \mathbf{s}^\infty, \mathbf{c}^\infty) \in D(\mathcal{A}_\nu)$ and that $\mathcal{A}_\nu(\mathbf{u}^\infty, \mathbf{s}^\infty, \mathbf{c}^\infty) = (\psi^\infty, \mathbf{c}^\infty, \mathbf{s}^\infty)$. This proves that $\mathcal{A}_\nu$ is closed.

Third part: Finally, we study the spectral problem of $\mathcal{A}_\nu$:

$$\begin{aligned}
&\text{find } w \in \mathbb{C},\ (\mathbf{u}, \mathbf{s}, \mathbf{c}) \in D(\mathcal{A}_\nu),\ (\mathbf{u}, \mathbf{s}, \mathbf{c}) \not\equiv (\mathbf{0}, \mathbf{0}, \mathbf{0}), \text{ such that} \\
&\mathcal{A}_\nu(\mathbf{u}, \mathbf{s}, \mathbf{c}) = w(\mathbf{u}, \mathbf{s}, \mathbf{c}) = (\psi, \mathbf{c}, \mathbf{t});
\end{aligned}$$

that is,

$$\begin{aligned}
&\text{find } w \in \mathbb{C},\ (\mathbf{u}, \mathbf{s}, \mathbf{c}) \in D(\mathcal{A}_\nu),\ (\mathbf{u}, \mathbf{s}, \mathbf{c}) \not\equiv (\mathbf{0}, \mathbf{0}, \mathbf{0}),\ p \in L^2(\Omega) \text{ such that} \\
&w\mathbf{u} = \psi = \nu \Delta \mathbf{u} - \nabla p \quad \text{in } \Omega, \\
&w\mathbf{s}_i = \mathbf{c}_i, \qquad i = 1, \ldots, K, \\
&w\mathbf{c}_i = -\frac{1}{m}\left(\int_{\Gamma_i} \sigma(\mathbf{u}, p)\mathbf{n}\, ds + k\mathbf{s}_i\right), \qquad i = 1, \ldots, K.
\end{aligned}$$

Using the fact that $\mathbf{c}_i = \gamma_i(\mathbf{u}),\ i = 1, \ldots, K$, we can rewrite this problem as follows:

$$\begin{aligned}
&\text{find } w \in \mathbb{C},\ \mathbf{u} \in H^1(\Omega)^2,\ \mathbf{u} \not\equiv \mathbf{0}\ \ p \in L^2(\Omega), \text{ such that} \\
&-\nu \Delta \mathbf{u} + \nabla p = -w\mathbf{u} \quad \text{in } \Omega, \\
&\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \\
&\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_0, \\
&\mathbf{u} = -\frac{w}{k + mw^2}\left(\int_{\Gamma_i} \sigma(\mathbf{u}, p)\mathbf{n}\, ds\right) \quad \text{on } \Gamma_i\ \forall i = 1, \ldots, K,
\end{aligned}$$

and this is exactly the spectral problem corresponding to the Stokes model (1.6).

Conversely, if we define $\mathbf{c}_i = \gamma_i(\mathbf{u}),\quad i = 1, \ldots, K$, and $\mathbf{s}_i$ by $\omega\mathbf{s} = \mathbf{c}$, from the Stokes model we deduce the spectral problem for $\mathcal{A}_\nu$. ∎

3.3. The operator $\mathcal{A}_0$ and its properties. We define the operator $\mathcal{A}_0$ as

$$\begin{aligned}
&\mathcal{A}_0 : U_H \longrightarrow U_H, \\
&\mathcal{A}_0(\mathbf{u}, \mathbf{s}, \mathbf{c}) = \left(-\nabla q, \mathbf{c}, -\frac{1}{m}(-\mathbf{T}(q) + k\mathbf{s})\right),
\end{aligned} \tag{3.21}$$

where $\mathbf{T}$ is the operator defined in (2.1) and, given $\mathbf{s} \in \mathbb{C}^{2K}$, q is the unique solution of

$$\Delta q = 0 \qquad \text{in } \Omega, \tag{3.22a}$$

$$\frac{\partial q}{\partial n} = 0 \qquad \text{on } \Gamma_0, \qquad \int_\Omega q\, dx = 0, \tag{3.22b}$$

$$\frac{\partial q}{\partial n} = \frac{1}{m}(-\mathbf{T}_i(q) + k\mathbf{s}_i) \cdot \mathbf{n} \qquad \text{on } \Gamma_i, \quad i = 1, \ldots, K, \tag{3.22c}$$

whose variational formulation in the space V_0 is:

$$\text{find } q \in V_0 \text{ such that } \forall \tilde{q} \in V_0 \tag{3.23}$$
$$\int_\Omega \nabla q \cdot \nabla \tilde{q}\, dx + \frac{1}{m}(\mathbf{T}(q), \mathbf{T}(\tilde{q}))_{2K} = \frac{k}{m}(\mathbf{s}, \mathbf{T}(\tilde{q}))_{2K}.$$

LEMMA 3.5. *$\mathcal{A}_0$ is a well-defined linear bounded operator and its eigenvalues and eigenvectors are exactly the eigenfrequencies and eigenfunctions of the Laplace model.*

Proof. To see that $\mathcal{A}_0$ is well defined it is only necessary to remark that for each $\mathbf{s}$ in $\mathbb{C}^{2K}$ fixed, the solution of (3.23) is unique (Lax–Milgram). The linearity of $\mathcal{A}_0$ is easily verified and we see that $\mathcal{A}_0$ is bounded. Effectively, from (3.23) with $\tilde{q} = q$, we have

$$\begin{aligned}\|\nabla q\|^2_{0,\Omega} &\le \|\nabla q\|^2_{0,\Omega} + \frac{1}{m}\,|\mathbf{T}(q)|^2_{2K} \\ &\le \frac{k}{m}\,|\mathbf{s}|_{2K}\,|\mathbf{T}(q)|_{2K} \le \frac{Kk}{m}\,|\mathbf{s}|_{2K}\,\|q\|_{L^2(\Gamma)} \\ &\le \frac{CKk}{m}\,|\mathbf{s}|_{2K}\,\|\nabla q\|_{0,\Omega}\,.\end{aligned}$$

Then,

$$\begin{aligned}\|\mathcal{A}_0(\mathbf{u}, \mathbf{s}, \mathbf{c})\|^2 &= \|\nabla q\|^2_{0,\Omega} + |\mathbf{s}|^2_{2K} + |\mathbf{c}|^2_{2K} \\ &\le (1 + CKk/m)(\|\mathbf{u}\|^2_{0,\Omega} + |\mathbf{s}|^2_{2K} + |\mathbf{c}|^2_{2K}).\end{aligned}$$

To analyze the spectrum of $\mathcal{A}_0$, we note that

$$\text{Ker } \mathcal{A}_0 = H \times \{\mathbf{0}\} \times \{\mathbf{0}\} \tag{3.24}$$

since, if $\mathcal{A}_0(\mathbf{u}, \mathbf{s}, \mathbf{c}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$, then $\mathbf{c} = 0$ and also

$$-\mathbf{T}(q) + k\mathbf{s} = \mathbf{0}. \tag{3.25}$$

From (3.25) the solution of (3.22) is $q = 0$, which, as of (3.25), will imply $\mathbf{s} = 0$. So $w = 0$ is an eigenvalue of $\mathcal{A}_0$ with associated eigenspace given by (3.24). This space can be interpreted as resonance solutions of (1.1) of free-divergence velocity and constant pressure. They had not been included in (1.1) by simplicity, but they are also valid.

Let us look for eigenvalues $w \neq 0$ with associated eigenspaces in $B \equiv (\text{Ker } \mathcal{A}_0)^\perp$. We know that $w \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of $\mathcal{A}_0$ if and only if

$$\exists (\mathbf{u}, \mathbf{s}, \mathbf{c}) \in B,\ (\mathbf{u}, \mathbf{s}, \mathbf{c}) \not\equiv (\mathbf{0}, \mathbf{0}, \mathbf{0}) \text{ such that}$$
$$-\nabla q = w\mathbf{u} \quad \text{in } \Omega, \tag{3.26a}$$
$$\mathbf{c} = w\mathbf{s}, \tag{3.26b}$$
$$-\frac{1}{m}(-\mathbf{T}(q) + k\mathbf{s}) = w\mathbf{c}, \tag{3.26c}$$

where q is the solution of (3.22) for $\mathbf{s} = \mathbf{s}(w)$. The solution of (3.26b) and (3.26c) gives

$$\mathbf{s} = \frac{1}{k + mw^2}\mathbf{T}(q), \tag{3.27}$$

and, hence,

$$-\frac{1}{m}(-\mathbf{T}(q)+k\mathbf{s}) = -\frac{w^2}{k+mw^2}\mathbf{T}(q).$$

Thus, if $w \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of $\mathcal{A}_0$, we have

$$\exists q \in V_0,\ q \not\equiv 0,\ \text{such that}$$

$$\Delta q = 0 \quad \text{in } \Omega, \tag{3.28a}$$

$$\frac{\partial q}{\partial n} = 0 \quad \text{on } \Gamma_0, \tag{3.28b}$$

$$\frac{\partial q}{\partial n} = -\frac{w^2}{k+mw^2}\ \mathbf{T}_i(q)\cdot\mathbf{n} \quad \text{on } \Gamma_i\ \forall i = 1,\dots,K, \tag{3.28c}$$

which is nothing but the Laplace model. Conversely, if w satisfies (3.28) then $\mathbf{u} = -\frac{1}{w}\nabla q$, $\mathbf{s}$ is given by (3.27), and $\mathbf{c} = w\mathbf{s}$ satisfy (3.26). □

3.4. The strong resolvent convergence. In section 1 we summarized some location properties of the spectra for the Stokes and Laplace models. By Lemmas 3.4 and 3.5 we know that these spectra correspond to $\mathcal{A}_\nu$ and $\mathcal{A}_0$, respectively. If $\mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z)$ denote the real and imaginary parts of the complex number z, it is clear from (1.7) and (1.2) that the complex region

$$E \equiv \{z \in \mathbb{C} \mid \mathrm{Re}\,(z) > 0\} \cup \{z \in \mathbb{C} \mid \mathrm{Im}\,(z) \neq 0 \text{ and } |z| > \sqrt{k/m}\}$$

lies in the resolvent of both operators $\mathcal{A}_\nu$ and $\mathcal{A}_0$. Thus, the resolvent operators

$$R_\mu(\mathcal{A}_\nu) \equiv (\mathcal{A}_\nu - \mu I)^{-1}, \tag{3.29}$$

$$R_\mu(\mathcal{A}_0) \equiv (\mathcal{A}_0 - \mu I)^{-1} \tag{3.30}$$

are well defined in $\mathcal{L}(U_H)$ for all $\mu \in E$ (here $\mathcal{L}(U_H)$ denote the set of all linear bounded operators from U_H on U_H).

Using the definitions (3.1)–(3.2) of $\mathcal{A}_\nu$ and (3.21)–(3.22) of $\mathcal{A}_0$ and Lemma 3.3, we can obtain explicit characterizations of $R_\mu(\mathcal{A}_\nu)$ and $R_\mu(\mathcal{A}_0)$.

LEMMA 3.6. (i) *If $R_\mu(\mathcal{A}_\nu)(\varphi,\mathbf{x},\mathbf{y}) = (\mathbf{u}^\nu,\mathbf{s}^\nu,\mathbf{c}^\nu)$, then $(\mathbf{u}^\nu,\mathbf{c}^\nu)$ is the unique solution of*

$$\text{find } (\mathbf{u}^\nu,\mathbf{c}^\nu) \in S_V \text{ such that } \forall(\phi,\mathbf{d}) \in S_V \tag{3.31a}$$

$$2\nu\int_\Omega e(\mathbf{u}^\nu) : e(\overline{\phi})\ dx + \mu\int_\Omega \mathbf{u}^\nu\cdot\overline{\phi}\ dx + \frac{k+m\mu^2}{\mu}(\mathbf{c}^\nu,\mathbf{d})_{2K}$$
$$= \left(\frac{k}{\mu}\mathbf{x} - m\mathbf{y},\mathbf{d}\right)_{2K} - \int_\Omega \varphi\cdot\overline{\phi}\ dx, \tag{3.31b}$$

and

$$\mathbf{s}^\nu = \frac{1}{\mu}(\mathbf{c}^\nu - \mathbf{x}). \tag{3.32}$$

(ii) *If $R_\mu(\mathcal{A}_0)(\varphi,\mathbf{x},\mathbf{y}) = (\mathbf{u}^0,\mathbf{s}^0,\mathbf{c}^0)$ then $(\mathbf{u}^0,\mathbf{c}^0)$ is the unique solution of*

$$\text{find } (\mathbf{u}^0,\mathbf{c}^0) \in S_H \text{ such that } \forall(\phi,\mathbf{d}) \in S_H \tag{3.33a}$$

$$\mu\int_\Omega \mathbf{u}^0\cdot\overline{\phi}\ dx + \frac{k+m\mu^2}{\mu}(\mathbf{c}^0,\mathbf{d})_{2K} = \left(\frac{k}{\mu}\mathbf{x} - m\mathbf{y},\mathbf{d}\right)_{2K} - \int_\Omega \varphi\cdot\overline{\phi}\ dx, \tag{3.33b}$$

and

$$\mathbf{s}^0 = \frac{1}{\mu}(\mathbf{c}^0 - \mathbf{x}). \tag{3.34}$$

Proof. (i) By the definition of $\mathcal{A}_\nu$, using the notations introduced in (3.2), we have

$$\begin{aligned}(\varphi, \mathbf{x}, \mathbf{y}) &= \mathcal{A}_\nu(\mathbf{u}^\nu, \mathbf{s}^\nu, \mathbf{c}^\nu) - \mu(\mathbf{u}^\nu, \mathbf{s}^\nu, \mathbf{c}^\nu)\\ &= (\psi^\nu, \mathbf{c}^\nu, \mathbf{t}^\nu) - \mu(\mathbf{u}^\nu, \mathbf{s}^\nu, \mathbf{c}^\nu)\\ &= (\psi^\nu - \mu\mathbf{u}^\nu, \mathbf{c}^\nu - \mu\mathbf{s}^\nu, \mathbf{t}^\nu - \mu\mathbf{c}^\nu).\end{aligned}$$

Then we obtain that $(\mathbf{u}^\nu, \mathbf{c}^\nu) \in S_V$ satisfies the following equations:

$$\nu\Delta\mathbf{u}^\nu - \nabla p^\nu - \mu\mathbf{u}^\nu = \varphi, \tag{3.35a}$$

$$(k + m\mu^2)\mathbf{c}^\nu + \mu\mathbf{T}(\sigma(\mathbf{u}^\nu, p^\nu)) = k\mathbf{x} - m\mu\mathbf{y}, \tag{3.35b}$$

$$\text{and } \mathbf{s}^\nu = \frac{1}{\mu}(\mathbf{c}^\nu - \mathbf{x}).$$

The variational formulation of (3.35) is exactly the problem (3.31) which admits a unique solution by the Lax–Milgram lemma. The proof is analogous to that in Lemma 3.3.

(ii) We have

$$\begin{aligned}(\varphi, \mathbf{x}, \mathbf{y}) &= \mathcal{A}_0(\mathbf{u}^0, \mathbf{s}^0, \mathbf{c}^0) - \mu(\mathbf{u}^0, \mathbf{s}^0, \mathbf{c}^0)\\ &= \left(-\nabla q^0, \mathbf{c}^0, -\frac{1}{m}(-\mathbf{T}(q^0) + k\mathbf{s}^0)\right) - \mu(\mathbf{u}^0, \mathbf{s}^0, \mathbf{c}^0),\end{aligned}$$

where, for each $\mathbf{s}^0$, q^0 is the solution of (3.22). Therefore, we have that $(\mathbf{u}^0, \mathbf{c}^0) \in S_H$ solves

$$-\nabla q^0 - \mu\mathbf{u}^0 = \varphi, \tag{3.36a}$$

$$(k + m\mu^2)\mathbf{c}^0 + \mu\mathbf{T}(q^0) = k\mathbf{x} - m\mu\mathbf{y}, \tag{3.36b}$$

$$\text{and } \mathbf{s}^0 = \frac{1}{\mu}(\mathbf{c}^0 - \mathbf{x}).$$

It is easy to show that (3.36) has the equivalent formulation (3.33), which admits a unique solution by the Lax–Milgram lemma. ∎

Remark. There are two useful identities we will use later. Taking $(\phi, \mathbf{d}) = (\mathbf{u}^\nu, \mathbf{c}^\nu)$ in (3.31) we obtain that $(\mathbf{u}^\nu, \mathbf{c}^\nu)$ satisfies

(3.37)

$$2\nu \|e(\mathbf{u}^\nu)\|^2_{0,\Omega} + \mu \|\mathbf{u}^\nu\|^2_{0,\Omega} + \frac{k + m\mu^2}{\mu} \|\mathbf{c}^\nu\|^2_{2K} = \left(\frac{k}{\mu}\mathbf{x} - m\mathbf{y}, \mathbf{c}^\nu\right)_{2K} - \int_\Omega \varphi \cdot \overline{\mathbf{u}}^\nu \, dx,$$

and taking $(\phi, \mathbf{d}) = (\mathbf{u}^0, \mathbf{c}^0)$ in (3.33) we obtain that $(\mathbf{u}^0, \mathbf{c}^0)$ satisfies

$$\mu \left\|\mathbf{u}^0\right\|^2_{0,\Omega} + \frac{k + m\mu^2}{\mu} \left\|\mathbf{c}^0\right\|^2_{2K} = \left(\frac{k}{\mu}\mathbf{x} - m\mathbf{y}, \mathbf{c}^0\right)_{2K} - \int_\Omega \varphi \cdot \overline{\mathbf{u}}^0 \, dx. \tag{3.38}$$

LEMMA 3.7.

$$R_\mu(\mathcal{A}_\nu) \longrightarrow R_\mu(\mathcal{A}_0)$$

in the strong sense of $\mathcal{L}(U_H)$ $\forall \mu \in E$.

Proof. Let $(\varphi, \mathbf{x}, \mathbf{y}) \in U_H$ be fixed and let us consider the sequence $(\mathbf{u}^\nu, \mathbf{s}^\nu, \mathbf{c}^\nu)$ defined by

$$(\mathbf{u}^\nu, \mathbf{s}^\nu, \mathbf{c}^\nu) = R_\mu(\mathcal{A}_\nu)(\varphi, \mathbf{x}, \mathbf{y}).$$

We will prove that, as $\nu \to 0$,

$$(\mathbf{u}^\nu, \mathbf{s}^\nu, \mathbf{c}^\nu) \longrightarrow (\mathbf{u}^0, \mathbf{s}^0, \mathbf{c}^0) \text{ in } U_H \text{ strongly,}$$

where

$$(\mathbf{u}^0, \mathbf{s}^0, \mathbf{c}^0) = R_\mu(\mathcal{A}_0)(\varphi, \mathbf{x}, \mathbf{y}).$$

To simplify the notations, we also define the linear functional $L : U_H \times \mathbb{C}^{2K} \to \mathbb{C}$ as

$$L(\mathbf{u}, \mathbf{c}) = \left(\frac{k}{\mu}\mathbf{x} - m\mathbf{y}, \mathbf{c}\right)_{2K} - \int_\Omega \varphi \cdot \overline{\mathbf{u}}\, dx.$$

We remark that, if $\mu \in E \cap \mathbb{R}$, from (3.37) and using the Cauchy–Schwarz and Young inequalities, then $\sqrt{\nu}\ \|e(\mathbf{u}^\nu)\|_{0,\Omega}$, $\|\mathbf{u}^\nu\|_{0,\Omega}$ and $\|\mathbf{c}^\nu\|_{2K}$ are bounded independently of ν. If $\mu \in E$ in general, taking real and imaginary parts in (3.37) we obtain

$$2\nu \|e(\mathbf{u}^\nu)\|_{0,\Omega}^2 + \operatorname{Re}\mu \left(\|\mathbf{u}^\nu\|_{0,\Omega}^2 + \frac{m}{|\mu|^2}\left(|\mu|^2 + \frac{k}{m}\right)\|\mathbf{c}^\nu\|_{2K}^2\right) = \operatorname{Re} L(\mathbf{u}^\nu, \mathbf{c}^\nu), \tag{3.39}$$

$$\operatorname{Im}\mu \left(\|\mathbf{u}^\nu\|_{0,\Omega}^2 + \frac{m}{|\mu|^2}\left(|\mu|^2 - \frac{k}{m}\right)\|\mathbf{c}^\nu\|_{2K}^2\right) = \operatorname{Im} L(\mathbf{u}^\nu, \mathbf{c}^\nu), \tag{3.40}$$

so we find the same uniform bounds in both cases (i) $\operatorname{Re}\mu > 0$ or (ii) $\operatorname{Im}\mu \neq 0$ and $|\mu| > \sqrt{k/m}$. Hence, except for a subsequence,

$$\mathbf{u}^\nu \rightharpoonup \mathbf{u}^* \qquad \text{in } L^2(\Omega)^2 \text{ weakly,} \tag{3.41}$$

$$\mathbf{c}_i^\nu = \gamma_i(\mathbf{u}^\nu) \longrightarrow \mathbf{c}_i^* \qquad \text{in } \mathbb{C}^2 \quad \forall i = 1, \dots, K, \tag{3.42}$$

$$\sqrt{\nu}\ \mathbf{u}^\nu \rightharpoonup \mathbf{v} \qquad \text{in } H^1(\Omega)^2 \text{ weakly.} \tag{3.43}$$

But, from (3.41) and (3.43), we necessarily have

$$\mathbf{v} = \mathbf{0}. \tag{3.44}$$

Taking the limit in (3.31), and by virtue of (3.41)–(3.44), we obtain

$$\mu \int_\Omega \mathbf{u}^* \cdot \overline{\phi}\, dx + \frac{k + m\mu^2}{\mu}(\mathbf{c}^*, \mathbf{d})_{2K} = L(\phi, \mathbf{d}) \qquad \forall (\phi, \mathbf{d}) \in S_V, \tag{3.45}$$

and by denseness the equation (3.45) is also verified for all $(\phi, \mathbf{d}) \in S_H$. Thus $(\mathbf{u}^*, \mathbf{c}^*)$ satisfies (3.33b). We have also that $(\mathbf{u}^*, \mathbf{c}^*) \in S_H$. Effectively, from (3.41) and (3.42), since $(\mathbf{u}^\nu, \mathbf{c}^\nu) \in S_V$ $\forall \nu > 0$, we have

$$\operatorname{div}\mathbf{u}^* = 0 \text{ in } \Omega, \quad \mathbf{u}^* \cdot \mathbf{n}|_{\Gamma_0} = 0 \quad \text{and} \quad \mathbf{u}^* \cdot \mathbf{n}|_{\Gamma_i} = \mathbf{c}_i^* \cdot \mathbf{n} \quad \forall i = 1, \dots, K.$$

Then, by uniqueness, $(\mathbf{u}^*, \mathbf{c}^*) = (\mathbf{u}^0, \mathbf{c}^0)$. Also, since $\mathbf{s}^\nu$ is done by (3.32), we have

$$\mathbf{s}^\nu = \frac{1}{\mu}(\mathbf{c}^\nu - \mathbf{x}) \longrightarrow \frac{1}{\mu}(\mathbf{c}^0 - \mathbf{x}) = \mathbf{s}^0 \quad \text{in } \mathbb{C}^{2K}. \tag{3.46}$$

Thus, summarizing the previous steps,

$$(\mathbf{u}^\nu, \mathbf{s}^\nu, \mathbf{c}^\nu) \rightharpoonup (\mathbf{u}^0, \mathbf{s}^0, \mathbf{c}^0) \quad \text{in } U_H \text{ weakly,} \tag{3.47}$$

and the entire sequence converges.

Let us now prove the strong convergence. Taking real and imaginary parts in (3.33) with $(\phi, \mathbf{d}) = (\mathbf{u}^\nu, \mathbf{c}^\nu)$, using the properties

$$\operatorname{Re}(z_1 z_2) = \operatorname{Re} z_1 \operatorname{Re} z_2 - \operatorname{Im} z_1 \operatorname{Im} z_2,$$
$$\operatorname{Im}(z_1 z_2) = \operatorname{Re} z_1 \operatorname{Im} z_2 + \operatorname{Im} z_1 \operatorname{Re} z_2,$$

and combining with (3.39) and (3.40), we deduce the identities

$$\begin{aligned} 2\nu \|e(\mathbf{u}^\nu)\|^2_{0,\Omega} &+ \operatorname{Re}\mu \left(\left\|\mathbf{u}^\nu - \mathbf{u}^0\right\|^2_{0,\Omega} + \frac{m}{|\mu|^2}\left(|\mu|^2 + \frac{k}{m}\right) \left\|\mathbf{c}^\nu - \mathbf{c}^0\right\|^2_{2K} \right) \\ &= \operatorname{Re}\mu \left(\left\|\mathbf{u}^0\right\|^2_{0,\Omega} + \frac{m}{|\mu|^2}\left(|\mu|^2 + \frac{k}{m}\right) \left\|\mathbf{c}^0\right\|^2_{2K} \right) - \operatorname{Re} L(\mathbf{u}^\nu, \mathbf{c}^\nu) \\ &- 2\operatorname{Im}\mu \operatorname{Im} \int_\Omega \mathbf{u}^0 \cdot \overline{\mathbf{u}}^\nu \, dx - 2\operatorname{Im}\left(\frac{k + m\mu^2}{\mu}\right) \operatorname{Im}(\mathbf{c}^0, \mathbf{c}^\nu)_{2K}, \end{aligned} \tag{3.48}$$

$$\begin{aligned} \operatorname{Im}\mu &\left(\left\|\mathbf{u}^\nu - \mathbf{u}^0\right\|^2_{0,\Omega} + \frac{m}{|\mu|^2}\left(|\mu|^2 - \frac{k}{m}\right) \left\|\mathbf{c}^\nu - \mathbf{c}^0\right\|^2_{2K} \right) \\ &= \operatorname{Im}\mu \left(\left\|\mathbf{u}^0\right\|^2_{0,\Omega} + \frac{m}{|\mu|^2}\left(|\mu|^2 - \frac{k}{m}\right) \left\|\mathbf{c}^0\right\|^2_{2K} \right) - \operatorname{Im} L(\mathbf{u}^\nu, \mathbf{c}^\nu) \\ &+ 2\operatorname{Re}\mu \operatorname{Im} \int_\Omega \mathbf{u}^0 \cdot \overline{\mathbf{u}}^\nu \, dx + 2\operatorname{Re}\left(\frac{k + m\mu^2}{\mu}\right) \operatorname{Im}(\mathbf{c}^0, \mathbf{c}^\nu)_{2K}. \end{aligned} \tag{3.49}$$

Also, taking real and imaginary parts in (3.38), we find exactly the formulas (3.39) and (3.40) for $\nu = 0$ (without the term in $e(\cdot)$, of course). Therefore, in both (3.48) and (3.49) the right-hand side converges to zero as $\nu \to 0$. Hence, in both cases (i) $\operatorname{Re}\mu > 0$ or (ii) $\operatorname{Im}\mu \neq 0$ and $|\mu| > \sqrt{k/m}$, we have that

$$\left\|\mathbf{u}^\nu - \mathbf{u}^0\right\| \to 0. \tag{3.50}$$

Finally, from (3.47) and (3.50), we conclude the strong convergence in U_H. □

Remark. In the self-adjoint case, this result does not imply the continuity, but the noncontraction of the spectrum (see [22, Chapter VIII, Theorem 1.14]). Here, the operator $\mathcal{A}_\nu$ is non-self-adjoint, and we cannot directly deduce a similar result. In spite of this, the existence of a subsequence of real eigenfrequencies of the Stokes model which converge to zero has been proved in [13] (see the proof of Theorem 4 in Chapter 6) or [10] (Theorem 4.5) by using interlacing inequalities with the eigenfrequencies of a problem with fixed tubes. For the nonreal eigenfrequencies, there exists a stronger convergence result, and in the next section we restrict our analysis to this case.

3.5. The analysis of the nonreal eigenfrequencies. The following result shows that the eigenvalues of $\mathcal{A}_0$ different from the eigenvalue zero are isolated under the perturbation of ν in the sense of Kato (see [22, section VIII.2.4]).

THEOREM 3.8. *Let w_L be a nonzero eigenvalue of $\mathcal{A}_0$. Then, there exists $\delta > 0$ such that every μ with $0 < |\mu - w_L| < \delta$ belongs to the resolvent set of $\mathcal{A}_\nu$ for* *sufficiently small* ν *(*$\nu < \nu(\mu)$*) and* $\|R_\mu(\mathcal{A}_\nu)\| \leq C\ \forall \nu < \nu(\mu)$*, where the constant* C *depends only on* μ*.*

Proof. We start by establishing an explicit formula for the numerical image of $\mathcal{A}_\nu$. We work with the inner product introduced in section 3.1 for U_H:

$$((\mathbf{u}^1, \mathbf{s}^1, \mathbf{c}^1), (\mathbf{u}^2, \mathbf{s}^2, \mathbf{c}^2))_{U_H} = \frac{1}{m}\int \mathbf{u}^1 \cdot \overline{\mathbf{u}}^2\, dx + \frac{k}{m}(\mathbf{s}^1, \mathbf{s}^2)_{2K} + (\mathbf{c}^1, \mathbf{c}^2)_{2K}$$

and the associated norm $\|\cdot\|_{U_H}$. From the definition (3.1)–(3.2) of $\mathcal{A}_\nu$, for each $(\mathbf{u}, \mathbf{s}, \mathbf{c}) \in D(\mathcal{A}_\nu)$,

$$(\mathcal{A}_\nu(\mathbf{u}, \mathbf{s}, \mathbf{c}), (\mathbf{u}, \mathbf{s}, \mathbf{c}))_{U_H} = \frac{1}{m}\int \psi \cdot \overline{\mathbf{u}}\, dx + \frac{k}{m}(\mathbf{c}, \mathbf{s})_{2K} + (\mathbf{t}, \mathbf{c})_{2K},$$

and using the characterization of Lemma 3.3, we obtain

$$\begin{aligned} -2\nu\, \|e(\mathbf{u})\|^2_{0,\Omega} &= \int \psi \cdot \overline{\mathbf{u}}\, dx + k(\mathbf{c}, \mathbf{s})_{2K} + m(\mathbf{t}, \mathbf{c})_{2K} - k(\mathbf{c}, \mathbf{s})_{2K} + k(\mathbf{s}, \mathbf{c})_{2K} \\ &= m(\mathcal{A}_\nu(\mathbf{u}, \mathbf{s}, \mathbf{c}), (\mathbf{u}, \mathbf{s}, \mathbf{c}))_{U_H} - 2ik\,\mathrm{Im}\,(\mathbf{c}, \mathbf{s})_{2K}. \end{aligned}$$

Hence, for all $(\mathbf{u}, \mathbf{s}, \mathbf{c}) \in D(\mathcal{A}_\nu)$, we deduce the identity

$$(\mathcal{A}_\nu(\mathbf{u}, \mathbf{s}, \mathbf{c}), (\mathbf{u}, \mathbf{s}, \mathbf{c}))_{U_H} = -\frac{2\nu}{m}\, \|e(\mathbf{u})\|^2_{0,\Omega} + i\frac{2k}{m}\mathrm{Im}\,(\mathbf{c}, \mathbf{s})_{2K}. \tag{3.51}$$

Let $\delta > 0$ be such that

$$B = \{\mu \mid 0 < |\mu - w_L| < \delta\}$$

does not contain any eigenvalue of $\mathcal{A}_0$ and

$$0 < \mathrm{Im}\,\mu < \sqrt{k/m} \qquad \forall \mu \in B.$$

Here we denote by $\sigma(\cdot)$ and $\rho(\cdot)$ the spectrum and resolvent sets, respectively. Let us prove the statement of Theorem 3.8 by contradiction. We suppose that there exists $\mu \in B$ and a sequence $\{\nu_n\}_{n \geq 0}$, $\nu_n \to 0$ as $n \to \infty$, such that either

(i) $\mu \in \sigma(\mathcal{A}_{\nu n})\quad \forall n \geq 0$ or

(ii) $\mu \in \rho(\mathcal{A}_{\nu n})$ for sufficiently large n and $\|R_\mu(\mathcal{A}_{\nu n})\| \to \infty$.

In the case (i), if $\mathcal{A}_{\nu n}(\mathbf{u}_n, \mathbf{s}_n, \mathbf{c}_n) = \mu(\mathbf{u}_n, \mathbf{s}_n, \mathbf{c}_n)$ with $\|(\mathbf{u}_n, \mathbf{s}_n, \mathbf{c}_n)\|_{U_H} = 1$, arguing as in Lemma 3.6 with $(\varphi, \mathbf{x}, \mathbf{y}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$, we easily see that $s_n = \frac{1}{\mu}\mathbf{c}_n$ and

$$2\nu \int_\Omega e(\mathbf{u}_n) : e(\overline{\phi})\, dx + \mu \int_\Omega \mathbf{u}_n \cdot \overline{\phi}\, dx + \frac{k + m\mu^2}{\mu}(\mathbf{c}_n, \mathbf{d})_{2K} = 0 \qquad \forall (\phi, \mathbf{d}) \in S_V.$$

But, from (3.51), we obtain

$$\mathrm{Re}\,\mu = -\frac{2\nu}{m}\, \|e(\mathbf{u}_n)\|^2_{0,\Omega}, \tag{3.52a}$$

$$\mathrm{Im}\,\mu = \frac{2k}{m}\mathrm{Im}\,(\mathbf{c}_n, \mathbf{s}_n). \tag{3.52b}$$

From (3.52a), $\sqrt{\nu}\, \|e(\mathbf{u}_n)\|_{0,\Omega}$ is uniformly bounded on ν, and since $(\mathbf{u}_n, \mathbf{s}_n, \mathbf{c}_n)$ converges (up to a subsequence) to a certain $(\mathbf{u}, \mathbf{s}, \mathbf{c})$ in U_H weakly, by using the same techniques as in Lemma 3.6 we deduce that $(\mathbf{u}, \mathbf{s}, \mathbf{c}) \in U_H$ and that it is a solution of

$$\mu \int_\Omega \mathbf{u} \cdot \overline{\phi}\, dx + \frac{k + m\mu^2}{\mu}(\mathbf{c}, \mathbf{d})_{2K} = 0 \quad \forall (\phi, \mathbf{d}) \in S_H$$

and $\mathbf{s} = \frac{1}{\mu}\mathbf{c}$. Taking the limit as $\nu \to 0$ in (3.52b) we see that $(\mathbf{u}, \mathbf{s}, \mathbf{c}) \neq (\mathbf{0}, \mathbf{0}, \mathbf{0})$, so $\mu \in \sigma(\mathcal{A}_0)$ and this is a contradiction.

In the case (ii), there is a sequence $\{(\varphi_n, \mathbf{x}_n, \mathbf{y}_n)\} \subset U_H$ with $\|(\varphi_n, \mathbf{x}_n, \mathbf{y}_n)\|_{U_H} = 1$, and a $\{(\mathbf{u}_n, \mathbf{s}_n, \mathbf{c}_n)\}$ solution of

$$\mathcal{A}_{\nu n}(\mathbf{u}_n, \mathbf{s}_n, \mathbf{c}_n) - \mu(\mathbf{u}_n, \mathbf{s}_n, \mathbf{c}_n) = (\varphi_n, \mathbf{x}_n, \mathbf{y}_n), \tag{3.53}$$

such that

$$\|(\mathbf{u}_n, \mathbf{s}_n, \mathbf{c}_n)\|_{U_H} \to \infty. \tag{3.54}$$

Multiplying (3.53) by $(\mathbf{u}_n, \mathbf{s}_n, \mathbf{c}_n)$ in U_H and defining

$$(\widehat{\mathbf{u}}_n, \widehat{\mathbf{s}}_n, \widehat{\mathbf{c}}_n) = (\mathbf{u}_n, \mathbf{s}_n, \mathbf{c}_n) / \|(\mathbf{u}_n, \mathbf{s}_n, \mathbf{c}_n)\|_{U_H}$$

we have

$$(\mathcal{A}_{\nu n}(\widehat{\mathbf{u}}_n, \widehat{\mathbf{s}}_n, \widehat{\mathbf{c}}_n), (\widehat{\mathbf{u}}_n, \widehat{\mathbf{s}}_n, \widehat{\mathbf{c}}_n))_{U_H} \to \mu.$$

Hence, by the identity (3.51),

$$-\frac{2\nu}{m} \|e(\widehat{\mathbf{u}}_n)\|^2_{0,\Omega} \to \operatorname{Re}\mu, \tag{3.55a}$$

$$\frac{2k}{m} \operatorname{Im}(\widehat{\mathbf{c}}_n, \widehat{\mathbf{s}}_n) \to \operatorname{Im}\mu. \tag{3.55b}$$

We can then argue as in (i), since from (3.53) we obtain a formula similar to (3.31), and taking the limit in this formula with

$$(\varphi, \mathbf{x}, \mathbf{y}) = (\varphi_n, \mathbf{x}_n, \mathbf{y}_n) / \|(\mathbf{u}_n, \mathbf{s}_n, \mathbf{c}_n)\|_{U_H},$$

we obtain that $\mu \in \sigma(\mathcal{A}_0)$, which is a contradiction. ∎

The property of isolation of the nonzero eigenvalues of $\mathcal{A}_0$ under the perturbation on ν (Theorem 3.8) and the fact that for each $\nu > 0$, $\mathcal{A}_\nu$ has at most $2K$ conjugate pairs of nonreal eigenvalues, lead us to establish the following result.

THEOREM 3.9. *Let w be an eigenvalue of $\mathcal{A}_0$ of multiplicity m. Then, as $\nu \to 0$, there are exactly m nonreal eigenvalues of $\mathcal{A}_\nu$ converging to w.*

For the proof, we utilize the following technical lemma.

LEMMA 3.10. *Let P_n, P be given projections in $\mathcal{L}(\mathcal{H})$; $\mathcal{H}$ being a Banach space, and assume that* $\dim P$ *is finite. If $P_n\varphi \to P\varphi$ for all $\varphi \in \mathcal{H}$ and* $\dim P_n \le \dim P$ *for all n, then* $\dim P_n = \dim P$ *for sufficiently large n.*

Proof. See [22, section VIII.2.4].

Proof of Theorem 3.9. Let $\{\pm i w_{j,L}\}$ be the nonzero eigenvalues of $\mathcal{A}_0$ ordered as follows:

$$0 < w^2_{1,L} < w^2_{2,L} < \cdots < w^2_{r,L} < k/m, \qquad 1 \le r \le 2K,$$

and let m_i be the multiplicity of $i w_{j,L}$ (or $-i w_{j,L}$). We will restrict the proof for the eigenvalues $i w_{j,L}$ with $w_{j,L} > 0$, but the analysis is the same for $-i w_{j,L}$.

If $B_j = \{\mu \mid 0 < |\mu - w_{j,L}| < \delta_j\}$ with the notations of Theorem 3.8, and $\mu_0 \in E$ (E was defined in section 3.4), then Lemma 3.7, Theorem 3.8, and the formula

$$R_\mu(\mathcal{A}_\nu) - R_\mu(\mathcal{A}_0) = (I + (\mu - \mu_0)R_\mu(\mathcal{A}_\nu))(R_{\mu_0}(\mathcal{A}_\nu) - R_{\mu_0}(\mathcal{A}_0))(I + (\mu - \mu_0)R_\mu(\mathcal{A}_0))$$

imply that for each j,

$$R_\mu(\mathcal{A}_\nu) \to R_\mu(\mathcal{A}_0) \quad \forall \mu \in B_j$$

in the strong sense of $\mathcal{L}(U_H)$, and the convergence is uniform in each compact subset of B_j (see [22, Chapter VIII, Theorems 1.2 and 1.3]). Then, if $\gamma_1, \dots, \gamma_r$ with $\gamma_j \subset B_j$ are simple curves which isolate $iw_{1,L}, \dots, iw_{r,L}$ in the complex plane, then for $j = 1, \dots, r$, the projections

$$P_j = -\frac{1}{2\pi i} \int_{\gamma_j} (\mathcal{A}_0 - \mu I)^{-1} d\mu,$$
$$P_j^\nu = -\frac{1}{2\pi i} \int_{\gamma_j} (\mathcal{A}_\nu - \mu I)^{-1} d\mu$$

are well defined (both integrals are taken in the direct sense). Moreover, since the strong convergence of the resolvents is uniform on each γ_j, we have

$$P_j^\nu \varphi \to P_j \varphi \qquad \forall \varphi \in U_H.$$

We know that the operator $\mathcal{A}_\nu$ has at most $2K$ eigenvalues on the upper open complex semiplane, so

$$\sum_{j=1}^{r} \dim P_j^\nu \le 2K = \sum_{j=1}^{r} \dim P_j. \tag{3.56}$$

Then, it is clear that there exists $j_1 \in \{1, \dots, r\}$ such that

$$\dim P_{j_1}^\nu \le \dim P_{j_1};$$

otherwise, we contradict (3.56). Therefore, applying Lemma 3.10, we deduce that for sufficiently small ν $(\nu < \nu(j_1))$,

$$\dim P_{j_1}^\nu = \dim P_{j_1}.$$

Now we have

$$\sum_{\substack{j=1 \\ j \ne j_1}}^{r} \dim P_j^\nu \le 2K - \dim P_{j_1} = \sum_{\substack{j=1 \\ j \ne j_1}}^{r} \dim P_j,$$

and then there exists $j_2 \in \{1, \dots, r\} \setminus \{j_1\}$ such that

$$\dim P_{j_2}^\nu \le \dim P_{j_2},$$

and then by Lemma 3.10, for sufficiently small ν $(\nu < \min\{\nu(j_1), \nu(j_2)\})$, we have

$$\dim P_{j_2}^\nu = \dim P_{j_2}.$$

We can repeat the argument to conclude that

$$\dim P_j^\nu = \dim P_j \qquad \forall j \in \{1, \dots, r\}$$

for sufficiently small viscosity uniformly on j $(\nu < \min\{\nu(j_1), \dots, \nu(j_r)\})$. This implies that the spectrum of $\mathcal{A}_\nu$ inside γ_j consists of isolated eigenvalues with total multiplicity m_j. As the curves γ_j can be taken as circles with vanishing radii, provided that ν is sufficiently small, then these enclosed eigenvalues converge to the respective eigenvalue $iw_{j,L}$ of $\mathcal{A}_0$. ∎

Finally, the proof of Theorem 1.2 follows easily from Lemmas 3.4 and 3.5 and Theorem 3.9. ∎

Remark (convergence of the eigenvectors). It is easy to calculate the adjoint operators of $\mathcal{A}_0$ and $\mathcal{A}_\nu$ with the inner product in U_H. Indeed, $\mathcal{A}_0$ is skew-adjoint; i.e., ${\mathcal{A}_0}^* = -\mathcal{A}_0$ and if

$$\mathcal{A}_\nu(\mathbf{u}, \mathbf{s}, \mathbf{c}) = (\psi, \mathbf{c}, \mathbf{t})$$

as in definition (3.1)–(3.2), then ${\mathcal{A}_\nu}^*$ is done by the relation

$${\mathcal{A}_\nu}^*(\mathbf{u}, -\mathbf{s}, \mathbf{c}) = (\psi, -\mathbf{c}, \mathbf{t}).$$

Therefore, Lemma 3.3 (changing $\mathbf{s}$ to $-\mathbf{s}$ in (3.3)), Lemmas 3.4, 3.5, and 3.6 (changing $\mathbf{s}^\nu$ to $-\mathbf{s}^\nu$, $\mathbf{s}^0$ to $-\mathbf{s}^0$, and $\mathbf{x}$ to $-\mathbf{x}$ in (3.31)–(3.34)), and Lemma 3.7 are also valid for ${\mathcal{A}_\nu}^*$ and ${\mathcal{A}_0}^*$. In such a way, the same steps of this section can be repeated to obtain that

$$P_j^* = -\frac{1}{2\pi i}\int_{\overline{\gamma_j}} ({\mathcal{A}_0}^* - \mu I)^{-1} d\mu,$$
$$P_j^{\nu *} = -\frac{1}{2\pi i}\int_{\overline{\gamma_j}} ({\mathcal{A}_\nu}^* - \mu I)^{-1} d\mu$$

can be defined ($\overline{\gamma_j}$ denote the mirror image of γ_j with respect to the real axis and both integrals are taken in the direct sense) and

$$P_j^{\nu *}\varphi \to P_j^*\varphi \qquad \forall \varphi \in U_H.$$

From [22, Chapter VIII, Lemma 1.24], we obtain that

$$\left\| P_j^\nu - P_j \right\|_{\mathcal{L}(U_H)} \to 0.$$

Hence, the total m_j-dimensional eigenprojection associated to the eigenvalues of $\mathcal{A}_\nu$ in γ_j converges in norm to the eigenprojection for $w_{j,L}$ of $\mathcal{A}_0$ as $\nu \to 0$.

4. Denseness of the generalized eigenfunctions of the Stokes model.

4.1. A denseness theorem. Let A be an unbounded operator in a Hilbert space H, and let the symbol $\overline{sp(A)}$ denote the closed subspace spanned by all $v \in H$ which satisfy an equation of the form $(\lambda I - A)^n v = 0$ for some complex λ and for some nonnegative integer n. That is, $\overline{sp(A)}$ is the closed subspace of H spanned by the generalized eigenvectors of A.

The following result (see [23]) establishes sufficient conditions for $\overline{sp(A)}$ to coincide with H.

THEOREM 4.1. *Let A be a closed, densely defined linear operator in a Hilbert space H, and assume there exists a point λ_0 in the resolvent set $\rho(A)$ of A, such that $R_{\lambda_0}(A)$ is a Hilbert–Schmidt operator. Let $\gamma_i = \{\mu \in \mathbb{C} \mid \arg\mu = \theta_j\}$ for $j = 1, \ldots, 5$ be rays from the origin in the complex plane, such that*

(i) *the angles between adjacent rays are less than $\pi/2$,*

(ii) *for $|\mu|$ sufficiently large all the points on the five rays belong to $\rho(A)$,*

(iii) $\|R_\mu(A)\|$ *is bounded for these* μ, *and*

(iv) *on at least one of the rays,* $\|R_\mu(A)\| \to 0$ *as* $\mu \to \infty$.

Then

$$\overline{sp(A)} = H.$$

N. Dunford and J. Schwartz use Carleman's inequality and the Phragmén–Linderlöf theorem to establish the first denseness result of this kind (see [15, pp. 1038–1044]). Also, there exists a version for operators with a compact resolvent (see [1]).

4.2. Proof of Theorem 1.3. Our goal is to use Theorem 4.1 to prove the denseness of the generalized eigenfunctions of the Stokes model. A first approach is done in [34].

THEOREM 4.2. *The generalized eigenvectors of* $\mathcal{A}_\nu$ *are dense in* U_H.

Proof. Following the notations of section 4.1, we take $A = \mathcal{A}_\nu$ and $H = U_H$. We recall that $\mathcal{A}_\nu$ has a dense domain in U_H by Lemma 3.4. We choose $\lambda_0 = 0$, which lies in $\rho(\mathcal{A}_\nu)$ (see section 1.3), and we will now prove that $R_0(\mathcal{A}_\nu) = \mathcal{A}_\nu^{-1}$ is a Hilbert–Schmidt operator.

Let G_0 be the inverse of the Stokes operator in Ω with Dirichlet conditions on Γ; i.e.,

$$\begin{aligned} G_0 :\ & L^2(\Omega)^2 \to H^1(\Omega)^2 \times L^2(\Omega), \\ & G_0 f = (\mathbf{u}_0, p_0), \end{aligned}$$

where $(\mathbf{u}_0, p_0)$ is the unique solution of

$$\begin{aligned} -\Delta \mathbf{u}_0 + \nabla p_0 &= f \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u}_0 &= 0 \quad \text{in } \Omega, \\ \mathbf{u}_0 &= 0 \quad \text{on } \Gamma. \end{aligned}$$

We denote by $G_0^{(1)} f = \mathbf{u}$ the first component operator of G_0. It is known that $G_0^{(1)}$ is a Hilbert–Schmidt operator. Indeed, if $\{(\lambda_n^{St}, \mathbf{u}_n)\}$ are the characteristic values and the corresponding orthogonalized eigenvectors of $G_0^{(1)}$, then the following identity holds for the Hilbert–Schmidt norm of $G_0^{(1)}$:

$$|||\, G_0^{(1)} \,|||^2 = \sum_{n=1}^{\infty} |(G_0^{(1)} \mathbf{u}_n, \mathbf{u}_n)| = \sum_{n=1}^{\infty} \frac{1}{|\lambda_n^{St}|^2}.$$

But $\lambda_n^{St} \geq \lambda_n$, where λ_n are the eigenvalues of the Laplace operator in Ω with Dirichlet conditions on Γ. And $\lambda_n = O(n)$, so that

$$|||\, G_0^{(1)} \,|||^2 \leq \sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^2} \leq \sum_{n=1}^{\infty} \frac{1}{O(n^2)} < +\infty.$$

Next we define the finite rank operator

$$\begin{aligned} G_{2K} :\ & \mathbb{C}^{2K} \to H^1(\Omega)^2 \times L^2(\Omega), \\ & G_{2K} f = (\mathbf{u}_{2K}, p_{2K}), \end{aligned}$$

where $(\mathbf{u}_{2K}, p_{2K})$ is the unique solution of

$$\begin{aligned} -\Delta \mathbf{u}_{2K} + \nabla p_{2K} &= 0 \quad \text{in } \Omega,\\ \operatorname{div} \mathbf{u}_{2K} &= 0 \quad \text{in } \Omega,\\ \mathbf{u}_{2K} &= 0 \quad \text{on } \Gamma_0,\\ \mathbf{u}_{2K} &= \mathbf{c}_i \quad \text{on } \Gamma_i \ \forall i = 1, \dots, K, \end{aligned}$$

and we denote it by $G_{2K}^{(1)} = \mathbf{u}_{2K}$.

It is easily verified from the definition (3.1) of $\mathcal{A}_\nu$ that

$$\begin{aligned} \mathcal{A}_\nu^{-1} &= \begin{pmatrix} G_0^{(1)} & G_{2K}^{(1)} & 0 \\ -\frac{1}{k}(\mathbf{T} \circ \sigma \circ G_0) & -\frac{1}{k}(\mathbf{T} \circ \sigma \circ G_{2K}) & -\frac{m}{k} I_{2K} \\ 0 & I_{2K} & 0 \end{pmatrix} \\ &= \begin{pmatrix} G_0^{(1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & G_{2K}^{(1)} & 0 \\ -\frac{1}{k}(\mathbf{T} \circ \sigma \circ G_0) & -\frac{1}{k}(\mathbf{T} \circ \sigma \circ G_{2K}) & -\frac{m}{k} I_{2K} \\ 0 & I_{2K} & 0 \end{pmatrix}, \end{aligned}$$

where $\mathbf{T}$ is the finite rank operator defined in (3.2) and I_{2K} is the identity in $\mathbb{C}^{2K}$. Therefore, $\mathcal{A}_\nu^{-1}$ is a Hilbert–Schmidt operator since it is the sum of a Hilbert–Schmidt operator and an operator of finite rank.

We recall that only the straight line $\arg \mu = \pi$ does not satisfy hypothesis (ii) of Theorem 4.1, since there exists a sequence of real eigenfrequencies of the Stokes model which diverges to $-\infty$. If we take, for example, $\theta_j = \frac{\pi}{20} + (j-1)\frac{2\pi}{5}$, $j = 1, \dots, 5$, and $\gamma_j = \{\mu \mid \arg \mu = \theta_j\}$, then hypotheses (i) and (ii) of Theorem 4.1 are fulfilled, and for each $\mu \in \gamma_j$ we have

$$\begin{aligned} \zeta &\equiv \operatorname{Re} \mu \neq 0, \\ \eta &\equiv \operatorname{Im} \mu \neq 0. \end{aligned}$$

Now we can check the other hypotheses of Theorem 4.1. With this goal, we utilize the resolvent operator for the operator $\mathcal{A}_\nu$ introduced in the previous section, and we recall the following identity: if $R_\mu(\mathcal{A}_\nu)(\mathbf{u}, \mathbf{s}, \mathbf{c}) = (\phi, \mathbf{x}, \mathbf{y})$ then

$$(4.1) \quad 2\nu \left\| e(\mathbf{u}) \right\|_{0,\Omega}^2 + \mu \left\| \mathbf{u} \right\|_{0,\Omega}^2 + \frac{k + m\mu^2}{\mu} \left\| \mathbf{c} \right\|_{2K}^2 = \left(\frac{k}{\mu} \mathbf{x} - m\mathbf{y}, \mathbf{c} \right)_{2K} - \int_\Omega \varphi \cdot \overline{\mathbf{u}} \, dx$$

and

$$(4.2) \qquad \mathbf{s} = \frac{1}{\mu}(\mathbf{c} - \mathbf{x}).$$

Let us fix $j \in \{1, \dots, 5\}$ and $\mu \in \gamma_j$ such that $|\mu| \geq \sqrt{k/m}$. We note

$$R_\mu(\mathcal{A}_\nu)(\varphi, \mathbf{x}, \mathbf{y}) = (\mathbf{u}, \mathbf{s}, \mathbf{c}).$$

From (4.5), taking real and imaginary parts,

$$\begin{aligned} 2\nu\zeta \left\| e(\mathbf{u}) \right\|_{0,\Omega}^2 + (\zeta^2 - \eta^2) \left\| \mathbf{u} \right\|_{0,\Omega}^2 + (k + m(\zeta^2 - \eta^2)) \left| \mathbf{c} \right|_{2K}^2 \\ = \operatorname{Re} \left((k\mathbf{x} - m\mu\mathbf{y}, \mathbf{c}) - \mu \int_\Omega \varphi \cdot \overline{\mathbf{u}} dx \right), \\ 2\nu\eta \left\| e(\mathbf{u}) \right\|_{0,\Omega}^2 + 2\zeta\eta \left\| \mathbf{u} \right\|_{0,\Omega}^2 + 2m\zeta\eta \left| \mathbf{c} \right|_{2K}^2 = \operatorname{Im} \left((k\mathbf{x} - m\mu\mathbf{y}, \mathbf{c}) - \mu \int_\Omega \varphi \cdot \overline{\mathbf{u}} dx \right). \end{aligned}$$

Multiplying the first equation by η and the second one by ζ and subtracting, we obtain

$$\eta\,|\mu|^2(\|\mathbf{u}\|_{0,\Omega}^2 + m\,|\mathbf{c}|_{2K}^2) - k\eta\,|\mathbf{c}|_{2K}^2 = -\,\eta\mathrm{Re}\left((k\mathbf{x} - m\mu\mathbf{y}, \mathbf{c}) - \mu\int_\Omega \varphi\cdot\overline{\mathbf{u}}dx\right) + \zeta\mathrm{Im}\left((k\mathbf{x} - m\mu\mathbf{y}, \mathbf{c}) - \mu\int_\Omega \varphi\cdot\overline{\mathbf{u}}dx\right).$$

But

$$\mathrm{Re}\left((k\mathbf{x} - m\mu\mathbf{y}, \mathbf{c}) - \mu\int_\Omega \varphi\cdot\overline{\mathbf{u}}dx\right) \le (k\,|\mathbf{x}|_{2K} + m\,|\mu|\,|\mathbf{y}|_{2K})\,|\mathbf{c}|_{2K} + |\mu|\,\|\varphi\|_{0,\Omega}\,\|\mathbf{u}\|_{0,\Omega}$$

and analogously for the imaginary part. Dividing by $\eta\,|\mu|^2$ and remarking that

$$\frac{\eta}{\zeta} = \tan\,\theta_j,$$

we obtain

$$\|\mathbf{u}\|_{0,\Omega}^2 + m\,|\mathbf{c}|_{2K}^2 \le \frac{k}{|\mu|^2}\,|\mathbf{c}|_{2K}^2 + \frac{1}{|\mu|^2}\Big(1 + \frac{1}{|\tan\,\theta_j|}\Big)\Big((k\,|\mathbf{x}|_{2K} + m\,|\mu|\,|\mathbf{y}|_{2K})\,|\mathbf{c}|_{2K} + |\mu|\,\|\varphi\|_{0,\Omega}\,\|\mathbf{u}\|_{0,\Omega}\Big).$$

If $|\mu| \ge 1$ and $|\mu| \ge \sqrt{\frac{2k}{m}}$, defining $L_j = \Big(1 + \frac{1}{|\tan\,\theta_j|}\Big)$, we have

$$\begin{aligned}\|\mathbf{u}\|_{0,\Omega}^2 + \frac{m}{2}\,|\mathbf{c}|_{2K}^2 &\le \frac{L_j}{|\mu|}\max\{1,k,m\}\Big(\|\varphi\|_{0,\Omega}\,\|\mathbf{u}\|_{0,\Omega} + (|\mathbf{x}|_{2K} + |\mathbf{y}|_{2K})\,|\mathbf{c}|_{2K}\Big)\\ &\le \frac{L_j}{|\mu|}\max\{1,k,m\}\Big(\|\varphi\|_{0,\Omega}^2 + \frac{4}{m}(|\mathbf{x}|_{2K}^2 + |\mathbf{y}|_{2K}^2)\Big)^{\frac{1}{2}}\Big(\|\mathbf{u}\|_{0,\Omega}^2 + \frac{m}{2}\,|\mathbf{c}|_{2K}^2\Big)^{\frac{1}{2}},\end{aligned}$$

hence there exists a constant C_j such that

$$\|\mathbf{u}\|_{0,\Omega}^2 + |\mathbf{c}|_{2K}^2 \le \frac{C_j}{|\mu|^2}\big(\|\varphi\|_{0,\Omega}^2 + |\mathbf{x}|_{2K}^2 + |\mathbf{y}|_{2K}^2\big).$$

From (4.3), we have

$$\begin{aligned}|\mathbf{s}|_{2K}^2 &\le \frac{1}{|\mu|^2}(|\mathbf{c}|_{2K}^2 + |\mathbf{x}|_{2K}^2)\\ &\le \frac{1}{|\mu|^4}\big(\|\varphi\|_{0,\Omega}^2 + |\mathbf{x}|_{2K}^2 + |\mathbf{y}|_{2K}^2\big) + \frac{1}{|\mu|^2}\,|\mathbf{x}|_{2K}^2\\ &\le \frac{2}{|\mu|^2}\big(\|\varphi\|_{0,\Omega}^2 + |\mathbf{x}|_{2K}^2 + |\mathbf{y}|_{2K}^2\big).\end{aligned}$$

Therefore, from the previous inequalities, we conclude with the existence of a constant C_j' such that

$$\big(\|\mathbf{u}\|_{0,\Omega}^2 + |\mathbf{c}|_{2K}^2 + |\mathbf{s}|_{2K}^2\big)^{\frac{1}{2}} \le \frac{C_j'}{|\mu|}\big(\|\varphi\|_{0,\Omega}^2 + |\mathbf{x}|_{2K}^2 + |\mathbf{y}|_{2K}^2\big)^{\frac{1}{2}},$$

and this finally implies that, for sufficiently large $|\mu|$,

$$\|R_\mu(\mathcal{A}_\nu)\| \le \frac{C_j'}{|\mu|}.$$

This result proves hypotheses (iii) and (iv) of Theorem 4.1. Thus we finish the proof of Theorem 4.2. ∎

TABLE 1
Main characteristics of the test problem.

Triangles in τ_h	128
Size of the Sylvester system	746
Number of calculated eigenfrequencies	373
Maximum number of nonreal eigenfrequencies	4

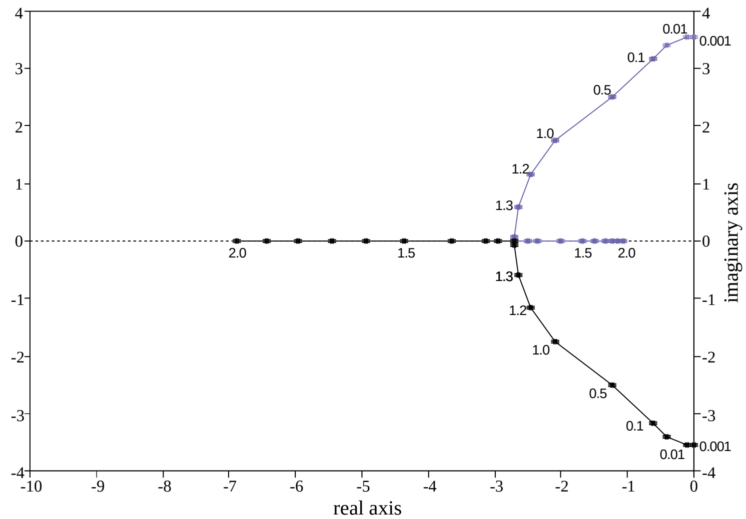

FIG. 2. *Behavior of the nonreal spectrum of the Stokes model as the viscosity (labeled next to each point) tends to zero. After bifurcating from the real axis, the nonreal eigenfrequencies of the Stokes model converge to the pure imaginary Laplace model eigenfrequencies.*

Remark. In fact, the hypotheses are sufficiently strong to use the original theorem of Dunford and Schwartz.

From Theorem 4.2, the definition of U_H (see section 3.1), and Lemma 3.4 we conclude the proof of Theorem 1.3. ∎

5. Numerical results. We have carried out numerical experiments for the Stokes model with the aim of observing the asymptotic behavior of the nonreal eigenfrequencies as $\nu \to 0$. The test problem and the numerical schema are the same as in [7]. We work with a single square tube ($K = 1$) located in the center of a square cavity. More precisely,

$$\Omega = [0, 6]^2 \setminus [2, 4]^2.$$

The classical Lagrange finite element method on triangles was used to discretize the Stokes model. Conformal P_2 elements vanishing on Γ_0 and which are constant on Γ_1 were used to discretize the velocity field. The pressure was approximated by continuous functions which are degree-one polynomials with a reference node.

As shown in [7], we obtain a generalized eigenvalue problem of Sylvester type, which can be solved by using a standard numerical library. In this particular case, we also take advantage of the geometrical symmetries to increase the numerical precision. For the effective calculus, we consider a triangulation τ_h of Ω, and we summarize the principal characteristics of the problem in Table 1.

Figure 2 and Table 2 provide the computed behavior of the nonreal eigenfrequencies as $\nu \to 0$. We have fixed the rigidity to $k = 100$ and the mass to $m = 1$. We

TABLE 2
Convergence of the nonreal spectrum $w(\nu)$ of the Stokes model to w_L.

ν	$w(\nu)$	$\lvert w\rvert(\nu) - w_L$
1.3	$-2.62 \pm i\,0.60$	3.95
1.2	$-2.43 \pm i\,1.15$	3.42
1.0	$-2.05 \pm i\,1.75$	2.72
0.5	$-1.20 \pm i\,2.50$	1.60
0.1	$-0.60 \pm i\,3.15$	0.72
0.05	$-0.40 \pm i\,3.40$	0.43
0.01	$-0.10 \pm i\,3.55$	0.10
0.001	$-0.01 \pm i\,3.55$	0.01

see the trajectory of two eigenfrequencies, each with double multiplicity. We have marked next to each point the corresponding value of ν. These two eigenfrequencies bifurcate from the real axis and, as $\nu \to 0$, move towards the imaginary axis. The convergence points are approximately $\pm 3.5\, i$, which correspond to the Laplace model eigenfrequencies for these values of the parameters k and m. We remark that the eigenfrequencies of the Laplace model are easily calculated as the eigenvalues of a suitable $2K \times 2K$ matrix (see, for example, [13]).

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