

EFFECTIVE CHEMICAL PROCESSES IN POROUS MEDIA

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In the book by Hornung, Chap. 6, the author proposes a homogenization strategy for the effective behavior of some chemical processes involving adsorption and reactions arising in porous media. Rigorous proofs of the convergence results are given in the case of linear adsorption rates and linear chemical reactions. The author leaves as an open question the case of a nonlinear adsorption rate. Our goal in this paper is to study two well-known examples of such nonlinear models, namely the so-called Freundlich and Langmuir kinetics.

Keywords: Nonlinear reaction-diffusion coupling; reactive flows; adsorption.

1. Introduction

The object of this paper is to find the effective behavior of chemical reactive flows involving diffusion, different types of adsorption rates and chemical reactions which take place at the boundary of a periodically perforated material. For a nice presentation of the chemical aspects involved in our model (and also for some mathematical and historical backgrounds) we refer to Refs. 1, 4, 14–16 and 18.

Let Ω be an open bounded set in \mathbb{R}^n and let us perforate it by holes. As a result, we obtain an open set Ω^ε which will be referred to as being the *perforated domain*; ε represents a small parameter related to the characteristic size of the perforations.

The nonlinear problem studied in this paper concerns the asymptotic behavior, as $\varepsilon \rightarrow 0$, of the microscopic models (V^ε) and (W^ε) below. In the chemical situation behind this model the domain Ω consists of two parts: a fluid phase Ω^ε and a solid skeleton (grains or pores), $\Omega \setminus \bar{\Omega}^\varepsilon$. We assume that chemical substances are dissolved in the fluid part Ω^ε . They are transported by diffusion and also, by adsorption, they can change from being dissolved in the fluid to residing on the surface of the pores. Here, on the boundary, chemical reactions (which can be influenced by catalysts) take place. Hence, the model consists of a diffusion system in the fluid phase Ω^ε , a reaction system on the pore surface and a boundary condition coupling them (see (1.2)). A simplified modelling of this situation is as follows:

$$(V^\varepsilon) \begin{cases} \frac{\partial v^\varepsilon}{\partial t}(t, x) - D\Delta v^\varepsilon(t, x) = h(t, x), & x \in \Omega^\varepsilon, t > 0, \\ v^\varepsilon(t, x) = 0, & x \in \partial\Omega, t > 0, \\ v^\varepsilon(t, x) = v_1(x), & x \in \Omega^\varepsilon, t = 0, \end{cases} \quad (1.1)$$

$$-D \frac{\partial v^\varepsilon}{\partial \nu}(t, x) = \varepsilon f^\varepsilon(t, x), \quad x \in S^\varepsilon, t > 0, \quad (1.2)$$

and

$$(W^\varepsilon) \begin{cases} \frac{\partial w^\varepsilon}{\partial t}(t, x) + aw^\varepsilon(t, x) = f^\varepsilon(t, x), & x \in S^\varepsilon, t > 0, \\ w^\varepsilon(t, x) = w_1(x), & x \in S^\varepsilon, t = 0, \end{cases} \quad (1.3)$$

where

$$f^\varepsilon(t, x) = \gamma(g(v^\varepsilon(t, x)) - w^\varepsilon(t, x)). \quad (1.4)$$

Here, ν is the exterior unit normal to Ω^ε , $a, \gamma > 0$, h is a given function representing an external source of energy, $v_1, w_1 \in H_0^1(\Omega)$ and S^ε is the boundary of our porous medium $\Omega \setminus \bar{\Omega}^\varepsilon$. Moreover, the fluid is assumed to be homogeneous and isotropic, with a constant diffusion coefficient $D > 0$. In (1.1)–(1.5), v^ε can be interpreted as being the concentration of the solute in the fluid region, w^ε as the concentration of the solute on the surface of the skeleton $\Omega \setminus \bar{\Omega}^\varepsilon$, v_1 as the initial concentration of the solute and w_1 as the initial concentration of the reactants on the surface S^ε of the skeleton; a and γ are called *the reaction factor* and *the adsorption factor*, respectively.

The semilinear boundary condition on S^ε (see (1.2), (1.4)) in problem (1.1)–(1.4) describes the interchanges of chemical flows across the boundary S^ε . They are governed by a general nonlinear balance law involving two main ingredients, namely an adsorption factor γ (which, in a first step, will be assumed to be constant) and the so-called *adsorption rate* which shows the nonlinear behavior of this equilibrium; it is represented by g in (1.2)–(1.4), which is assumed to be given. Two situations will be considered: the case in which g is a monotone smooth function satisfying the condition $g(0) = 0$ and the case of a maximal monotone graph with $g(0) = 0$, i.e. the

case in which g is the subdifferential of a convex lower semicontinuous function G . These two general situations are well illustrated by the following important practical examples (see Refs. 13 and 18):

$$(a) \quad g(v) = \frac{\alpha v}{1 + \beta v}, \quad \alpha, \beta > 0 \quad (\text{Langmuir kinetics})$$

and

$$(b) \quad g(v) = |v|^{p-1}v, \quad 0 < p < 1 \quad (\text{Freundlich kinetics}).$$

The exponent p in example (b) is called *the order of the reaction*. In some applications the limit case ($p = 0$) is of great relevance (see Remark 3.5). It is worth remarking that if $v^\varepsilon \geq 0$ in $\bar{\Omega}^\varepsilon$ and $v^\varepsilon > 0$ in Ω^ε , then the function g in example (a) is indeed a particular example of our first model situation (g is a monotone smooth function satisfying the condition $g(0) = 0$). Also, let us note that, instead of (1.4), we could consider a more general boundary condition, given in terms of

$$f^\varepsilon(t, x) = \gamma_1 g(v^\varepsilon(t, x)) - \gamma_2 w^\varepsilon(t, x),$$

where $\gamma_1 > 0$ is called *adsorption factor* and $\gamma_2 > 0$ is called *desorption factor* (see Ref. 19).

The existence and uniqueness of a weak solution of the system (1.1)–(1.4) can be settled by using the classical theory of semilinear monotone problems (see, for instance, Refs. 7 and 20). As a result, we know that there exists a unique weak solution $u^\varepsilon = (v^\varepsilon, w^\varepsilon)$.

From a geometrical point of view, we shall just consider periodic structures obtained by removing periodically from Ω , with period εY (where Y is a given hyper-rectangle in \mathbb{R}^n), an elementary hole F which has been appropriately rescaled and which is strictly included in Y , i.e. $\bar{F} \subset Y$.

As usual in homogenization, we shall be interested in obtaining a suitable description of the asymptotic behavior, as ε tends to zero, of the solution u^ε in such domains. We will wonder, for example, whether the solution u^ε converges to a limit u as $\varepsilon \rightarrow 0$. And if this limit exists, can it be characterized?

If we denote by \star the convolution with respect to time and if

$$r(\rho) = e^{-(a+\gamma)\rho}, \tag{1.5}$$

then we prove that the solution v^ε , properly extended to the whole of Ω , converges to the unique solution v (effective behavior) of the following problem:

$$(V) \quad \begin{cases} \frac{\partial v}{\partial t}(t, x) - D \sum_{i,j=1}^n q_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j}(t, x) + F_0(t, x) = h(t, x), & t > 0, x \in \Omega, \\ v(t, x) = 0, & t > 0, x \in \partial\Omega, \\ v(t, x) = v_1(x), & t = 0, x \in \Omega, \end{cases} \tag{1.6}$$

with

$$F_0(t, x) = \frac{|\partial F|}{|Y \setminus \bar{F}|} \gamma \left[g(v(t, x)) - w_1(x) e^{-(a+\gamma)t} - \gamma r(\cdot) \star g(v(\cdot, x))(t) \right]. \tag{1.7}$$

In (1.6), $Q = ((q_{ij}))$ is the classical homogenized matrix, whose entries are defined as follows:

$$q_{ij} = \delta_{ij} + \frac{1}{|Y \setminus \bar{F}|} \int_{Y \setminus \bar{F}} \frac{\partial \chi_j}{\partial y_i} dy, \tag{1.8}$$

in terms of the functions $\chi_i, i = 1, \dots, n$, solutions of the so-called cell problems

$$\begin{cases} -\Delta \chi_i = 0 \text{ in } Y \setminus \bar{F}, \\ \frac{\partial(\chi_i + y_i)}{\partial \nu} = 0 \text{ on } \partial F, \\ \chi_i \text{ } Y\text{-periodic.} \end{cases} \tag{1.9}$$

Moreover, let us notice that the limit problem for the surface concentration w is

$$\begin{cases} \frac{\partial w}{\partial t}(t, x) + (a + \gamma)w(t, x) = \gamma g(v(t, x)), & t > 0, x \in \Omega, \\ w(t, x) = w_1(x), & t = 0, x \in \Omega \end{cases} \tag{1.10}$$

and obviously w can be written as

$$w(t, x) = w_1(x) e^{-(a+\gamma)t} + \gamma r(\cdot) \star g(v(\cdot, x))(t). \tag{1.11}$$

The macroscopic problem (1.6) arises from the homogenization of a boundary-value problem in a periodically perforated domain and the zeroth-order term occurring in (1.6) has its origin in this particular structure of the model. The influence of the adsorption and chemical reactions taking place on the boundaries of the perforations is reflected by the appearance of this zeroth-order extra-term.

In problem (1.1)–(1.4), the rate of chemical reactions on S^ε , namely a , and the adsorption coefficient γ were assumed to be constant. From a practical point of view, a more realistic model would be to assume that the surface ∂F is chemically and physically heterogeneous, which means that a and γ are rapidly oscillating functions. Moreover, one can consider a more general model, including the diffusion of the chemical species on the surface S^ε . In fact, the chemical substances can creep on the surface and this effect is similar to a surface-like diffusion. From a mathematical point of view, we can model this phenomenon by introducing a diffusion term in the law governing the evolution of the surface concentration w^ε (see (1.3)). This new term is nothing but the Laplace–Beltrami operator properly rescaled. Taking into account all these generalizations, we are led to system (4.5)–(4.8) (see Sec. 4). The limit problem in this case is almost the same as before, except that it involves the solution of a reaction–diffusion system with respect to an additional microvariable. Also, notice that the local behavior is no longer governed by an ordinary differential equation, but by a partial differential one (see (4.12)).

The structure of our paper is as follows: first, let us mention that we shall just focus on the case $n \geq 3$, which will be treated explicitly. The case $n = 2$ is much more simpler and we shall omit to treat it. In Sec. 2 we consider the simpler case of chemical flows just involving homogeneous adsorption and chemical reactions. We start by stating some preliminary notation and assumptions, we give a rigorous setting of the problem and we formulate the main convergence result, the proof of which is given in Sec. 3. Section 4 is devoted to treat a more general model, namely the case where the surface of the grains is heterogeneous and we have also diffusion thereon.

Finally, notice that throughout the paper, we shall denote by C a generic fixed strictly positive constant, whose value can change from line to line.

2. Preliminaries and Main Result

In this section, we will be concerned with some preliminary notation and assumptions, as well as with the rigorous setting of our main model, involving the simpler case of adsorption and chemical reactions. As already mentioned in the Introduction, our main result provides the effective behavior of the reactive flows (see Sec. 2.3).

2.1. Notation and assumptions

2.1.1. The geometry of the problem

Let Ω be a smooth bounded connected open subset of \mathbb{R}^n ($n \geq 3$) and let $Y = [0, l_1[\times \dots \times [0, l_n[$ be the representative cell in \mathbb{R}^n . Denote by F an open subset of Y with smooth boundary ∂F such that $\bar{F} \subset Y$. We shall refer to F as being *the elementary hole*. Also, let $[0, T]$, with $T > 0$, be the time interval of interest.

Let ε be a real parameter taking values in a sequence of positive numbers converging to zero. For each ε and for any integer vector $k \in \mathbb{Z}^n$, set T_k^ε the translated image of εF by the vector $kl = (k_1 l_1, \dots, k_n l_n)$:

$$F_k^\varepsilon = \varepsilon(kl + F).$$

The set F_k^ε represents the holes in \mathbb{R}^n . Also, let us denote by F^ε the set of all the holes contained in Ω , i.e.

$$F^\varepsilon = \bigcup \{F_k^\varepsilon \mid \bar{F}_k^\varepsilon \subset \Omega, k \in \mathbb{Z}^n\}.$$

Set

$$\Omega^\varepsilon = \Omega \setminus \bar{F}^\varepsilon.$$

Hence, Ω^ε is a periodically perforated domain with holes of size of the same order as the period. Remark that the holes do not intersect the boundary $\partial\Omega$.

Let

$$S^\varepsilon = \bigcup \{\partial F_k^\varepsilon \mid \bar{F}_k^\varepsilon \subset \Omega, k \in \mathbb{Z}^n\}.$$

So

$$\partial\Omega^\varepsilon = \partial\Omega \cup S^\varepsilon.$$

We shall also use the following notations:

$|\omega|$ = the Lebesgue measure of any measurable subset ω of \mathbb{R}^n ,

χ_ω = the characteristic function of the set ω ,

$$Y^* = Y \setminus \bar{F}$$

and

$$\theta = \frac{|Y^*|}{|Y|}. \tag{2.1}$$

2.1.2. *Function spaces and norms*

In the sequel, we shall use the following notations:

$$H = L^2(\Omega),$$

with the classical scalar product and norm:

$$(u, v)_\Omega = \int_\Omega u(x)v(x)dx, \quad \|u\|_\Omega^2 = (u, u)_\Omega,$$

$$\mathcal{H} = L^2(0, T; H),$$

with

$$(u, v)_{\Omega, T} = \int_0^T (u(t), v(t))_\Omega dt,$$

where $u(t) = u(t, \cdot)$, $v(t) = v(t, \cdot)$, $\|u\|_{\Omega, T}^2 = (u, u)_{\Omega, T}$,

$$V = H^1(\Omega),$$

with

$$(u, v)_V = (u, v)_\Omega + (\nabla u, \nabla v)_\Omega,$$

$$\mathcal{V} = L^2(0, T; V),$$

with

$$(u, v)_\mathcal{V} = \int_0^T (u(t), v(t))_V dt,$$

$$\mathcal{W} = \left\{ v \in \mathcal{V} \left| \frac{dv}{dt} \in \mathcal{V}' \right. \right\} \quad \text{where } \mathcal{V}' \text{ is the dual space of } \mathcal{V},$$

$$\mathcal{V}_0 = \{ v \in \mathcal{V} | v = 0 \text{ on } \partial\Omega \text{ a.e. on } (0, T) \},$$

$$\mathcal{W}_0 = \mathcal{V}_0 \cap \mathcal{W}.$$

Similarly, we define the spaces $V(\Omega^\varepsilon)$, $\mathcal{V}(\Omega^\varepsilon)$, $V(S^\varepsilon)$ and $\mathcal{V}(S^\varepsilon)$. For the latter we write

$$\langle u, v \rangle_{S^\varepsilon} = \int_{S^\varepsilon} g^\varepsilon u v d\sigma,$$

where g^ε is the metric tensor on S^ε ; the rule of partial integration on S^ε applies and, if we denote the gradient on S^ε by ∇^ε and the Laplace–Beltrami operator on S^ε by Δ^ε , we have

$$-(\Delta^\varepsilon u, v)_{S^\varepsilon} = \langle \nabla^\varepsilon u, \nabla^\varepsilon v \rangle_{S^\varepsilon}.$$

Also, for the space of test functions we use the notation

$$\mathcal{D} = C_0^\infty((0, T) \times \Omega).$$

Moreover, for an arbitrary function $\psi \in L^2(\Omega^\varepsilon)$, we shall denote by $\tilde{\psi}$ its extension by zero inside the holes:

$$\tilde{\psi} = \begin{cases} \psi & \text{in } \Omega^\varepsilon, \\ 0 & \text{in } \Omega \setminus \bar{\Omega}^\varepsilon. \end{cases}$$

and, also, for any open subset $D \subset \mathbb{R}^n$ and for any function $g \in L^1(D)$, we set

$$\mathcal{M}_D(g) = \frac{1}{|D|} \int_D g dx. \tag{2.2}$$

2.2. Setting of the problem

As already mentioned, we are interested in studying the behavior of the solution $u^\varepsilon = (v^\varepsilon, w^\varepsilon)$, in such a perforated domain, of the following problem:

$$(V^\varepsilon) \begin{cases} \frac{\partial v^\varepsilon}{\partial t}(t, x) - D \Delta v^\varepsilon(t, x) = h(t, x), & x \in \Omega^\varepsilon, t > 0, \\ v^\varepsilon(t, x) = 0, & x \in \partial\Omega, t > 0, \\ v^\varepsilon(t, x) = v_1(x), & x \in \Omega^\varepsilon, t = 0, \end{cases} \tag{2.3}$$

$$-D \frac{\partial v^\varepsilon}{\partial \nu}(t, x) = \varepsilon f^\varepsilon(t, x), \quad x \in S^\varepsilon, t > 0, \tag{2.4}$$

and

$$(W^\varepsilon) \begin{cases} \frac{\partial w^\varepsilon}{\partial t}(t, x) + a w^\varepsilon(t, x) = f^\varepsilon(t, x), & x \in S^\varepsilon, t > 0, \\ w^\varepsilon(t, x) = w_1(x), & x \in S^\varepsilon, t = 0, \end{cases} \tag{2.5}$$

where

$$f^\varepsilon(t, x) = \gamma(g(v^\varepsilon(t, x)) - w^\varepsilon(t, x)). \tag{2.6}$$

Here, ν is the exterior unit normal to Ω^ε , $a, \gamma > 0$, $h \in \mathcal{H}$, $v_1, w_1 \in H_0^1(\Omega)$ and S^ε is the boundary of our porous medium $\Omega \setminus \bar{\Omega}^\varepsilon$. Moreover, the fluid is assumed to be homogeneous and isotropic, with a constant diffusion coefficient $D > 0$.

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The function g in (2.6) is assumed to be given. Two situations will be considered; the case in which g is a monotone smooth function satisfying the condition $g(0) = 0$ and the case of a maximal monotone graph with $g(0) = 0$, i.e. the case in which g is the subdifferential of a convex lower semicontinuous function G . These two general situations are well illustrated by the following important practical examples:

$$(a) \quad g(v) = \frac{\alpha v}{1 + \beta v}, \quad \alpha, \beta > 0 \quad (\text{Langmuir kinetics})$$

and

$$(b) \quad g(v) = |v|^{p-1}v, \quad 0 < p < 1 \quad (\text{Freundlich kinetics}).$$

The existence and uniqueness of a weak solution of the system (2.3)–(2.6) can be settled by using the classical theory of semilinear monotone problems (see, for instance, Refs. 7 and 20).

We shall first consider the case in which g is a continuously differentiable function, monotonously nondecreasing and such that $g(v) = 0$ iff $v = 0$. Also, we shall suppose that there exist a positive constant C and an exponent q , with $0 \leq q < n/(n - 2)$, such that

$$\left| \frac{\partial g}{\partial v} \right| \leq C(1 + |v|^q). \tag{2.7}$$

As we will see later on this hypothesis concerning the smoothness of the nonlinearity g is not fundamental and can be easily overcome by using a regularization technique, for example Yosida approximation (see Sec. 3.4).

The weak formulation of problem (2.3)–(2.6) is:

$$\left\{ \begin{array}{l} \text{find } v^\varepsilon \in \mathcal{W}_0(\Omega^\varepsilon), v^\varepsilon(0) = v_1|_{\Omega^\varepsilon} \text{ such that} \\ - \left(v^\varepsilon, \frac{d\varphi}{dt} \right)_{\Omega^\varepsilon, T} + \varepsilon(f^\varepsilon, \varphi)_{\Omega^\varepsilon, T} = -D(\nabla v^\varepsilon, \nabla \varphi)_{\Omega^\varepsilon, T} + (h, \varphi)_{\Omega^\varepsilon, T}, \\ \forall \varphi \in \mathcal{W}_0(\Omega^\varepsilon) \end{array} \right. \tag{2.8}$$

and

$$\left\{ \begin{array}{l} \text{find } w^\varepsilon \in \mathcal{W}(S^\varepsilon), w^\varepsilon(0) = w_1|_{S^\varepsilon} \text{ such that} \\ - \left(w^\varepsilon, \frac{d\varphi}{dt} \right)_{S^\varepsilon, T} + a(w^\varepsilon, \varphi)_{S^\varepsilon, T} = (f^\varepsilon, \varphi)_{S^\varepsilon, T}, \quad \forall \varphi \in \mathcal{W}(S^\varepsilon). \end{array} \right. \tag{2.9}$$

By classical existence results (see Refs. 7 and 20), there exists a unique weak solution $u^\varepsilon = (v^\varepsilon, w^\varepsilon)$ of the system (2.8)–(2.9).

Remark 2.1. Let us notice that the solution of (2.5) can be written as

$$w^\varepsilon(t, x) = w_1(x)e^{-(a+\gamma)t} + \gamma \int_0^t e^{-(a+\gamma)(t-s)} g(v^\varepsilon(s, x)) ds \tag{2.10}$$

or, if we denote by \star the convolution with respect to time, as

$$w^\varepsilon(\cdot, x) = w_1(x)e^{-(a+\gamma)t} + \gamma r(\cdot) \star g(v^\varepsilon(\cdot, x)), \tag{2.11}$$

where

$$r(\rho) = e^{-(a+\gamma)\rho}.$$

The solution v^ε of problem (V^ε) being defined only on Ω^ε , we need to extend it to the whole of Ω to be able to state the convergence result. In order to do that, let us recall the following well-known extension result^{10,12}:

Lemma 2.2. *There exists a linear continuous extension operator $P^\varepsilon \in \mathcal{L}(L^2(\Omega^\varepsilon); L^2(\Omega)) \cap \mathcal{L}(V^\varepsilon; H_0^1(\Omega))$ and a positive constant C , independent of ε , such that*

$$\|P^\varepsilon v\|_{L^2(\Omega)} \leq C\|v\|_{L^2(\Omega^\varepsilon)}$$

and

$$\|\nabla P^\varepsilon v\|_{L^2(\Omega)} \leq C\|\nabla v\|_{L^2(\Omega^\varepsilon)},$$

for any $v \in V^\varepsilon$, where

$$V^\varepsilon = \{v \in H^1(\Omega^\varepsilon) | v = 0 \text{ on } \partial\Omega\},$$

with

$$\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(\Omega^\varepsilon)}.$$

An immediate consequence of the previous lemma is the following Poincaré’s inequality in V^ε :

Lemma 2.3. *There exists a positive constant C , independent of ε , such that*

$$\|v\|_{L^2(\Omega^\varepsilon)} \leq C\|\nabla v\|_{L^2(\Omega^\varepsilon)},$$

for any $v \in V^\varepsilon$.

We also recall the following well-known result (see Ref. 11):

Lemma 2.4. *There exists a positive constant C , independent of ε , such that*

$$\|v\|_{L^2(S^\varepsilon)}^2 \leq C(\varepsilon^{-1}\|v\|_{L^2(\Omega^\varepsilon)}^2 + \varepsilon\|\nabla v\|_{L^2(\Omega^\varepsilon)}^2), \tag{2.12}$$

for any $v \in V^\varepsilon$.

2.3. The main result

The main result of this paper is the following:

Theorem 2.5. *One can construct an extension $P^\varepsilon v^\varepsilon$ of the solution v^ε of the problem (V^ε) such that*

$$P^\varepsilon v^\varepsilon \rightharpoonup v \quad \text{weakly in } \mathcal{V},$$

where v is the unique solution of the following limit problem:

$$\begin{cases} \frac{\partial v}{\partial t}(t, x) + F_0(t, x) - D \sum_{i,j=1}^n q_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j}(t, x) = h(t, x), & t > 0, x \in \Omega, \\ v(t, x) = 0, & t > 0, x \in \partial\Omega, \\ v(t, x) = v_1(x), & t = 0, x \in \Omega, \end{cases} \quad (2.13)$$

with

$$F_0(t, x) = \frac{|\partial F|}{|Y^*|} \gamma [g(v(t, x)) - w_1(x)e^{-(a+\gamma)t} - \gamma r(\cdot) \star g(v(\cdot, x))(t)].$$

In (2.13), $Q = ((q_{ij}))$ is the classical homogenized matrix, whose entries are defined as follows:

$$q_{ij} = \delta_{ij} + \frac{1}{|Y^*|} \int_{Y^*} \frac{\partial \chi_j}{\partial y_i} dy \quad (2.14)$$

in terms of the functions $\chi_i, i = 1, \dots, n$, solutions of the so-called cell problems

$$\begin{cases} -\Delta \chi_i = 0 \text{ in } Y^*, \\ \frac{\partial(\chi_i + y_i)}{\partial \nu} = 0 \text{ on } \partial F, \\ \chi_i \text{ } Y\text{-periodic.} \end{cases} \quad (2.15)$$

The constant matrix Q is symmetric and positive-definite. Moreover, the limit problem for the surface concentration is:

$$\begin{cases} \frac{\partial w}{\partial t}(t, x) + (a + \gamma)w(t, x) = \gamma g(v(t, x)), & t > 0, x \in \Omega, \\ w(t, x) = w_1(x), & t = 0, x \in \Omega \end{cases} \quad (2.16)$$

and obviously, w can be written as

$$w(t, x) = w_1(x)e^{-(a+\gamma)t} + \gamma r(t) \star g(v(t, x)). \quad (2.17)$$

Remark 2.6. The weak formulation of problem (2.13) is:

$$\begin{cases} \text{find } v \in \mathcal{W}_0(\Omega), v(0) = v_1 \text{ such that} \\ - \left(v, \frac{d\varphi}{dt} \right)_{\Omega, T} + (F_0, \varphi)_{\Omega, T} = -D(Q \nabla v, \nabla \varphi)_{\Omega, T} + (h, \varphi)_{\Omega, T} \\ \forall \varphi \in \mathcal{W}_0(\Omega). \end{cases} \quad (2.18)$$

3. Proof of the Main Result

3.1. Uniqueness of the limit problem (2.18)

The proof of Theorem 2.5 consists of several different steps, the first one being to prove the uniqueness of the limit problem (2.18). This is stated in the following:

Proposition 3.1. *There is at most one solution of the weak problem (2.18).*

Proof. Let us suppose that there exist two solutions $v^1, v^2 \in \mathcal{W}_0(\Omega)$ of the weak problem (2.18) and let $v = v^1 - v^2$. Denote

$$\alpha = \gamma \frac{|\partial F|}{|Y^*|}.$$

Obviously $v \in \mathcal{W}_0(\Omega)$, $v(0) = 0$ and v satisfies

$$\begin{cases} \left(\frac{\partial v}{\partial t}, \varphi \right)_\Omega + \alpha(g(v^1) - g(v^2), \varphi)_\Omega \\ -\alpha\gamma(r \star (g(v^1) - g(v^2)), \varphi)_\Omega = -D(Q\nabla v, \nabla \varphi)_\Omega, \end{cases} \quad (3.1)$$

a.e. on $(0, T)$ and for all $\varphi \in \mathcal{V}_0(\Omega)$. We intent to prove that $v = 0$. Taking v as a test function in (3.1), we get

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_\Omega^2 + \alpha(g(v^1) - g(v^2), v)_\Omega - \alpha\gamma(r \star (g(v^1) - g(v^2)), v)_\Omega = -D(Q\nabla v, \nabla v)_\Omega.$$

Therefore

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_\Omega^2 + \alpha(g(v^1) - g(v^2), v^1 - v^2)_\Omega \leq \alpha\gamma(r \star (g(v^1) - g(v^2)), v^1 - v^2)_\Omega.$$

Integrating from 0 to t and taking into account the assumptions we have made for g , we obtain

$$\frac{1}{2} \|v(t)\|_\Omega^2 \leq \alpha\gamma \int_0^t \left[\int_\Omega (r \star (g(v^1) - g(v^2)))(s)(v^1 - v^2)(s) dx \right] ds. \quad (3.2)$$

Let us evaluate now the right-hand side of (3.2). We have

$$\begin{aligned} & \left| \int_\Omega \left(\int_0^s r(s - \sigma)(g(v^1(\sigma, x)) - g(v^2(\sigma, x))) d\sigma \right) (v^1 - v^2)(s, x) dx \right| \\ & \leq C \int_0^s \|(g(v^1) - g(v^2))(\sigma)\|_\Omega \|v(s)\|_\Omega d\sigma \\ & \leq C \left\{ \frac{1}{2} \int_0^s \|(g(v^1) - g(v^2))(\sigma)\|_\Omega^2 + \frac{s}{2} \|v(s)\|_\Omega^2 \right\}. \end{aligned}$$

Integrating over s between 0 and t , we have

$$\begin{aligned} & \int_0^t \left| \int_\Omega \left(\int_0^s r(s - \sigma)(g(v^1(\sigma, x)) - g(v^2(\sigma, x))) d\sigma \right) (v^1 - v^2)(s, x) dx \right| ds \\ & \leq C \left\{ \int_0^t \int_0^s \|(g(v^1) - g(v^2))(\sigma)\|_\Omega^2 d\sigma ds + s \|v(s)\|_\Omega^2 ds \right\} \\ & = C \left\{ \int_0^t \int_\sigma^t ds \|(g(v^1) - g(v^2))(\sigma)\|_\Omega^2 d\sigma + \int_0^t s \|v(s)\|_\Omega^2 ds \right\} \end{aligned}$$

$$\begin{aligned} &\leq C \left\{ \int_0^t \int_{\sigma}^t ds \| (v^1 - v^2)(\sigma) \|_{\Omega}^2 d\sigma + \int_0^t s \| v(s) \|_{\Omega}^2 ds \right\} \\ &\leq Ct \int_0^t \| v(s) \|_{\Omega}^2 ds, \quad \forall t \leq T. \end{aligned}$$

Finally, from (3.2) we obtain

$$\| v(t) \|_{\Omega}^2 \leq C \int_0^t \| v(s) \|_{\Omega}^2 ds$$

and hence, using Gronwall’s inequality, we get $v \equiv 0$. □

3.2. A priori estimates

The second step of the proof of Theorem 2.5 consists of describing the effective behavior of v^ε and w^ε , as $\varepsilon \rightarrow 0$. To this end, some *a priori* estimates on these solutions are required.

Proposition 3.2. *Let v^ε and w^ε be the solutions of the problem (2.3)–(2.6). There exists a positive constant C , independent of ε , such that*

$$\| w^\varepsilon(t) \|_{S^\varepsilon}^2 \leq \left(\| w^\varepsilon(0) \|_{S^\varepsilon}^2 + \frac{\gamma}{\delta} \| g(v^\varepsilon) \|_{S^\varepsilon,t}^2 \right) e^{\gamma\delta t}, \quad \forall t \geq 0, \quad \forall \delta > 0, \quad (3.3)$$

$$\left\| \frac{\partial w^\varepsilon}{\partial t} \right\|_{S^\varepsilon,t}^2 \leq C (\| w^\varepsilon(0) \|_{S^\varepsilon}^2 + \| g(v^\varepsilon) \|_{S^\varepsilon,t}^2), \quad \forall t \geq 0, \quad (3.4)$$

$$\| v^\varepsilon(t) \|_{\Omega^\varepsilon}^2 \leq C, \quad \| \nabla v^\varepsilon(t) \|_{\Omega^\varepsilon,t}^2 \leq C \quad (3.5)$$

and

$$\left\| \frac{\partial v^\varepsilon}{\partial t}(t) \right\|_{\Omega^\varepsilon}^2 \leq C. \quad (3.6)$$

Proof. From (2.5) we obtain

$$\int_{S^\varepsilon} \frac{\partial w^\varepsilon}{\partial t} w^\varepsilon d\sigma + \int_{S^\varepsilon} (a + \gamma)(w^\varepsilon)^2 d\sigma = \int_{S^\varepsilon} \gamma g(v^\varepsilon) w^\varepsilon d\sigma.$$

Therefore

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \| w^\varepsilon \|_{S^\varepsilon}^2 + (a + \gamma) \| w^\varepsilon \|_{S^\varepsilon}^2 &\leq \gamma \| g(v^\varepsilon) \|_{S^\varepsilon} \| w^\varepsilon \|_{S^\varepsilon} \\ &\leq \gamma \left(\frac{\| g(v^\varepsilon) \|_{S^\varepsilon}^2}{2\delta} + \frac{\delta}{2} \| w^\varepsilon \|_{S^\varepsilon}^2 \right), \quad \forall \delta > 0. \end{aligned}$$

Integrating with respect to t and using Gronwall’s inequality, we have

$$\| w^\varepsilon(t) \|_{S^\varepsilon}^2 \leq \left(\| w^\varepsilon(0) \|_{S^\varepsilon}^2 + \frac{\gamma}{\delta} \| g(v^\varepsilon) \|_{S^\varepsilon,t}^2 \right) e^{\gamma\delta t}, \quad \forall t \geq 0, \quad \forall \delta > 0. \quad (3.7)$$

In a similar manner we can obtain

$$\left\| \frac{\partial w^\varepsilon}{\partial t} \right\|_{S^\varepsilon, t}^2 \leq C(\|w^\varepsilon(0)\|_{S^\varepsilon}^2 + \|g(v^\varepsilon)\|_{S^\varepsilon, t}^2), \quad \forall t \geq 0. \tag{3.8}$$

Let us now prove (3.5). Multiplying (2.3) by v^ε , using (2.4)–(2.6) and integrating over Ω^ε , we have

$$\int_{\Omega^\varepsilon} \frac{\partial v^\varepsilon}{\partial t} v^\varepsilon dx + D \int_{\Omega^\varepsilon} \nabla v^\varepsilon \nabla v^\varepsilon dx + \varepsilon \int_{S^\varepsilon} \gamma(g(v^\varepsilon) - w^\varepsilon) v^\varepsilon d\sigma = \int_{\Omega^\varepsilon} h v^\varepsilon dx.$$

Therefore

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|v^\varepsilon\|_{\Omega^\varepsilon}^2 + D \|\nabla v^\varepsilon\|_{\Omega^\varepsilon}^2 + \varepsilon \gamma \int_{S^\varepsilon} g(v^\varepsilon) v^\varepsilon d\sigma \\ &= \varepsilon \gamma \int_{S^\varepsilon} w^\varepsilon v^\varepsilon d\sigma + \int_{\Omega^\varepsilon} h v^\varepsilon dx \leq \varepsilon \gamma \|w^\varepsilon\|_{S^\varepsilon} \|v^\varepsilon\|_{S^\varepsilon} + C \|v^\varepsilon\|_{\Omega^\varepsilon} \\ &\leq \varepsilon \gamma \|w^\varepsilon\|_{S^\varepsilon} \|v^\varepsilon\|_{S^\varepsilon} + \frac{D}{2} \|\nabla v^\varepsilon\|_{\Omega^\varepsilon}^2 + C. \end{aligned}$$

Hence

$$\frac{1}{2} \frac{d}{dt} \|v^\varepsilon\|_{\Omega^\varepsilon}^2 + \frac{D}{2} \|\nabla v^\varepsilon\|_{\Omega^\varepsilon}^2 + \varepsilon \gamma \int_{S^\varepsilon} g(v^\varepsilon) v^\varepsilon d\sigma \leq \varepsilon \gamma \|w^\varepsilon\|_{S^\varepsilon} \|v^\varepsilon\|_{S^\varepsilon} + C.$$

Using Young’s inequality and (3.3), we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|v^\varepsilon\|_{\Omega^\varepsilon}^2 + \frac{D}{2} \|\nabla v^\varepsilon\|_{\Omega^\varepsilon}^2 + \varepsilon \gamma \int_{S^\varepsilon} g(v^\varepsilon) v^\varepsilon d\sigma \\ &\leq C_1 \varepsilon \|w^\varepsilon\|_{S^\varepsilon}^2 + C_2 \varepsilon \|v^\varepsilon\|_{S^\varepsilon}^2 + C \\ &\leq C_3 \varepsilon \|w^\varepsilon(0)\|_{S^\varepsilon}^2 + C_4 \varepsilon \|g(v^\varepsilon)\|_{S^\varepsilon}^2 + C_2 \varepsilon \|v^\varepsilon\|_{S^\varepsilon}^2 + C. \end{aligned}$$

Using Lemma 2.4 and our hypotheses for g and w_1 , we easily get

$$\frac{1}{2} \frac{d}{dt} \|v^\varepsilon\|_{\Omega^\varepsilon}^2 + \frac{D}{2} \|\nabla v^\varepsilon\|_{\Omega^\varepsilon}^2 + \varepsilon \gamma \int_{S^\varepsilon} g(v^\varepsilon) v^\varepsilon d\sigma \leq C_5 \|v^\varepsilon\|_{\Omega^\varepsilon}^2 + C_0 \varepsilon^2 \|\nabla v^\varepsilon\|_{\Omega^\varepsilon}^2 + K.$$

Then

$$\frac{1}{2} \frac{d}{dt} \|v^\varepsilon\|_{\Omega^\varepsilon}^2 + \left(\frac{D}{2} - C_0 \varepsilon^2 \right) \|\nabla v^\varepsilon\|_{\Omega^\varepsilon}^2 \leq C_5 \|v^\varepsilon\|_{\Omega^\varepsilon}^2 + K.$$

Integrating with respect to time, we obtain

$$\|v^\varepsilon(t)\|_{\Omega^\varepsilon}^2 + \frac{D}{2} \|\nabla v^\varepsilon\|_{\Omega^\varepsilon, t}^2 \leq C.$$

Hence

$$\|v^\varepsilon(t)\|_{\Omega^\varepsilon}^2 \leq C \tag{3.9}$$

and

$$\|\nabla v^\varepsilon(t)\|_{\Omega^\varepsilon, t}^2 \leq C. \tag{3.10}$$

In a similar manner, we obtain

$$\left\| \frac{\partial v^\varepsilon}{\partial t}(t) \right\|_{\Omega^\varepsilon}^2 \leq C. \tag{3.11}$$

□

3.3. Limit passage

The remaining step in the proof of Theorem 2.5 will be divided into four new steps.

First step. Let $v^\varepsilon \in W_0(\Omega^\varepsilon)$ be the solution of the variational problem (2.8) and let $P^\varepsilon v^\varepsilon$ be the extension of v^ε inside the holes given by Lemma 2.2. Using our *a priori* estimates (3.5) and (3.6), we can easily see that there exists a constant C depending on T and the data, but independent of ε such that

$$\|P^\varepsilon v^\varepsilon(t)\|_\Omega + \|\nabla P^\varepsilon v^\varepsilon\|_{\Omega,t} + \|\partial_t(P^\varepsilon v^\varepsilon)(t)\|_\Omega \leq C, \tag{3.12}$$

for all $t \leq T$. Consequently, by passing to a subsequence, still denoted by $P^\varepsilon v^\varepsilon$, we can assume that there exists $v \in \mathcal{V}$ such that the following convergence properties hold:

$$P^\varepsilon v^\varepsilon \rightharpoonup v \quad \text{weakly in } \mathcal{V}, \tag{3.13}$$

$$\partial_t(P^\varepsilon v^\varepsilon) \rightharpoonup \partial_t v \quad \text{weakly in } \mathcal{H}, \tag{3.14}$$

$$P^\varepsilon v^\varepsilon \rightarrow v \quad \text{strongly in } \mathcal{H}. \tag{3.15}$$

It remains to identify the limit equation satisfied by v .

Second step. In order to get the limit equation satisfied by v , we have to pass to the limit in (2.8). For getting the limit of the second term on the left-hand side of (2.8), let us introduce, for any $h \in L^{s'}(\partial T)$, $1 \leq s' \leq \infty$, the linear form μ_h^ε on $W_0^{1,s}(\Omega)$ defined by

$$\langle \mu_h^\varepsilon, \varphi \rangle = \varepsilon \int_{S^\varepsilon} h\left(\frac{x}{\varepsilon}\right) \varphi d\sigma \quad \forall \varphi \in W_0^{1,s}(\Omega),$$

with $1/s + 1/s' = 1$. It is proved in Ref. 8 that

$$\mu_h^\varepsilon \rightarrow \mu_h \quad \text{strongly in } (W_0^{1,s}(\Omega))', \tag{3.16}$$

where

$$\langle \mu_h, \varphi \rangle = \mu_h \int_\Omega \varphi dx,$$

with

$$\mu_h = \frac{1}{|Y|} \int_{\partial F} h(y) d\sigma.$$

In the particular case in which $h \in L^\infty(\partial F)$ or even h is constant, we have

$$\mu_h^\varepsilon \rightarrow \mu_h \quad \text{strongly in } W^{-1,\infty}(\Omega).$$

In what follows, we shall denote by μ^ε the above introduced measure in the particular case in which $h = 1$. Notice that in this case μ_h becomes $\mu_1 = \frac{|\partial F|}{|Y|}$.

Moreover, if $z^\varepsilon \in H_0^1(\Omega)$ is such that $z^\varepsilon \rightharpoonup z$ weakly in $H_0^1(\Omega)$, then (see Ref. 9)

$$\langle \mu_h^\varepsilon, z^\varepsilon |_{\Omega^\varepsilon} \rangle \rightarrow \mu_h \int_{\Omega} z dx. \tag{3.17}$$

Let us prove now that for any $\varphi \in C_0^\infty(\Omega)$ and for any $z^\varepsilon \rightharpoonup z$ weakly in $H_0^1(\Omega)$, we get

$$\varphi g(z^\varepsilon) \rightharpoonup \varphi g(z) \quad \text{weakly in } W_0^{1, \bar{q}}(\Omega), \tag{3.18}$$

where

$$\bar{q} = \frac{2n}{q(n-2) + n}.$$

To prove (3.18), let us first note that

$$\sup \|\nabla g(z^\varepsilon)\|_{L^{\bar{q}}(\Omega)} < \infty. \tag{3.19}$$

Indeed, from the growth condition (2.7) imposed to g , we get

$$\begin{aligned} \int_{\Omega} \left| \frac{\partial g}{\partial x_i}(z^\varepsilon) \right|^{\bar{q}} dx &\leq C \int_{\Omega} (1 + |z^\varepsilon|^{q\bar{q}}) \left| \frac{\partial z^\varepsilon}{\partial x_i} \right|^{\bar{q}} dx \\ &\leq C \left(1 + \left(\int_{\Omega} |z^\varepsilon|^{q\bar{q}\gamma} dx \right)^{1/\gamma} \right) \left(\int_{\Omega} |\nabla z^\varepsilon|^{\bar{q}\delta} dx \right)^{1/\delta}, \end{aligned}$$

where we took γ and δ such that $\bar{q}\delta = 2$, $1/\gamma + 1/\delta = 1$ and $q\bar{q}\gamma = 2n/(n-2)$. Notice that, since $0 \leq q < n/(n-2)$, we have $\bar{q} > 1$. Now, since

$$\sup \|z^\varepsilon\|_{L^{\frac{2n}{n-2}}(\Omega)} < \infty,$$

we get immediately (3.19). Hence, to obtain (3.18), it remains only to prove that

$$g(z^\varepsilon) \rightarrow g(z) \quad \text{strongly in } L^{\bar{q}}(\Omega). \tag{3.20}$$

But this is just a consequence of the following well-known result (see Ref. 20):

Theorem 3.3. *Let $G : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function, i.e.*

- (a) *for every z the function $G(\cdot, z)$ is measurable with respect to $x \in \Omega$.*
 - (b) *for every (a.e.) $x \in \Omega$, the function $G(x, \cdot)$ is continuous with respect to z .*
- Moreover, if we assume that there exists a positive constant C such that*

$$|G(x, z)| \leq C(1 + |z|^{r/t}),$$

with $r \geq 1$ and $t < \infty$, then the map $z \in L^r(\Omega) \mapsto G(x, z(x)) \in L^t(\Omega)$ is continuous in the strong topologies.

Indeed, since

$$|g(z)| \leq C(1 + |z|^{q+1}),$$

applying the above theorem for $G(x, z) = g(z)$, $t = \bar{q}$ and $r = (2n/(n - 2)) - r'$, with $r' > 0$ such that $q + 1 < r/t$ and using the compact injection $H^1(\Omega) \hookrightarrow L^r(\Omega)$ we easily obtain (3.20).

Finally, from (3.16) (with $h = 1$) and (3.18) written for $z^\varepsilon = P^\varepsilon v^\varepsilon(t)$, we conclude

$$\langle \mu^\varepsilon, \varphi g(P^\varepsilon v^\varepsilon(t)) \rangle \rightarrow \frac{|\partial F|}{|Y|} \int_\Omega \varphi g(v(t)) dx \quad \forall \varphi \in \mathcal{D}. \tag{3.21}$$

We are now in a position to use Lebesgue’s convergence theorem. To this end, we use the above pointwise convergence, the *a priori* estimates (3.5) and the growth condition (2.7). As a result, we get

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \gamma(g(v^\varepsilon), \varphi)_{S^\varepsilon, T} = \frac{|\partial F|}{|Y|} \gamma(g(v), \varphi)_{\Omega, T}, \tag{3.22}$$

which is the desired result. This ends the second step of the proof.

Third step. Let ξ^ε be the gradient of v^ε in Ω^ε and let us denote by $\tilde{\xi}^\varepsilon$ its extension with zero to the whole of Ω , i.e.

$$\tilde{\xi}^\varepsilon = \begin{cases} \xi^\varepsilon & \text{in } \Omega^\varepsilon, \\ 0 & \text{in } \Omega \setminus \bar{\Omega}^\varepsilon. \end{cases}$$

Obviously, $\tilde{\xi}^\varepsilon$ is bounded in $(\mathcal{H}(\Omega))^n$ and hence there exists $\xi \in (\mathcal{H}(\Omega))^n$ such that

$$\tilde{\xi}^\varepsilon \rightharpoonup \xi \quad \text{weakly in } (\mathcal{H}(\Omega))^n. \tag{3.23}$$

Let us see now which is the equation satisfied by ξ . Take $\varphi \in \mathcal{D}$. From (2.8) we get

$$- \left(\chi_{\Omega^\varepsilon} P^\varepsilon v^\varepsilon, \frac{d\varphi}{dt} \right)_{\Omega, T} + D(\tilde{\xi}^\varepsilon, \nabla \varphi)_{\Omega, T} + \varepsilon (f^\varepsilon, \varphi)_{S^\varepsilon, T} = (\chi_{\Omega^\varepsilon} h, \varphi)_{\Omega, T}. \tag{3.24}$$

Now, we can pass to the limit, with $\varepsilon \rightarrow 0$, in all the terms of (3.24). For the first one, we have

$$- \lim_{\varepsilon \rightarrow 0} \left(\chi_{\Omega^\varepsilon} P^\varepsilon v^\varepsilon, \frac{d\varphi}{dt} \right)_{\Omega, T} = - \frac{|Y^*|}{|Y|} \left(v, \frac{d\varphi}{dt} \right)_{\Omega, T}. \tag{3.25}$$

For the second term we get

$$\lim_{\varepsilon \rightarrow 0} D(\tilde{\xi}^\varepsilon, \nabla \varphi)_{\Omega, T} = D(\xi, \nabla \varphi)_{\Omega, T}. \tag{3.26}$$

To get the limit of the third term, let us notice that using (3.22), we get

$$\lim_{\varepsilon \rightarrow 0} \varepsilon (g(v^\varepsilon), \varphi)_{S^\varepsilon, T} = \frac{|\partial F|}{|Y|} (g(v), \varphi)_{\Omega, T}. \tag{3.27}$$

Let us prove now that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon (w^\varepsilon, \varphi)_{S^\varepsilon, T} = \frac{|\partial F|}{|Y|} (\gamma r \star g(v), \varphi)_{\Omega, T}, \quad \forall \varphi \in \mathcal{D}. \tag{3.28}$$

Indeed, we have

$$\begin{aligned} \varepsilon(w^\varepsilon, \varphi)_{S^\varepsilon, T} &= \varepsilon \int_0^T \int_{S^\varepsilon} w^\varepsilon(t, x) \varphi(t, x) dx dt \\ &= \varepsilon \int_0^T \int_{S^\varepsilon} \left(w_1(x) e^{-(a+\gamma)t} + \int_0^t \gamma g(v^\varepsilon(s, x)) e^{-(a+\gamma)(t-s)} ds \right) \varphi(t, x) dx dt \\ &= \int_0^T e^{-(a+\gamma)t} \left[\varepsilon \int_{S^\varepsilon} w_1(x) \varphi(t, x) dx \right] dt \\ &\quad + \int_0^T \left[\int_0^t \gamma e^{-(a+\gamma)(t-s)} \left(\varepsilon \int_{S^\varepsilon} g(v^\varepsilon(s, x)) \varphi(t, x) dx \right) ds \right] dt. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon(w^\varepsilon, \varphi)_{S^\varepsilon, T} &= \lim_{\varepsilon \rightarrow 0} \left\{ \int_0^T e^{-(a+\gamma)t} \left[\varepsilon \int_{S^\varepsilon} w_1(x) \varphi(t, x) dx \right] dt \right. \\ &\quad \left. + \int_0^T \left[\int_0^t \gamma e^{-(a+\gamma)(t-s)} \left(\varepsilon \int_{S^\varepsilon} g(v^\varepsilon(s, x)) \varphi(t, x) dx \right) ds \right] dt \right\} \\ &= \int_0^T e^{-(a+\gamma)t} \left[\frac{|\partial F|}{|Y|} \int_\Omega w_1(x) \varphi(t, x) dx \right] dt \\ &\quad + \int_0^T \left[\int_0^t \gamma e^{-(a+\gamma)(t-s)} \left(\frac{|\partial F|}{|Y|} \int_\Omega g(v(s, x)) \varphi(t, x) dx \right) ds \right] dt \\ &= \int_0^T e^{-(a+\gamma)t} \left[\frac{|\partial F|}{|Y|} \int_\Omega w_1(x) \varphi(t, x) dx \right] dt \\ &\quad + \int_0^T \int_\Omega \frac{|\partial F|}{|Y|} \gamma \left[\int_0^t e^{-(a+\gamma)(t-s)} g(v(s, x)) ds \right] \varphi(t, x) dx dt \\ &= \frac{|\partial F|}{|Y|} \int_0^T \left\{ \int_\Omega [w_1(x) e^{-(a+\gamma)t} + \gamma r \star g(v(\cdot, x))(t)] \varphi(t, x) dx \right\} dt. \end{aligned}$$

From (3.27) and (3.28) we get

$$\lim_{\varepsilon \rightarrow 0} \varepsilon(f^\varepsilon, \varphi)_{S^\varepsilon, T} = \frac{|Y^*|}{|Y|} (F_0, \varphi)_{\Omega, T}, \quad \forall \varphi \in \mathcal{D}. \tag{3.29}$$

It is not difficult to pass to the limit on the right-hand side of (3.24). Since

$$\chi_{\Omega^\varepsilon} h \rightharpoonup \frac{|Y^*|}{|Y|} h \quad \text{weakly in } \mathcal{H}(\Omega),$$

we obtain

$$\lim_{\varepsilon \rightarrow 0} (\chi_{\Omega^\varepsilon} h, \varphi)_{\Omega, T} = \frac{|Y^*|}{|Y|} (h, \varphi)_{\Omega, T}. \tag{3.30}$$

Putting together (3.25)–(3.26) and (3.29)–(3.30), we have

$$-\frac{|Y^*|}{|Y|} \left(v, \frac{d\varphi}{dt} \right)_{\Omega, T} + D(\xi, \nabla\varphi)_{\Omega, T} + \frac{|Y^*|}{|Y|} (F_0, \varphi)_{\Omega, T} = \frac{|Y^*|}{|Y|} (h, \varphi)_{\Omega, T} \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Hence ξ verifies

$$\frac{|Y^*|}{|Y|} \frac{\partial v}{\partial t} - D \operatorname{div} \xi + \frac{|Y^*|}{|Y|} F_0 = \frac{|Y^*|}{|Y|} h, \quad t > 0, \quad x \in \Omega. \tag{3.31}$$

It remains now to identify ξ .

Fourth step. In order to identify ξ , we shall make use of the solutions of the cell-problems (2.15). For any fixed $i = 1, \dots, n$, let us define

$$\Phi_{i\varepsilon}(x) = \varepsilon \left(\chi_i \left(\frac{x}{\varepsilon} \right) + y_i \right) \quad \forall x \in \Omega^\varepsilon, \tag{3.32}$$

where

$$y = \frac{x}{\varepsilon}.$$

By periodicity

$$P^\varepsilon \Phi_{i\varepsilon} \rightharpoonup x_i \quad \text{weakly in } H^1(\Omega). \tag{3.33}$$

Let η_i^ε be the gradient of $\Phi_{i\varepsilon}$ in Ω^ε . Denote by $\widetilde{\eta}_i^\varepsilon$ the extension by zero of η_i^ε inside the holes.

From (3.32), for the j -component of $\widetilde{\eta}_i^\varepsilon$ we get

$$(\widetilde{\eta}_i^\varepsilon)_j = \left(\frac{\partial \widetilde{\Phi}_{i\varepsilon}}{\partial x_j} \right) = \left(\frac{\partial \widetilde{\chi}_i}{\partial y_j}(y) \right) + \delta_{ij} \chi_{Y^*}$$

and hence

$$(\widetilde{\eta}_i^\varepsilon)_j \rightharpoonup \frac{1}{|Y|} \left(\int_{Y^*} \frac{\partial \chi_i}{\partial y_j} dy + |Y^*| \delta_{ij} \right) = \frac{|Y^*|}{|Y|} q_{ij} \quad \text{weakly in } L^2(\Omega). \tag{3.34}$$

On the other hand, it is not difficult to see that η_i^ε satisfies

$$\begin{cases} -\operatorname{div} \eta_i^\varepsilon = 0 & \text{in } \Omega^\varepsilon, \\ \eta_i^\varepsilon \cdot \nu = 0 & \text{on } S^\varepsilon. \end{cases} \tag{3.35}$$

Now, let $\varphi \in \mathcal{D}$. Multiplying the first equation in (3.35) by φv^ε and integrating by parts over Ω^ε we get

$$(\eta_i^\varepsilon, \nabla \varphi v^\varepsilon)_{\Omega^\varepsilon, T} + (\eta_i^\varepsilon, \nabla v^\varepsilon \varphi)_{\Omega^\varepsilon, T} = 0.$$

So

$$(\widetilde{\eta}_i^\varepsilon, \nabla \varphi P^\varepsilon v^\varepsilon)_{\Omega, T} + (\eta_i^\varepsilon, \nabla v^\varepsilon \varphi)_{\Omega^\varepsilon, T} = 0. \tag{3.36}$$

On the other hand, taking $\varphi\Phi_{i\varepsilon}$ as a test function in (2.8) we obtain

$$\begin{aligned}
 & - \left(v^\varepsilon, \Phi_{i\varepsilon} \frac{d\varphi}{dt} \right)_{\Omega^\varepsilon, T} + D(\nabla v^\varepsilon, \nabla \varphi \Phi_{i\varepsilon})_{\Omega^\varepsilon, T} + D(\nabla v^\varepsilon, \nabla \Phi_{i\varepsilon} \varphi)_{\Omega^\varepsilon, T} \\
 & + \varepsilon(f^\varepsilon, \varphi \Phi_{i\varepsilon})_{S^\varepsilon, T} = (h, \varphi \Phi_{i\varepsilon})_{\Omega^\varepsilon, T},
 \end{aligned}$$

which, using the definitions of $\tilde{\xi}^\varepsilon$ and $\tilde{\eta}_i^\varepsilon$, gives

$$\begin{aligned}
 & - \left(\chi_{\Omega^\varepsilon} P^\varepsilon v^\varepsilon, P^\varepsilon \Phi_{i\varepsilon} \frac{d\varphi}{dt} \right)_{\Omega, T} + D(\tilde{\xi}^\varepsilon, \nabla \varphi P^\varepsilon \Phi_{i\varepsilon})_{\Omega, T} + D(\nabla v^\varepsilon, \eta_i^\varepsilon \varphi)_{\Omega^\varepsilon, T} \\
 & + \varepsilon(f^\varepsilon, \varphi \Phi_{i\varepsilon})_{S^\varepsilon, T} = (h \chi_{\Omega^\varepsilon}, \varphi P^\varepsilon \Phi_{i\varepsilon})_{\Omega, T}.
 \end{aligned}$$

Now, using (3.36), we get

$$\begin{aligned}
 & - \left(\chi_{\Omega^\varepsilon} P^\varepsilon v^\varepsilon, P^\varepsilon \Phi_{i\varepsilon} \frac{d\varphi}{dt} \right)_{\Omega, T} + D(\tilde{\xi}^\varepsilon, \nabla \varphi P^\varepsilon \Phi_{i\varepsilon})_{\Omega, T} - D(\tilde{\eta}_i^\varepsilon, \nabla \varphi P^\varepsilon \Phi_{i\varepsilon})_{\Omega, T} \\
 & + \varepsilon(f^\varepsilon, \varphi \Phi_{i\varepsilon})_{S^\varepsilon, T} = (h \chi_{\Omega^\varepsilon}, \varphi P^\varepsilon \Phi_{i\varepsilon})_{\Omega, T}. \tag{3.37}
 \end{aligned}$$

Let us pass to the limit in (3.37). For the first term we obviously have

$$- \lim_{\varepsilon \rightarrow 0} \left(\chi_{\Omega^\varepsilon} P^\varepsilon v^\varepsilon, P^\varepsilon \Phi_{i\varepsilon} \frac{d\varphi}{dt} \right)_{\Omega, T} = - \frac{|Y^*|}{|Y|} \left(v, x_i \frac{d\varphi}{dt} \right)_{\Omega, T}. \tag{3.38}$$

For the second one, using (3.23) and (3.33), we have

$$\lim_{\varepsilon \rightarrow 0} (\tilde{\xi}^\varepsilon, \nabla \varphi P^\varepsilon \Phi_{i\varepsilon})_{\Omega, T} = (\xi, \nabla \varphi x_i)_{\Omega, T}. \tag{3.39}$$

On the other hand, (3.15) and (3.34) imply that

$$\lim_{\varepsilon \rightarrow 0} (\tilde{\eta}_i^\varepsilon, \nabla \varphi P^\varepsilon v^\varepsilon)_{\Omega, T} = \frac{|Y^*|}{|Y|} (q_i, \nabla \varphi v)_{\Omega, T}, \tag{3.40}$$

where q_i is the vector having the j th-component equal to q_{ij} .

As the boundary of F is smooth, of class C^2 , $P^\varepsilon \Phi_{i\varepsilon} \in W^{1,\infty}(\Omega)$ and $P^\varepsilon \Phi_{i\varepsilon} \rightarrow x_i$ strongly in $L^\infty(\Omega)$. Then

$$\lim_{\varepsilon \rightarrow 0} \varepsilon(g(v^\varepsilon), \varphi \Phi_{i\varepsilon})_{S^\varepsilon, T} = \frac{|\partial F|}{|Y|} (g(v), \varphi x_i)_{\Omega, T} \tag{3.41}$$

and hence

$$\lim_{\varepsilon \rightarrow 0} \varepsilon(f^\varepsilon, \varphi \Phi_{i\varepsilon})_{S^\varepsilon, T} = \frac{|Y^*|}{|Y|} (F_0, \varphi x_i)_{\Omega, T}. \tag{3.42}$$

Finally, for the limit of the right-hand side of (3.37), since $\chi_{\Omega^\varepsilon} h \rightharpoonup \frac{|Y^*|}{|Y|} h$ weakly in $\mathcal{H}(\Omega)$, using again (3.33) we have

$$\lim_{\varepsilon \rightarrow 0} (h \chi_{\Omega^\varepsilon}, \varphi P^\varepsilon \Phi_{i\varepsilon})_{\Omega, T} = \frac{|Y^*|}{|Y|} (h, \varphi x_i)_{\Omega, T}. \tag{3.43}$$

Hence we get

$$\begin{aligned}
 & -\frac{|Y^*|}{|Y|} \left(v, x_i \frac{d\varphi}{dt} \right)_{\Omega, T} + D(\xi, \nabla\varphi x_i)_{\Omega, T} - D\frac{|Y^*|}{|Y|}(q_i, \nabla\varphi v)_{\Omega, T} \\
 & \quad + \frac{|Y^*|}{|Y|}(F_0, \varphi x_i)_{\Omega, T} = \frac{|Y^*|}{|Y|}(h, \varphi x_i)_{\Omega, T}. \tag{3.44}
 \end{aligned}$$

Using Green’s formula and Eq. (3.31), we have

$$-D(\xi, \nabla x_i \varphi)_{\Omega, T} + D\frac{|Y^*|}{|Y|}(q_i, \nabla v \varphi)_{\Omega, T} = 0.$$

The above equality holds true for any $\varphi \in \mathcal{D}(\Omega)$. This implies that

$$-D\xi \cdot \nabla x_i + D\frac{|Y^*|}{|Y|}q_i \cdot \nabla v = 0, \quad t > 0, \quad x \in \Omega. \tag{3.45}$$

Writing (3.45) by components, derivating with respect to x_i , summing after i and using (3.31), we conclude that

$$D\frac{|Y^*|}{|Y|} \sum_{i,j=1}^n q_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} = D \operatorname{div} \xi = \frac{|Y^*|}{|Y|} \frac{\partial v}{\partial t} + \frac{|Y^*|}{|Y|} F_0 - \frac{|Y^*|}{|Y|} h,$$

which means that v satisfies

$$\frac{\partial v}{\partial t} - D \sum_{i,j=1}^n q_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} + F_0(t, x) = h, \quad t > 0, \quad x \in \Omega.$$

Since $v \in \mathcal{W}_0(\Omega)$ (i.e. $v = 0$ on $\partial\Omega$) and v is uniquely determined, the whole sequence $P^\varepsilon v^\varepsilon$ converges to v and Theorem 2.5 is proved. \square

3.4. The case of a non-smooth boundary condition

In this section we want to cover Example (b) in the Introduction, namely the case in which the function g appearing in (1.4) is given by

$$g(v) = |v|^{p-1}v, \quad 0 < p < 1 \quad (\text{Freundlich kinetics}).$$

For this case, we see that g is a single-valued maximal monotone graph in $\mathbb{R} \times \mathbb{R}$, satisfying the condition $g(0) = 0$. Also, if we denote by $D(g)$ the domain of g , i.e. $D(g) = \{\xi \in \mathbb{R} | g(\xi) \neq \emptyset\}$, then $D(g) = \mathbb{R}$. Moreover, g is continuous and satisfies

$$|g(v)| \leq C(1 + |v|).$$

We know that in this case there exists a lower semicontinuous convex function G from \mathbb{R} to $] -\infty, +\infty]$, G proper, i.e. $G \not\equiv +\infty$ such that g is the subdifferential of G , $g = \partial G$ (G is an indefinite “integral” of g).

The main result of this section is the following:

Theorem 3.4. *One can construct an extension $P^\varepsilon v^\varepsilon$ of the solution v^ε of the problem (V^ε) such that*

$$P^\varepsilon v^\varepsilon \rightharpoonup v \quad \text{weakly in } \mathcal{V},$$

where v is the unique solution of the following limit problem:

$$\begin{cases} \frac{\partial v}{\partial t}(t, x) + F_0(t, x) - D \sum_{i,j=1}^n q_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j}(t, x) = h(t, x), & t > 0, x \in \Omega, \\ v(t, x) = 0, & t > 0, x \in \partial\Omega, \\ v(t, x) = v_1(x), & t = 0, x \in \Omega, \end{cases} \quad (3.46)$$

with

$$F_0(t, x) = \frac{|\partial F|}{|Y^*|} \gamma [g(v(t, x)) - w_1(x)e^{-(a+\gamma)t} - \gamma r(\cdot) \star g(v(\cdot, x))](t).$$

In (3.46), $Q = ((q_{ij}))$ is the classical homogenized matrix, whose entries were defined by (2.14) and (2.15). Moreover, the limit problem for the surface concentration is:

$$\begin{cases} \frac{\partial w}{\partial t}(t, x) + (a + \gamma)w(t, x) = \gamma g(v(t, x)), & t > 0, x \in \Omega, \\ w(t, x) = w_1(x), & t = 0, x \in \Omega \end{cases} \quad (3.47)$$

and obviously

$$w(t, x) = w_1(x)e^{-(a+\gamma)t} + \gamma r(t) \star g(v(t, x)). \quad (3.48)$$

Proof. To deal with this case, we can use an approximation technique, namely Yosida regularization technique. Let $\lambda > 0$ be given. We consider the approximating problems:

$$\begin{cases} \text{find } v_\lambda^\varepsilon \in \mathcal{W}_0(\Omega^\varepsilon), v_\lambda^\varepsilon(0) = v_1|_{\Omega^\varepsilon} \text{ such that} \\ - \left(v_\lambda^\varepsilon, \frac{d\varphi}{dt} \right)_{\Omega^\varepsilon, T} + \varepsilon (f_\lambda^\varepsilon, \varphi)_{\Omega^\varepsilon, T} = -D(\nabla v_\lambda^\varepsilon, \nabla \varphi)_{\Omega^\varepsilon, T} + (h, \varphi)_{\Omega^\varepsilon, T}, \\ \forall \varphi \in \mathcal{W}_0(\Omega^\varepsilon) \end{cases} \quad (3.49)$$

and

$$\begin{cases} \text{find } w_\lambda^\varepsilon \in \mathcal{W}(S^\varepsilon), w_\lambda^\varepsilon(0) = w_1|_{S^\varepsilon} \text{ such that} \\ - \left(w_\lambda^\varepsilon, \frac{d\varphi}{dt} \right)_{S^\varepsilon, T} + a(w_\lambda^\varepsilon, \varphi)_{S^\varepsilon, T} = (f_\lambda^\varepsilon, \varphi)_{S^\varepsilon, T}, \quad \forall \varphi \in \mathcal{W}(S^\varepsilon), \end{cases} \quad (3.50)$$

where

$$f_\lambda^\varepsilon = \gamma(g_\lambda(v_\lambda^\varepsilon) - w_\lambda^\varepsilon)$$

and

$$g_\lambda = \frac{I - J_\lambda}{\lambda},$$

with

$$J_\lambda = (I + \lambda \partial G)^{-1}.$$

Note that g_λ is a Lipschitz function, which is nondecreasing and satisfies the condition $g_\lambda(0) = 0$.

Problem (3.49) and (3.50) has a unique solution $(v_\lambda^\varepsilon, w_\lambda^\varepsilon)$, for every $\lambda > 0$.^{7,20} As we saw in Sec. 2, we can express w_λ^ε in terms of v_λ^ε ; therefore, it is enough to get a problem only for v_λ^ε and in what follows we shall focus our attention only on getting the limit problem for v_λ^ε .

Mollifying g_λ to make it a smooth function (see Ref. 5) and using the results of the previous section, for any $\lambda > 0$, we get

$$P^\varepsilon v_\lambda^\varepsilon \rightarrow v_\lambda \text{ strongly in } \mathcal{H}(\Omega).$$

Then, it is not difficult to see that, proving suitable *a priori* estimates (classical energy estimates) on the solutions v_λ , we can ensure, via compactness arguments (see Ref. 3), the strong convergence of v_λ , as $\lambda \rightarrow 0$, to v , the unique solution of problem (3.46). Hence

$$v_\lambda \rightarrow v \text{ strongly in } \mathcal{H}(\Omega).$$

Finally, since

$$\|P^\varepsilon v^\varepsilon - v\|_{\Omega,T} \leq \|P^\varepsilon v^\varepsilon - P^\varepsilon v_\lambda^\varepsilon\|_{\Omega,T} + \|P^\varepsilon v_\lambda^\varepsilon - v_\lambda\|_{\Omega,T} + \|v_\lambda - v\|_{\Omega,T},$$

we get the strong convergence of $P^\varepsilon v^\varepsilon$ to v in $\mathcal{H}(\Omega)$. □

Remark 3.5. The conclusion of the above theorem remains true for more general situations. It is the case of the so-called zeroth-order reactions, in which, formally, g is given by the discontinuous function $g(v) = 0$, if $v \leq 0$ and $g(v) = 1$ if $v > 0$ (see, for instance, Ref. 2). The correct mathematical treatment needs the problem to be reformulated by using the maximal monotone graph of \mathbb{R}^2 associated to the Heaviside function $\beta(v) = \{0\}$ if $v < 0$, $\beta(0) = [0, 1]$ and $\beta(v) = \{1\}$ if $v > 0$. The existence and uniqueness of a solution can be found, for instance, in Ref. 7. The solution is obtained by passing to the limit in a sequence of problems associated to a monotone sequence of Lipschitz functions approximating β and so the results of this section remain true.

4. Laplace–Beltrami Model with Oscillating Coefficients

In a similar manner we can treat the case in which the surface ∂F is physically and chemically heterogeneous and more precisely, the case in which the reaction and the adsorption coefficients a and γ , respectively, are rapidly oscillating functions, i.e.

$$a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right), \quad \gamma^\varepsilon(x) = \gamma\left(\frac{x}{\varepsilon}\right),$$

with a and γ positive functions in $W^{1,\infty}(\Omega)$ which are Y -periodic (for linear adsorption rates, see Ref. 19).

In this case, v^ε and w^ε satisfy the following system of equations:

$$(\bar{V}^\varepsilon) \begin{cases} \frac{\partial v^\varepsilon}{\partial t}(t, x) - D\Delta v^\varepsilon(t, x) = h(t, x), & x \in \Omega^\varepsilon, t > 0, \\ v^\varepsilon(t, x) = 0, & x \in \partial\Omega, t > 0, \\ v^\varepsilon(t, x) = v_1(x), & x \in \Omega^\varepsilon, t = 0, \end{cases} \quad (4.1)$$

$$-D\frac{\partial v^\varepsilon}{\partial \nu}(t, x) = \varepsilon f^\varepsilon(t, x), \quad x \in S^\varepsilon, t > 0, \quad (4.2)$$

and

$$(\bar{W}^\varepsilon) \begin{cases} \frac{\partial w^\varepsilon}{\partial t}(t, x) + a^\varepsilon(x)w^\varepsilon(t, x) = f^\varepsilon(t, x), & x \in S^\varepsilon, t > 0, \\ w^\varepsilon(t, x) = w_1(x), & x \in S^\varepsilon, t = 0, \end{cases} \quad (4.3)$$

where

$$f^\varepsilon(t, x) = \gamma^\varepsilon(x)(g(v^\varepsilon(t, x)) - w^\varepsilon(t, x)). \quad (4.4)$$

Denote

$$y = \frac{x}{\varepsilon}.$$

The main result of this section is the following one:

Theorem 4.1. *The effective behavior of v and w is governed by the following systems:*

$$\begin{cases} \frac{\partial v}{\partial t}(t, x) + G_0(t, x) - D \sum_{i,j=1}^n q_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} = h(t, x), & t > 0, x \in \Omega, \\ v(t, x) = 0 & t > 0, x \in \partial\Omega, \\ v(t, x) = v_1(x) & t = 0, x \in \Omega, \end{cases} \quad (4.5)$$

and

$$\begin{cases} \frac{\partial w}{\partial t}(t, x, y) + (a(y) + \gamma(y))w(t, x, y) = \gamma(y)g(v(t, x)), & t > 0, x \in \Omega, y \in \partial F \\ w(t, x, y) = w_1(x) & t = 0, x \in \Omega, y \in \partial F, \end{cases} \quad (4.6)$$

where

$$G_0(t, x) = \frac{1}{|Y^*|} \int_{\partial F} f_0(t, x, y) d\sigma \quad (4.7)$$

and

$$f_0 = \gamma(y)(g(v(t, x)) - w(t, x, y)). \quad (4.8)$$

In (4.5), $Q = ((q_{ij}))$ is the classical homogenized matrix, whose entries were defined by (2.14) and (2.15).

Obviously, the solution of (4.6) can be found using the method of “variation of constants”. Hence, we get

$$w(t, x, y) = w_1(x)e^{-(a(y)+\gamma(y))t} + \gamma(y) \int_0^t e^{-(a(y)+\gamma(y))(t-s)} g(v(s, x)) ds,$$

or, using the convolution notation

$$w(t, x, y) = w_1(x)e^{-(a(y)+\gamma(y))t} + \gamma(y)r(\cdot, y) \star g(v(\cdot, x))(t),$$

with

$$r(\tau, y) = e^{-(a(y)+\gamma(y))\tau}.$$

Moreover, let us notice that (4.5)–(4.8) imply that $v(t, x)$ satisfies the following equation

$$\frac{\partial v}{\partial t}(t, x) - D \sum_{i,j=1}^n q_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j}(t, x) + \bar{F}_0(t, x) = h(t, x), \quad t > 0, x \in \Omega, \quad (4.9)$$

with

$$\begin{aligned} \bar{F}_0(t, x) = & \frac{1}{|Y^\star|} \int_{\partial F} \{ \gamma(y)[g(v(t, x)) - w_1(x)e^{-(a(y)+\gamma(y))t} \\ & - \gamma(y)r(\cdot, y) \star g(v(\cdot, x))(t)] \} d\sigma. \end{aligned} \quad (4.10)$$

Proof. We shall not go into the details of the proof of this theorem, since it follows exactly the same steps in the proof of Theorem 2.5. The only difference we have to tackle is the way we treat the coefficients a^ε and γ^ε . To get rid of the difficulties coming from the fact that they are rapidly oscillating let us notice that in fact they are both uniformly bounded in $L^\infty(\Omega)$ and converge strongly therein. \square

Remark 4.2. The above adsorption model can be slightly generalized by allowing surface diffusion on S^ε . This implies that the first equation in (4.3) has to be replaced by

$$\frac{\partial w^\varepsilon}{\partial t}(t, x) - \varepsilon^2 E \Delta^\varepsilon w^\varepsilon(t, x) + a^\varepsilon(x)w^\varepsilon(t, x) = f^\varepsilon(t, x), \quad x \in S^\varepsilon, t > 0, \quad (4.11)$$

where $E > 0$ is the diffusion constant on the surface S^ε and Δ^ε is the Laplace–Beltrami operator on S^ε (see Sec. 2.1.2).

In this case, the homogenized limit is almost the same as before, the only difference being that now, instead of (4.6), we get the following local partial differential equation:

$$\begin{aligned} \frac{\partial w}{\partial t}(t, x, y) - E \Delta_y^{\partial F} w(t, x, y) + (a(y) + \gamma(y))w(t, x, y) \\ = \gamma(y)g(v(t, x)), \quad t > 0, x \in \Omega, y \in \partial F, \end{aligned} \quad (4.12)$$

where $\Delta^{\partial F}$ denotes the Laplace–Beltrami operator on ∂F and the subscript y indicates the fact that the derivatives are taken with respect to the local variable y .

Problems containing a superficial diffusion arise also in the context of climatology: see, for instance, Refs. 6 and 17.

The macroscopic behavior of these more general models appeals to some comments that we gather together below.

First of all, it is worth noticing that the bulk behavior of system (\bar{V}^ε) – (\bar{W}^ε) involves an additional microvariable y . This local phenomenon yields a more complicated microstructure of the effective medium; one can say that in Eqs. (4.5) and (4.6) x plays the role of a macroscopic variable, whereas y is a microscopic one.

Secondly, let us observe that the zeroth-order term in (4.9), namely \bar{F}_0 involves the convolution $\gamma r \star g$, which shows that we clearly have a memory term in the principal part of our diffusion–reaction equation (4.9).

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