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The Bloch Approximation in Periodically Perforated Media*

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Abstract. We consider a periodically heterogeneous and perforated medium filling an open domain Ω of $\mathbb{R}^N$. Assuming that the size of the periodicity of the structure and of the holes is $O(\varepsilon)$, we study the asymptotic behavior, as $\varepsilon \to 0$, of the solution of an elliptic boundary value problem with strongly oscillating coefficients posed in Ω^ε (Ω^ε being Ω minus the holes) with a Neumann condition on the boundary of the holes. We use Bloch wave decomposition to introduce an approximation of the solution in the energy norm which can be computed from the homogenized solution and the first Bloch eigenfunction. We first consider the case where Ω is $\mathbb{R}^N$ and then localize the problem for a bounded domain Ω, considering a homogeneous Dirichlet condition on the boundary of Ω.

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1. Introduction

In this paper we consider the Bloch wave decomposition technique to study a homogenization problem posed in an open domain containing many small periodically distributed holes. As is well known, Bloch wave decomposition provides the spectral resolution of certain partial differential operators with periodic coefficients; this is the standard tool for transforming partial differential equations with periodic coefficients into a set of algebraic equations (see [2], [11], [12] and [17] for example).

Although Bloch waves were first used by Bloch [3] in solid state physics, the basic idea was introduced in the mathematical literature by Floquet [13]. More recently, this method has been used by several authors in different mathematical problems. Let us mention, for example, the studies on the analyticity of Bloch eigenvalues and eigenvectors for periodic media [22] and [20]; linear thermoelasticity [21]; dispersive effective media for wave propagation in periodic composites [19]; spectral problems in fluid–solid structure [1]; and homogenization of elliptic operators with periodic coefficients in domains without holes [9], [10] and [12]. In this last regard, see, for instance, [15] for another technique related to an integral representation formula for the solution.

We deal here with the asymptotic behavior of the solution of a classical homogenization problem. Specifically, we consider a heterogeneous material filling an open domain Ω of $\mathbb{R}^N$ which contains periodically distributed perforations. We assume that the size of the holes and the periodicity depend on a small parameter ε, and both are of the same order of magnitude $O(\varepsilon)$. We denote by Ω^ε the domain Ω minus these holes, and study the asymptotic behavior, as $\varepsilon \to 0$, of the solution of an elliptic boundary value problem posed in Ω^ε with a Neumann condition on the boundary of the holes. In the case where Ω is a bounded domain, we also consider a homogeneous Dirichlet condition on the boundary of Ω.

In this paper we introduce what we call the *Bloch approximation* θ^ε of the solution u^ε. This is a function which can be obtained from the homogenized solution and the first Bloch eigenfunction, and provides an approximation in the energy norm for u^ε. As is well known in homogenization, the homogenized solution u^0 is merely the weak limit of the solutions u^ε in H^1. This homogenized solution has been obtained by several authors using different techniques; we refer to [6], [7] and [16] for the different methods and further references. In particular, in [7] the homogenized solution has been obtained by using Bloch wave decomposition. Nevertheless, one can introduce certain *corrector terms* which, when added to u^0, provide an approximation of u^ε in the energy norm. Usually, these corrector terms are obtained from a formal asymptotic expansion of u^ε in which a multiscale structure of the solution is assumed. Even though error estimates for correctors are sometimes found in the literature, they are usually obtained using the maximum principle with important regularity hypotheses on the coefficients of the operator and the datum f. Here, we obtain a first corrector term under optimal hypotheses, namely bounded measurable coefficients. In addition, we do not need to use any problem or function previously obtained from formal methods (see Remark 6.1).

In fact, using Bloch wave decomposition, we completely avoid asymptotic expansions since, firstly, the homogenized problem is obtained directly from the Hessian of the first Bloch eigenvalue (see Proposition 3.3) and, secondly, the new approximation in the energy norm depends only on the solution of the homogenized problem u^0, the first

Bloch eigenvector and the Fourier transform of u^0 (see (5.2) for $\Omega = \mathbb{R}^N$ and (6.2) for Ω a bounded domain). Roughly speaking, we can assert that this new approximation contains the homogenized solution and the classical first-order corrector (see Theorem 6.2 and Remark 6.1).

Bloch wave decomposition is used in [12], [9] and [10] to study homogenization problems in heterogeneous media without holes, providing both the corresponding homogenized solution and the Bloch approximation. This approximation can be useful for numerical computations (see [8] for a detailed comparison between the Bloch approach and the classical one). It should be noted that the methods of the above-mentioned paper cannot be applied directly to our problem (namely, problem (2.3)). Since we are dealing with a perforated domain, we are forced to develop an alternative approach which involves, among other techniques, a different spectral resolution of the operators under consideration and prolongation operators.

In Section 2 we pose the homogenization problem and introduce the notations used throughout the paper. We also introduce without any proof some classical results, obtained by different authors, in order to compare our results with those obtained by other methods.

In Section 3, in order to make our discussion self-contained, we summarize the main results obtained in [7] which are used throughout the paper. As a matter of fact, the results stated in this section allow us to assert that all the information on the homogenized problem is contained in the first Bloch eigenvalue and the first Bloch coefficient (see Proposition 3.1) and that there is a connection between the first Bloch coefficient (defined in Theorem 3.1) and the Fourier transform (see Proposition 3.4).

In Section 4 we prove some useful results necessary for our approach. Among other things, the results in Section 4.1 give an extension of the Bloch coefficients and the Parseval–Plancherel identity for functions of L^2 to distributions in H^{-1} (see Theorems 3.1 and 4.1 for comparison). Moreover, results in Section 4.2 improve those stated in Section 3. Specifically, we prove that not only are the higher-order Bloch modes negligible for the homogenized problem, but also for the energy norm (see Proposition 4.2). Besides, we characterize the asymptotic behavior of the first Bloch coefficient in $L^2(\mathbb{R}^N)$ (see Propositions 3.4 and 4.3 for comparison). We also establish some estimates for the Bloch eigenelements which will be used in Sections 5 and 6.

Sections 5 and 6 contain the main results of this paper. Section 5 is devoted to the case where Ω is the whole $\mathbb{R}^N$. We introduce the new approximation θ^ε, the *Bloch approximation* (see (5.2)), and prove the convergence in the energy norm of the difference between the solution of the ε-depending problem and the new approximation (see Theorems 5.1 and 5.2).

Finally, in Section 6, we study the case where Ω is a bounded domain. We introduce a modified Bloch approximation $\breve{\theta}^\varepsilon$ (see (6.2)) and extend the results in Section 5 to a bounded domain Ω (see Theorems 6.1 and 6.2). As a matter of fact, the proofs are rather more complicated because of the localization process and the extension of the Parseval–Plancherel identity in Section 4.1 becomes essential.

To conclude, since a corrector of the solutions has been obtained in previous works (see [6] and [16]), we compare our Bloch approximation with this classical first-order corrector and verify that they are asymptotically close functions, as $\varepsilon \to 0$, in the norm of H^1 (see Theorems 5.3 and 6.2, and Remark 6.1). Nevertheless, we emphasize that our approach is independent either of formal asymptotic expansions or of the other methods used in classical homogenization to prove convergence for periodically perforated media.

2. Setting the Problem

Let Y be the unit cell in $\mathbb{R}^N$, $Y = [0, 2\pi)^N$, and let T be an open bounded domain with a smooth boundary, ∂T; $\bar{T} \subset Y$ and $d(\bar{T}, \partial Y) > 0$. We denote by Y^* the domain $Y^* = Y - \bar{T}$ and by θ the volume fraction material constant, $\theta = |Y^*|/|Y|$. Let Y_{k} denote the translation of Y to the point $\tilde{x}_{\mathrm{k}} \in 2\pi\mathbb{Z}^N$ and let T_{k} denote the translation of T to the point $\tilde{x}_{\mathrm{k}}$, $\bar{T}_{\mathrm{k}} \subset Y_{\mathrm{k}}$ and $Y^*_{\mathrm{k}} = Y_{\mathrm{k}} - \bar{T}_{\mathrm{k}}$. Let $Y^\varepsilon_{\mathrm{k}}$ ($T^\varepsilon_{\mathrm{k}}$, $Y^{\varepsilon *}_{\mathrm{k}}$) be the homothetics of Y_{k} (T_{k}, Y^*_{k}), $\varepsilon Y_{\mathrm{k}}$ ($\varepsilon T_{\mathrm{k}}$, $\varepsilon Y^*_{\mathrm{k}}$) where ε is a small parameter that converges to zero.

Let $\{a_{ij}\}_{i,j=1,\ldots,N}$ be Y-periodic bounded measurable real functions defined on $\mathbb{R}^N$ minus the holes, $a_{ij} \in L^\infty_\#(Y^*)$ satisfying the symmetry and ellipticity conditions

$$a_{ij} = a_{ji}, \qquad \forall i, j = 1, \ldots, N, \quad \text{and} \tag{2.1}$$

$$a_{ij}(y)\xi_i\xi_j \geq \alpha|\xi|^2 \qquad \text{for some} \quad \alpha > 0, \quad \forall \xi \in \mathbb{R}^N. \tag{2.2}$$

Here, and henceforth, the classical assumption of the summation over repeated indices is performed.

Let Ω be an open domain of $\mathbb{R}^N$ with a smooth boundary. For each $\varepsilon > 0$, we consider $\Omega^\varepsilon = \Omega - (\bigcup_{\mathrm{k}} \bar{T}^\varepsilon_{\mathrm{k}} \cap \Omega) = \Omega - \bigcup_{\mathrm{k}=1}^{N(\varepsilon)} \bar{T}^\varepsilon_{\mathrm{k}}$ where $N(\varepsilon)$ is the number of cells $Y^\varepsilon_{\mathrm{k}}$ contained in Ω. We denote by $a^\varepsilon_{ij}(x)$ the value of the coefficient $a_{ij}(y)$ at the point x/ε.

To fix ideas, we additionally assume that Ω is bounded. In this case the boundary value problem which forms the goal of this paper can be written as follows:

$$\begin{cases} -\dfrac{\partial}{\partial x_j}\left(a^\varepsilon_{ij}(x)\dfrac{\partial u^\varepsilon}{\partial x_i}\right) = f & \text{in } \Omega^\varepsilon, \\ a^\varepsilon_{ij}\dfrac{\partial u^\varepsilon}{\partial x_i}n_j = 0 & \text{on } \partial T^\varepsilon_{\mathrm{k}}, \ \forall \mathrm{k}, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where $f \in L^2(\Omega)$, and n denotes the outward unit normal to the boundary of the hole $T^\varepsilon_{\mathrm{k}}$.

The asymptotic behavior of the solution of (2.3) has already been studied in [6] and [7] by using the energy method and Bloch waves, respectively (see [16] for other techniques). Since we are dealing with a perforated domain, most of the results are obtained on the basis that there is an extension operator P^ε, P^ε: $V^\varepsilon \to H^1_0(\Omega)$ such that

$$\|\nabla_x P^\varepsilon v\|_{L^2(\Omega)} \leq C\|\nabla_x v\|_{L^2(\Omega^\varepsilon)}, \qquad \forall v \in V^\varepsilon, \tag{2.4}$$

where V^ε is the space $V^\varepsilon = \{u \in H^1(\Omega^\varepsilon) \mid u = 0 \text{ on } \partial\Omega\}$. Here, and in what follows, C denotes different constants independent of ε. The following homogenization result holds:

Theorem 2.1. *Let u^ε be the solution of* (2.3) *with coefficient $a_{ij} \in L^\infty_\#(Y^*)$ satisfying assumptions* (2.1) *and* (2.2). *Let f be a function of $L^2(\Omega)$. Then $P^\varepsilon u^\varepsilon$ converges weakly in $H^1_0(\Omega)$ towards u^0, as $\varepsilon \to 0$, where u^0 is the solution of the* homogenized problem:

$$\begin{cases} -a^h_{ij}\dfrac{\partial^2 u^0}{\partial x_i\,\partial x_j} = f\theta & \text{in } \Omega, \\ u^0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

The homogenized coefficients a^h_{ij} *in* (2.5) *satisfy the symmetry and ellipticity conditions and are given*[1] *by*

$$a^h_{ij} = \frac{1}{|Y|}\int_{Y^*}\left(a_{ij}(y) + a_{im}(y)\frac{\partial v^j}{\partial y_m}\right)\,dy, \qquad i, j = 1, 2, \dots, N, \tag{2.6}$$

where y is the local variable, $y = x/\varepsilon$, *and for each $k = 1, 2, \dots, N$, v^k is the solution of the* local problem *on the unit cell Y*:

$$\begin{cases} -\dfrac{\partial}{\partial y_j}\left(a_{ij}(y)\dfrac{\partial v^k}{\partial y_i}\right) = \dfrac{\partial a_{kj}}{\partial y_j} & \text{in } Y^*, \\ a_{ij}\dfrac{\partial v^k}{\partial y_i}n_j = -a_{kj}n_j & \text{on } \partial T, \\ v^k \quad Y\text{-periodic}, & \mathcal{M}_{Y^*}(v^k) \equiv \dfrac{1}{|Y^*|}\displaystyle\int_{Y^*} v^k\,dy = 0. \end{cases} \tag{2.7}$$

We refer to [6], [7] and [16] for different proofs of results in Theorem 2.1, for further references, and for references which avoid extension operators.

A classical approximation of u^ε is constructed by means of the solution of the homogenized problem and the correcting terms, which are obtained from a formal asymptotic expansion of the solution of (2.3). In fact, by using a two-scale asymptotic expansion (see [18] for the technique), we have

$$u^\varepsilon(x) = u^0(x) + \varepsilon v^k\left(\frac{x}{\varepsilon}\right)\frac{\partial u^0}{\partial x_k}(x) + \varepsilon^2 v^{kl}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u^0}{\partial x_k\,\partial x_l} + \cdots, \tag{2.8}$$

where u^0 and v^k are the functions defined in Theorem 2.1 and v^{kl} are the solutions of the problem

$$\begin{cases} -\dfrac{\partial}{\partial y_j}\left(a_{ij}(y)\dfrac{\partial v^{kl}}{\partial y_i}\right) = a_{kl} - \dfrac{1}{\theta}a^h_{kl} + \dfrac{\partial}{\partial y_j}(a_{jk}v^l) + a_{kj}\dfrac{\partial v^l}{\partial y_j} & \text{in } Y^*, \\ a_{ij}\dfrac{\partial v^{kl}}{\partial y_i}n_j = -a_{kj}v^l\,n_j \quad \text{on } \partial T, \\ v^{kl} \quad Y\text{-periodic}, \qquad \mathcal{M}_{Y^*}(v^{kl}) = 0. \end{cases} \tag{2.9}$$

A justification of (2.8) is given by the following result:

[1] Another characterization of the homogenized coefficients is obtained from the Hessian of the first Bloch eigenvalue (see (3.10)).

Theorem 2.2. *Let u^ε be the solution of* (2.3) *with $f \in L^2(\Omega)$ and the coefficients $a_{ij} \in L^\infty_\#(Y^*)$ satisfying the symmetry and ellipticity conditions* (2.1) *and* (2.2). *We assume that the solutions v^k of problem* (2.7) *satisfy $v^k \in W^{1,\infty}_\#(Y^*)$. Then*

$$\left\| u^\varepsilon(x) - u^0(x) - \varepsilon v^k\left(\frac{x}{\varepsilon}\right)\frac{\partial u^0}{\partial x_k}(x)\right\|_{H^1(\Omega^\varepsilon)} \xrightarrow{\varepsilon\to 0} 0.$$

The proof of Theorem 2.2 can be obtained using the technique in [2], for nonperforated domains, suitably modified (see also Remark 6.1). We refer to [6] and [16] for another proof of this result with stronger restrictive hypotheses on the coefficients a_{ij} and the datum f. See [5] for a very different technique to obtain correctors in periodic media.

The purpose of this paper is to provide a new approximation, θ^ε, of u^ε which has the following property:

$$\|u^\varepsilon - \theta^\varepsilon\|_{H^1(\Omega^\varepsilon)} \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

This approach can be found in Theorems 5.1 and 5.2 for the case where Ω is the whole $\mathbb{R}^N$ and in Theorem 6.1 for Ω a bounded domain. We use a different approach to that in (2.8), namely, the Bloch approximation θ^ε, which has been introduced in [9] for domains without holes. The basic tool of this new approximation is the Bloch wave decomposition, which allows us to obtain the result for u^ε avoiding any a priori assumption on the structure of the solution. In fact, in Sections 5 and 6 we give the formula for θ^ε in $\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}^\varepsilon_{\mathrm{k}}$ (see (5.2)) and $\Omega - \bigcup_{\mathrm{k}} \bar{T}^\varepsilon_{\mathrm{k}}$ (see (6.2)), respectively, which depends only on the solution of the homogenized problem and the first Bloch eigenvector.

We denote by $H^1_\#(Y^*)$ and $L^2_\#(Y^*)$ the spaces

$$H^1_\#(Y^*) = \left\{ u \in H^1_{\mathrm{loc}}\left(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}\right) \mid u \text{ is } Y\text{-periodic}\right\},$$

$$L^2_\#(Y^*) = \left\{ u \in L^2_{\mathrm{loc}}\left(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}\right) \mid u \text{ is } Y\text{-periodic}\right\},$$

where the union is extended to all the holes T_{k} in $\mathbb{R}^N$.

3. The Bloch Transform. Previous Results

In this section we introduce the basic tool of this new approximation: the Bloch wave decomposition for the case of domains with holes. This technique is used in [7] to prove the classical homogenization result stated in Theorem 2.1. Here we outline the main results proved in [7], which will be of great use throughout the paper.

Let A be the operator,

$$A = -\frac{\partial}{\partial y_j}\left(a_{ij}(y)\frac{\partial}{\partial y_i}\right) \qquad \text{in } \left(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}\right)$$

with a homogeneous Neumann condition on the boundary of the holes, i.e.

$$a_{ij}(y)\frac{\partial}{\partial y_i}n_j = 0 \qquad \text{on } \partial T_{\mathrm{k}}, \quad \forall \mathrm{k}.$$

For each η, $\eta \in Y^{'}$, $Y^{'}$ being the dual cell to Y, $Y^{'} = [-\frac{1}{2}, \frac{1}{2})^N$, we consider $A(\eta)$ the shifted operator,

$$A(\eta) = -\left(\frac{\partial}{\partial y_i} + \mathbf{i}\eta_i\right)\left[a_{ij}(y)\left(\frac{\partial}{\partial y_j} + \mathbf{i}\eta_j\right)\right] \qquad \text{in } \left(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}\right),$$

acting on the Y-periodic functions, with the boundary conditions

$$a_{ij}\left(\frac{\partial}{\partial y_i} + \mathbf{i}\eta_i\right)n_j = 0 \qquad \text{on } \partial T_{\mathrm{k}}, \ \forall \mathrm{k}.$$

For each $\eta \in Y^{'}$, the operator $A(\eta)$ is a self-adjoint operator on $L^2_{\#}(Y^*)$ with a compact resolvent. Thus, the spectral problem

$$\begin{cases} A(\eta)\varphi = \lambda\varphi & \text{in } (\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}), \\ \varphi \quad Y\text{-periodic}, \end{cases} \tag{3.1}$$

has a discrete spectrum. Let $0 \leq \lambda_1(\eta) \leq \lambda_2(\eta) \leq \cdots \leq \lambda_m(\eta) \leq \cdots \to \infty$ be the sequence of eigenvalues of problem (3.1) with the classical convention of repeated eigenvalues. Let $\{\varphi_m(\cdot, \eta)\}_{m=1}^{\infty}$ denote the corresponding eigenfunctions that form an orthonormal basis in $L^2_{\#}(Y^*)$. As usual, $\{\lambda_m(\eta)\}_{m=1}^{\infty}$ are referred to as the *Bloch eigenvalues* and $\{\varphi_m(\cdot, \eta)\}_{m=1}^{\infty}$ as the *Bloch eigenvectors* or the *Bloch waves*.

Notice that for $\eta = 0$, the first eigenvalue of problem (3.1) is $\lambda_1(0) = 0$ and the corresponding eigenfunctions are the constants. We have chosen $\varphi_1(\cdot, 0)$ as the normalized function in $L^2_{\#}(Y^*)$,

$$\varphi_1(\cdot, 0) = \frac{1}{|Y^*|^{1/2}}. \tag{3.2}$$

It is worth noting that the functions $\psi(y, \eta) = e^{\mathbf{i}\eta\cdot y}\varphi_m(y, \eta)$ are the so-called generalized eigenfunctions of A associated with the generalized eigenvalues $\lambda_m(\eta)$:

$$\begin{cases} A\psi(\cdot, \eta) = \lambda(\eta)\psi(\cdot, \eta) & \text{in } (\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}), \\ \psi(\cdot, \eta) \text{ is } (\eta, Y)\text{-periodic, i.e.} \\ \forall m \in \mathbb{Z}^N, \quad y \in \mathbb{R}^N, \qquad \psi(y + 2\pi m, \eta) = e^{2\pi \mathbf{i} m\cdot\eta}\psi(y, \eta), \end{cases}$$

with the Neumann condition on the boundary of the holes.

We consider A^ε the operator defined by

$$A^\varepsilon = -\frac{\partial}{\partial x_i}\left(a^\varepsilon_{ij}(x)\frac{\partial}{\partial x_j}\right) \qquad \text{in } \left(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}^\varepsilon_{\mathrm{k}}\right),$$

with the boundary conditions on the holes $T^\varepsilon_{\mathrm{k}}$:

$$a^\varepsilon_{ij}\frac{\partial}{\partial x_i}n_j = 0 \qquad \text{on } \partial T^\varepsilon_{\mathrm{k}}, \quad \forall \mathrm{k}.$$

In order to state the spectral resolution of A^ε in the x variable, we introduce here the notations:

$$y = \frac{x}{\varepsilon}, \qquad \xi = \frac{\eta}{\varepsilon}, \tag{3.3}$$

$$\lambda_m^\varepsilon(\xi) = \frac{1}{\varepsilon^2}\lambda_m(\eta), \qquad \varphi_m^\varepsilon(x,\xi) = \varphi_m(y,\eta), \tag{3.4}$$

where $y \in (\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}})$, $x \in (\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)$, $\eta \in Y'$ and $\xi \in Y'/\varepsilon$. Using the spectral resolution of A as an unbounded self-adjoint operator on $L^2(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}})$ in terms of the Bloch eigenelements (see [7]), the following theorem holds:

Theorem 3.1. *Let $g \in L^2(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)$. Then g admits the expansion*:

$$g(x) = \sum_{m=1}^{\infty} \int_{Y'/\varepsilon} (B_m^\varepsilon g)(\xi) e^{\mathbf{i}x\cdot\xi} \varphi_m^\varepsilon(x,\xi)\, d\xi,$$

*where $(B_m^\varepsilon g)(\xi)$ denotes the m*th *Bloch coefficient*

$$(B_m^\varepsilon g)(\xi) = \int_{(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)} g(x) e^{-\mathbf{i}x\cdot\xi} \bar{\varphi}_m^\varepsilon(x,\xi)\, dx.$$

Moreover, the Parseval–Plancherel identity

$$\int_{(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)} g(x)\overline{h(x)}\, dx = \int_{Y'/\varepsilon} \sum_{m=1}^{\infty} (B_m^\varepsilon g)(\xi)\overline{(B_m^\varepsilon h)(\xi)}\, d\xi \tag{3.5}$$

holds for all $g, h \in L^2(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)$. In addition, we have

$$A^\varepsilon u(x) = \sum_{m=1}^{\infty} \int_{Y'/\varepsilon} \lambda_m^\varepsilon(\xi)(B_m^\varepsilon u)(\xi) e^{\mathbf{i}x\cdot\xi} \varphi_m^\varepsilon(x,\xi)\, d\xi\,, \tag{3.6}$$

for each $u \in \{v \in L^2(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon) \mid A^\varepsilon v \in L^2(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)\}$.

Using the above theorem, the classical homogenization result in Theorem 2.1 is proved in [7]. In order to be self-contained, we outline the main results used for the proof. Thanks to (3.6), solving equation $A^\varepsilon u^\varepsilon = f$ in $(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)$ is equivalent to solving

$$\lambda_m^\varepsilon(\xi)(B_m^\varepsilon u^\varepsilon)(\xi) = (B_m^\varepsilon f)(\xi), \qquad m \geq 1, \quad \xi \in Y'/\varepsilon. \tag{3.7}$$

We observe that the information on the homogenized problem is contained in the first Bloch eigenvalue (see Proposition 3.1). Then Propositions 3.2–3.4 lead us to obtain the homogenization result for $\Omega = \mathbb{R}^N$. For Ω a bounded domain, techniques of localization lead us to Theorem 2.1.

Proposition 3.1. *Let f be a function of $L^2(\mathbb{R}^N)$ and let u^ε be the solution of $A^\varepsilon u^\varepsilon = f$ in $(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)$, $u^\varepsilon \in H^1(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)$. Then*

$$\left\| \sum_{m=2}^{\infty} \int_{Y'/\varepsilon} (B_m^\varepsilon u^\varepsilon)(\xi) e^{\mathbf{i}x\cdot\xi} \varphi_m^\varepsilon(x,\xi)\, d\xi \right\|_{L^2(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)} \leq C\varepsilon^2,$$

where C is a constant which does not depend on ε.

Proposition 3.2. *There is a neighborhood $\mathcal{D}$ of the origin where the first eigenvalue $\lambda_1(\eta)$ of problem* (3.1) *remains simple and defines an analytic function of η. Besides, the first eigenvector $\varphi_1(\cdot,\eta)$ can be chosen in such a way that the map*

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & H^1_\#(Y^*) \\ \eta & \longrightarrow & \varphi_1(\cdot,\eta), \end{array} \tag{3.8}$$

is analytic and $\varphi_1(\cdot,0) = |Y^*|^{-1/2}$.

Proposition 3.3. *The first Bloch eigenvalue $\lambda_1(\eta)$ satisfies the following relations*:

$$\frac{\partial \lambda_1}{\partial \eta_k}(0) = 0, \qquad k = 1,2,\ldots,N, \tag{3.9}$$

$$\frac{1}{2}\frac{\partial^2 \lambda_1}{\partial \eta_k\, \partial \eta_l}(0) = \frac{a_{kl}^h}{\theta}, \qquad k,l = 1,2,\ldots,N, \tag{3.10}$$

where a_{kl}^h are the homogenized coefficients defined in (2.6) *and $\theta = |Y^*|/|Y|$.*

Proposition 3.4. *Let $\{g^\varepsilon\}_\varepsilon$ be a sequence $g^\varepsilon \in L^2(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)$. Let $\tilde{g}^\varepsilon$ be the extension by zero on $T_{\mathrm{k}}^\varepsilon$ of g^ε. We assume that $\tilde{g}^\varepsilon$ converges weakly in $L^2(\mathbb{R}^N)$ towards some function $g \in L^2(\mathbb{R}^N)$, as $\varepsilon \to 0$, and* $\operatorname{supp} g^\varepsilon \subseteq K$; *$K$ being some fixed compact set of $\mathbb{R}^N$. Then*

$$\chi_{Y'/\varepsilon} B_1^\varepsilon \tilde{g}^\varepsilon \longrightarrow \frac{1}{\theta^{1/2}}\hat{g} \qquad \text{weakly in } L^2_{\mathrm{loc}}(\mathbb{R}^N), \quad \text{as } \varepsilon \to 0,$$

where χ_B is the characteristic function of B and $\hat{g}$ is the Fourier transform of g.

4. Some Useful Results

In this section we prove certain results which will be useful in Sections 5 and 6 in order to obtain the new approximation θ^ε mentioned in Section 2.

4.1. *Extension of the Parseval–Plancherel Identity*

Results in the following theorem extend the Parseval–Plancherel identity stated in Theorem 3.1 to distributions in $H^{-1}(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)$ providing a precise definition for Bloch coefficients of these distributions (see Remark 4.1). We also obtain certain general estimates involving oscillatory integrals. All these results prove to be essential for the results in the rest of the paper.

Theorem 4.1. *Let $h \in H^{-1}(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$. Let $u \in H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$ be the unique solution of $A^\varepsilon u + u = h$ in $(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$. Then, for all $v \in H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$,*

$$\langle h, v\rangle_{H^{-1}(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)\times H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} = \int_{Y'/\varepsilon} \sum_{m=1}^{\infty} (B_m^\varepsilon h)(\xi)\overline{(B_m^\varepsilon v)(\xi)}\, d\xi, \tag{4.1}$$

where $(B_m^\varepsilon h)(\xi)$ is defined by

$$(B_m^\varepsilon h)(\xi) = (\lambda_m^\varepsilon(\xi) + 1)(B_m^\varepsilon u)(\xi), \qquad m \geq 1, \quad \xi \in Y'/\varepsilon, \tag{4.2}$$

and $\langle \cdot, \cdot\rangle_{H^{-1}(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)\times H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)}$ denotes the product of duality in the spaces $H^{-1}(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$ and $H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$.

Proof. Since $u \in H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$ is the unique solution of $A^\varepsilon u + u = h$ in $(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$, it is clear that for all $v \in H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$,

$$\langle h, v\rangle_{H^{-1}(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)\times H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} = \int_{(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} a_{ij}^\varepsilon \frac{\partial u}{\partial x_i}\overline{\frac{\partial v}{\partial x_j}}\, dx + \int_{(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} u\bar{v}\, dx.$$

Then we apply the Second Representation Theorem for sesquilinear and symmetric forms (see, for example, Section VI.2.6 of [14]) and we have

$$\begin{aligned}\int_{(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} a_{ij}^\varepsilon \frac{\partial u}{\partial x_i}\overline{\frac{\partial v}{\partial x_j}}\, dx + \int_{(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} u\bar{v}\, dx \\ = ((A^\varepsilon + I)^{1/2}u, (A^\varepsilon + I)^{1/2}v)_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)}.\end{aligned}$$

Now, the spectral resolution of $(A^\varepsilon + I)^{1/2}$ gives

$$\begin{aligned}&((A^\varepsilon + I)^{1/2}u, (A^\varepsilon + I)^{1/2}v)_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \\ &\quad = \int_{Y'/\varepsilon} \sum_{m=1}^{\infty} (\lambda_m^\varepsilon(\xi) + 1)(B_m^\varepsilon u)(\xi)\overline{(B_m^\varepsilon v)(\xi)}\, d\xi,\end{aligned}$$

that is, (4.1) is proved. □

Remark 4.1. We observe that the isomorphism $(A^\varepsilon + I)$: $H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon) \to H^{-1}(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$ allows us to extend the definition of the mth Bloch coefficient to elements of $H^{-1}(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$. As a matter of fact, (4.2) is referred to as *the mth Bloch coefficient of the distribution $h \in H^{-1}(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$.*

Lemma 4.1. *For all $g \in H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$, we have*

$$C_1 \|\nabla_x g\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \leq \int_{Y'/\varepsilon} \sum_{m=1}^{\infty} \lambda_m^\varepsilon(\xi) |(B_m^\varepsilon g)(\xi)|^2 d\xi \leq C_2 \|\nabla_x g\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)},$$

where the constants C_1, C_2 are independent of ε and g.

Proof. On account of the properties of the coefficients a_{ij}, we have

$$C_1 \int_{(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} |\nabla_x g|^2 \, dx \leq \int_{(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} a_{ij}^\varepsilon \frac{\partial g}{\partial x_i} \overline{\frac{\partial g}{\partial x_j}} \, dx \leq C_2 \int_{(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} |\nabla_x g|^2 \, dx.$$

Then, applying Theorem 4.1, we obtain

$$\begin{aligned} \int_{(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} a_{ij}^\varepsilon \frac{\partial g}{\partial x_i} \overline{\frac{\partial g}{\partial x_j}} \, dx &= \langle A^\varepsilon g, g \rangle_{H^{-1}(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon) \times H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \\ &= \int_{Y'/\varepsilon} \lambda_m^\varepsilon(\xi) |(B_m^\varepsilon g)(\xi)|^2 \, d\xi, \end{aligned}$$

and the lemma is proved. □

We consider $g^\varepsilon = g^\varepsilon(\xi)$ a measurable function defined on Y'/ε and $\rho = \rho(y, \eta)$ another measurable function defined on $Y^* \times Y'$ which is Y-periodic in y. We define the function

$$G^\varepsilon(x) = \int_{Y'/\varepsilon} g^\varepsilon(\xi) \, e^{\mathbf{i} x \cdot \xi} \, \rho(x/\varepsilon, \varepsilon\xi) \, d\xi, \qquad x \in \left(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon \right). \tag{4.3}$$

The following estimates of G^ε in $L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$ and $H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$ are obtained:

Lemma 4.2. *Let $g^\varepsilon \in L^2(Y'/\varepsilon)$ and $\rho \in L^\infty(Y', H^1_\#(Y^*))$. Then*

$$\|G^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} = \int_{Y'/\varepsilon} |g^\varepsilon(\xi)|^2 \, \|\rho(\cdot, \varepsilon\xi)\|^2_{L^2(Y^*)} \, d\xi,$$
$$\|\nabla_x G^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} = \int_{Y'/\varepsilon} |g^\varepsilon(\xi)|^2 \, \left\| \mathbf{i}\, \xi \, \rho(\cdot, \varepsilon\xi) + \tfrac{1}{\varepsilon} \nabla_y \rho(\cdot, \varepsilon\xi) \right\|^2_{L^2(Y^*)} \, d\xi.$$

Proof. For each $\eta \in Y'$, we expand $\rho(y, \eta) \in H^1_\#(Y^*)$ in the orthonormal basis $\{\varphi_m(y, \eta)\}_{n=1}^\infty$:

$$\rho(y, \eta) = \sum_{m=1}^\infty a_m(\eta) \varphi_m(y, \eta) \quad \text{and} \quad \|\rho(y, \eta)\|^2_{L^2(Y^*)} = \sum_{m=1}^\infty |a_m(\eta)|^2.$$

Introducing this expansion in (4.3),

$$G^\varepsilon(x) = \int_{Y'/\varepsilon} g^\varepsilon(\xi) \sum_{m=1}^\infty a_m(\varepsilon\xi) \varphi_m^\varepsilon(x, \xi) \, d\xi$$

and applying (3.5), we obtain

$$\begin{aligned} \|G^\varepsilon(x)\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} &= \int_{Y'/\varepsilon} |g^\varepsilon(\xi)|^2 \sum_{m=1}^\infty |a_m(\varepsilon\xi)|^2 \, d\xi \\ &= \int_{Y'/\varepsilon} |g^\varepsilon(\xi)|^2 \|\rho(\cdot, \varepsilon\xi)\|^2_{L^2(Y^*)} \, d\xi. \end{aligned}$$

To estimate $\nabla_x G^\varepsilon$ in the $L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$ norm, we differentiate G^ε with respect to x,

$$\frac{\partial G^\varepsilon}{\partial x_k}(x) = \int_{Y'/\varepsilon} g^\varepsilon(\xi) e^{\mathbf{i}x\cdot\xi} \left(\mathbf{i}\xi_k \rho\left(\frac{x}{\varepsilon}, \varepsilon\xi\right) + \frac{1}{\varepsilon}\frac{\partial\rho}{\partial y_k}\left(\frac{x}{\varepsilon}, \varepsilon\xi\right)\right) d\xi, \quad x \in \left(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon\right),$$

and apply the same arguments as in G^ε. □

We consider $g = g(x)$ a measurable function defined on $(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$ and $\rho = \rho(y, \eta)$ another measurable function defined on $Y^* \times Y'$ which is Y-periodic in the y variable. We define the function

$$J^\varepsilon g(\xi) = \int_{(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} g(x)\, e^{-\mathbf{i}x\cdot\xi}\, \rho(x/\varepsilon, \varepsilon\xi)\, dx, \qquad \xi \in Y'/\varepsilon \tag{4.4}$$

(see (4.3) for comparison). For $J^\varepsilon g$, we obtain the following estimate in $L^2(Y'/\varepsilon)$:

Lemma 4.3. *If $g \in L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$ and $\rho \in L^\infty(Y', H^1_\#(Y^*))$, then*

$$\|J^\varepsilon g\|_{L^2(Y'/\varepsilon)} \le \|g\|_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \|\rho\|_{L^\infty(Y', H^1_\#(Y^*))}.$$

Proof. As in the proof of Lemma 4.2, for each η, we expand $\rho(y, \eta)$ as a function of y in the orthonormal basis $\{\bar{\varphi}_m(y, \eta)\}_{m=1}^\infty$, and we have

$$J^\varepsilon g(\xi) = \int_{(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} g(x) \left(\sum_{m=1}^\infty a_m(\varepsilon\xi) \overline{\varphi}_m(x/\varepsilon, \varepsilon\xi) \right) e^{-\mathbf{i}x\cdot\xi} dx = \sum_{m=1}^\infty a_m(\varepsilon\xi) B_m^\varepsilon g(\xi).$$

By the Cauchy–Schwarz inequality,

$$|J^\varepsilon g(\xi)|^2 \le \sum_{m=1}^\infty |a_m(\varepsilon\xi)|^2 \sum_{m=1}^\infty |B_m^\varepsilon g(\xi)|^2 = \|\rho(\cdot, \varepsilon\xi)\|_{L^2(Y^*)} \sum_{m=1}^\infty |B_m^\varepsilon g(\xi)|^2,$$

and therefore, integrating with respect to $\xi \in Y'/\varepsilon$ and using (3.5), we obtain the result. □

4.2. *Bloch Eigenelements*

In this section we improve the general results given in Section 3. By Proposition 3.2, there is a neighborhood $\mathcal{D}$ of the origin where the first eigenvector $\varphi_1(\cdot . \eta)$ can be chosen in such a way that the map defined by (3.8) is analytic and $\varphi_1(\cdot, 0) = |Y^*|^{-1/2}$. In order to prove Proposition 4.1 it is necessary to add a new condition, namely

$$\mathrm{Im} \int_{Y^*} \varphi_1(y, \eta)\, dy = 0, \qquad \forall \eta \in \mathcal{D}. \tag{4.5}$$

This condition can be achieved by reducing the neighborhood $\mathcal{D}$. In fact, if we multiply

$\varphi_1(\cdot, \eta)$ by the complex number

$$d(\eta) = -\mathrm{Re} \int_{Y^*} \varphi_1(y, \eta)\, dy + \mathbf{i}\ \mathrm{Im} \int_{Y^*} \varphi_1(y, \eta)\, dy,$$

which is analytic with respect to η, then

$$\mathrm{Im} \int_{Y^*} d(\eta)\varphi_1(y, \eta)\, dy = 0, \qquad \forall \eta \in \mathcal{D}.$$

It is evident that this procedure has destroyed the normalization condition but not the analyticity. It is enough to divide by $|d(\eta)|$, which is different from zero in a neighborhood of the origin because $d(0) \neq 0$, to regain (4.5).

Note that the three conditions about $\varphi_1(\cdot, \eta)$, namely, that the map defined by (3.8) is an analytic function, $\varphi_1(\cdot, 0) = |Y^*|^{-1/2}$ and (4.5), only fix the eigenvector $\varphi_1(\cdot, \eta)$.

We denote by $\mathcal{D}_\delta$ a δ-neighborhood of the origin where Proposition 3.2 and (4.5) are satisfied. Let $a(\eta; \cdot, \cdot)$ be the bilinear form associated with the operator $A(\eta)$:

$$a(\eta; u, v) = \int_{Y^*} a_{ij}(y) \left(\frac{\partial u}{\partial y_i} + \mathbf{i}\eta_i u \right) \left(\frac{\partial \bar{v}}{\partial y_j} - \mathbf{i}\eta_j \bar{v} \right) dy, \qquad u, v \in H^1_\#(Y^*).$$

Proposition 4.1. *The first Bloch eigenvector $\varphi_1(\cdot, \eta)$ of problem* (3.1) *satisfies the following relations*:

$$\frac{\partial \varphi_1}{\partial \eta_k}(\cdot, 0) = \mathbf{i}|Y^*|^{-1/2} v^k, \qquad k = 1, 2, \ldots, N, \tag{4.6}$$

$$\frac{1}{2} \frac{\partial^2 \varphi_1}{\partial \eta_k\, \partial \eta_l}(\cdot, 0) = -|Y^*|^{-1/2}(\tilde{v}^{kl} + \beta^{kl}), \qquad k, l = 1, 2, \ldots, N, \tag{4.7}$$

where $\tilde{v}^{kl} = (v^{kl} + v^{lk})/2$, v^k and v^{kl} are the solutions of (2.7) *and* (2.9), *respectively, and $\beta^{kl} = (2|Y^*|)^{-1} \int_{Y^*} v^k v^l\, dy$.*

Proof. We take derivatives with respect to η_k in the variational formulation of $A(\eta)\varphi_1(\eta) = \lambda_1(\eta)\varphi_1(\eta)$ and we obtain, $\forall \psi \in H^1_\#(Y^*)$,

$$\begin{aligned}
&\left(\frac{\partial \lambda_1}{\partial \eta_k} \varphi_1 . \frac{\partial \varphi_1}{\partial \eta_k} \lambda_1, \psi \right)_{L^2(Y^*)} \\
&\quad = a\left(\eta; \frac{\partial \varphi_1}{\partial \eta_k}, \psi \right) + 2\eta_j \int_{Y^*} a_{jk} \varphi_1 \bar{\psi}\, dy \\
&\qquad + \mathbf{i} \int_{Y^*} \left(a_{kj} \varphi_1 \frac{\partial \bar{\psi}}{\partial y_j} - a_{ki} \frac{\partial \varphi_1}{\partial y_i} \bar{\psi} \right) dy.
\end{aligned} \tag{4.8}$$

Making $\eta = 0$ and taking into account $\lambda_1(0) = 0$, (3.9), (3.10) and (3.2), we have

$$\int_{Y^*} a_{ij} \frac{\partial}{\partial y_i} \left(\frac{\partial \varphi_1}{\partial \eta_k}(\cdot, 0) \right) \frac{\partial \overline{\psi}}{\partial y_j}\, dy = -\frac{\mathbf{i}}{|Y^*|^{1/2}} \int_{Y^*} a_{kj} \frac{\partial \bar{\psi}}{\partial y_j} dy, \qquad \forall \psi \in H^1_\#(Y^*),$$

and $(\partial \varphi_1 / \partial \eta_k)(\cdot, 0)$ is determined up to a constant.

Taking derivatives with respect to η_k in $\|\varphi_1(\cdot,\eta)\|^2_{L^2(Y^*)} = 1$ and (4.5), yields

$$\operatorname{Re}\int_{Y^*}\frac{\partial\varphi_1}{\partial\eta_k}(y,\eta)\overline{\varphi_1}(y,\eta)\,dy=0 \quad\text{and}\quad \operatorname{Im}\int_{Y^*}\frac{\partial\varphi_1}{\partial\eta_k}(y,\eta)dy=0,\quad \forall\eta\in\mathcal{D}_\delta. \tag{4.9}$$

Now, thanks to (3.2) and (4.9) for $\eta = 0$,

$$\int_{Y^*}\frac{\partial\varphi_1}{\partial\eta_k}(y,0)\,dy = |Y^*|^{1/2}\operatorname{Re}\int_{Y^*}\frac{\partial\varphi_1}{\partial\eta_k}(y,0)\overline{\varphi_1}(y,0)\,dy = 0,$$

and (4.6) holds. Note that $(\partial\varphi_1/\partial\eta_k)(\cdot,0) = \mathbf{i}|Y^*|^{-1/2}v^k$ is purely imaginary.

In order to compute the second derivatives of $\varphi_1(\cdot,\eta)$ at $\eta = 0$, we take derivatives with respect to η_l in (4.8) and make $\eta = 0$; thus, since $\lambda_1(0) = 0$, (3.9), (3.10), (3.2) and (4.6), we obtain, $\forall\psi\in H^1_\#(Y^*)$,

$$\begin{aligned}
&\int_{Y^*} a_{ij}\frac{\partial}{\partial y_i}\left(\frac{\partial^2\varphi_1}{\partial\eta_k\,\partial\eta_l}(\cdot,0)\right)\frac{\partial\overline{\psi}}{\partial y_j}\,dy\\
&\quad = \frac{1}{|Y^*|^{1/2}}\left[\int_{Y^*} -2\left(a_{kl}-\frac{1}{\theta}a^h_{kl}\right)\overline{\psi}\,dy\right.\\
&\qquad\qquad \left.+\int_{Y^*}\left(a_{kj}v^l\frac{\partial\overline{\psi}}{\partial y_j}+a_{lj}v^k\frac{\partial\overline{\psi}}{\partial y_j}-a_{il}\frac{\partial v^k}{\partial y_i}\overline{\psi}-a_{ik}\frac{\partial v^l}{\partial y_i}\overline{\psi}\right)dy\right],
\end{aligned}$$

and $(\partial^2\varphi_1/\partial\eta_k\partial\eta_l)(\cdot,0)$ is determined up to a constant.

Taking derivatives with respect to η_l in (4.9), we have

$$\operatorname{Re}\int_{Y^*}\frac{\partial^2\varphi_1}{\partial\eta_k\,\partial\eta_l}(y,\eta)\overline{\varphi_1}(y,\eta)\,dy + \operatorname{Re}\int_{Y^*}\frac{\partial\varphi_1}{\partial\eta_k}(y,\eta)\frac{\partial\overline{\varphi_1}}{\partial\eta_l}(y,\eta)\,dy = 0,$$

and

$$\operatorname{Im}\int_{Y^*}\frac{\partial^2\varphi_1}{\partial\eta_k\,\partial\eta_l}(y,\eta)\,dy = 0,\qquad \forall\eta\in\mathcal{D}_\delta.$$

Then, by virtue of the last two expressions for $\eta = 0$, (3.2) and (4.6), it follows that

$$\begin{aligned}
\int_{Y^*}\frac{\partial^2\varphi_1}{\partial\eta_k\,\partial\eta_l}(y,0)\,dy &= |Y^*|^{1/2}\operatorname{Re}\int_{Y^*}\frac{\partial^2\varphi_1}{\partial\eta_k\,\partial\eta_l}(y,0)\overline{\varphi_1}(y,0)\,dy\\
&= -|Y^*|^{1/2}\operatorname{Re}\int_{Y^*}\frac{\partial\varphi_1}{\partial\eta_k}(y,0)\frac{\partial\overline{\varphi_1}}{\partial\eta_l}(y,0)\,dy\\
&= -\frac{1}{|Y^*|^{1/2}}\int_{Y^*}v^kv^l\,dy = -2\beta^{kl}|Y^*|^{1/2},
\end{aligned}$$

which completes the proof. □

Lemma 4.4. *Let $a_{ij}\in L^\infty_\#(Y^*)$ satisfying the symmetry and ellipticity conditions* (2.1) *and* (2.2). *Then, the Bloch eigenvalues and eigenvectors of problem* (3.1) *satisfy the* *following relations*:

$$C_1|\eta|^2 \le \lambda_1(\eta) \le C_2|\eta|^2, \qquad \forall \eta \in Y', \tag{4.10}$$

$$\lambda_2^* \le \lambda_2(\eta) \le \lambda_m(\eta), \qquad \forall m \ge 2, \quad \forall \eta \in Y', \tag{4.11}$$

$$\left\| \frac{\partial \varphi_m}{\partial y_k}(\cdot, \eta) \right\|_{L^2(Y^*)} \le C_3 \lambda_m(\eta)^{1/2}, \qquad \forall \eta \in Y', \quad m \ge 1, \quad k = 1, \dots, N, \tag{4.12}$$

where the constants C_i are independent of m and η and λ_2^* is the second eigenvalue of the Neumann problem on Y^*.

Proof. It is well known (see [11], for example) that there exists a constant C independent of η such that

$$\|\nabla_y u\|^2_{L^2(Y)} + |\eta|^2 \|u\|^2_{L^2(Y)} \le C \|\nabla_y u + \mathbf{i} u \eta\|^2_{L^2(Y)}, \qquad \forall u \in H^1_\#(Y). \tag{4.13}$$

Besides, there exists an extension operator $P : H^1(Y^*) \to H^1(Y)$ such that for any $\varphi \in H^1(Y^*)$,

$$\|P\varphi\|_{L^2(Y)} \le C\|\varphi\|_{L^2(Y^*)},$$
$$\|\nabla_y (P\varphi)\|_{L^2(Y)} \le C\|\nabla\varphi\|_{L^2(Y^*)}.$$

Hence, using the extension operator P, formula (4.13) and the ellipticity of the coefficients a_{ij}, we obtain

$$\|\nabla_x \varphi\|^2_{L^2(Y^*)} + |\eta|^2 \|\varphi\|^2_{L^2(Y^*)} \le C a(\eta; \varphi, \varphi), \qquad \forall \varphi \in H^1_\#(Y^*).$$

Now, the variational formulation of (3.1) and the normalization of φ_m lead us to obtain (4.12) and the left-hand side of (4.10).

In order to prove the right-hand side of (4.10), we show that for all $m \ge 1$, $\lambda_m(\eta)$ is a Lipschitz function of η. We write

$$a(\eta, v, v) = a(\eta', v, v) + R(\eta, \eta', v, v), \qquad \eta, \eta' \in Y', \quad v \in H^1_\#(Y^*),$$

where

$$R(\eta, \eta', v, v) = \int_{Y^*} a_{ij}(y) \frac{\partial v}{\partial y_j} \overline{\mathbf{i}\,(\eta_i - \eta_i')\, v}\, dy + \int_{Y^*} a_{ij}(y)\, \mathbf{i}\,(\eta_j - \eta_j')\, v \frac{\partial \bar{v}}{\partial y_i}\, dy$$
$$+ \int_{Y^*} a_{ij}(y)(\eta_j\eta_i - \eta_j'\eta_i')|v|^2\, dy, \qquad v \in H^1_\#(Y^*).$$

By the Cauchy–Schwarz inequality, R can be estimated by $|R| \le C|\eta - \eta'|\, \|v\|^2_{H^1(Y^*)}$. Using the above estimate on R and the minimax principle:

$$\lambda_m(\eta) = \min_{\substack{E_m \subset H^1_\#(Y^*) \\ \dim(E_m) = m}} \ \max_{\substack{v \in E_m \\ v \ne 0}} \frac{a(\eta; v, v)}{\|v\|^2_{L^2(Y^*)}},$$

we deduce that $\lambda_m(\eta) \le \lambda_m(\eta') + C_m|\eta - \eta'|$. Next, interchanging η and η', we obtain that $|\lambda_m(\eta) - \lambda_m(\eta')| \le C_m|\eta - \eta'|$, which is our assertion.

If $|\eta| < \delta$, the right-hand side of (4.10) holds from Propositions 3.2 and 3.3. If $|\eta| > \delta$, as $\lambda_1(\eta)$ is a Lipschitz function of η, we have

$$\lambda_1(\eta) \le C|\eta| \le C\delta^{-1}|\eta|^2,$$

and the right-hand side of estimate (4.10) also holds.

Finally, estimate (4.11) is a direct consequence of the minimax principle. □

Proposition 4.2. *Let u^ε be the solution of problem $A^\varepsilon u^\varepsilon = f$ in $(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$ with f a function of $L^2(\mathbb{R}^N)$, $u^\varepsilon \in H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$. Let v^ε be the function defined by*

$$v^\varepsilon(x) = \sum_{m=2}^{\infty} \int_{Y'/\varepsilon} (B_m^\varepsilon u^\varepsilon)(\xi) e^{\mathbf{i}x\cdot\xi} \varphi_m^\varepsilon(x,\xi)\, d\xi, \qquad x \in \left(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon\right). \tag{4.14}$$

Then we have the estimates

$$\|v^\varepsilon\|_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \le C\varepsilon^2 \|f\|_{L^2(\mathbb{R}^N)} \quad \textit{and} \quad \|\nabla_x v^\varepsilon\|_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \le C\varepsilon \|f\|_{L^2(\mathbb{R}^N)},$$

where C is a constant that does not depend on ε.

Proof. Because of (3.5), (3.7), (3.4) and (4.11), we have

$$\begin{aligned}\|v^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} &= \int_{Y'/\varepsilon} \sum_{m=2}^{\infty} \left| \frac{(B_m^\varepsilon f)(\xi)}{\lambda_m^\varepsilon(\xi)} \right|^2 d\xi \\ &\le \int_{Y'/\varepsilon} \varepsilon^4 \sum_{m=2}^{\infty} \left| \frac{(B_m^\varepsilon f)(\xi)}{\lambda_2^*} \right|^2 d\xi \le C\varepsilon^4 \|f\|^2_{L^2(\mathbb{R}^N)},\end{aligned}$$

and the first estimate holds. In order to prove the estimate of the gradient of v^ε, we apply Lemma 4.1 with $g = v^\varepsilon$ and obtain

$$\|\nabla_x v^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k)} \le C \int_{Y'/\varepsilon} \sum_{m=2}^{\infty} \lambda_m^\varepsilon(\xi) |B_m^\varepsilon u^\varepsilon(\xi)|^2\, d\xi.$$

It is enough to use again (3.7), (3.4), (4.11) and (3.5) to obtain the result. □

Finally, we show the relation between the first Bloch coefficient and the Fourier transform (see Proposition 3.4 for comparison):

Proposition 4.3. *For all $g \in L^2(\mathbb{R}^N)$, we have*

$$\chi_{Y'/\varepsilon} B_1^\varepsilon g \to \theta^{1/2} \hat{g} \qquad \textit{strongly in } L^2(\mathbb{R}^N), \quad \textit{as } \varepsilon \to 0. \tag{4.15}$$

Moreover, if g^ε is a sequence in $L^2(\mathbb{R}^N)$ which converges towards g strongly, as $\varepsilon \to 0$, *then*

$$\chi_{Y'/\varepsilon} B_1^\varepsilon g^\varepsilon \to \theta^{1/2}\hat{g} \qquad \textit{strongly in } L^2(\mathbb{R}^N), \quad \textit{as } \varepsilon \to 0.$$

We use some properties of the *discrete Fourier transform* to prove Proposition 4.3. In order to make the exposition self-contained, we introduce here some notations and lemmas that we use later.

Let $\{Y_l^\varepsilon\}_{l\in\mathbb{Z}^N}$ be the mesh of $\mathbb{R}^N$ generated by the cell εY. More precisely, $Y_l^\varepsilon = x_l^\varepsilon + \varepsilon Y$ where $x_l^\varepsilon = 2\pi\varepsilon l$ is the origin of the cell Y_l^ε. Corresponding to this mesh, we can introduce the discrete Fourier transform of a function as follows: For $g \in W^{1,p}(\mathbb{R}^N)$ with compact support and $p > N$ we define

$$F^\varepsilon g(\xi) = \sum_{l\in\mathbb{Z}^N} g(x_l^\varepsilon) e^{-\mathbf{i} x_l^\varepsilon\cdot\xi}, \qquad \forall \xi \in Y'/\varepsilon.$$

Note that $F^\varepsilon g$ is well defined since for $p > N$, $W^{1,p}(\mathbb{R}^N)$ is embedded in $C(\mathbb{R}^N)$.

Lemma 4.5. *For $g \in W^{1,p}(\mathbb{R}^N)$ with compact support K, $p > N$, we have*

$$\varepsilon^N \chi_{Y'/\varepsilon} F^\varepsilon g \to \frac{1}{(2\pi)^{N/2}}\hat{g} \qquad \textit{in } L^2(\mathbb{R}^N), \quad \textit{as } \varepsilon \to 0.$$

The proof of Lemma 4.5 can be found in [9].

We consider $\rho(y,\eta)$ a measurable function defined on $Y^* \times Y'$ which is Y-periodic in the variable y. We define $\tilde{\rho}^{(0)}$ as the function

$$\tilde{\rho}^{(0)}(\eta) = \frac{1}{|Y|}\int_{Y^*} \rho(y,\eta) e^{-\mathbf{i} y\cdot\eta}\, dy, \qquad \eta \in Y'.$$

With this notation, we obtain the following result.

Lemma 4.6. *We assume $\rho \in L^\infty(Y', L^2_\#(Y^*))$. Then, for all $g \in W^{1,p}(\mathbb{R}^N)$ with compact support K and $p > N$, we have*

$$\chi_{Y'/\varepsilon}(\xi)(J^\varepsilon g(\xi) - (2\pi)^{N/2}\tilde{\rho}^{(0)}(\varepsilon\xi)\hat{g}(\xi)) \to 0 \qquad \textit{strongly in } L^2(\mathbb{R}^N), \quad \textit{as } \varepsilon \to 0.$$

Proof. For each fixed ε, we consider the ε-mesh $\{Y_l^{\varepsilon *}\}$ of $(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)$ and decompose $J^\varepsilon g$ as

$$\begin{aligned} J^\varepsilon g(\xi) = &\sum_{l\in\mathbb{Z}^N} g(x_l^\varepsilon) \int_{Y_l^{\varepsilon *}} e^{-\mathbf{i} x\cdot\xi} \rho(x/\varepsilon, \varepsilon\xi)\, dx \\ &+ \sum_{l\in\mathbb{Z}^N} (g(x) - g(x_l^\varepsilon)) \int_{Y_l^{\varepsilon *}} e^{-\mathbf{i} x\cdot\xi} \rho(x/\varepsilon, \varepsilon\xi)\, dx. \end{aligned} \tag{4.16}$$

By introducing the change of variable $x = x_l^\varepsilon + \varepsilon y$ in the first term of the right-hand side of (4.16) and taking into account Lemma 4.5, we obtain

$$\left\| \chi_{Y'/\varepsilon}(\xi) \left[\sum_{l \in \mathbb{Z}^N} g(x_l^\varepsilon) \int_{Y_l^{\varepsilon *}} e^{-\mathbf{i}x\cdot\xi} \rho(x/\varepsilon, \varepsilon\xi)\, dx - (2\pi)^{N/2} \tilde{\rho}^{(0)}(\varepsilon\xi) \hat{g}(\xi) \right] \right\|_{L^2(\mathbb{R}^N)}$$
$$= \left\| \chi_{Y'/\varepsilon}(\xi) [\varepsilon^N |Y| \tilde{\rho}^{(0)}(\varepsilon\xi) F^\varepsilon g(\xi) - |Y|^{1/2} \tilde{\rho}^{(0)}(\varepsilon\xi) \hat{g}(\xi)] \right\|_{L^2(\mathbb{R}^N)} \xrightarrow{\varepsilon\to 0} 0.$$

Hence, we have established the lemma if we prove that the second term of the right-hand side of (4.16), which we denote by $r^\varepsilon(\xi)$, converges towards zero strongly in $L^2(\mathbb{R}^N)$.

We denote by $\tilde{g}_1^\varepsilon$ and $\tilde{g}_2^\varepsilon$ the functions

$$\tilde{g}_1^\varepsilon(x) = \sum_{l \in \mathbb{Z}^N} (g(x) - g(x_l^\varepsilon)) \chi_{Y_l^{\varepsilon *}}(x) \quad \text{and} \quad \tilde{g}_2^\varepsilon(x) = \sum_{l \in \mathbb{Z}^N} (g(x) - g(x_l^\varepsilon)) \chi_{Y_l^\varepsilon}(x).$$

It is clear that $\tilde{g}_1^\varepsilon = \tilde{g}_2^\varepsilon \chi_{(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)}$ and

$$r^\varepsilon(\xi) = \int_{(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \tilde{g}_1^\varepsilon(x) e^{-\mathbf{i}x\cdot\xi} \rho(x/\varepsilon, \varepsilon\xi)\, dx = J^\varepsilon \tilde{g}_1^\varepsilon.$$

Thus, applying Lemma 4.3,

$$\begin{aligned} \|r^\varepsilon\|_{L^2(Y'/\varepsilon)} &\leq \|\rho\|_{L^\infty(Y', H^1_\#(Y^*))} \|\tilde{g}_1^\varepsilon\|_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \\ &\leq \|\rho\|_{L^\infty(Y', H^1_\#(Y^*))} \|\tilde{g}_2^\varepsilon\|_{L^2(\mathbb{R}^N)}. \end{aligned}$$

By Morrey's estimate (see [4], for example) and the Holder inequality, it follows that

$$\|\tilde{g}_2^\varepsilon\|^2_{L^2(\mathbb{R}^N)} \leq C(p, N, K) \varepsilon^2 \|\nabla_x g\|^2_{L^p(\mathbb{R}^N)},$$

where $C(p, N, K)$ is a constant which depends on p, N, and K. This completes the proof of Lemma 4.6. □

Proof of Proposition 4.3. First, we prove (4.15) for $g \in \mathcal{D}(\mathbb{R}^N)$. By the definition of J^ε with $\rho(\cdot, \eta) = \varphi_1(\cdot, \eta)$, $J^\varepsilon g = B_1^\varepsilon g$. Thus, thanks to Lemma 4.6, it is enough to show that

$$\chi_{Y'/\varepsilon}(\xi) (2\pi)^{N/2} \tilde{\rho}^{(0)}(\varepsilon\xi) \hat{g}(\xi) \to \theta^{1/2} \hat{g}(\xi) \qquad \text{strongly in } L^2(\mathbb{R}^N), \quad \text{as } \varepsilon \to 0. \tag{4.17}$$

However, on account of Proposition 3.2, it is easy to check the pointwise convergence

$$(2\pi)^{N/2} \tilde{\rho}^{(0)}(\varepsilon\xi) \to \theta^{1/2}, \qquad \forall \xi \in \mathbb{R}^N, \quad \text{as } \varepsilon \to 0,$$

and consequently, (4.17) holds.

Now, on account of estimate

$$\int_{Y'/\varepsilon} |B_1^\varepsilon g(\xi)|^2\, d\xi \leq \int_{(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} |g(x)|^2\, dx \leq \int_{\mathbb{R}^N} |g(x)|^2\, dx, \qquad \forall g \in L^2(\mathbb{R}^N), \tag{4.18}$$

and by density, we obtain (4.15) for all $g \in L^2(\mathbb{R}^N)$.

The second assertion follows from (4.15) and the uniform estimate (4.18). □

5. The Corrector and the Bloch Approximation in $\mathbb{R}^N$

Throughout this section we consider $\Omega = \mathbb{R}^N$. We use the results in Section 4 concerning the Bloch transform to obtain correctors in $\mathbb{R}^N$. We assume $u^\varepsilon \in H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$ is the solution of

$$A^\varepsilon u^\varepsilon = f \qquad \text{in } \left(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon\right), \tag{5.1}$$

with $\operatorname{supp} u^\varepsilon \subseteq K$; K being a fixed compact. It is known that the solution u^ε, extended to $H^1(\mathbb{R}^N)$ by the operator defined in (2.4), converges weakly in $H^1(\mathbb{R}^N)$ towards the solution of the homogenized problem (2.5) with $\Omega = \mathbb{R}^N$ (see [7] for a proof). Our aim is to find, by means of Bloch waves, a function θ^ε which approximates the solution u^ε of problem (5.1) in the norm of $H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$, as $\varepsilon \to 0$.

We define the following function:

$$\theta^\varepsilon(x) = \int_{Y'/\varepsilon} \theta^{1/2}\widehat{u^0}(\xi) e^{\mathbf{i}x\cdot\xi} \varphi_1^\varepsilon(x,\xi)\, d\xi, \qquad x \in \left(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon\right), \tag{5.2}$$

where u^0 is the solution of the homogenized problem (2.5) with $\Omega = \mathbb{R}^N$. θ^ε is the function that we call *Bloch approximation for periodically perforated media* since, as stated in the following theorem, $\theta^\varepsilon - u^0$ provide a corrector of the homogenized solution u^0.

Theorem 5.1. *Let u^ε be the solution of problem* (5.1) *with $f \in L^2(\mathbb{R}^N)$ and the coefficients $a_{ij} \in L^\infty_\#(Y^*)$ satisfying* (2.1) *and* (2.2). *We assume that*

$$\|u^\varepsilon - u^0\|_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \to 0 \qquad as \quad \varepsilon \to 0. \tag{5.3}$$

Then

$$\|u^\varepsilon - \theta^\varepsilon\|_{H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \to 0 \qquad as \quad \varepsilon \to 0.$$

Proof. We denote by $\mathcal{D}_\delta$ a δ-neigbourhood of the origin where Proposition 3.2 and (4.5) are satisfied. We write the functions u^ε and θ^ε as

$$u^\varepsilon = u_1^\varepsilon + u_2^\varepsilon + v^\varepsilon \quad \text{and} \quad \theta^\varepsilon = \theta_1^\varepsilon + \theta_2^\varepsilon,$$

where

$$
\begin{aligned}
u_1^\varepsilon(x) &= \int_{\substack{\xi \in Y'/\varepsilon \\ |\xi| > \delta/\varepsilon}} (B_1^\varepsilon u^\varepsilon)(\xi) e^{\mathbf{i}x\cdot\xi} \varphi_1^\varepsilon(x,\xi)\, d\xi, && x \in \left(\mathbb{R}^N - \textstyle\bigcup_k \bar{T}_k^\varepsilon\right), \\
u_2^\varepsilon(x) &= \int_{|\xi|<\delta/\varepsilon} (B_1^\varepsilon u^\varepsilon)(\xi) e^{\mathbf{i}x\cdot\xi} \varphi_1^\varepsilon(x,\xi)\, d\xi, && x \in \left(\mathbb{R}^N - \textstyle\bigcup_k \bar{T}_k^\varepsilon\right), \\
\theta_1^\varepsilon(x) &= \int_{\substack{\xi \in Y'/\varepsilon \\ |\xi| > \delta/\varepsilon}} \theta^{1/2}\widehat{u^0}(\xi) e^{\mathbf{i}x\cdot\xi} \varphi_1^\varepsilon(x,\xi)\, d\xi, && x \in \left(\mathbb{R}^N - \textstyle\bigcup_k \bar{T}_k^\varepsilon\right), \\
\theta_2^\varepsilon(x) &= \int_{|\xi|<\delta/\varepsilon} \theta^{1/2}\widehat{u^0}(\xi) e^{\mathbf{i}x\cdot\xi} \varphi_1^\varepsilon(x,\xi)\, d\xi, && x \in \left(\mathbb{R}^N - \textstyle\bigcup_k \bar{T}_k^\varepsilon\right),
\end{aligned}
\tag{5.4}
$$

and v^ε is defined by (4.14). By Proposition 4.2 we have that $\|v^\varepsilon\|_{H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \leq C\varepsilon\|f\|_{L^2(\mathbb{R}^N)}$. We estimate the functions u_1^ε and θ_1^ε in $H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$.

Applying Lemma 4.2 with $\rho(y,\eta) = \varphi_1(y,\eta)$, and taking into account the normalization of the eigenfuctions $\varphi_m(\cdot,\eta)$, we obtain

$$
\|u_1^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} = \int_{\substack{\xi \in Y'/\varepsilon \\ |\xi| > \delta/\varepsilon}} |(B_1^\varepsilon u^\varepsilon)(\xi)|^2\, d\xi. \tag{5.5}
$$

Now, formulas (3.7), (4.10) and (3.5) lead us to

$$
\begin{aligned}
\|u_1^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} &\leq C \int_{\substack{\xi \in Y'/\varepsilon \\ |\xi| > \delta/\varepsilon}} \frac{|(B_1^\varepsilon f)(\xi)|^2}{|\xi|^4}\, d\xi \\
&\leq C\varepsilon^4\delta^{-4} \int_{\substack{\xi \in Y'/\varepsilon \\ |\xi| > \delta/\varepsilon}} |(B_1^\varepsilon f)(\xi)|^2\, d\xi \leq C\varepsilon^4\delta^{-4}\|f\|^2_{L^2(\mathbb{R}^N)}.
\end{aligned}
$$

In order to prove the estimate of the gradient of u_1^ε, we apply again Lemma 4.2 and take into account formulas (4.10) and (4.12):

$$
\|\nabla_x u_1^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \leq C \int_{\substack{\xi \in Y'/\varepsilon \\ |\xi| > \delta/\varepsilon}} |(B_1^\varepsilon u^\varepsilon)(\xi)|^2 \lambda_1^\varepsilon(\xi)\, d\xi. \tag{5.6}
$$

Combining (3.7), (4.10) and (3.5), we have

$$
\|\nabla_x u_1^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \leq C \int_{\substack{\xi \in Y'/\varepsilon \\ |\xi| > \delta/\varepsilon}} \frac{|(B_1^\varepsilon f)(\xi)|^2}{|\xi|^2}\, d\xi \leq C\varepsilon^2\delta^{-2}\|f\|^2_{L^2(\mathbb{R}^N)}.
$$

Similar considerations give us estimates for θ_1^ε. As a matter of fact, by Lemma 4.2, we have

$$
\|\theta_1^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} = \int_{\substack{\xi \in Y'/\varepsilon \\ |\xi| > \delta/\varepsilon}} \theta|\widehat{u^0}(\xi)|^2\, d\xi \leq C\varepsilon^4\delta^{-4} \int_{\substack{\xi \in Y'/\varepsilon \\ |\xi| > \delta/\varepsilon}} |\xi|^4 |\widehat{u^0}(\xi)|^2\, d\xi,
$$

and, since $f \in L^2(\mathbb{R}^N)$, it is well known that $u^0 \in H^2(\mathbb{R}^N)$ and

$$
\|\theta_1^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \leq C\varepsilon^4\delta^{-4}\|u^0\|^2_{H^2(\mathbb{R}^N)} \leq C\varepsilon^4\delta^{-4}\|f\|^2_{L^2(\mathbb{R}^N)}. \tag{5.7}
$$

Following the same procedure as that for u_1^ε with minor modifications (see (5.6)), we obtain

$$\|\nabla_x \theta_1^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \le C \int_{\substack{\xi \in Y'/\varepsilon \\ |\xi| > \delta/\varepsilon}} |\widehat{u^0}(\xi)|^2 \lambda_1^\varepsilon(\xi)\, d\xi.$$

Now, from estimate (4.10) it follows that

$$\begin{aligned} \|\nabla_x \theta_1^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} &\le C \int_{\substack{\xi \in Y'/\varepsilon \\ |\xi| > \delta/\varepsilon}} |\widehat{u^0}(\xi)|^2 |\xi|^2\, d\xi \\ &\le C\varepsilon^2\delta^{-2}\|u^0\|^2_{H^2(\mathbb{R}^N)} \le C\varepsilon^2\delta^{-2}\|f\|^2_{L^2(\mathbb{R}^N)}. \end{aligned} \tag{5.8}$$

The proof is completed by showing that

$$\|u_2^\varepsilon - \theta_2^\varepsilon\|_{H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

An analysis similar to (5.5) leads us to

$$\begin{aligned} \|u_2^\varepsilon - \theta_2^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} &\le \int_{|\xi|<\delta/\varepsilon} |(B_1^\varepsilon u^\varepsilon)(\xi) - \theta^{1/2}\widehat{u^0}(\xi)|^2\, d\xi \\ &\le \int_{|\xi|<\delta/\varepsilon} |B_1^\varepsilon (u^\varepsilon - u^0)(\xi)|^2\, d\xi \\ &\quad + \int_{|\xi|<\delta/\varepsilon} |B_1^\varepsilon u^0(\xi) - \theta^{1/2}\widehat{u^0}(\xi)|^2\, d\xi. \end{aligned} \tag{5.9}$$

Taking into account (4.18), (5.3) and Proposition 4.3, we have that both terms of the right-hand side of (5.9) converge towards zero as ε tends to zero, and, consequently,

$$\|u_2^\varepsilon - \theta_2^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \le \int_{|\xi|<\delta/\varepsilon} |(B_1^\varepsilon u^\varepsilon)(\xi) - \theta^{1/2}\widehat{u^0}(\xi)|^2\, d\xi \xrightarrow{\varepsilon\to 0} 0. \tag{5.10}$$

Finally, as in (5.6), we show that

$$\|\nabla(u_2^\varepsilon - \theta_2^\varepsilon)\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \le C \int_{|\xi|<\delta/\varepsilon} |(B_1^\varepsilon u^\varepsilon)(\xi) - \theta^{1/2}\widehat{u^0}(\xi)|^2 \lambda_1^\varepsilon(\xi)\, d\xi. \tag{5.11}$$

By virtue of (3.7) and the homogenized equation (2.5) in the Fourier space, we can write

$$\begin{aligned} &\lambda_1^\varepsilon(\xi)|(B_1^\varepsilon u^\varepsilon)(\xi) - \theta^{1/2}\widehat{u^0}(\xi)|^2 \\ &\quad = ((B_1^\varepsilon f)(\xi) - \theta^{1/2}\hat{f}(\xi))\overline{((B_1^\varepsilon u^\varepsilon)(\xi) - \theta^{1/2}\widehat{u^0}(\xi))} \\ &\qquad + (\theta^{-1}a_{ij}^h\xi_i\xi_j - \lambda_1^\varepsilon(\xi))\theta^{1/2}\widehat{u^0}(\xi)\overline{((B_1^\varepsilon u^\varepsilon)(\xi) - \theta^{1/2}\widehat{u^0}(\xi))}. \end{aligned} \tag{5.12}$$

Moreover, by Propositions 3.2 and 3.3 and (3.3) and (3.4),

$$|\lambda_1^\varepsilon(\xi) - \theta^{-1}a_{ij}^h\xi_i\xi_j| = \left|\frac{\lambda_1(\varepsilon\xi)}{\varepsilon^2} - \frac{1}{2}\frac{\partial^2\lambda_1}{\partial\eta_i\,\partial\eta_j}(0)\xi_i\xi_j\right| \le C\varepsilon|\xi|^3, \qquad |\xi| \le \frac{\delta}{\varepsilon}, \tag{5.13}$$

and since $u^0 \in H^2(\mathbb{R}^N)$, we have

$$\int_{|\xi|<\delta/\varepsilon} (\theta^{-1} a_{ij}^h \xi_i \xi_j - \lambda_1^\varepsilon(\xi))^2 |\widehat{u^0}(\xi)|^2 \, d\xi \le C \int_{|\xi|<\delta/\varepsilon} |\xi|^6 \varepsilon^2 |\widehat{u^0}(\xi)|^2 \, d\xi$$
$$\le C\delta^2 \int_{|\xi|<\delta/\varepsilon} |\xi|^4 |\widehat{u^0}(\xi)|^2 \, d\xi$$
$$\le C\delta^2 \|\widehat{u^0}\|^2_{H^2(\mathbb{R}^N)}. \tag{5.14}$$

Therefore, using the Cauchy–Schwarz inequality and combining (5.11), (5.12), (5.14), (5.10) and Proposition 4.3, we can assert that $\|\nabla(u_2^\varepsilon - \theta_2^\varepsilon)\|_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)}$ converges towards zero as $\varepsilon \to 0$ and the proof of the theorem is complete. □

Remark 5.1. Note that unlike the case of a nonperforated domain (see [9]), we have not managed to obtain a precise bound of the type

$$\|\nabla(u^\varepsilon - \theta^\varepsilon)\|_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \le C\varepsilon^\alpha \qquad \text{with} \quad \alpha > 0,$$

mainly because the Bloch coefficients in our problem are defined in $\mathbb{R}^N$ minus the holes.

We have proved that the function θ^ε is an approximation of first order for the problem (5.1). Notice that hypothesis (5.3) may, at first sight, look artificial. Nevertheless, we observe that this is not the case. Indeed, if Ω is a bounded domain, Theorem 2.1 shows that $P^\varepsilon u^\varepsilon$ converges weakly in $H_0^1(\Omega)$ towards u^0, and consequently the strong convergence in $L^2(\Omega)$ holds. On the other hand, if $\Omega = \mathbb{R}^N$, it is natural to replace the operator A^ε by $(A^\varepsilon + I)$ and, in this case also, the convergence of $P^\varepsilon u^\varepsilon$ towards u^0 follows. More precisely, the following result holds:

Theorem 5.2. *Let w^ε be the solution of problem $(A^\varepsilon + I)w^\varepsilon = f$ in $(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$ with $f \in L^2(\mathbb{R}^N)$ and the coefficients $a_{ij} \in L^\infty_\#(Y^*)$ satisfying* (2.1) *and* (2.2)*. Let w^0 be the weak limit in $H^1(\mathbb{R}^N)$ of $P^\varepsilon w^\varepsilon$ as $\varepsilon \to 0$. Then*

$$\|w^\varepsilon - w^0\|_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \to 0 \qquad \text{as} \quad \varepsilon \to 0, \quad \text{and}$$

$$\|w^\varepsilon - \theta^\varepsilon\|_{H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \to 0 \qquad \text{as} \quad \varepsilon \to 0,$$

where θ^ε is given by (5.2) *with $u^0 \equiv w^0$.*

Proof. We first consider

$$w^\varepsilon(x) - w^0(x)$$
$$= \int_{Y'/\varepsilon} B_1^\varepsilon(w^\varepsilon - w^0)(\xi) e^{\mathbf{i}x\cdot\xi} \varphi_1^\varepsilon(x,\xi) \, d\xi + \sum_{m=2}^\infty \int_{Y'/\varepsilon} (B_m^\varepsilon w^\varepsilon)(\xi) e^{\mathbf{i}x\cdot\xi} \varphi_m^\varepsilon(x,\xi) \, d\xi$$
$$- \sum_{m=2}^\infty \int_{Y'/\varepsilon} (B_m^\varepsilon w^0)(\xi) e^{\mathbf{i}x\cdot\xi} \varphi_m^\varepsilon(x,\xi) d\xi, \qquad x \in \left(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon\right). \tag{5.15}$$

By rewriting the arguments in the proof of Proposition 4.2 with minor modifications we obtain that the norm in $L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$ of the last two terms in the right-hand side of (5.15) converges towards zero as $\varepsilon \to 0$. Hence, by Lemma 4.2, we have only to show that

$$\int_{Y'/\varepsilon} |B_1^\varepsilon(w^\varepsilon - w^0)(\xi)|^2\, d\xi \xrightarrow{\varepsilon\to 0} 0.$$

In order to do this, we write

$$\begin{aligned}&\chi_{Y'/\varepsilon}(\xi)[B_1^\varepsilon(w^\varepsilon - w^0)(\xi)]\\ &\quad = [\chi_{Y'/\varepsilon}(\xi)(B_1^\varepsilon w^\varepsilon)(\xi) - \theta^{1/2}\widehat{w^0}(\xi)] + [\theta^{1/2}\widehat{w^0}(\xi) - \chi_{Y'/\varepsilon}(\xi)(B_1^\varepsilon w^0)(\xi)].\end{aligned}$$

Proposition 4.3 leads us to assert that the norm in $L^2_\xi(\mathbb{R}^N)$ of the last term tends to zero as $\varepsilon \to 0$. Hence, it remains to be proven that

$$\chi_{Y'/\varepsilon}(B_1^\varepsilon w^\varepsilon) - \theta^{1/2}\widehat{w^0} \to 0 \qquad \text{strongly in } L^2_\xi(\mathbb{R}^N).$$

By virtue of Theorem 3.1, it is clear that

$$(\lambda_1^\varepsilon(\xi) + 1)(B_1^\varepsilon w^\varepsilon)(\xi) = (B_1^\varepsilon f)(\xi), \qquad \xi \in Y'/\varepsilon.$$

Besides, since $P^\varepsilon w^\varepsilon$ converges weakly in $H^1(\mathbb{R}^N)$ towards w^0, as $\varepsilon \to 0$, $\widehat{w^0}$ verifies the homogenized equation in the Fourier space

$$a_{ij}^h \xi_i \xi_j \widehat{w^0}(\xi) + \theta\widehat{w^0}(\xi) = \theta\hat{f}(\xi), \qquad \xi \in \mathbb{R}^N.$$

Using both expressions, we have

$$\chi_{Y'/\varepsilon}(\xi)(B_1^\varepsilon w^\varepsilon)(\xi) - \theta^{1/2}\widehat{w^0}(\xi) = s_1^\varepsilon(\xi) + s_2^\varepsilon(\xi), \qquad \xi \in \mathbb{R}^N,$$

where

$$s_1^\varepsilon(\xi) = \frac{\chi_{Y'/\varepsilon}(\xi)(B_1^\varepsilon f)(\xi) - \theta^{1/2}\hat{f}(\xi)}{\lambda_1^\varepsilon(\xi) + 1} \quad \text{and}$$
$$s_2^\varepsilon(\xi) = \frac{\theta^{-1}a_{ij}^h\xi_i\xi_j - \lambda_1^\varepsilon(\xi)}{(\lambda_1^\varepsilon(\xi) + 1)(\theta^{-1}a_{ij}^h\xi_i\xi_j + 1)}\theta^{1/2}\hat{f}(\xi).$$

Since $\lambda_1^\varepsilon(\xi) + 1 \geq 1$, Proposition 4.3 allows us to assert that s_1^ε converges to zero in $L^2(\mathbb{R}^N)$. The convergence of s_2^ε is not straightforward. To prove this, we take a fixed constant γ and write

$$\int_{\mathbb{R}^N} |s_2^\varepsilon|^2\, d\xi = \int_{\substack{|\xi| < \gamma\\ |\xi| < \delta/\varepsilon}} |s_2^\varepsilon|^2\, d\xi + \int_{\substack{|\xi| > \gamma\\ |\xi| < \delta/\varepsilon}} |s_2^\varepsilon|^2\, d\xi + \int_{|\xi|>\delta/\varepsilon} |s_2^\varepsilon|^2\, d\xi.$$

Thanks to (5.13) and (4.10), we have

$$\int_{\substack{|\xi| < \gamma \\ |\xi| < \delta/\varepsilon}} |s_2^\varepsilon|^2 \, d\xi \le \int_{\substack{|\xi| < \gamma \\ |\xi| < \delta/\varepsilon}} |\lambda_1^\varepsilon(\xi) - \theta^{-1} a_{ij}^h \xi_i \xi_j|^2 \theta |\hat{f}(\xi)|^2 \, d\xi \le C\varepsilon^2 \gamma^6 \|f\|_{L^2(\mathbb{R}^N)},$$

$$\int_{\substack{|\xi| > \gamma \\ |\xi| < \delta/\varepsilon}} |s_2^\varepsilon|^2 d\xi \le \int_{\substack{|\xi| > \gamma \\ |\xi| < \delta/\varepsilon}} \frac{|\theta^{-1} a_{ij}^h \xi_i \xi_j - \lambda_1^\varepsilon(\xi)|^2}{|\lambda_1^\varepsilon(\xi) \theta^{-1} a_{ij}^h \xi_i \xi_j|^2} \theta |\hat{f}(\xi)|^2 d\xi \le C\varepsilon^2 \gamma^{-2} \|f\|^2_{L^2(\mathbb{R}^N)}.$$

Finally, the third integral can be estimated by

$$\int_{|\xi| > \delta/\varepsilon} |s_2^\varepsilon|^2 \, d\xi \le \int_{|\xi| > \delta/\varepsilon} \frac{|\theta^{-1} a_{ij}^h \xi_i \xi_j - \lambda_1^\varepsilon(\xi)|^2}{|\lambda_1^\varepsilon(\xi) + \theta^{-1} a_{ij}^h \xi_i \xi_j|^2} \theta |\hat{f}(\xi)|^2 d\xi \le C \int_{|\xi| > \delta/\varepsilon} |\hat{f}(\xi)|^2 d\xi,$$

which tends to zero because $f \in L^2(\mathbb{R}^N)$. This completes the proof of the first assertion in the theorem.

The second assertion holds as that in Theorem 5.1 with minor modifications. □

In the following theorem we obtain the first terms of the asymptotic expansion of the Bloch approximation. We verify that this asymptotic expansion coincides up to the second order with the asymptotic expansion of u^ε (2.8).

Theorem 5.3. *Let u^0 be the solution of problem* (2.5) *with $\Omega = \mathbb{R}^N$. We assume that the functions $a_{ij} \in L^\infty_\#(Y^*)$ satisfy* (2.1) *and* (2.2). *We denote by $v_\varepsilon^k(x)$ and $v_\varepsilon^{kl}(x)$ the functions $v^k(x/\varepsilon)$ and $v^{kl}(x/\varepsilon)$ where v^k and v^{kl} are the solutions of* (2.7) *and* (2.9), *respectively. We have*:

(i) *If $f \in L^2(\mathbb{R}^N)$ and $v^k \in W^{1,\infty}(Y^*)$, then*

$$\left\| \theta^\varepsilon - u^0 - \varepsilon v_\varepsilon^k \frac{\partial u^0}{\partial x_k} \right\|_{H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \le C\varepsilon \|f\|_{L^2(\mathbb{R}^N)}.$$

(ii) *If $f \in H^1(\mathbb{R}^N)$ and $v^k, v^{kl} \in W^{1,\infty}(Y^*)$, then*

$$\left\| \theta^\varepsilon - u^0 - \varepsilon v_\varepsilon^k \frac{\partial u^0}{\partial x_k} - \varepsilon^2 (v_\varepsilon^{kl} + \beta^{kl}) \frac{\partial^2 u^0}{\partial x_k \, \partial x_l} \right\|_{H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \le C\varepsilon^2 \|f\|_{H^1(\mathbb{R}^N)}.$$

Proof. We first prove that

$$\|\theta^\varepsilon - u^0\|_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \le C\varepsilon \|f\|_{L^2(\mathbb{R}^N)}. \tag{5.16}$$

To show this inequality, we write $\theta^\varepsilon - u^0$ as

$$\theta^\varepsilon - u^0 = \theta_1^\varepsilon - u_1^0 + w^\varepsilon,$$

where θ_1^ε is defined by (5.4),

$$u_1^0(x) = \frac{1}{(2\pi)^{N/2}} \int_{|\xi|>\delta/\varepsilon} \widehat{u^0}(\xi) e^{\mathbf{i}x\cdot\xi} d\xi, \qquad x \in \left(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon\right) \tag{5.17}$$

$$w^\varepsilon(x) = \int_{|\xi|<\delta/\varepsilon} \theta^{1/2} \widehat{u^0}(\xi) e^{\mathbf{i}x\cdot\xi} (\varphi_1(x/\varepsilon, \varepsilon\xi) - \varphi_1(x/\varepsilon, 0))\, d\xi, \qquad x \in \left(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon\right)$$

and we estimate each term in the $L^2(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)$-norm.

θ_1^ε has already been estimated in (5.7). On account of the Plancherel identity and $u^0 \in H^2(\mathbb{R}^N)$, we have

$$\|u_1^0\|^2_{L^2(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)} \le \varepsilon^4 \delta^{-4} \int_{|\xi|>\delta/\varepsilon} |\widehat{u^0}(\xi)|^2 |\xi|^4 \, d\xi \le C\delta^{-4}\varepsilon^4 \|f\|^2_{L^2(\mathbb{R}^N)}. \tag{5.18}$$

Besides, applying Lemma 4.2 and using the analyticity of $\varphi_1(\cdot, \eta)$ in B_δ, we obtain

$$\begin{aligned}\|w^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)} &= \int_{|\xi|<\delta/\varepsilon} \theta |\widehat{u^0}(\xi)|^2 \|\varphi_1(\cdot, \varepsilon\xi) - \varphi_1(\cdot, 0)\|^2_{L^2(Y^*)} \, d\xi \\ &\le C\varepsilon^2 \int_{|\xi|<\delta/\varepsilon} |\widehat{u^0}(\xi)|^2 |\xi|^2 \, d\xi \le C\varepsilon^2 \|f\|^2_{L^2(\mathbb{R}^N)},\end{aligned}$$

and (5.16) holds.

It is clear that the proof of (i) is completed by proving that

$$\left\| \nabla_x \left(\theta^\varepsilon - u^0 - \varepsilon v_\varepsilon^k \frac{\partial u^0}{\partial x_k} \right) \right\|_{L^2(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)} \le C\varepsilon \|f\|_{L^2(\mathbb{R}^N)}. \tag{5.19}$$

To this end, we write

$$\frac{\partial u^0}{\partial x_k}(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \mathbf{i}\xi_k \widehat{u^0}(\xi) e^{\mathbf{i}x\cdot\xi} \, d\xi$$

and then, by (4.6),

$$\theta^\varepsilon - u^0 - \varepsilon v_\varepsilon^k \frac{\partial u^0}{\partial x_k} = \theta_1^\varepsilon - u_1^0 + w_1^\varepsilon - w_2^\varepsilon, \tag{5.20}$$

where θ_1^ε and u_1^0 are defined by (5.4) and (5.17), respectively, and

$$w_1^\varepsilon(x) = \int_{|\xi|<\delta/\varepsilon} \theta^{1/2} \widehat{u^0}(\xi) e^{\mathbf{i}x\cdot\xi} \left(\varphi_1\left(\frac{x}{\varepsilon}, \varepsilon\xi\right) - \varphi_1\left(\frac{x}{\varepsilon}, 0\right) - \varepsilon\xi_k \frac{\partial \varphi_1}{\partial \eta_k}\left(\frac{x}{\varepsilon}, 0\right) \right) d\xi \quad \text{and}$$

$$w_2^\varepsilon(x) = \varepsilon v_\varepsilon^k(x) \frac{1}{(2\pi)^{N/2}} \int_{|\xi|>\delta/\varepsilon} \mathbf{i}\xi_k \widehat{u^0}(\xi) e^{\mathbf{i}x\cdot\xi} \, d\xi, \qquad x \in \left(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon\right). \tag{5.21}$$

For θ_1^ε, we have estimate (5.8). For u_1^0, the Plancherel identity leads us to

$$\|\nabla_x u_1^0\|^2_{L^2(\mathbb{R}^N - \bigcup_{\mathrm{k}} \bar{T}_{\mathrm{k}}^\varepsilon)} \le \varepsilon^2 \delta^{-2} \int_{|\xi|>\delta/\varepsilon} |\widehat{u^0}(\xi)|^2 |\xi|^4 \, d\xi \le C\varepsilon^2 \delta^{-2} \|f\|^2_{L^2(\mathbb{R}^N)}. \tag{5.22}$$

For w_1^ε, applying Lemma 4.2 and the Taylor expansion for $\varphi_1(\cdot,\eta)$ in $\eta=\varepsilon\xi$,

$$\left\|\varphi_1(\cdot,\varepsilon\xi)-\varphi_1(\cdot,0)-\varepsilon\xi_k\frac{\partial\varphi_1}{\partial\eta_k}(\cdot,0)\right\|_{H^1(Y^*)}\le C\varepsilon^2|\xi|^2, \tag{5.23}$$

we have

$$\begin{aligned}\|\nabla_x w_1^\varepsilon\|^2_{L^2(\mathbb{R}^N-\bigcup_k \bar{T}_k^\varepsilon)} &\le C\int_{|\xi|<\delta/\varepsilon}|\widehat{u^0}(\xi)|^2[\varepsilon^4|\xi|^6+\varepsilon^2|\xi|^4]\,d\xi\\ &\le C_\delta\varepsilon^2\int_{|\xi|<\delta/\varepsilon}|\widehat{u^0}(\xi)|^2|\xi|^4\,d\xi\le C_\delta\varepsilon^2\|f\|^2_{L^2(\mathbb{R}^N)}.\end{aligned}$$

Besides, since $v_k\in W^1_\#(Y^*)$,

$$\begin{aligned}\|\nabla_x w_2^\varepsilon\|^2_{L^2(\mathbb{R}^N-\bigcup_k \bar{T}_k^\varepsilon)} &\le C\int_{|\xi|>\delta/\varepsilon}|\widehat{u^0}(\xi)|^2|\xi|^2\,d\xi+C\varepsilon^2\int_{|\xi|>\delta/\varepsilon}|\widehat{u^0}(\xi)|^2|\xi|^4\,d\xi\\ &\le C\varepsilon^2\delta^{-2}\int_{|\xi|>\delta/\varepsilon}|\widehat{u^0}(\xi)|^2|\xi|^4\,d\xi\le C\varepsilon^2\delta^{-2}\|f\|^2_{L^2(\mathbb{R}^N)} \quad (5.24)\end{aligned}$$

and (5.19) holds, which completes the proof of (i).

In order to prove (ii) we consider again the decomposition (5.20). For θ_1^ε and u_1^0, we have estimates (5.7) and (5.18) in $L^2(\mathbb{R}^N-\bigcup_k \bar{T}_k^\varepsilon)$, respectively. Moreover, applying Lemma 4.2, (5.23) and the Plancherel identity, we obtain

$$\|w_1^\varepsilon\|^2_{L^2(\mathbb{R}^N-\bigcup_k \bar{T}_k^\varepsilon)}\le C\varepsilon^4\int_{|\xi|<\delta/\varepsilon}|\widehat{u^0}(\xi)|^2|\xi|^4\,d\xi\le C\varepsilon^4\|f\|^2_{L^2(\mathbb{R}^N)} \quad \text{and}$$

$$\|w_2^\varepsilon\|^2_{L^2(\mathbb{R}^N-\bigcup_k \bar{T}_k^\varepsilon)}\le C\varepsilon^2\int_{|\xi|>\delta/\varepsilon}|\widehat{u^0}(\xi)|^2|\xi|^2\,d\xi\le C\varepsilon^4\delta^{-2}\|f\|^2_{L^2(\mathbb{R}^N)}.$$

Therefore, the remaining part is to prove that

$$\begin{aligned}&\left\|\nabla_x\left(\theta^\varepsilon-u^0-\varepsilon v_\varepsilon^k\frac{\partial u^0}{\partial x_k}-\varepsilon^2(v_\varepsilon^{kl}+\beta^{kl})\frac{\partial^2u^0}{\partial x_k\,\partial x_l}\right)\right\|_{L^2(\mathbb{R}^N-\bigcup_k \bar{T}_k^\varepsilon)}\\ &\quad\le C\varepsilon^2\|f\|_{H^1(\mathbb{R}^N)}.\end{aligned} \tag{5.25}$$

To do this, thanks to (4.6) and (4.7), we write

$$\theta^\varepsilon-u^0-\varepsilon v_\varepsilon^k\frac{\partial u^0}{\partial x_k}-\varepsilon^2(v_\varepsilon^{kl}+\beta^{kl})\frac{\partial^2u^0}{\partial x_k\,\partial x_l}=\theta_1^\varepsilon-u_1^0+\tilde{w}_1^\varepsilon-w_2^\varepsilon+w_3^\varepsilon,$$

where θ_1^ε, u_1^0 and w_2^ε are defined by (5.4), (5.17) and (5.21), respectively, and

$$\begin{aligned}\tilde{w}_1^\varepsilon(x)=\int_{|\xi|<\delta/\varepsilon}\theta^{1/2}\widehat{u^0}(\xi)e^{\mathbf{i}x\cdot\xi}\Bigg(&\varphi_1\left(\frac{x}{\varepsilon},\varepsilon\xi\right)-\varphi_1\left(\frac{x}{\varepsilon},0\right)-\varepsilon\xi_k\frac{\partial\varphi_1}{\partial\eta_k}\left(\frac{x}{\varepsilon},0\right)\\ &-\varepsilon^2\xi_k\xi_l\frac{1}{2}\frac{\partial^2\varphi_1}{\partial\eta_k\,\partial\eta_l}\left(\frac{x}{\varepsilon},0\right)\Bigg)\,d\xi,\end{aligned}$$

$$w_3^\varepsilon(x) = \varepsilon^2(v_\varepsilon^{kl}(x) + \beta^{kl})\frac{1}{(2\pi)^{N/2}}\int_{|\xi|>\delta/\varepsilon} \xi_k\xi_l\widehat{u^0}(\xi)e^{\mathbf{i}x\cdot\xi}\,d\xi,$$

for $x \in (\mathbb{R}^N - \bigcup_{\rm k} \bar{T}_{\rm k}^\varepsilon)$. Since $f \in H^1(\Omega)$, from (5.8), (5.22) and (5.24), we obtain the estimates

$$\begin{aligned}
&\|\nabla_x\theta_1^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_{\rm k}\bar{T}_{\rm k}^\varepsilon)} \le C\varepsilon^4\delta^{-4}\|f\|^2_{H^1(\mathbb{R}^N)},\\
&\|\nabla_x u_1^0\|^2_{L^2(\mathbb{R}^N - \bigcup_{\rm k}\bar{T}_{\rm k}^\varepsilon)} \le C\varepsilon^4\delta^{-4}\|f\|^2_{H^1(\mathbb{R}^N)} \quad \text{and}\\
&\|\nabla_x w_2^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_{\rm k}\bar{T}_{\rm k}^\varepsilon)} \le C\varepsilon^4\delta^{-4}\|f\|^2_{H^1(\mathbb{R}^N)}.
\end{aligned}$$

For $\tilde{w}_1^\varepsilon$, we apply Lemma 4.2 and the Taylor expansion for $\varphi_1(\cdot,\eta)$ in $\eta = \varepsilon\xi$,

$$\left\|\varphi_1(\cdot,\varepsilon\xi) - \varphi_1(\cdot,0) - \varepsilon\xi_k\frac{\partial\varphi_1}{\partial\eta_k}(\cdot,0) - \varepsilon^2\xi_k\xi_l\frac{1}{2}\frac{\partial^2\varphi_1}{\partial\eta_k\,\partial\eta_l}(\cdot,0)\right\|_{H^1(Y^*)} \le C\varepsilon^3|\xi|^3,$$

and we obtain

$$\|\nabla_x\tilde{w}_1^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_{\rm k}\bar{T}_{\rm k}^\varepsilon)} \le C\int_{|\xi|<\delta/\varepsilon}|\widehat{u^0}(\xi)|^2[\varepsilon^6|\xi|^8 + \varepsilon^4|\xi|^6]\,d\xi \le C_\delta\varepsilon^4\|f\|^2_{H^1(\mathbb{R}^N)}.$$

Finally, since $v^{kl} \in W^1_\#(Y^*)$, similar considerations as in (5.24) lead us to

$$\|\nabla_x w_3^\varepsilon\|^2_{L^2(\mathbb{R}^N - \bigcup_{\rm k}\bar{T}_{\rm k}^\varepsilon)} \le C\varepsilon^4\delta^{-2}\|f\|^2_{H^1(\mathbb{R}^N)},$$

and the theorem is proved. □

6. The Corrector and the Bloch Approximation in a Bounded Domain

Once we have studied the problem in $\mathbb{R}^N$, we extend the result to the case in Section 2, where Ω is a bounded domain and u^ε is the solution of problem (2.3). For the sake of simplicity, we assume that this boundary does not cut any hole $T_{\rm k}^\varepsilon$ (see Remark 6.3).

For each $\varepsilon > 0$, we introduce a cut-off function m^ε that satisfies the following properties:

$$\begin{cases}
m^\varepsilon \in \mathcal{D}(\Omega), \qquad 0 \le m^\varepsilon(x) \le 1, \quad \forall x \in \Omega,\\
m^\varepsilon(x) = 0 \qquad \text{if} \quad \mathrm{dist}(x,\partial\Omega) \le \varepsilon,\\
m^\varepsilon(x) = 1 \qquad \text{if} \quad \mathrm{dist}(x,\partial\Omega) \ge 2\varepsilon,\\
\varepsilon^{|\alpha|}|D_x^\alpha m^\varepsilon(x)| \le C_\alpha, \qquad \forall\alpha \in \mathbb{Z}_+^N.
\end{cases} \tag{6.1}$$

Obviously, this function exists provided that $\partial\Omega$ is sufficiently smooth.

We define the following function:

$$\breve{\theta}^\varepsilon(x) = \int_{Y'/\varepsilon}\theta^{1/2}\widehat{(m^\varepsilon u^0)}(\xi)e^{\mathbf{i}x\cdot\xi}\varphi_1^\varepsilon(x,\xi)d\xi, \qquad x \in \left(\mathbb{R}^N - \bigcup_{\rm k}\bar{T}_{\rm k}^\varepsilon\right), \tag{6.2}$$

where u^0 is the solution of the homogenized problem (2.5). The following theorem assures that $\breve{\theta}^\varepsilon$ is an approximation at first order for the solution of problem (2.3).

Theorem 6.1. *Let u^ε be the solution of problem* (2.3) *with $f \in L^2(\Omega)$ and the coefficients $a_{ij} \in L^\infty_\#(Y^*)$ satisfying* (2.1) *and* (2.2). *We assume that the solutions v^k of problem* (2.7) *verify $v^k \in W^{1,\infty}_\#(Y^*)$. Then*

$$\|u^\varepsilon - \breve{\theta}^\varepsilon\|_{H^1(\Omega - \bigcup_k \bar{T}^\varepsilon_k)} \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

To prove Theorem 6.1, we first deduce some properties of the cut-off functions m^ε and obtain an asymptotic approximation of the Bloch approximation $\breve{\theta}^\varepsilon$. These results are respectively in Lemma 6.1 and Theorem 6.2 below.

Lemma 6.1. *Let $u \in H^1_0(\Omega)$ and let us consider the functions m^ε defined by* (6.1). *Then we have*

$$m^\varepsilon u \to u \qquad \text{in } H^1_0(\Omega), \quad \text{as } \varepsilon \to 0.$$

Moreover, if $u \in H^2(\Omega)$,

$$\|m^\varepsilon u - u\|_{H^1(\Omega)} \le C\varepsilon^{1/2}\|u\|_{H^2(\Omega)} \quad \text{and} \quad \varepsilon^{1/2}\left\|\frac{\partial^2(m^\varepsilon u)}{\partial x_k \partial x_l}\right\|_{L^2(\Omega)} \le C\|u\|_{H^2(\Omega)}. \quad (6.3)$$

Proof. We denote by $\omega_{2\varepsilon}$ the domain $\omega_{2\varepsilon} = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \le 2\varepsilon\}$. By the definition of m^ε, it is clear that

$$\|\nabla_x(m^\varepsilon u - u)\|^2_{L^2(\Omega)} \le C[\varepsilon^{-2}\|u\|^2_{L^2(\omega_{2\varepsilon})} + \|\nabla_x u\|^2_{L^2(\omega_{2\varepsilon})}].$$

Besides, since u vanishes on $\partial\Omega$, we have

$$\|u\|_{L^2(\omega_{2\varepsilon})} \le C\varepsilon\|\nabla_x u\|_{L^2(\omega_{2\varepsilon})} \tag{6.4}$$

which allows us to assert that

$$\|\nabla_x(m^\varepsilon u - u)\|_{L^2(\Omega)} \le C\|\nabla_x u\|_{L^2(\omega_{2\varepsilon})},$$

and $m^\varepsilon u - u$ converges strongly in $H^1_0(\Omega)$ towards zero, as $\varepsilon \to 0$.

Now, we prove (6.3) under the assumptions that $u \in H^1_0(\Omega) \cap H^2(\Omega)$. For ν small enough ($\nu \le \nu_0$), we denote by S_ν the boundary of the domain defined by $\{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \ge \nu\}$. By virtue of the embedding theorem we have

$$\int_{S_\nu} |w|^2\, dS \le C\|w\|^2_{H^1(\{x\in\Omega \mid \operatorname{dist}(x,\partial\Omega)\ge\nu\})} \le C\|w\|^2_{H^1(\Omega)}, \qquad \forall w \in H^1(\Omega).$$

Integrating this inequality with respect to n from 0 to 2ε, we prove

$$\|w\|^2_{L^2(\omega_{2\varepsilon})} \le C\varepsilon\|w\|^2_{H^1(\Omega)}, \qquad \forall w \in H^1(\Omega), \tag{6.5}$$

in particular, for $w = \partial u/\partial x_k$. Consequently, we obtain

$$\|\nabla_x(m^\varepsilon u - u)\|^2_{L^2(\Omega)} \le C\|\nabla_x u\|^2_{L^2(\omega_{2\varepsilon})} \le C\varepsilon\|u\|^2_{H^2(\Omega)}.$$

Finally, the properties of m^ε and (6.4) and (6.5) lead us to

$$\begin{aligned}\left\|\frac{\partial^2(m^\varepsilon u)}{\partial x_k\,\partial x_l}\right\|^2_{L^2(\Omega)} &\le C\left[\varepsilon^{-4}\|u\|^2_{L^2(\omega_{2\varepsilon})} + \varepsilon^{-2}\|\nabla_x u\|^2_{L^2(\omega_{2\varepsilon})} + \left\|\frac{\partial^2 u}{\partial x_k\,\partial x_l}\right\|^2_{L^2(\Omega)}\right]\\ &\le C\varepsilon^{-1}\|u\|^2_{H^2(\Omega)},\end{aligned}$$

and the proof is complete. □

Theorem 6.2. *Let u^0 be the solution of problem* (2.5) *with $f \in L^2(\Omega)$. We assume that the functions $a_{ij} \in L^\infty_\#(Y^*)$ verify* (2.1) *and* (2.2) *and the solutions v^k of problem* (2.7) *satisfy $v^k \in W^{1,\infty}_\#(Y^*)$. Then*

$$\left\|\breve{\theta}^\varepsilon - m^\varepsilon u^0 - \varepsilon v^k_\varepsilon \frac{\partial(m^\varepsilon u^0)}{\partial x_k}\right\|_{H^1(\Omega - \bigcup_k \bar{T}^\varepsilon_k)} \le C\varepsilon^{1/2}\|f\|_{L^2(\Omega)},$$

where $\breve{\theta}^\varepsilon$ is the function defined by (6.2) *and $v^k_\varepsilon(x) = v^k(x/\varepsilon)$.*

Proof. We follow the proof in Theorem 5.3 with suitable modifications that we outline here. First, we prove that

$$\|\breve{\theta}^\varepsilon - m^\varepsilon u^0\|_{L^2(\Omega - \bigcup_k \bar{T}^\varepsilon_k)} \le C\varepsilon\|f\|_{L^2(\Omega)}. \tag{6.6}$$

To this end, we write $\breve{\theta}^\varepsilon - m^\varepsilon u^0 = z_1 + z_2 + z_3$ where

$$z_1(x) = \int_{|\xi|<\delta/\varepsilon} \theta^{1/2}\widehat{(m^\varepsilon u^0)}(\xi)e^{\mathbf{i}x\cdot\xi}(\varphi_1(x/\varepsilon,\xi) - \varphi_1(x/\varepsilon,0))d\xi, \qquad x \in \left(\mathbb{R}^N - \bigcup_k \bar{T}^\varepsilon_k\right),$$

$$z_2(x) = \int_{\substack{\xi\,\in\, Y'/\varepsilon\\ |\xi|\,>\,\delta/\varepsilon}} \theta^{1/2}\widehat{(m^\varepsilon u^0)}(\xi)e^{\mathbf{i}x\cdot\xi}\varphi^\varepsilon_1(x,\xi)d\xi, \qquad x \in \left(\mathbb{R}^N - \bigcup_k \bar{T}^\varepsilon_k\right), \tag{6.7}$$

$$z_3(x) = -\frac{1}{(2\pi)^{N/2}}\int_{|\xi|>\delta/\varepsilon} \widehat{(m^\varepsilon u^0)}(\xi)e^{\mathbf{i}x\cdot\xi}d\xi, \qquad x \in \left(\mathbb{R}^N - \bigcup_k \bar{T}^\varepsilon_k\right). \tag{6.8}$$

Applying Lemma 4.2, and taking into account the analyticity of the first Bloch eigenvector in B_δ and the normalization of the Bloch eigenvectors, we have

$$\|z_1\|^2_{L^2(\Omega - \bigcup_k \bar{T}^\varepsilon_k)} \leq C\varepsilon^2 \int_{|\xi|<\delta/\varepsilon} |\widehat{(m^\varepsilon u^0)}(\xi)|^2 |\xi|^2 \, d\xi \quad \text{and} \tag{6.9}$$

$$\|z_i\|^2_{L^2(\Omega - \bigcup_k \bar{T}^\varepsilon_k)} \leq C\varepsilon^2 \delta^{-2} \int_{|\xi|>\delta/\varepsilon} |\widehat{(m^\varepsilon u^0)}(\xi)|^2 |\xi|^2 \, d\xi, \qquad i = 2, 3. \tag{6.10}$$

Moreover, as $u^0 \in H^1(\Omega)$, Lemma 6.1 gives us

$$\|z_i\|^2_{L^2(\Omega - \bigcup_k \bar{T}^\varepsilon_k)} \leq C\varepsilon^2 \|\nabla_x (m^\varepsilon u^0)\|^2_{L^2(\mathbb{R}^N)} \leq C\varepsilon^2 \|\nabla_x u^0\|^2_{L^2(\Omega)} \leq C\varepsilon^2 \|f\|^2_{L^2(\Omega)},$$

for $i = 1, 2, 3$ and (6.6) holds.

It remains to be proven that

$$\left\| \nabla_x \left(\theta^\varepsilon - m^\varepsilon u^0 - \varepsilon v^k_\varepsilon \frac{\partial (m^\varepsilon u^0)}{\partial x_k} \right) \right\|_{L^2(\Omega - \bigcup_k \bar{T}^\varepsilon_k)} \leq C\varepsilon^{1/2} \|f\|_{L^2(\Omega)}. \tag{6.11}$$

Now, we write

$$\theta^\varepsilon - m^\varepsilon u^0 - \varepsilon v^k_\varepsilon \frac{\partial (m^\varepsilon u^0)}{\partial x_k} = s_1 + s_2 + z_2 + z_3,$$

where z_2 and z_3 are defined by (6.7) and (6.8), respectively, and

$$s_1(x) = \int_{|\xi|<\delta/\varepsilon} \theta^{1/2} \widehat{(m^\varepsilon u^0)}(\xi) e^{\mathbf{i}x\cdot\xi} \left(\varphi_1 \left(\frac{x}{\varepsilon}, \varepsilon\xi \right) - \varphi_1 \left(\frac{x}{\varepsilon}, 0 \right) - \varepsilon \xi_k \frac{\partial \varphi_1}{\partial \eta_k} \left(\frac{x}{\varepsilon}, 0) \right) \right) d\xi$$

and

$$s_2(x) = -\varepsilon v^k_\varepsilon(x) \frac{1}{(2\pi)^{N/2}} \int_{|\xi|>\delta/\varepsilon} \mathbf{i}\xi_k \widehat{(m^\varepsilon u^0)}(\xi) e^{\mathbf{i}x\cdot\xi} d\xi, \qquad x \in \left(\mathbb{R}^N - \bigcup_k \bar{T}^\varepsilon_k \right).$$

Then, as in Theorem 5.3, we apply Lemma 4.2, the Taylor expansion of order 2 for $\varphi_1(\cdot, \eta)$ in $\eta = \varepsilon\xi$ and the fact that $v_k \in W^{1,\infty}(Y^*)$, and with minor modifications we obtain

$$\|\nabla_x s_1\|^2_{L^2(\Omega - \bigcup_k \bar{T}^\varepsilon_k)} \leq C_\delta \varepsilon^2 \int_{|\xi|<\delta/\varepsilon} |\widehat{(m^\varepsilon u^0)}(\xi)|^2 |\xi|^4 \, d\xi,$$

$$\|\nabla_x s_2\|^2_{L^2(\Omega - \bigcup_k \bar{T}^\varepsilon_k)} \leq C\varepsilon^2 \delta^{-2} \int_{|\xi|>\delta/\varepsilon} |\widehat{(m^\varepsilon u^0)}(\xi)|^2 |\xi|^4 \, d\xi,$$

$$\|\nabla_x z_i\|^2_{L^2(\Omega - \bigcup_k \bar{T}^\varepsilon_k)} \leq C\varepsilon^2 \delta^{-2} \int_{|\xi|>\delta/\varepsilon} |\widehat{(m^\varepsilon u^0)}(\xi)|^2 |\xi|^4 \, d\xi, \qquad i = 2, 3.$$

Finally, as $f \in L^2(\Omega)$, $u^0 \in H^2(\Omega)$ and taking into account the properties of the test functions m^ε and Lemma 6.1, we can assert that

$$\begin{aligned} \int_{\mathbb{R}^N} |\widehat{(m^\varepsilon u^0)}(\xi)|^2 |\xi|^4 \, d\xi &\leq \int_\Omega \left| \frac{\partial^2 (m^\varepsilon u^0)}{\partial x_k \, \partial x_l}(x) \right|^2 dx \leq C\varepsilon^{-1} \|u^0\|^2_{H^2(\Omega)} \\ &\leq C\varepsilon^{-1} \|f\|^2_{L^2(\Omega)}, \end{aligned}$$

which completes the proof. □

Proof of Theorem 6.1. We first introduce some notation and results which will be useful throughout the proof. For $v \in L^2(\Omega - \bigcup_k \bar{T}_k^\varepsilon)$, $\widetilde{v}$ denotes the extension by zero of v outside Ω. For each $\varepsilon > 0$, we define $N^\varepsilon \widetilde{u^\varepsilon} \in H^{-1}(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$ by

$$\langle N^\varepsilon \widetilde{u^\varepsilon}, v\rangle_{H^{-1}(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)\times H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} = \int_{\Omega - \bigcup_k \bar{T}_k^\varepsilon} f \bar{v}\, dx - \int_{\Omega - \bigcup_k \bar{T}_k^\varepsilon} a_{ij}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial \bar{v}}{\partial x_j}\, dx$$

for all $v \in H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$. Since $u^\varepsilon \in H^1(\Omega - \bigcup_k \bar{T}_k^\varepsilon)$ is the solution of problem (2.3) with $f \in L^2(\Omega)$ and $a_{ij} \in L^\infty_\#(Y^*)$ satisfying (2.1) and (2.2), it is clear that

$$\|u^\varepsilon\|_{H^1(\Omega - \bigcup_k \bar{T}_k^\varepsilon)} \le C_1 \quad \text{and} \quad \|N^\varepsilon \widetilde{u^\varepsilon}\|_{H^{-1}(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} \le C_2, \tag{6.12}$$

where C_1 and C_2 are some constants independent of ε. Moreover, $\widetilde{u^\varepsilon} \in H^1\left(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon\right)$ is the solution of

$$A^\varepsilon \widetilde{u^\varepsilon} + N^\varepsilon \widetilde{u^\varepsilon} = \widetilde{f} \qquad \text{in } \left(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon\right).$$

Therefore, from Theorem 4.1, it follows that

$$\lambda_m^\varepsilon(\xi)(B_m^\varepsilon \widetilde{u^\varepsilon})(\xi) + (B_m^\varepsilon N^\varepsilon \widetilde{u^\varepsilon})(\xi) = (B_m^\varepsilon \widetilde{f})(\xi), \qquad m \ge 1, \xi \in Y'/\varepsilon \ \text{ and} \tag{6.13}$$

$$\begin{aligned}\int_{Y'/\varepsilon} \sum_{m=1}^\infty (B_m^\varepsilon N^\varepsilon \widetilde{u^\varepsilon})(\xi)\overline{(B_m^\varepsilon v)(\xi)} d\xi \\ = \langle N^\varepsilon \widetilde{u^\varepsilon}, v\rangle_{H^{-1}(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)\times H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)} = 0,\end{aligned} \tag{6.14}$$

for all $v \in H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$ such that $v = 0$ on $\partial\Omega$.

By the definition of $\widetilde{u^\varepsilon}$, equation (3.5) and Lemma 4.1, it is easy to check that Theorem 6.1 holds once we prove

$$\int_{Y'/\varepsilon} \sum_{m=1}^\infty |(B_m^\varepsilon \widetilde{u^\varepsilon})(\xi) - (B_m^\varepsilon \breve{\theta}^\varepsilon)(\xi)|^2\, d\xi \to 0 \qquad \text{as} \quad \varepsilon \to 0 \tag{6.15}$$

and

$$\int_{Y'/\varepsilon} \sum_{m=1}^\infty \lambda_m^\varepsilon(\xi) |(B_m^\varepsilon \widetilde{u^\varepsilon})(\xi) - (B_m^\varepsilon \breve{\theta}^\varepsilon)(\xi)|^2\, d\xi \to 0 \qquad \text{as} \quad \varepsilon \to 0. \tag{6.16}$$

In order to prove (6.15), we write

$$\begin{aligned}&\int_{Y'/\varepsilon} \sum_{m=1}^\infty |(B_m^\varepsilon \widetilde{u^\varepsilon})(\xi) - (B_m^\varepsilon \breve{\theta}^\varepsilon)(\xi)|^2\, d\xi \\ &= \int_{Y'/\varepsilon} |(B_1^\varepsilon \widetilde{u^\varepsilon})(\xi) - \theta^{1/2}\widehat{(m^\varepsilon u^0)}(\xi)|^2\, d\xi + \int_{Y'/\varepsilon} \sum_{m=2}^\infty |(B_m^\varepsilon \widetilde{u^\varepsilon})(\xi)|^2\, d\xi \\ &\le C\int_{Y'/\varepsilon} |(B_1^\varepsilon \widetilde{u^\varepsilon})(\xi) - (B_1^\varepsilon \widetilde{u^0})(\xi)|^2\, d\xi + C\int_{Y'/\varepsilon} |(B_1^\varepsilon \widetilde{u^0})(\xi) - \theta^{1/2}\widehat{\widetilde{u^0}}(\xi)|^2\, d\xi \\ &\quad + C\int_{Y'/\varepsilon} \theta |(\widehat{\widetilde{u^0} - m^\varepsilon u^0})(\xi)|^2\, d\xi + \int_{Y'/\varepsilon} \sum_{m=2}^\infty |(B_m^\varepsilon \widetilde{u^\varepsilon})(\xi)|^2\, d\xi.\end{aligned} \tag{6.17}$$

From (3.4), (4.11) and Lemma 4.1, we have

$$\int_{Y'/\varepsilon} \sum_{m=2}^{\infty} |(B_m^\varepsilon \widetilde{u^\varepsilon})(\xi)|^2 d\xi \leq \int_{Y'/\varepsilon} \sum_{m=2}^{\infty} \frac{\lambda_m^\varepsilon(\xi)\varepsilon^2}{\lambda_2^*} |(B_m^\varepsilon \widetilde{u^\varepsilon})(\xi)|^2 d\xi$$
$$\leq C\varepsilon^2 \|\nabla \widetilde{u^\varepsilon}\|^2_{L^2(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)}. \tag{6.18}$$

Then, using (6.18), (3.5), Proposition 4.3, Lemma 6.1 and the strong convergence of $P^\varepsilon u^\varepsilon$ towards u^0 in $L^2(\Omega)$ as $\varepsilon \to 0$, we obtain that the last four terms in (6.17) converge to zero as $\varepsilon \to 0$ and (6.15) holds

To prove (6.16), we use (6.13) and write

$$\int_{Y'/\varepsilon} \sum_{m=1}^{\infty} \lambda_m^\varepsilon |(B_m^\varepsilon \widetilde{u^\varepsilon}) - (B_m^\varepsilon \breve{\theta}^\varepsilon)|^2 \, d\xi$$
$$= \int_{Y'/\varepsilon} (B_1^\varepsilon \widetilde{f}) \overline{(B_1^\varepsilon \widetilde{u^\varepsilon})} d\xi - \int_{Y'/\varepsilon} (B_1^\varepsilon \widetilde{f}) \overline{\theta^{1/2} \widehat{(m^\varepsilon u^0)}} d\xi - \int_{Y'/\varepsilon} \overline{(B_1^\varepsilon \widetilde{f})} \theta^{1/2} \widehat{(m^\varepsilon u^0)} d\xi$$
$$+ \int_{Y'/\varepsilon} \lambda_1^\varepsilon \theta |\widehat{(m^\varepsilon u^0)}|^2 \, d\xi + \int_{Y'/\varepsilon} \sum_{m=2}^{\infty} (B_m^\varepsilon \widetilde{f}) \overline{(B_m^\varepsilon \widetilde{u^\varepsilon})} \, d\xi$$
$$- \int_{Y'/\varepsilon} \sum_{m=1}^{\infty} (B_m^\varepsilon N^\varepsilon \widetilde{u^\varepsilon}) \overline{(B_m^\varepsilon \widetilde{u^\varepsilon})} \, d\xi + \theta^{1/2} \int_{Y'/\varepsilon} (B_1^\varepsilon N^\varepsilon \widetilde{u^\varepsilon}) \overline{\widehat{(m^\varepsilon u^0)}} \, d\xi$$
$$+ \theta^{1/2} \int_{Y'/\varepsilon} \widehat{(m^\varepsilon u^0)} \overline{(B_1^\varepsilon N^\varepsilon \widetilde{u^\varepsilon})} \, d\xi. \tag{6.19}$$

On account of the decomposition

$$\int_{Y'/\varepsilon} \theta^{1/2} (B_1^\varepsilon N^\varepsilon \widetilde{u^\varepsilon}) \overline{\widehat{(m^\varepsilon u^0)}} \, d\xi$$
$$= \int_{Y'/\varepsilon} \sum_{m=1}^{\infty} (B_m^\varepsilon N^\varepsilon \widetilde{u^\varepsilon}) \overline{(B_m^\varepsilon \breve{\theta}^\varepsilon)} \, d\xi$$
$$= \int_{Y'/\varepsilon} \sum_{m=1}^{\infty} (B_m^\varepsilon N^\varepsilon \widetilde{u^\varepsilon}) \overline{B_m^\varepsilon \left(\breve{\theta}^\varepsilon - m^\varepsilon u^0 - \varepsilon v_\varepsilon^k \frac{\partial (m^\varepsilon u^0)}{\partial x_k} \right)} d\xi$$
$$+ \int_{Y'/\varepsilon} \sum_{m=1}^{\infty} (B_m^\varepsilon N^\varepsilon \widetilde{u^\varepsilon}) \overline{B_m^\varepsilon \left(m^\varepsilon u^0 + \varepsilon v_\varepsilon^k \frac{\partial (m^\varepsilon u^0)}{\partial x_k} \right)} d\xi,$$

(6.14), (4.1) and (6.12) and Theorem 6.2, it is easy to check that the last three terms in (6.19) tend to zero as $\varepsilon \to 0$. Besides, the Cauchy–Schwarz inequality, (3.5) and (6.18) lead us to

$$\int_{Y'/\varepsilon} \sum_{m=2}^{\infty} (B_m^\varepsilon \widetilde{f}) \overline{(B_m^\varepsilon \widetilde{u^\varepsilon})} \, d\xi \leq C\varepsilon \|f\|_{L^2(\Omega)} \|\nabla u^\varepsilon\|_{L^2(\Omega - \bigcup_k \bar{T}_k^\varepsilon)}.$$

Finally, the strong convergence of $P^\varepsilon u^\varepsilon$ towards u^0 in $L^2(\Omega)$ as $\varepsilon \to 0$, Proposition 4.3

and Lemma 6.1 allow us to assert that

$$\int_{Y'/\varepsilon} (B_1^\varepsilon \widetilde{f})\overline{(B_1^\varepsilon \widetilde{u^\varepsilon})}\, d\xi \xrightarrow{\varepsilon\to 0} \int_{\mathbb{R}^N} \theta^{1/2}\widehat{\widetilde{f}}\,\theta^{1/2}\widehat{\widetilde{u^0}}\, d\xi = \theta \int_\Omega f u^0\, dx,$$

$$\int_{Y'/\varepsilon} (B_1^\varepsilon \widetilde{f})\overline{\theta^{1/2}\widehat{(m^\varepsilon u^0)}}\, d\xi \xrightarrow{\varepsilon\to 0} \int_{\mathbb{R}^N} \theta^{1/2}\widehat{\widetilde{f}}\,\theta^{1/2}\widehat{\widetilde{u^0}}\, d\xi = \theta \int_\Omega f u^0\, dx.$$

Therefore, to complete the proof of (6.16) it remains to prove the convergence

$$\int_{Y'/\varepsilon} \lambda_1^\varepsilon \theta |\widehat{(m^\varepsilon u^0)}|^2\, d\xi \xrightarrow{\varepsilon\to 0} \theta \int_\Omega f u^0\, dx. \tag{6.20}$$

To do this, we use (2.5) and write

$$\begin{aligned}
&\int_{Y'/\varepsilon} \lambda_1^\varepsilon \theta |\widehat{(m^\varepsilon u^0)}|^2\, d\xi - \theta \int_\Omega f u^0\, dx \\
&\quad = \int_{\substack{\xi\,\in\, Y'/\varepsilon \\ |\xi|\,>\,\delta/\varepsilon}} \lambda_1^\varepsilon \theta |\widehat{(m^\varepsilon u^0)}|^2\, d\xi + \int_{|\xi|<\delta/\varepsilon} (\lambda_1^\varepsilon - \theta^{-1} a_{ij}^h \xi_i \xi_j)\theta |\widehat{(m^\varepsilon u^0)}|^2\, d\xi \\
&\qquad + \int_{|\xi|<\delta/\varepsilon} a_{ij}^h \xi_i\xi_j \widehat{(m^\varepsilon u^0 - \widetilde{u^0})}\overline{\widehat{(m^\varepsilon u^0)}}\, d\xi + \int_{|\xi|<\delta/\varepsilon} a_{ij}^h \xi_i\xi_j \widehat{\widetilde{u^0}}\overline{\widehat{(m^\varepsilon u^0 - \widetilde{u^0})}}\, d\xi \\
&\qquad + \int_{|\xi|<\delta/\varepsilon} a_{ij}^h \xi_i\xi_j |\widehat{\widetilde{u^0}}|^2\, d\xi - \int_{\mathbb{R}^N} a_{ij}^h \frac{\partial \widetilde{u^0}}{\partial x_i}\frac{\partial \widetilde{u^0}}{\partial x_j}\, dx.
\end{aligned} \tag{6.21}$$

Then, applying (3.4), (4.10), (5.13), Lemma 6.1 and the fact that $u^0 \in H^2(\Omega)$, we obtain (6.20), which ends the proof of the theorem. □

Remark 6.1. It should be noted that using the technique of [2] we can prove that the asymptotic approximation of $\breve{\theta}^\varepsilon$ in Theorem 6.2 is a first-order corrector for the solution u^ε of (2.3); that is, the convergence

$$\left\| u^\varepsilon - m^\varepsilon u^0 - \varepsilon v_\varepsilon^k \frac{\partial(m^\varepsilon u^0)}{\partial x_k} \right\|_{H^1(\Omega - \bigcup_k \bar{T}_k^\varepsilon)} \to 0 \qquad \text{as} \quad \varepsilon \to 0$$

holds (see Theorem 2.2 for comparison). Nevertheless, we point out that this fact only confirms the convergence of $\|u^\varepsilon - \breve{\theta}^\varepsilon\|_{H^1(\Omega - \bigcup_k \bar{T}_k^\varepsilon)}$ towards zero, as $\varepsilon \to 0$, obtained in Theorem 6.1.

We also note that the functions v^k, solutions of (2.7), appear in the Bloch approximation in a natural way as the partial derivatives of the first Bloch eigenvector (see (4.6)).

Remark 6.2. Note that in the case where Ω is a bounded domain, the difference of the new approximation $\breve{\theta}^\varepsilon$ and the solution u^ε in the H^1-norm converge towards zero, as $\varepsilon \to 0$, at a rate of $\varepsilon^{1/2}$ (see Remark 5.1 for comparison). This holds from Theorem 6.2, Lemma 6.1 and the error estimate

$$\left\| u^\varepsilon(x) - u^0(x) - \varepsilon v^k \left(\frac{x}{\varepsilon}\right) \frac{\partial u^0}{\partial x_k}(x) \right\|_{H^1(\Omega^\varepsilon)} \le C\varepsilon^{1/2}$$

(see, for example, Section II.1 of [16] and Section I.2 of [6]).

Remark 6.3. As we noted in [7], the technique of Bloch wave decomposition allows the assumption on the geometry of the domain Ω and the holes to be weakened. For example, if the holes meet the boundary of Ω and the Dirichlet condition is imposed on $\Gamma^\varepsilon = \partial\Omega - \bigcup T^\varepsilon$, the proofs in Section 6 still hold: we require the existence of the extension $\widetilde{u^\varepsilon} \in H^1(\mathbb{R}^N - \bigcup_k \bar{T}_k^\varepsilon)$ in the proof of Theorem 6.1, a uniformly bounded family of extension operators P^ε (see, for example, Section I.4 of [16]), and the bound for the solutions $\|u^\varepsilon\|_{H^1(\Omega^\varepsilon)} \le C$, with C a constant independent of ε, which obviously holds.

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