

An Extremal Eigenvalue Problem for a Two-Phase Conductor in a Ball

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Abstract The pioneering works of Murat and Tartar (Topics in the mathematical modeling of composite materials. PNLDE 31. Birkhäuser, Basel, 1997) go a long way in showing, in general, that problems of optimal design may not admit solutions if microstructural designs are excluded from consideration. Therefore, assuming, tacitly, that the problem of minimizing the first eigenvalue of a two-phase conducting material with the conducting phases to be distributed in a fixed proportion in a given domain has no true solution in general domains, Cox and Lipton only study conditions for an optimal microstructural design (Cox and Lipton in Arch. Ration. Mech. Anal. 136:101–117, 1996). Although, the problem in one dimension has a solution (cf. Kreĭn in AMS Transl. Ser. 2(1):163–187, 1955) and, in higher dimensions, the problem set in a ball can be deduced to have a radially symmetric solution (cf. Alvino et al. in Nonlinear Anal. TMA 13(2):185–220, 1989), these existence results have been regarded so far as being exceptional owing to complete symmetry. It is still not clear why the same problem in domains with partial symmetry should fail to have a solution which does not develop microstructure and respecting the symmetry of the

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domain. We hope to revive interest in this question by giving a new proof of the result in a ball using a simpler symmetrization result from Alvino and Trombetti (J. Math. Anal. Appl. 94:328–337, 1983).

Keywords First eigenvalue · Two-phase conductors · Optimal design

1 Introduction

Let Ω be a bounded domain in \mathbb{R}^n which is to be called the design region. Let m be a positive number, $0 < m < |\Omega|$, where $|\Omega|$ is the total volume (Lebesgue measure) of the region Ω . Two materials with conductivities α and β ($0 < \alpha < \beta$) are distributed in arbitrary disjoint measurable subsets A and B , respectively, of Ω so that $A \cup B = \Omega$ and $|B| = m$. For any such distribution, it is well known (cf. [4, 8, 15]) that the first eigenvalue in the spectral problem

$$\begin{cases} -\operatorname{div}((\alpha\chi_A + \beta\chi_B)\nabla u) = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \Omega \end{cases} \quad (1.1)$$

is given by

$$\lambda^1(B) := \min_{u \in H_0^1(\Omega)} \frac{\int_{\Omega} (\alpha\chi_A + \beta\chi_B) |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx}. \quad (1.2)$$

Let $\mathcal{A} := \{B : B \subset \Omega, B \text{ measurable}, |B| = m\}$ be the class of admissible domains for the material with conductivity β . We are interested in the following eigenvalue minimization problem

$$\inf\{\lambda^1(B) : B \in \mathcal{A}\}. \quad (1.3)$$

Starting from the works of Murat and Tartar on a control problem involving immiscible fluids [19] it is well known that, generally speaking, optimal design problems may not always have a solution if the development of microstructures is not taken into consideration. However, if microstructures are allowed as admissible designs then the infimum is reached corresponding to some microstructure. In the case of our problem such an approach was followed by Cox and Lipton [11] and a characterization of the optimal microstructure has been established. Nevertheless, the original problem in the one-dimensional case and, in the case of a ball admit true solutions with symmetry as has been shown by Kreĭn [13] and Alvino et al. [1], respectively. The one-dimensional problem was solved by Kreĭn [12, 13] by exploiting the equivalence between the original problem and a similar problem for a vibrating membrane involving the objective functional

$$\lambda^1(B) := \min_{u \in H_0^1(\Omega)} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} (\alpha\chi_A + \beta\chi_B) |u|^2 dx} \quad (1.4)$$

although this equivalence does not hold in higher dimensions. The works of Cox and McLaughlin [9, 10] show that the latter problem, in any dimension, has a true

solution. It still remains to answer the question whether the original problem admits a minimum. Our aim is to revive interest in this question by giving an almost self-contained and a vastly simplified treatment of the existence result in a ball originally due to Alvino et al. [1]. The main result of our paper reads as follows.

Theorem *Let Ω denote a ball in \mathbb{R}^n . The problem (1.3) of minimizing the first eigenvalue, defined by (1.2), given two conducting materials with conductivities α, β , in given ratio, admits a radially symmetric solution.*

It is worth observing that this is a kind of Faber-Krahn inequality for non-homogeneous elliptic operators. The paper of Alvino et al. [1] treats many other problems of this kind. We refer also to Burton [5] for some results on problems of a similar nature.

Our proof of the above theorem will be based on a symmetrization result from Alvino and Trombetti [2, Lemma 1.2] whereas the original proof given in [1] is based on a more fine comparison result for the solutions of Hamilton-Jacobi equations [1, Theorem 3.1].

Plan of the paper In the next section we shall introduce some notations and recall, briefly, the Schwarz symmetrization and some basic results on the Schwarz symmetrization. The problem will then be reformulated in a way that makes it possible to apply symmetrization techniques. Subsequently, we shall discuss some of the structural properties of the objective functional and the constraint set provided in [1, 2]. We then recall a symmetrization result [2, Lemma 1.2] of which we give a different but formal proof (see Appendix) which could be adapted to domains with partial symmetry.

In Sect. 3, we shall give a proof of the main theorem (cf. Corollaries 3.1 and 3.2 [1]) with the help of the above symmetrization result and some basic properties discussed in [1, 2] instead of the more intricate [1, Theorem 3.1].

2 Notations and Preliminaries

As the results of this article concern a ball, henceforth, Ω will refer to $B(0, 1)$, the n -dimensional unit ball in \mathbb{R}^n centered at the origin. We shall use f^{-1} to denote the reciprocal of a non-vanishing real valued function f . Given a measurable function $f : \Omega \rightarrow \mathbb{R}$ and a real number c , $\Omega_{f,c}$ will denote the level set

$$\Omega_{f,c} := \{x \in \Omega : f(x) \geq c\} \tag{2.1}$$

which is a measurable subset of Ω and, of course, depends on the function f . We denote by $\Omega_{f,c}^*$ a ball concentric to Ω and having the same Lebesgue measure as $\Omega_{f,c}$.

Schwarz symmetrization The Schwarz symmetrization of the function f is a radially symmetric decreasing function f^* defined on Ω through the relation

$$f^*(z) := \sup\{c \in \mathbb{R} : z \in \Omega_{f,c}^*\}. \tag{2.2}$$

It follows from the very definition of f^* that $\{f^* \geq c\} = \Omega_{f,c}^*$ and therefore, that the functions f and its Schwarz symmetrization f^* are equimeasurable in the sense that

$$|\{x \in \Omega : f(x) \geq c\}| = |\{z \in \Omega : f^*(z) \geq c\}|. \tag{2.3}$$

Remark 2.1 The \geq sign in (2.3) can be changed to \leq without changing any of the consequences. As a consequence, the relation in (2.3) holds with the $=$ sign replacing \geq , but this cannot be taken as a characterization of equi-measurability, except when we deal with simple functions.

The equimeasurability property has several important consequences such as, for any measurable function, $h : \mathbb{R} \rightarrow \mathbb{R}$, we have:

$$\int_{\Omega} h(f(x)) dx = \int_{\Omega} h(f^*(z)) dz. \tag{2.4}$$

In particular, one has

$$\int_{\Omega} |f(x)|^2 dx = \int_{\Omega} |f^*(z)|^2 dz. \tag{2.5}$$

The following inequality is also fundamental (cf. [16, Proposition 1.2.2]) :

$$\int_E f(x) dx \leq \int_{E^*} f^*(x) dx \tag{2.6}$$

for all measurable subsets $E \subset \Omega$. Another fundamental property of the Schwarz symmetrization is the iso-perimetric inequality

$$P(\{f \geq c\}) \geq P(\{f^* \geq c\}) \tag{2.7}$$

where $P(C)$ denotes the perimeter of a subset C in Ω , when it is defined.

Suitable forms of the above properties are also true of various other forms of symmetrization. An extensive treatment of the various forms of symmetrizations and their applications may be found in the monographs [3, 6, 14, 16, 20].

A reformulation of the minimization problem Let us begin by considering λ^1 , defined in (1.2), as a function of $v := \alpha \chi_{\Omega \setminus B} + \beta \chi_B$ instead of looking at it as a set function while writing $\lambda^1(v)$ for $\lambda^1(B)$. Let $\theta := \alpha \chi_{\Omega \setminus B_0} + \beta \chi_{B_0}$ be the simple function where B_0 is a ball centered at 0 having Lebesgue measure m . Note that θ is a radially symmetric and decreasing function.

Proposition 2.2 *The minimization problem (1.3) can be recast as*

$$\inf\{\lambda^1(v) : v^* = \theta\}. \tag{2.8}$$

Proof It is clear that if $v := \alpha \chi_{\Omega \setminus B} + \beta \chi_B$ for some $B \in \mathcal{A}$ then it's radially symmetric decreasing rearrangement is the function θ . We would like to establish the

converse now. By the last part of Remark 2.1 in the previous section, as θ is a simple function, if we have $\nu^* = \theta$ then ν is a simple function taking the same values as θ on sets of equal measure. In particular, $|\{x \in \Omega : \nu(x) = \beta\}| = |B_0| = m$. So, the one-one correspondence between the constraints in (1.3) and (2.8) is established. \square

In the same way, if we set $\eta(\xi) = \lambda^1(\xi^{-1})$, the minimization problem can also be written as

$$\inf\{\eta(\xi) : \xi^* = (\theta^{-1})^*\}. \tag{2.9}$$

The infimum in a minimization problem will be attained, by the direct methods of the calculus of variation, if it happens that the objective functional is lower semi-continuous and the constraint set is compact for some topology.

The constraint set in either formulation (2.8) or (2.9) is of the form

$$C(\varphi) = \{f : f^* = \varphi\} \tag{2.10}$$

given φ which is a non-negative, bounded, measurable, radially symmetric decreasing function on the ball Ω . This set is relatively compact for the weak-* topology as a subset of $L^\infty(\Omega)$ as all $f \in C(\varphi)$ have the same L^∞ norm as φ , being equimeasurable with φ and, as bounded sets in $L^\infty(\Omega)$ are weak-* compact. However, this is not closed as, in the first place, weak-* limits of simple functions need not be simple whereas, we have seen, in the arguments given in the proof of Proposition 2.2, that the Schwarz symmetrization of a simple function is also a simple function.

Remark 2.3 In general, in order to calculate the infimum, at first, the closure of the constraint set needs to be calculated with respect to a suitable topology and then, the lower semicontinuous envelope of the objective functional with respect to the same topology. In our problem, this is hard to achieve without the consideration of microstructural designs and, the results of Cox and Lipton [11] are in this spirit but lead further away from the study of a classical solution.

We now put together some observations which highlight some of the structure of the problem leading to the determination of a classical solution to our problem. A characterization of the weak-* closure of this set in $L^\infty(\Omega)$, to be denoted by $K(\varphi)$, was given by Migliaccio [18].

Proposition 2.4 *The set $K(\varphi)$ is a weak-* compact convex set characterized by the relation*

$$K(\varphi) = \left\{ f \in L^\infty(\Omega) : \int_{B(0,r)} f(x) dx \leq \int_{B(0,r)} \varphi(z) dz \forall r, \int_{\Omega} f(x) dx = \int_{\Omega} \varphi(z) dz \right\}. \tag{2.11}$$

Proposition 2.5 *The set $C(\varphi)$ is the set of extreme points of $K(\varphi)$.*

These results can be found in Alvino et al. [1, Sect. 2]. Let us now make the following simple observation.

Remark 2.6 It is quite easy to see that the above propositions continue to hold if we consider $C^s(\varphi)$ and $K^s(\varphi)$ consisting of the radially symmetric functions in $C(\varphi)$ and $K(\varphi)$, respectively.

The following proposition establishes the continuity of the first eigenvalue with respect to weak- $*$ convergence of the reciprocals of the coefficients, for radially symmetric coefficients. A similar convergence result is proved in [1, Corollary 3.2] but for minimizing sequences of the functional λ^1 . It is worth mentioning here that the objective functional λ^1 is not lower semi-continuous for the weak- $*$ convergence of the coefficients.

Proposition 2.7 *Let v_i be a sequence of radially symmetric functions in $K(\varphi)$ such that v_i^{-1} converges weakly- $*$ to a function v^{-1} as i tends to ∞ . Then, we have $\lambda^1(v_i)$ converges to $\lambda^1(v)$ as i tends to ∞ .*

Proof Let the sequence v_i and the function v satisfy the hypotheses of the proposition. We write $v_i(x) = \frac{1}{\xi_i(|x|)}$ and $v(x) = \frac{1}{\xi(|x|)}$. Then, by the hypothesis it follows that ξ_i weak- $*$ converges to ξ in $L^\infty(0, 1)$. Now, if u_i gives the minimum value in the definition of $\lambda^1(v_i)$ then it can be argued, using the Kreĭn-Rutman theorem [17], that this is radially symmetric. We may also assume that u_i is non-negative and further, normalize it so that its L^2 norm is 1. The Euler equation corresponding to the minimizing property of u_i reads

$$-\operatorname{div}(v_i \nabla u_i) = \lambda^1(v_i) u_i. \tag{2.12}$$

It can be checked from this that the sequence u_i is bounded in $H_0^1(\Omega)$ and a subsequence can be extracted converging weakly in $H_0^1(\Omega)$ to a radial function $u(x) = v(|x|)$. A further subsequence, indexed by i_k , may be extracted so that $\lambda^1(v_{i_k})$ converges to some λ as $k \rightarrow \infty$. Now, writing $u_{i_k}(x) = v_k(|x|)$, the Euler equation (2.12) in radial co-ordinates, for this subsequence, reads

$$-\left(r^{n-1} \frac{1}{\xi_{i_k}(r)} v'_k(r) \right)' = \lambda^1(v_{i_k}) r^{n-1} v_k(r). \tag{2.13}$$

By integration, we obtain

$$r^{n-1} \frac{1}{\xi_{i_k}(r)} v'_k(r) = -\lambda^1(v_{i_k}) \int_0^r s^{n-1} v_k(s) ds. \tag{2.14}$$

It can be checked that the sequence v_k converges weakly in $L^2(0, 1)$ to the function v . So, after transferring ξ_{i_k} to the right hand side of (2.14), it is possible to pass to the limit therein as $k \rightarrow \infty$ to obtain the relation

$$r^{n-1} v'(r) = -\lambda \xi(r) \int_0^r s^{n-1} v(s) ds. \tag{2.15}$$

We then divide by $\xi(r)$, differentiate with respect to r and write the equation that we obtain in original co-ordinates as

$$-\operatorname{div}(v\nabla u) = \lambda u. \tag{2.16}$$

The function u is non-zero as its L^2 norm is 1 and thus, is an eigenfunction and, being the limit of non-negative functions, is itself non-negative. So, by the Kreĭn-Rutman theorem, λ is the first eigenvalue in the above spectral problem. By the uniqueness of the limit, $\lambda = \lambda^1(v)$ it follows that the entire sequence $\lambda^1(v_i)$ converges to $\lambda^1(v)$. \square

Next, we make the observation that the objective functional λ^1 is concave in v being, by its definition, the infimum of linear functionals. It is interesting to know whether it is strictly concave in v .

In the proof of our main theorem, we shall employ the following symmetrization result, based on [2, Lemma 1.2] to limit our search for minimizers among radially symmetric functions. The older proof by Alvino et al. [1] achieves the same while it is based on a finer comparison result based for solutions of Hamilton-Jacobi equations [1, Theorem 3.1].

Proposition 2.8 *Given any $v \in C(\theta)$ and any $u \in H_0^1(\Omega)$, there exists a \tilde{v} which is radially symmetric with $\tilde{v}^{-1} \in K((\theta^{-1})^*)$ such that*

$$\int_{\Omega} v|\nabla u|^2 dx \geq \int_{\Omega} \tilde{v}|\nabla u^*|^2 dx. \tag{2.17}$$

Proof With the same hypothesis as in this proposition, the Lemma 1.2 in Alvino et al. [2] says that (2.17) holds for the radially symmetric function $\tilde{v}(z) = \varphi(C_n|z|^n)$ for φ defined below through the relation

$$\int_0^{|u \geq c|} \frac{1}{\varphi(r)} dr := \int_{\{u \geq c\}} \frac{1}{v(x)} dx \tag{2.18}$$

which holds for all $c \in \mathbb{R}$. This gives the relation

$$\int_{\Omega_{u,c}^*} \frac{1}{\tilde{v}(x)} dx = \int_{\Omega_{u,c}} \frac{1}{v(x)} dx \tag{2.19}$$

for all c real, where we recall that $\Omega_{u,c}$ is the level set of u at the level c and $\Omega_{u,c}^*$ is a ball centered at the origin having the same measure as $\Omega_{u,c}$. In particular the above identity holds on the full domain Ω . So, as $(v^{-1})^* = (\theta^{-1})^*$, by using the formula (2.4) we have,

$$\int_{\Omega} \frac{1}{\tilde{v}(x)} dx = \int_{\Omega} (\theta^{-1})^*(x) dx.$$

Once again as $(v^{-1})^* = (\theta^{-1})^*$, from the property (2.6) we obtain

$$\int_{\Omega_{u,c}} \frac{1}{v(x)} dx \leq \int_{\Omega_{u,c}^*} (\theta^{-1})^*(x) dx.$$

The above inequality combined with (2.19) gives the relation

$$\int_{\Omega_{u,c}^*} \frac{1}{\tilde{v}(x)} dx \leq \int_{\Omega_{u,c}^*} (\theta^{-1})^*(x) dx \tag{2.21}$$

for all c real. We then note that the two relations (2) and (2.21), by the characterization (2.11), imply that $\tilde{v}^{-1} \in K((\theta^{-1})^*)$. □

3 Proof of the Main Theorem

The proof of the main theorem is given in several steps.

Step 1 Let us recall that the constraint in the original problem can be written as $v \in C(\theta)$ or equivalently, as $v^{-1} \in C((\theta^{-1})^*)$. So the minimization problem reads

$$\inf\{\lambda^1(v) : v^{-1} \in C((\theta^{-1})^*)\}. \tag{3.1}$$

We shall denote by $C^s((\theta^{-1})^*)$ and $K^s((\theta^{-1})^*)$ the subset of radially symmetric functions in $C((\theta^{-1})^*)$ and $K((\theta^{-1})^*)$, respectively. We use Proposition 2.8 above to show that

$$|\inf\{\lambda^1(v) : v^{-1} \in C((\theta^{-1})^*)\} = \inf\{\lambda^1(v) : v^{-1} \in K^s((\theta^{-1})^*)\}. \tag{3.2}$$

Following Remark 2.6 we deduce that $K^s((\theta^{-1})^*)$ is the closed convex hull of $C^s((\theta^{-1})^*)$ for the weak-* topology. So, applying the continuity property in Proposition 2.7, we obtain first that

$$\inf\{\lambda^1(v) : v^{-1} \in C^s((\theta^{-1})^*)\} = \inf\{\lambda^1(v) : v^{-1} \in K^s((\theta^{-1})^*)\}.$$

So, it readily follows that

$$\inf\{\lambda^1(v) : v^{-1} \in K^s((\theta^{-1})^*)\} \geq \inf\{\lambda^1(v) : v^{-1} \in C((\theta^{-1})^*)\}. \tag{3.3}$$

To prove the reverse inequality, let $v^{-1} \in C((\theta^{-1})^*)$ be arbitrary and let u be the corresponding minimizer in the definition of $\lambda^1(v)$. Considering a $\tilde{v}^{-1} \in K^s((\theta^{-1})^*)$ and u^* associated to the pair (v, u) as given by Proposition 2.8 and using the property (2.5) we obtain

$$\begin{aligned} \lambda^1(v) &= \frac{\int_{\Omega} v |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx} \geq \frac{\int_{\Omega} \tilde{v} |\nabla u^*|^2 dx}{\int_{\Omega} |u^*|^2 dx} \geq \lambda^1(\tilde{v}) \\ &\geq \inf\{\lambda^1(v) : v^{-1} \in K^s((\theta^{-1})^*)\}. \end{aligned} \tag{3.4}$$

By the arbitrariness of v the reverse inequality to (3.3) follows.

Step 2 The inf on the right hand side of (3.2) is in fact a minimum, that is to say, the infimum value is achieved. To see this let us define a topology on the set $K := \{v : v^{-1} \in K^s((\theta^{-1})^*)\}$ by saying that v_i converges to v if and only if v_i^{-1} converges weakly- $*$ to v^{-1} in $L^\infty(\Omega)$. Then, with the knowledge that $K^s((\theta^{-1})^*)$ is a compact set for the weak- $*$ topology on $L^\infty(\Omega)$ as announced by Proposition 2.4, it follows that K is a compact set for the topology defined above. Besides, by Proposition 2.7, we know that λ^1 restricted to K is continuous for the above topology. Thus, our thesis follows.

Step 3 In the previous step, we have been able to show that the minimization problem admits a solution in a slightly enlarged class. Although, the functional λ^1 is concave, it is not clear whether the constraint set $\{v : v^{-1} \in K^s((\theta^{-1})^*)\}$ is convex. If this were so it is immediate to obtain a solution in the original class as, whenever a concave function admits a minimum over a compact convex set there is a minimizer which is an extreme point. So, in this problem, in order to show that there is a solution in the original class, we shall have to do differently as is done in Alvino et al. [1]. It can be shown that $J : v^{-1} \mapsto (\lambda^1(v))^{-1}$ is a convex map when restricted to $K^s((\theta^{-1})^*)$ (cf. [1, Corollary 3.2]). Indeed, it is shown that

$$\begin{aligned}
 J(\mu) &= \max \left\{ \int_{\Omega} \mu \left(|x|^{n-1} \int_0^{|x|} s^{n-1} v(s) ds \right)^2 dx : v \in L^2(\Omega), \int_{\Omega} v^2(x) dx \right. \\
 &\quad \left. = 1, v \text{ radial} \right\}.
 \end{aligned}
 \tag{3.5}$$

So, as the minimization problem on the right hand side of (3.2) is equivalent to maximizing the reciprocal functional J , the above mentioned convexity guarantees that there is a maximizer of J which is an extreme point of the compact convex set $K^s((\theta^{-1})^*)$ which, by Proposition 2.5 and Remark 2.6, has to belong to $C^s((\theta^{-1})^*)$. This permits us to conclude that the infimum in (3.1) is achieved for a radially symmetric function.

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Appendix

We remark that we only require Lemma 1.2 [2] in the form stated below for our applications. Now, we give a more flexible alternate proof of the same.

Proposition 4.9 *Given any $v \in C(\theta)$ and any non-negative $u \in H_0^1(\Omega)$, for \tilde{v} defined through the relation,*

$$\int_{\Omega_{u,c}^*} \frac{1}{\tilde{v}(x)} dx = \int_{\Omega_{u,c}} \frac{1}{v(x)} dx
 \tag{4.1}$$

we have

$$\int_{\Omega} v|\nabla u|^2 dx \geq \int_{\Omega} \tilde{v}|\nabla u^*|^2 dx. \tag{4.2}$$

Proof We shall make repeated use of the co-area formula (cf. formula (2.2.1) Kesavan [16])

$$\int_{\Omega} g(x) |\nabla u(x)| dx = \int_{-\infty}^{\infty} \int_{u=s} g(x) d\sigma(x) ds \tag{4.3}$$

where $d\sigma(x)$ is the surface element on the level surface $u = s$ at the point x . Applying (4.3), we obtain the identity

$$\int_{\{u \geq t\}} v(x) |\nabla u(x)|^2 dx = \int_t^{\infty} \int_{\{u=s\}} v(x) |\nabla u(x)| d\sigma(x) ds. \tag{4.4}$$

Therefore, it follows that,

$$-\frac{d}{dt} \left(\int_{\{u \geq t\}} v(x) |\nabla u(x)|^2 dx \right) = \int_{\{u=t\}} v(x) |\nabla u(x)| d\sigma(x). \tag{4.5}$$

We apply the fact that the arithmetic mean of a non-negative function is always greater than the harmonic mean, to the function $v|\nabla u|$ on the surface $\{u = t\}$ equipped with its surface measure, to conclude that

$$\begin{aligned} \int_{u=t} v(x) |\nabla u(x)| d\sigma(x) &= \left(\frac{\int_{\{u=t\}} v(x) |\nabla u(x)| d\sigma(x)}{\int_{\{u=t\}} d\sigma(x)} \right) \left(\int_{\{u=t\}} d\sigma(x) \right) \\ &\geq \left(\frac{\int_{\{u=t\}} d\sigma(x)}{\int_{\{u=t\}} \frac{1}{v(x) |\nabla u(x)|} d\sigma(x)} \right) \left(\int_{\{u=t\}} d\sigma(x) \right). \\ &= \left(\int_{\{u=t\}} \frac{1}{v(x) |\nabla u(x)|} d\sigma(x) \right)^{-1} (P(\{u \geq t\}))^2 \end{aligned} \tag{4.6}$$

$$\geq \left(\int_{\{u=t\}} \frac{1}{v(x) |\nabla u(x)|} d\sigma(x) \right)^{-1} (P(\{u^* \geq t\}))^2. \tag{4.7}$$

The last inequality above is due to the iso-perimetric inequality (2.7). Therefore, from (4.5) and (4.7) we have

$$\begin{aligned} &-\frac{d}{dt} \left(\int_{\{u \geq t\}} v(x) |\nabla u(x)|^2 dx \right) \\ &\geq \left(\int_{\{u=t\}} \frac{1}{v(x) |\nabla u(x)|} d\sigma(x) \right)^{-1} (P(\{u^* \geq t\}))^2. \end{aligned} \tag{4.8}$$

We remember that $\{u^* \geq t\}$ for $t \geq 0$ form a continuum of concentric balls, having radius r_t , whose union over $t \geq 0$ is the ball Ω . Observing that u^* is a radially symmetric function and consequently, so is $\nabla u^*(x)$, we may define a radially symmetric

function \tilde{v} as follows.

$$\tilde{v}(|x|) := \frac{\int_{\{u^*=t\}} d\sigma(x)}{\left(\int_{\{u=t\}} \frac{1}{v(x)|\nabla u(x)} d\sigma(x)\right) |\nabla u^*(x)|} \quad \text{for any } x, |x| = r_t. \tag{4.9}$$

We check, first, that \tilde{v} satisfies (4.1). To see this we use the co-area formula. We have

$$\begin{aligned} \int_{\{u^*\geq t\}} \frac{1}{\tilde{v}(x)} dx &= \int_t^\infty \int_{\{u^*=s\}} \frac{1}{\tilde{v}(x) |\nabla u^*(x)|} d\sigma(x) ds \\ &= \int_t^\infty \int_{\{u=s\}} \frac{1}{v(x) |\nabla u(x)|} d\sigma(x) ds \\ &= \int_{\{u\geq t\}} \frac{1}{v(x)} dx \end{aligned}$$

where in the penultimate expression we have plugged in (4.9). Then, (4.8) may be rewritten using \tilde{v} as

$$\begin{aligned} -\frac{d}{dt} \left(\int_{\{u\geq t\}} v(x) |\nabla u(x)|^2 dx \right) &\geq \int_{\{u^*=t\}} \tilde{v}(x) |\nabla u^*(x)| d\sigma(x) \\ &= -\frac{d}{dt} \left(\int_{\{u^*\geq t\}} \tilde{v}(x) |\nabla u^*(x)|^2 dx \right). \end{aligned} \tag{4.10}$$

Integrating (4.10) we obtain the needful. □

Remark 4.10 The definition (4.9) of the rearranged coefficient can be written entirely in terms of the coefficient v , the function u and the derivative of the corresponding distribution function $\mu_u(t) = |\{u \geq t\}| = \mu_{u^*}(t)$ as

$$\tilde{v}(|x|) := \frac{-(\mu_u)'(t)}{\left(\int_{\{u=t\}} \frac{1}{v(x)|\nabla u(x)} d\sigma(x)\right)} \quad \text{for any } x, |x| = r_t. \tag{4.11}$$

This is due to the fact that, by using the co-area formula, we have (see also [7, Lemma 4.1] for a similar result in the case of Steiner symmetrization)

$$(\mu_u)'(t) = (\mu_{u^*})'(t) = - \int_{\{u^*=t\}} \frac{1}{|\nabla u^*(x)|} d\sigma(x).$$

It is worthwhile to note from the above that the gradient of the rearranged function u^* can be written in terms of the distribution function μ_u and its derivative as

$$|\nabla u^*(x)| = -\frac{n C_n^{\frac{1}{n}} \mu_u(t)^{1-\frac{1}{n}}}{(\mu_u)'(t)} \quad \text{for any } x, |x| = r_t \tag{4.13}$$

since $P(\{u^* = t\}) = n C_n^{\frac{1}{n}} \mu_u(t)^{1-\frac{1}{n}}$, being C_n the volume of the unit sphere in \mathbb{R}^n .

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