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FOURIER HOMOGENIZATION METHOD AND THE PROPAGATION OF ACOUSTIC WAVES THROUGH A PERIODIC VORTEX ARRAY*

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Abstract. The classical problem of homogenization of elliptic operators in arbitrary domains with periodically oscillating coefficients is considered. As the period goes to zero, an asymptotic analysis of the corresponding sequence of operators is performed with the help of this new method which we call in a natural way the *Fourier homogenization method*, since it is based on the standard Fourier transform. This method offers an alternative way to view the classical results in homogenization. It works in the Fourier space and thus in a framework dual to the one used in most of the mathematical approaches to this subject.

The Fourier homogenization method is then used to derive an expression for the effective speed of sound for an acoustic wave that propagates through a background flow made up of a periodic array of vortices, in the limit of wavelength large compared with the lattice spacing. The main result is an effective speed of sound that depends on the relative orientation between wave vector and lattice. Examples in two and three dimensions are provided.

Key words. Fourier homogenization method, acoustic waves, vortex array

AMS subject classifications. 35B27, 76C99

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1. Introduction.

As is well known, homogenization process in classical examples is concerned with the study of the behavior of solutions of elliptic boundary value problems when the coefficients are periodic with small period $\varepsilon > 0$. From a more physical viewpoint, this means that we are interested in the *bulk effective constants* of a periodic heterogeneous medium as its period goes to zero. Thus posed, the homogenization process is therefore reminiscent of the problem of calculating the macroscopic properties of a periodic inhomogeneous medium from its basic structure, i.e., the microgeometry or microstructure of the medium (which, in our case, is represented by the period). For a nice introduction to this subject, the reader is referred to the book of Bensoussan, Lions, and Papanicolaou [3].

The main mathematical result says that the limit of such integrals resolves a suitable limit boundary value problem which has constant coefficients that represent what is known as homogenized medium. For the last 20 years or so, homogenization methods have proved to be powerful techniques for studying such heterogeneous media. Some of these classical tools today include Γ-convergence (these ideas have been expounded in Dal Maso [5]), multiple-scale expansions (see [3]), two-scale convergence methods (introduced by Nguetseng [11] and Allaire [1]), Bloch waves decomposition techniques (which can be found in the book by Conca, Planchard, and Vanninathan [4]), and energy methods and their variants (which were developed by Tartar [14] in large part in association with Murat [10]; see also [3]). Those readers who are

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interested in applications of homogenization theory to other branches of physics or mechanics should consult Sánchez-Palencia [13].

In this paper, we suggest a different approach based on Fourier analysis. The method works in the following way: First, the original operator is transformed to an equivalent one in the Fourier space (standard Fourier series is used to expand the coefficients of the operator and Fourier transform to decompose their integrals). Next, the Fourier transform of the integrals are expanded using a suitable two-scale expansion and the homogenized problem is finally deduced merely neglecting high-order terms in the above expansions when passing to the limit as the period ϵ goes to zero. Although the results are not new, the method offers an alternative way to view the classical results.

The Fourier homogenization method is used in this paper to study the propagation of sound through a moving, periodic background flow. It is well known that the only modification suffered by an acoustic wave propagating through a uniformly moving medium is a Doppler shift in its frequency. When the motion of the background flow is not uniform the situation can be considerably more complicated, and it has been the subject of much research since the classical work of Kraichnan [8].

An important motivation behind this effort lies in the desire to understand the propagation of sound through complex, including turbulent, flows. For reviews see Ostachev [12] and Karweit, Blanc-Benon, Juve, and Comte-Bellot [6]. As emphasized by Kerschen [7], a serious obstacle to progress in this subject is the lack of physical models for the medium through which the acoustic wave propagates. Lund [9] has suggested focusing attention on the interaction with localized vortical structures and the consequences, in the case of a flow consisting of many slender vortex filaments have been recently worked out in the case of wavelengths small compared with typical intervortex separation by Baffico, Boyer, and Lund [2]. In this paper we address the opposite limit, the wavelength large compared with intervortex separation, in the special case that the vortices are placed periodically in space.

Finally, a word about the notation adopted in this work: Summation with respect to repeated indices is understood throughout this paper.

2. Classical homogenization.

As an application along with a justification of our *Fourier method*, we are going to deduce a classical homogenization result in arbitrary domains. To announce the result, let us consider the operator

$$A \stackrel{\text{def}}{=} -\frac{\partial}{\partial y_i}\Big(a_{ij}(y)\frac{\partial}{\partial y_j}\Big),$$

where the coefficients a_{ij} are assumed to satisfy

$$\text{(2.1)}\quad \begin{cases} \text{each } a_{ij} \text{ is a } [0,\Lambda[^N\text{-periodic bounded measurable function defined on } \mathbb{R}^N, \\ \exists\alpha > 0 \quad \text{such that} \quad a_{ij}(y)\xi_i\xi_j \geq \alpha|\xi|^2 \quad \text{(ellipticity)}, \\ a_{ij} = a_{ji} \quad \forall i,j = 1,\dots,N \quad \text{(symmetry)}. \end{cases}$$

Associated with this operator, for each $\varepsilon > 0$, we consider also the operator A^ε, where

$$A^\varepsilon \stackrel{\text{def}}{=} -\frac{\partial}{\partial x_i}\Big(a^\varepsilon_{ij}(x)\frac{\partial}{\partial x_j}\Big) \quad \text{with} \quad a^\varepsilon_{ij}(x) = a_{ij}\left(\frac{x}{\varepsilon}\right).$$

From the theory of homogenization (see [3]), it is known that there is a corresponding homogenized operator A^* given by

$$A^* \stackrel{\text{def}}{=} -\frac{\partial}{\partial x_i}\Big(q_{ij}\frac{\partial}{\partial x_j}\Big).$$

The homogenized coefficients q_{ij} are constants and their definition can be found in [3, p. 17]; we will deduce it later (see (2.5)). It is known that (q_{ij}) is a symmetric, positive definite matrix: $q_{ij}\xi_i\xi_j \geq \alpha|\xi|^2\ \forall\xi \in \mathbb{R}^N$, where $\alpha > 0$ is the same constant appearing in (2.1).

To get the homogenized or effective operator A^*, for each $\epsilon > 0$, we consider the equation $A^\epsilon u^\epsilon = f$ in $\mathbb{R}^N$, where f is any given smooth function with compact support in $\mathbb{R}^N$. Since the coefficients $a^\epsilon_{ij}(x)$ are periodic with period $[0, \epsilon\Lambda[^N$, their Fourier series is

$$a^\epsilon_{ij}(x) = \sum_{\vec{n}\in\mathbb{Z}^N} a^n_{ij} e^{\imath \vec{k}^n\cdot\frac{x}{\epsilon}}, \qquad \vec{k}^n = \frac{2\pi\vec{n}}{\Lambda}.$$

Replacing this expansion of $a^\epsilon_{ij}(x)$ in the equation $A^\epsilon u^\epsilon = f$ and taking Fourier transform, we obtain

$$\widehat{A^\epsilon u^\epsilon} = \sum_{\vec{n}\in\mathbb{Z}^N} a^n_{ij} k_i \left(k_j - \frac{k^n_j}{\epsilon}\right)\widehat{u^\epsilon}\left(\vec{k} - \frac{\vec{k}^n}{\epsilon}\right) = \widehat{f}(\vec{k}),$$

where $\widehat{u^\epsilon}$ and $\widehat{f}$ are the Fourier transforms of u^ϵ and f, respectively. Let us now assume that $\widehat{A^\epsilon u^\epsilon}$ has a limit as $\epsilon \to 0$ and that this limit is the Fourier transform of the homogenized operator A^* acting on the limit u of the integrals u^ϵ. Thus the following identity holds:

$$q_{ij}k_ik_j\hat{u}(\vec{k}) = \lim_{\epsilon\to 0}\sum_{\vec{n}\in\mathbb{Z}^N} a^n_{ij}k_i\left(k_j - \frac{k^n_j}{\epsilon}\right)\widehat{u^\epsilon}\left(\vec{k} - \frac{\vec{k}^n}{\epsilon}\right). \tag{2.2}$$

Our next step consists in passing to the limit in the right-hand side of (2.2). To this end, we propose an *ansatz* for u_ϵ which takes care of periodic variations:

$$u^\epsilon(x) = u(x) + \epsilon\chi_\ell(y)\frac{\partial u}{\partial x_\ell}\Big|_{y=\frac{x}{\epsilon}} + 0(\epsilon^2), \tag{2.3}$$

where $\chi_1, \ldots, \chi_N$ are defined for $y \in [0, \Lambda[^N$ and are assumed to be periodic. Denoting by χ^n_ℓ the nth Fourier coefficient of χ_ℓ and Fourier transforming this ansatz, we get

$$\begin{cases} \widehat{u^\epsilon}(\vec{k}) = \widehat{u}(\vec{k}) + \epsilon\imath\displaystyle\sum_{\vec{n}\in\mathbb{Z}^N}\chi^n_\ell\left(k_\ell - \frac{k^n_\ell}{\epsilon}\right)\widehat{u}\left(\vec{k} - \frac{\vec{k}^n}{\epsilon}\right) + \widehat{0(\epsilon^2)}, \\ \widehat{u^\epsilon}\left(\vec{k} - \dfrac{\vec{k}^n}{\epsilon}\right) = \widehat{u}\left(\vec{k} - \dfrac{\vec{k}^n}{\epsilon}\right) + \epsilon\imath\displaystyle\sum_{\vec{m}\in\mathbb{Z}^N}\chi^m_\ell\left(k_\ell - \frac{k^n_\ell}{\epsilon} - \frac{\vec{k}^m_\ell}{\epsilon}\right)\widehat{u}\left(\vec{k} - \frac{\vec{k}^n}{\epsilon} - \frac{\vec{k}^m}{\epsilon}\right) \\ \qquad\qquad + \widehat{0(\epsilon^2)}. \end{cases} \tag{2.4}$$

Substituting (2.4) into the right-hand side of (2.3), there appears an expression with a single sum $\sum_{\vec{n}}$ which involves $\widehat{u}(\vec{k} - \frac{\vec{k}^n}{\epsilon})$ and a double sum $\sum_{\vec{n},\vec{m}}$ involving $\widehat{u}(\vec{k} -$

$\frac{\vec{k}^n}{\epsilon} - \frac{\vec{k}^m}{\epsilon}$). Here, we now use that $\hat{u}(\vec{k})$ is significantly different from zero only when $\vec{k}$ is small and conclude that in both sums the leading-order terms are those where the argument of $\hat{u}$ remains bounded as ϵ goes to zero. Therefore, as $\epsilon \to 0$, the first sum is dominated by the term for which $\vec{k}^n = 0$ (i.e., $\vec{n} = 0$) and the double sum by those for which $\vec{k}^n + \vec{k}^m = 0$ (i.e., $\vec{m} = -\vec{n}$). Using this and the fact that $\widehat{0(\epsilon^2)}$ tends to zero, we get

$$q_{ij} k_i k_j \widehat{u}(\vec{k}) = \left[a_{ij}^0 - \imath \sum_{\vec{n} \in \mathbb{Z}^N} a_{ip}^n k_p^n \chi_j^{(-n)} \right] k_i k_j \widehat{u}(\vec{k}).$$

Since $\imath a_{ip}^n k_p^n$ is the nth Fourier coefficient of $\frac{\partial a_{ip}}{\partial y_p}(y)$, this latter identity gives the following expression for the homogenized coefficients

$$q_{ij} = \frac{1}{\Lambda^N} \int\limits_{[0,\Lambda[^N} a_{ip}(y) \left[\delta_{pj} + \frac{\partial \chi_j}{\partial y_p} \right] dy \quad \forall i, j = 1, \ldots, N. \tag{2.5}$$

In addition, an easy (but tedious) calculation applying A^ϵ to the ansatz (2.3) yields

$$A^\epsilon u^\epsilon = \frac{1}{\epsilon} \left[A\chi_\ell(y) - \frac{\partial a_{i\ell}}{\partial y_i} \right] \frac{\partial u}{\partial x_\ell} + O(1).$$

Since the left-hand side of this identity has a limit as ϵ goes to zero and u varies among all solutions of the homogenized equation $A^* u = f$, we conclude that each χ_ℓ is the unique integral (defined up to an additive constant) of the following problem with periodic boundary conditions:

$$A\chi_\ell = \frac{\partial a_{i\ell}}{\partial y_i} \quad \text{in} \quad \mathbb{R}^N, \qquad \chi_\ell \text{ is } [0, \Lambda[^N\text{-periodic.}$$

Therefore, formula (2.5) coincides with the classical expression for the homogenized coefficients (see, as mentioned above, [3, p. 17]).

Remark. Observe that the homogenized operator A^* can perfectly represent the bulk effective constants of a nonisotropic medium even if $(a_{ij}(y))$ is a scalar tensor for all y.

3. Propagation of an acoustic wave through a background, periodic, vortical flow. We take an inviscid, isentropic fluid described by

$$\begin{cases} \vec{U} = \vec{u}_0 + \vec{u}' & \text{velocity,} \\ P = p_0 + p' & \text{pressure,} \\ \rho = \rho_0 + \rho' & \text{density.} \end{cases} \tag{3.1}$$

Primed quantities are small, and ρ_0 is a constant. Order zero equations are those corresponding to incompressible flow:

$$\begin{cases} \nabla \cdot \vec{u}_0 = 0, \\ \partial_t \vec{u}_0 + (\vec{u}_0 \cdot \nabla)\vec{u}_0 = -\dfrac{1}{\rho_0} \nabla p_0. \end{cases} \tag{3.2}$$

Equations to order one are

$$\begin{cases} \partial_t \rho' + (\vec{u}_0 \cdot \nabla)\rho' + \rho_0 \nabla \cdot \vec{u}' = 0, \\ \partial_t \vec{u}' + (\vec{u}_0 \cdot \nabla)\vec{u}' + (\vec{u}' \cdot \nabla)\vec{u}_0 = \dfrac{\rho'}{\rho_0^2} \nabla p_0 - \dfrac{1}{\rho_0} \nabla p'. \end{cases} \tag{3.3}$$

Taking time derivative of the first one, divergence of the second one, and subtracting gives, to leading order,

$$\partial_{tt}\rho' - \nabla^2 p' = 2\rho_0 \nabla_i \nabla_j (u_{0i} u'_j). \tag{3.4}$$

The fact that $\nabla \cdot \vec{u}_0 = 0$ has been used, as well as the estimate, obtained by assuming $\partial_t \sim \nu$ (frequency) and $\nabla \sim k$ (wave vector) for the primed quantities

$$\frac{\rho'}{\rho_0} \sim \frac{u'}{c},$$

where c is the speed of sound defined by

$$c^{-2} \equiv \frac{d\rho}{dp}.$$

In the absence of a background flow $\vec{u}_0$ it relates acoustic frequency ν and acoustic wavenumber k through $\nu = ck$, and it is assumed that $u_0 \ll c$; that is, the background flow is of low Mach number.

We thus have a wave equation for the disturbances with a right-hand side. Assume now that $\vec{u}_0$, and the corresponding p_0 are given by a periodic array of vortex disks in two dimensions. Question: Is it possible to write an "effective" wave equation in which the right-hand side is replaced by an "effective" speed of sound?

The right-hand side of (3.4) is supposed to be a small correction to the propagation of sound in the absence of a background flow, so that, to leading order, the following approximate forms of (3.3) can be used in that right-hand side:

$$\begin{cases} \partial_t \rho' + \rho_0 \nabla \cdot \vec{u}' = 0, \\ \partial_t \vec{u}' = -\dfrac{1}{\rho_0} \nabla p'. \end{cases} \tag{3.5}$$

Before getting an effective wave equation we try to get an effective dispersion relation. To this end we Fourier transform all quantities associated with the acoustic wave through the definition

$$f(\vec{x}, t) = \int (d^N k)(d\nu) e^{\imath(\vec{k}\cdot\vec{x} - \nu t)} \tilde{f}(\vec{k}, \nu).$$

Also, quantities characterizing the background flow are assumed to be periodic with period Λ, so that

$$\vec{u}_0(\vec{x}) = \sum_{\vec{n} \in \mathbb{Z}^N} \vec{u}_0^{\,n} e^{\imath \vec{k}_n \cdot \vec{x}}, \qquad \vec{k}_n = \frac{2\pi \vec{n}}{\Lambda}.$$

The basic intuition is that the period of the background flow will be small compared with the wavelength of the acoustic wave, so we expect wave vectors relevant for the acoustic wave to be much smaller than all k_n's.

Taking the Fourier transform of (3.4) and (3.5), using

$$(2\pi)^{-N} \int d^N x e^{\imath \vec{k}\cdot\vec{x}} = \delta(\vec{k})$$

and evaluating at wavenumbers $\vec{k} = \vec{k}'$, an "acoustic" wavenumber, and at $\vec{k} = \vec{k}_m$, a "background" wavenumber, we get

$$\left\{ \begin{aligned} &\left(-\frac{\nu^2}{c^2} + k'^2\right) \tilde{p}'(\vec{k}', \nu) = 2\rho_0 \sum_n [\vec{k}' \cdot \vec{u}_0^n][\vec{k}' \cdot \vec{u}'(\vec{k}' - \vec{k}_n, \nu)], \\ &\left(-\frac{\nu^2}{c^2} + k_m^2\right) \tilde{p}'(\vec{k}_m, \nu) = 2\rho_0 \sum_n [\vec{k}_m \cdot \vec{u}_0^n][\vec{k}_m \cdot \vec{u}'(\vec{k}_m - \vec{k}_n, \nu)]. \end{aligned} \right. \tag{3.6}$$

Using now (note that this relation is valid only to leading order so that we shall be allowed to keep whatever corrections we get below only to leading order)

$$-\nu \tilde{u}_j'(\vec{k}' - \vec{k}_n) = \frac{1}{\rho_0}(k_j' - k_{nj})\tilde{p}(\vec{k}' - \vec{k}_n), \tag{3.7}$$

we get a set of two equations that relate the acoustic pressure at acoustic wavenumbers $\tilde{p}'(\vec{k})$ with the acoustic pressure at background wavenumbers $\tilde{p}'(\vec{k}_n)$:

$$\left\{ \begin{aligned} &\left(-\frac{\nu^2}{c^2} + k'^2\right) \tilde{p}'(\vec{k}', \nu) = \frac{-2}{\nu} \sum_n [\vec{k}' \cdot \vec{u}_0^n][\vec{k}' \cdot (\vec{k}' - \vec{k}_n)] \tilde{p}'(\vec{k}' - \vec{k}_n, \nu), \\ &\quad \left(-\frac{\nu^2}{c^2} + |\vec{k}' - \vec{k}_m|^2\right) \tilde{p}'(\vec{k}' - \vec{k}_m, \nu) \\ &\qquad\qquad = \frac{-2}{\nu} \sum_n [(\vec{k}' - \vec{k}_m) \cdot \vec{u}_0^n][(\vec{k}' - \vec{k}_m) \cdot \vec{k}''] \tilde{p}'(\vec{k}'', \nu), \end{aligned} \right. \tag{3.8}$$

where $\vec{k}'' \equiv \vec{k}' - \vec{k}_m - \vec{k}_n$. We wish to have the acoustic pressure $\tilde{p}'$ as a function of the same argument in both sides of the equation so we can extract a dispersion relation. To this end we iterate (3.6) to obtain

$$\begin{aligned} \left(-\frac{\nu^2}{c^2} + k'^2\right) \tilde{p}'(\vec{k}', \nu) =& \frac{-2}{\nu} [\vec{k}' \cdot \vec{u}_0^0] k'^2 \tilde{p}'(\vec{k}', \nu) \\ &+ \left(\frac{2}{\nu}\right)^2 \sum_{n \neq 0} \frac{[\vec{k}' \cdot \vec{u}_0^n][\vec{k}' \cdot (\vec{k}' - \vec{k}_n)]}{-\frac{\nu^2}{c^2} + |\vec{k}' - \vec{k}_n|^2} S_n, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} S_n =& [(\vec{k}' - \vec{k}_n) \cdot \vec{u}_0^0] |\vec{k}' - \vec{k}_n|^2 \tilde{p}'(\vec{k}' - \vec{k}_n, \nu) \\ &+ \sum_{m \neq 0} [(\vec{k}' - \vec{k}_n) \cdot \vec{u}_m^0][(\vec{k}' - \vec{k}_n) \cdot (\vec{k}' - \vec{k}_n - \vec{k}_m)] \tilde{p}'(\vec{k}' - \vec{k}_n - \vec{k}_m, \nu). \end{aligned} \tag{3.10}$$

There appears an expression that involves the acoustic pressure evaluated at three different arguments: $\tilde{p}'(\vec{k}')$, which is the same one appearing in the left-hand side, $\tilde{p}'(\vec{k}' - \vec{k}_n)$ with $n \neq 0$, and $\tilde{p}'(\vec{k}' - \vec{k}_n - \vec{k}_m)$ with $n, m \neq 0$. As in section 2, we wish to argue that $\tilde{p}'(\vec{k})$ is significantly different from zero only when $\vec{k}$ is small. This is to implement the requirement that the acoustic wave will have a wavelength that is long compared with the period of the flow, and the latter will introduce small short wavelength components. If this is true, then the contribution of $\tilde{p}'(\vec{k}' - \vec{k}_n)$ with $n \neq 0$ will be negligible, and the double sum will be dominated by the term for which

$\vec{k}_m + \vec{k}_n = 0$. Using this and the fact that in the second term in the right-hand side we can use, to leading order, $\nu^2 = c^2 k'^2$, we get, remembering $\vec{k}_n \cdot \vec{u}_0^n = \vec{k}_n \cdot \vec{u}_0^{-n} = 0$,

(3.11)
$$\left(-\frac{\nu^2}{c^2} + k'^2\right) \tilde{p}'(\vec{k}') = \frac{-2}{\nu}[\vec{k}' \cdot \vec{u}_0^0] k'^2 \tilde{p}'(\vec{k}', \nu) + 4\left(\sum_n \frac{(\vec{k}' \cdot \vec{k}_n)^2 |\vec{k}' \cdot \vec{u}_0^n|^2}{|\vec{k}'|^2 |\vec{k}_n|^2 c^2}\right) \tilde{p}'(\vec{k}').$$

There are two cases to consider:

$$\begin{aligned} &\text{Case I:} && \vec{u}_0^0 \neq 0, \\ &\text{Case II:} && \vec{u}_0^0 = 0, \end{aligned} \tag{3.12}$$

according to whether the background flow has, or doesn't have, a steady component. In Case I the correction that the background flow brings to acoustic wave propagation is completely given, to the accuracy with which we have been working, by the first term in the right-hand side which is linear in the background velocity. As noted above, higher-order corrections are not accurate. In this case the dispersion relation is

$$-\frac{\nu^2}{c^2} + k'^2 = \frac{-2}{\nu}[\vec{k}' \cdot \vec{u}_0^0] k'^2,$$

which is the well-known Doppler shift

$$\nu_{\text{Doppler}} = \nu - \vec{u}_0^0 \cdot \frac{\vec{k}'}{c},$$

accurate to terms linear in the uniform background flow. Case II is more interesting. Corrections to acoustic wave propagation are described by the second term in the right-hand side of (3.11), leading to more interesting effects. The dispersion relation in this case is

$$\nu^2 = c^2 k'^2 \left(1 - 4 \sum_n \frac{(\hat{k}' \cdot \vec{k}_n)^2 |\hat{k}' \cdot \vec{u}_0^n|^2}{|\vec{k}_n|^2 c^2}\right),$$

where $\hat{k}'$ is a unit vector on the direction of $\vec{k}'$. The periodic vortex structure thus gives rise to a new, homogenized speed of sound c^*:

$$(c^*)^2 = c^2 \left(1 - 4 \sum_n \frac{(\hat{k}' \cdot \vec{k}_n)^2 |\hat{k}' \cdot \vec{u}_0^n|^2}{|\vec{k}_n|^2 c^2}\right). \tag{3.13}$$

This can also be written

$$(c^*)^2 = c^2 (1 - C_{ijrs} \hat{k}'_i \hat{k}'_j \hat{k}'_r \hat{k}'_s),$$

with

$$C_{ijrs} = 4 \sum_n \frac{k_{ni} k_{nj}}{|\vec{k}_n|^2} \frac{u_{0r}^n u_{0s}^{-n}}{c^2} \tag{3.14}$$

a tensor with the symmetries $C_{ijrs} = C_{jirs} = C_{ijsr}$.

It is interesting to note that the homogenized velocity depends on the direction of the acoustic propagation, given by $\hat{k}'$, but not on its wavelength, given by $|\vec{k}'|$. This was to be expected, since we took the leading-order correction in wavelengths large compared with the intervortex spacing. Note also that the presence of the vortex lattice breaks the isotropy of the acoustic propagation. This was also to be expected since the lattice is not isotropic.

4. Examples. A simple example of a two-dimensional divergenceless field is given by

$$\vec{u}_0 = u_1(x)\hat{y} + u_2(y)\hat{x}.$$

This flow has nonvanishing vorticity. Periodicity is imposed through

$$\begin{cases} u_1(x) = \displaystyle\sum_n u_0^{(n,0)} e^{2\pi\imath nx/a}, \\ u_2(y) = \displaystyle\sum_m u_0^{(0,m)} e^{2\pi\imath my/b}, \end{cases}$$

and the wave numbers of this periodic flow are

$$\vec{k}_{(n,m)} = \frac{2\pi n}{a}\hat{x} + \frac{2\pi m}{b}\hat{y}.$$

Consequently we have

$$\begin{cases} \vec{k}' \cdot \vec{k}_n = \dfrac{2\pi n}{a} k'_x + \dfrac{2\pi m}{b} k'_y, \\ \vec{k}' \cdot \vec{u}_0^{(n,m)} = u_0^{(m,0)} k'_x + u_0^{(0,n)} k'_y. \end{cases}$$

Writing now

$$\begin{cases} k'_x = |\vec{k}'| \cos\theta, \\ k'_y = |\vec{k}'| \sin\theta, \end{cases}$$

we have

$$(c^*)^2 = c^2 \left[1 - 4 \sum_{n,m} \frac{\left(\frac{2\pi n}{a}\cos\theta + \frac{2\pi m}{b}\sin\theta\right)^2 \left| u_0^{(m,0)}\cos\theta + u_0^{(0,n)}\sin\theta \right|^2}{\left(\left(\frac{2\pi n}{a}\right)^2 + \left(\frac{2\pi m}{b}\right)^2\right) c^2} \right].$$

Note that the correction to the speed of sound depends on the orientation of the wave vector but not on its magnitude.

More generally, a two-dimensional divergenceless velocity field can be written in terms of a velocity potential Ψ_0:

$$\begin{cases} u_{0x} = \partial_y \Psi_0, \\ u_{0y} = -\partial_x \Psi_0, \end{cases}$$

with

$$\Psi_0(x, y) = \sum_{n,m} \Psi_0^{(n,m)} e^{2\pi\imath\left(\frac{nx}{a} + \frac{my}{b}\right)}.$$

With this definition the homogenized speed of sound is

$$(c^*)^2 = c^2 \left[1 - 16\pi^2 \sum_{n,m} \frac{|\Psi_0^{(n,m)}|^2 \left(\frac{2\pi n}{a}\cos\theta + \frac{2\pi m}{b}\sin\theta\right)^2 \left(\frac{m}{b}\cos\theta - \frac{n}{a}\sin\theta\right)^2}{\left(\left(\frac{2\pi n}{a}\right)^2 + \left(\frac{2\pi m}{b}\right)^2\right) c^2} \right].$$

Similarly, in three dimensions it is possible to describe a divergenceless velocity field $\vec{u}_0$ in terms of a vector potential $\vec{A}_0$:

$$\vec{u}_0 = \nabla \times \vec{A}_0.$$

The vector potential $\vec{A}_0$ is not uniquely defined. A number of choices are possible, a popular one being $\nabla \cdot \vec{A}_0 = 0$. Imposing periodicity through

$$\vec{A}_0 = \sum_n \vec{A}_0^n e^{\imath \vec{k}_n \cdot \vec{x}},$$

we have

$$\vec{u}_0^n = \imath \vec{k}_n \times \vec{A}_0^n \qquad \text{and} \qquad \vec{k}_n \cdot \vec{A}_0^n = 0,$$

and the coefficients C_{ijrs} are obtained after substitution into (3.14).

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