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Ultrametric and Tree Potential

**Claude Dellacherie · Servet Martinez ·
Jaime San Martin**

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Abstract In this article we study which infinite matrices are potential matrices. We tackle this problem in the ultrametric framework by studying infinite tree matrices and ultrametric matrices. For each tree matrix, we show the existence of an associated symmetric random walk and study its Green potential. We provide a representation theorem for harmonic functions that includes simple expressions for any increasing harmonic function and the Martin kernel. For ultrametric matrices, we supply probabilistic conditions to study its potential properties when immersed in its minimal tree matrix extension.

Keywords Potential theory · Ultrametricity · Harmonic functions · Martin kernel

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C. Dellacherie
Laboratoire Raphaël Salem, UMR 6085, Université de Rouen, Site Colbert, 76821 Mont Saint Aignan Cedex, France
e-mail: Claude.Dellacherie@univ-rouen.fr

S. Martinez (✉) · J. San Martin
CMM-DIM; UMI 2807 CNRS-UCHILE, FCFM, Universidad de Chile, Casilla 170-3 Correo 3, Santiago, Chile
e-mail: smartine@dim.uchile.cl

J. San Martin
e-mail: jsanmart@dim.uchile.cl

1 Introduction and Basic Notation

1.1 Introduction

Here we study ultrametric and tree matrices, the associated random walks on trees, their potentials, and the exit measure on the boundary.

There exists a broad literature in this field (a complete state-of-the-art study can be found in [21]). The main difference between our work and most part of this literature is that our starting point is not a random walk on a tree but a tree matrix or, more generally, an ultrametric matrix. In this viewpoint, the random walk is constructed from the matrix, a nontrivial fact even in the finite case. Hence, most of the concepts must be expressed with respect to the matrix that turns out to be the sum of a potential and a harmonic basis. Our results are not a simple translation of well-known results from walks on trees to matrices. New phenomena appear: the formula for monotone harmonic functions and the predictable representation property of tree matrices that turns out to be the keystone for a wide class of relations including the Martin kernel at ∞.

Below we give the framework of our work and summarize some of the main results.

An ultrametric matrix $U=(U_{ij}:i,j\in I)$ is a symmetric nonnegative matrix verifying the ultrametric inequality $U_{ij}\ge \min\{U_{ik},U_{kj}\}$ for all $i,j,k\in I$. When I is finite, it was shown in [12, 23] that the inverse U^{-1} of a nonsingular ultrametric matrix U is a diagonal dominant Stieltjes matrix (see also [26]). Then, U is proportional to the Green potential of a subMarkov kernel P, that is, $U=\alpha\sum_{n\in\mathbb{N}}P^n$. Thus, $d(i,j)=1/U_{ij}$ for $i\neq j$ is an ultrametric distance, and $1/d$ is a Green potential. A similar relation happens in $\mathbb{R}^3$ between the Newtonian potential and the Euclidean distance, or in $\mathbb{R}^d$ with $d\ge 4$ when we allow an increasing function of the Euclidean distance.

Tree matrices are a special case of ultrametric matrices. They are defined by a rooted tree $(I,\mathcal{T})$ (with root r) and a strictly increasing function $w:\{|k|:k\in I\}\to\mathbb{R}_+$, where $|k|$, the level of k, is the length of the geodesic from a site k to r. The tree matrix U is defined as $U_{ij}=w_{|i\wedge j|}$ with $i\wedge j$ being the farthest vertex from r that is common to the geodesic from i and j to r. When I is finite, U is the potential of a Markov process, whose skeleton is a simple symmetric random walk on the tree, only defective at the root. Let us mention that every ultrametric matrix can be obtained by restriction of a matrix in this class (see [12]). That is, for every ultrametric matrix U, there exists a minimal extension tree matrix $\widetilde{U}$, defined on $(\widetilde{I},\widetilde{\mathcal{T}})$, such that $U=\widetilde{U}|_I$. This minimal tree $\widetilde{\mathcal{T}}$ contains all the information that is required to understand the one-step transitions of the Markov process associated to U. In fact, $P_{ij}>0$ if and only if the geodesic in $\widetilde{\mathcal{T}}$ joining i and j does not contain other points in I.

One of the purposes of this paper is to extend this study to countably infinite ultrametric and tree matrices. At this respect we use that every tree matrix U defines a natural kernel W in the boundary ∂_∞ of the tree. This class of operators was already considered in [19] and [20], where a deep study of potential properties is done, mainly in connection to dimension and capacity on the boundary. We show that W is a stochastic integral operator whose associated filtration $\mathcal{F}=(\mathcal{F}_k)$ is given by the tree structure, see Proposition 3.2. The operator W allows us to represent harmonic functions in the infinite tree (see Corollary 3.1). This representation is alternative to the well-known Martin kernel representation supplied, for example, in [8] and [30]. In Theorem 3.1 we describe the set of harmonic functions that are increasing along the branches as the functions that are convex combinations of U. Also, we characterize the set of bounded harmonic functions which are the differences of two harmonic increasing functions (see Theorem 3.2).

In the finite setting, a tree matrix U is the potential of a continuous-time Markov chain, the leaves of the tree being reflecting states (see Proposition 2.2). Nevertheless, in the infinite transient case, each column of U is the sum of a potential and a nontrivial harmonic function, as follows from relation (3.3). This last result uses two main ingredients. The first one comes from the finite-case analysis: when imposing Dirichlet boundary conditions at the boundary, a finite tree matrix is the sum of the potential matrix and a matrix whose columns generate the harmonic functions (see Proposition 2.4). The second element is the exit measure at the boundary.

We mainly consider the potential of tree matrices for Markov semigroups defective at the root, because this is natural in the finite case. However, in the transient infinite case we can reflect the process at the root as we do in Sect. 4 and by a limit procedure we can represent the Martin kernel in a similar way as for the absorbed chain, see Theorem 4.1. Also explicit computations for homogeneous trees are done, retrieving some known formulae [9, 30].

In Sect. 5 we study ultrametric matrices $U=(U_{ij}:i,j\in I)$. Under some explicit hypotheses, we associate to U a minimal tree matrix $\widetilde{U}=(\widetilde{U}_{\tilde{\imath}\tilde{\jmath}}:\tilde{\imath},\tilde{\jmath}\in\widetilde{I})$ extending it, with a natural immersion of the sites I into $\widetilde{I}$. In Theorem 5.1 we show that a canonical generator Q can be associated to U with the help of the generator $\widetilde{Q}$ associated to $\widetilde{U}$; in Theorem 5.2 it is shown that the harmonic functions defined by Q can be retrieved from the harmonic functions defined by $\widetilde{Q}$. The key hypothesis is that a random walk starting from $\widetilde{I}\setminus I$ is trapped at the cemetery or it reaches I with probability one.

We note that the main assumption on the tree $(I,\mathcal{T})$ is local finiteness. No other hypothesis is needed; in particular no kind of homogeneity is required. In this generality, the exit measure at ∞ fulfills the requirements allowing us to describe the process at the boundary. In the case of stochastic process on the p-adic field or p-adic tree (see, for example, [1–3, 18], and the references therein), there is a natural measure in the boundary, the Haar measure for the p-adic tree, or an absolutely continuous probability measure with respect to the Haar measure for the p-adic field.

In our work we use strongly the notion of a stochastic integral operator that is the natural framework in which ultrametricity appears in stochastic analysis. We recall that an operator Y acting on a space L^2 is a stochastic integral operator (see [14]) if for some filtration $\mathcal{F}=(\mathcal{F}_t)$, Y can be written as $Yf=\int_0^\infty H_t\,d\mathbb{E}(f|\mathcal{F}_t)$, where $H=(H_t)$ is an $\mathcal{F}$-predictable process. The fact that H is predictable plays a fundamental role in the analysis of W. In a previous work, the stochastic integral operators defined on countable spaces were characterized by using the relations between ultrametric matrices and filtrations (see [11]). We point out that the continuous version of ultrametric matrices needs to consider operators of the form $V=\int_0^\infty \mathbb{E}(|\mathcal{F}_t)\,dG_t$, where (G_t) is a bounded increasing and adapted process. In [13] it is shown that these operators are Markov potential kernels (see also [15]), which generalizes some of the results obtained in [7].

Finally, we mention that ultrametricity is an important tool in applied areas like: Taxonomy (see [5]); the problem of maximal flow on finite graphs, namely the theorem of Gomory-Hu (see [6]); statistical physics to explore the ultrametric Parisi solution to spin-glass models (see [29] and references therein).

1.2 Trees

Here we fix notation and recall some well-known notions on trees. Let $(I,\mathcal{T})$ be a connected, nonoriented, and locally finite tree. I is the set of sites, and $\mathcal{T}\subset I\times I$ is the set of links. Two sites i,j are neighbors if $(i,j)\in\mathcal{T}$. The set of sites with a unique neighbor is called the extremal set and is denoted by $\mathcal{E}$. The geodesic joining i and j is denoted by $geod(i,j)$, and its length is written $|i-j|$. In particular $geod(i,i)$ only contains i, and its length is 0. We assume that the tree is rooted by $r\in I$ and write $|i|=|i-r|$. We introduce the following order relation on I:

$$i\preceq j\quad \text{if } i\in geod(r,j). \tag{1.1}$$

The element $i\wedge j=\max(geod(r,i)\cap geod(r,j))$ denotes the $\preceq$-minimum between i and j. For $i\in I\setminus\{r\}$, there is a unique element i^- verifying $(i^-,i)\in\mathcal{T}$ and $i^-\preceq i$, called the predecessor of i. It verifies $|i^-|=|i|-1$. The set of successors of $i\in I$ is denoted by $S_i=\{j\in I: j^-=i\}$. This is a finite set that can be empty. By i^+ we mean a generic element of S_i, and $\mathcal{L}=\{i\in I: S_i=\emptyset\}$ is the set of leaves of the tree. We notice that $\mathcal{L}\subseteq\mathcal{E}$ and that r is the only point that can be extremal without being a leaf. The branch of the tree born at $i\in I$ is denoted by $[i,\infty)=\{j\in I: i\preceq j\}$.

Assume that I is countably infinite. Any sequence $(i_n\in I: n\in\mathbb{N})$ such that $(i_n,i_{n+1})\in\mathcal{T}$ for every $n\in\mathbb{N}=\{0,1,2,\ldots\}$ is called an infinite path in the tree with origin i_0. If all i_n are different, this path is called an infinite chain. The relation

$$(i_n:n\in\mathbb{N})\sim(j_n:n\in\mathbb{N})\quad\Longleftrightarrow\quad \bigl|\{i_n:n\in\mathbb{N}\}\cap\{j_n:n\in\mathbb{N}\}\bigr|=\infty$$

is an equivalence relation in the set of chains. The quotient set is the boundary of the tree $(I,\mathcal{T})$ (see [8]), and we denote it by ∂_∞.

For every $i\in I$ and every $\xi\in\partial_\infty$, there exists a unique chain with origin i which is in the equivalence class ξ. This chain is called the *geodesic* between i and ξ and denoted by $geod(i,\xi)$. For fixed $\xi\in\partial_\infty$ and $n\in\mathbb{N}$, we denote by $\xi(n)$ the unique point in the geodesic $geod(r,\xi)$ such that $|\xi(n)|=n$. In particular, $\xi(0)=r$.

For $i\in I$ and $\xi\in\partial_\infty$, we write $i\preceq\xi$ if $i\in geod(r,\xi)$. Let us extend $\wedge$ to $I\cup\partial_\infty$. Let $i\in I$, and let $\eta,\xi\in\partial_\infty$ be such that $\eta\neq\xi$. We put

$$\begin{aligned} &i\wedge\eta=\eta\wedge i=\max\bigl(geod(r,i)\cap geod(r,\eta)\bigr),\\ &\xi\wedge\eta=\max\bigl(geod(r,\xi)\cap geod(r,\eta)\bigr),\\ &\eta\wedge\eta=\eta. \end{aligned}\tag{1.2}$$

Note that $\xi\wedge\eta\in I$ and $\xi\wedge\eta=j$ if and only if $\xi(|j|)=j=\eta(|j|)$ and $\xi(n)\neq\eta(n)$ for $n>|j|$.

The extended subtree hanging from $i \in I$ is $[i, \infty] = \{z \in I \cup \partial_\infty : i \preceq z\}$. The set $I \cup \partial_\infty$ is endowed with the topology $\mathbf{T}$ generated by the basis of open sets $\mathcal{A} = \{[i, \infty] : i \in I\} \cup \{\{i\} : i \in I\}$. The sets in $\mathcal{A}$ are open and closed in $\mathbf{T}$. The topological space $(I \cup \partial_\infty, \mathbf{T})$ is compact, totally discontinuous, and metrically generated; the trace topology on I is the discrete one, and I is an open dense subset in $I \cup \partial_\infty$. Then, when $\xi \in \partial_\infty$, it holds $\xi = \lim_{n\to\infty} \xi(n)$.

The open basis $\mathcal{A}$ is also a semi-algebra generating the Borel σ-algebra $\sigma(\mathbf{T})$. We will use the notation

$$\partial_\infty(i) = [i, \infty] \cap \partial_\infty = \{\eta \in \partial_\infty : i \preceq \eta\}. \tag{1.3}$$

The class of sets $\mathcal{C} = \{\partial_\infty(i) : i \in I\}$ is a basis of open (and closed) sets generating $\mathbf{T} \cap \partial_\infty$ and also is a semi-algebra generating the trace of $\sigma(\mathbf{T})$ on ∂_∞. We have $\partial_\infty(\xi(n)) = \{\eta \in \partial_\infty : \ |\xi \wedge \eta| \geq n\big\}$. We introduce the notation

$$C^n(\xi) = \partial_\infty\big(\xi(n)\big) \quad \text{for } n \in \mathbb{N} \text{ and } \xi \in \partial_\infty. \tag{1.4}$$

The set function $C^n(\bullet)$ with domain ∂_∞ takes a finite number of values. We denote by $\mathcal{F}_n$ the σ-field in ∂_∞ generated by the finite family of sets $\{C^n(\xi) : \xi \in \partial_\infty\}$. This sequence of σ-fields is increasing and generating, that is, $\mathcal{F}_\infty = \sigma(\mathbf{T})$.

The following criterion stated in [8] is useful to establish convergence to a point in the boundary. Let $(i_n : n \in \mathbb{N})$ be an infinite path; then

$$\Big(\forall j \in I : \big|\{n \in \mathbb{N} : i_n = j\}\big| < \infty\Big) \quad \Longrightarrow \quad \exists! \xi = \lim_{n\to\infty} i_n \in \partial_\infty. \tag{1.5}$$

In this case there exists a subsequence $(k_n : n \in \mathbb{N})$ verifying $geod(i_0, \xi) = (i_{k_n} : n \in \mathbb{N})$.

It will be useful to add a state $\partial_r \notin I$ and the oriented link (r, ∂_r). We put $r^- = \partial_r$ and $|\partial_r| = -1$.

In the sequel we adopt the following notation. For any nonempty countable set J, we denote by $\mathbb{I}_J$ the identity $J \times J$ matrix. If M is an $I \times I$ matrix and $J, K \subseteq I$ are nonempty, the matrix $M_{JK} = (M_{jk} : j \in J, k \in K)$ (also denoted by $M_{J,K}$) is the restriction of M to $J \times K$.

By $\mathbf{1}_A$ we mean the characteristic function of a set A, and $\mathbf{1}$ is the constant function taking the value 1 in its domain of definition. As used frequently for the mean expected value operator, by $\mathbb{E}(f, A)$ we mean $\mathbb{E}(f\mathbf{1}_A)$.

2 Tree Matrices

In [12] we have introduced the notion of tree matrices in the finite case. Here we give a general version of it. Let $(I, \mathcal{T})$ be a locally finite tree with root r. Put $\mathbf{N} = \{|i| : i \in I\}$, which is equal to $\mathbb{N}$ when the tree is infinite.

Definition 2.1 A tree matrix $U = (U_{ij} : i, j \in I)$ is defined by a strictly positive and strictly increasing function $w : \mathbf{N} \to (0, \infty)$ as follows:

$$U_{ij} = w_{|i \wedge j|} \quad \text{for } i, j \in I.$$

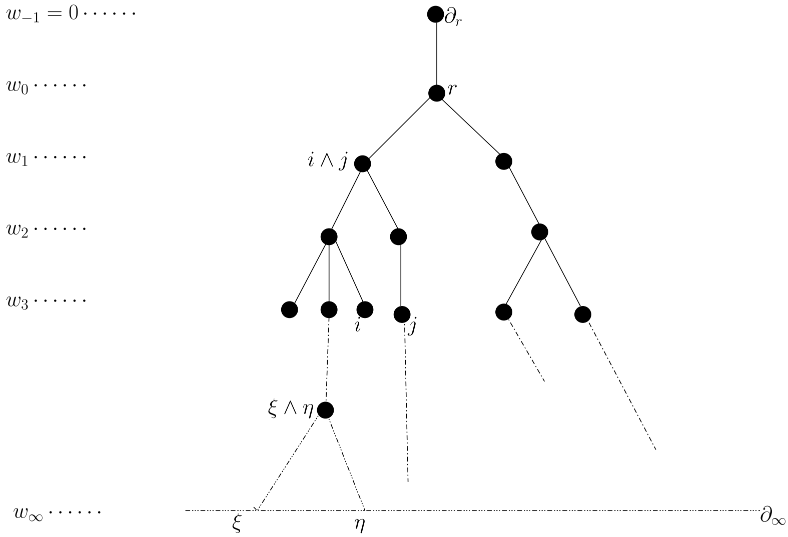

Fig. 1 Tree matrix

See Fig. 1 for a depiction of this notion.

The matrix U is strictly positive and symmetric, and it verifies $U_{ij} = U_{i\wedge j, i\wedge j}$. In particular, $U_{i^-i} = U_{i^-i^-} = w_{|i|-1}$ when $i^- \in I$. Notice that $U_{i^+i^+} = w_{|i|+1}$ does not depend on the particular element $i^+ \in S_i$. We extend U to $I \cup \{\partial_r\}$ by putting

$$U_{\partial_r \partial_r} = U_{i\partial_r} = U_{\partial_r i} = w_{-1} = 0, \quad \text{for } i \in I.$$

In what follows, it is useful to define $w_\infty = \lim_{n\to\infty} w_n$. Let us use (1.2) to extend U to $I \cup \partial_\infty$. For $i \in I$ and $\eta, \xi \in \partial_\infty$ such that $\eta \neq \xi$, we put

$$\begin{aligned} U_{i\eta} &= U_{\eta i} = w_{|i\wedge\eta|}, \\ U_{\xi\eta} &= U_{\eta\xi} = w_{|\xi\wedge\eta|}, \\ U_{\eta\eta} &= w_\infty. \end{aligned} \tag{2.1}$$

This extension is continuous in both variables: $U_{\xi\eta} = \lim_{n\to\infty, m\to\infty} U_{\xi(n)\eta(m)}$ for $\xi, \eta \in \partial_\infty$.

We associate to U a symmetric matrix $Q = (Q_{ij} : i, j \in I)$ supported by the tree and the diagonal, that is, $Q_{ij} = 0$ if $i \neq j$ and $(i, j) \notin \mathcal{T}$. This matrix Q is given by

$$\begin{aligned} Q_{ii^-} &= Q_{i^-i} = (w_{|i|} - w_{|i|-1})^{-1} \quad \text{for } i^-, i \in I; \\ Q_{ii} &= -\big((w_{|i|} - w_{|i|-1})^{-1} + |S_i|(w_{|i|+1} - w_{|i|})^{-1}\big) \quad \text{for } i \in I. \end{aligned} \tag{2.2}$$

Observe that $Q_{ii^+} = Q_{i^+i} = (w_{|i|+1} - w_{|i|})^{-1}$ does not depend on $i^+ \in S_i$. When $i \in \mathcal{L}$ is a leaf, then $Q_{ii} = -Q_{ii^-}$.

The matrix Q verifies: $Q_{ij} \geq 0$ if $j \neq i$; $Q_{ii} \leq 0$ and $\sum_{j\in I} Q_{ij} \leq 0$ for $i \in I$, so Q is a q-matrix. Q is conservative at $i \in I \setminus \{r\}$, that is, $\sum_{j\in I} Q_{ij} = 0$, and defective at r since $\sum_{j\in I} Q_{rj} = -w_0^{-1}$.

Observe that if M is an $I \times I$ matrix, then the formal products of matrices QM and MQ are well defined because each line and column of Q has finite support.

Proposition 2.1 *The q-matrix Q verifies $(-Q)U = U(-Q) = \mathbb{I}_I$.*

Proof By symmetry it suffices to show that $(-Q)U = \mathbb{I}_I$. For $i, k \in I$, we have

$$(QU)_{ik} = Q_{ii^-} U_{i^-k} + Q_{ii} U_{ik} + Q_{ii^+} \sum_{j\in S_i} U_{jk}.$$

If $k \wedge i \preceq i^-$, we have $i \neq r$ and $k \wedge i = k \wedge i^- = k \wedge i^+$. Then $(QU)_{ik} = 0$ because Q is conservative at $i \in I$.

For $k = i$, we have

$$\begin{aligned}(QU)_{ii} &= Q_{ii^-} U_{i^-i} + Q_{ii} U_{ii} + |S_i| Q_{ii^+} U_{ii} \\ &= Q_{ii^-} U_{i^-i} - Q_{ii^-} U_{ii} - |S_i| Q_{ii^+} U_{ii} + |S_i| Q_{ii^+} U_{ii} \\ &= -Q_{ii^-}(U_{ii} - U_{i^-i}) = -1.\end{aligned}$$

The last case left to analyze is when $k \wedge i^+ = i^+$ for some unique $i^+ \in S_i$. Then $k \wedge i^- = i^-, k \wedge i = i = k \wedge j$ for $j \in S_i \setminus \{i^+\}$. Hence

$$\begin{aligned}(QU)_{ik} &= Q_{ii^-} U_{i^-i^-} + Q_{ii} U_{ii} + \big(|S_i| - 1\big) Q_{ii^+} U_{ii} + Q_{ii^+} U_{i^+i^+} \\ &= (QU)_{ii} + Q_{ii^+}(U_{i^+i^+} - U_{ii}) = -1 + 1.\end{aligned}$$

□

When $(I, \mathcal{T})$ is a finite tree rooted at r, the matrix Q is the generator of a sub-Markov process with semigroup $(e^{Qt} : t \geq 0)$. We denote its lifetime by ζ, so the process is $X = (X_t : 0 \leq t < \zeta)$. We point out that this chain is irreducible and ζ is finite $\mathbb{P}_i$-a.s. for any $i \in I$. For simplicity, in some of the computations below we put $X_t = \partial_r$ for $t \in [\zeta, \infty]$. In particular, $X_\infty = \partial_r$ $\mathbb{P}_i$-a.s. for any $i \in I$. Since the state space is finite, the equation $QU = -\mathbb{I}_I$ is equivalent to $Q = -U^{-1}$. This implies that $U = -Q^{-1} = \int_0^\infty e^{tQ}\, dt$. Thus we have shown the following result.

Proposition 2.2 *Let $(I, \mathcal{T})$ be a finite tree rooted at r. Then U is the potential matrix of the chain $(X_t : 0 \leq t < \zeta)$ on I, that is, $U = \int_0^\infty e^{tQ}\, dt$ or equivalently $U_{ij} = \mathbb{E}_i(\int_0^\zeta \mathbf{1}_{\{X_t = j\}}\, dt)$ for $i, j \in I$.*

Continuing in the finite setting, we put $n + 1 = \max\{|i| : i \in I\}$. Consider the sets

$$B^{n+1} = \big\{i \in I : |i| = n + 1\big\} \quad \text{and} \quad \widetilde{B}^n = \big\{i \in I : |i| = n,\, S_i \neq \emptyset\big\}.$$

Hence $B^{n+1} = \bigcup_{i\in\widetilde{B}^n} S_i$. To avoid trivial situations we assume $n \geq 1$. We will also set $I^m = \{i \in I : |i| \leq m\}$, so $I = I^{n+1}$.

We denote by $T_i = \inf\{t \geq 0 : X_t = i\}$ the hitting time of $i \in I$, and by $T_{\widetilde{B}^n} := \inf\{T_i : i \in \widetilde{B}^n\}$ and $T_{B^{n+1}} := \inf\{T_i : i \in B^{n+1}\}$ we denote the hitting times of $\widetilde{B}^n$ and B^{n+1}, respectively.

Let $Q_{I^n I^n}$ be the restriction of Q to $I^n \times I^n$. The chain $(X_t : t < \zeta \wedge T_{B^{n+1}})$ has generator $Q_{I^n I^n}$ and semigroup $(e^{tQ_{I^n I^n}} : t \geq 0)$. Its potential $V^{(n)} := -(Q_{I^n I^n})^{-1}$ verifies

$$V_{ij}^{(n)} = \mathbb{E}_i\left(\int_0^{T_{B^{n+1}}} \mathbf{1}_{\{X_t = j\}}\, dt\right) \quad \text{for } i, j \in I^n.$$

Definition 2.2 Given a q-matrix Q on the set I, we say that a function $h : I \to \mathbb{R}$ is Q-harmonic if it verifies $Qh = 0$.

From the definition it is clear that h is Q-harmonic if and only if $e^{tQ}h = h$ for all $t \geq 0$.

Consider the q-matrix $\bar{Q}^{(n)}$ defined in I_n by

$$\bar{Q}^{(n)}_{I^n \setminus \widetilde{B}^n, I^n} = Q_{I^n \setminus \widetilde{B}^n, I^n} \quad \text{and} \quad \bar{Q}^{(n)}_{\widetilde{B}^n I^n} = 0.$$

The next proposition is a characterization of the $\bar{Q}^{(n)}$-harmonic functions, and its proof is based on Doob' sampling theorem.

Proposition 2.3 *A function $h : I^n \to \mathbb{R}$ is $\bar{Q}^{(n)}$-harmonic if and only if*

$$\mathbb{E}_i\big(h(X_{\tau \wedge T_{\widetilde{B}^n}})\big) = h(i) \quad \textit{for } i \in I^n \textit{ and any stopping time } \tau,$$

where we put $h(\partial_r) = 0$.

The class of $\bar{Q}^{(n)}$-harmonic functions, denoted by $\mathcal{H}^n$, is a linear space with dimension $\dim \mathcal{H}^n = |\widetilde{B}^n|$. Indeed, for each $k \in \widetilde{B}^n$, the function $h^k(i) = \mathbb{E}_i(\mathbf{1}_{\{k\}}(X_{T_{\widetilde{B}^n}}))$ is the unique harmonic function which verifies $h^k(j) = \delta_{kj}$ for $j \in \widetilde{B}^n$. The class of these harmonic functions constitutes a basis for $\mathcal{H}^n$.

Proposition 2.4 *The matrix $H := U_{I^n I^n} - V^{(n)}$ is symmetric, and its columns generate the space $\mathcal{H}^n$ of $\bar{Q}^{(n)}$-harmonic functions. Moreover, the columns of $U_{I^n \widetilde{B}^n}$ is a basis of this space.*

Proof First, let us introduce the matrices $W = (W_{ik} : i \in I^n, k \in \widetilde{B}^n)$, $E = (E_{i\ell} : i \in I^n, \ell \in B^{n+1})$, and $D = (D_{ik} : i \in I^n, k \in \widetilde{B}^n)$ with terms

$$W_{ik} = \mathbb{P}_i\{X_{T_{\widetilde{B}^n}} = k\}, \qquad E_{i\ell} = \mathbb{P}_i\{X_{T_{B^{n+1}}} = \ell\}, \qquad D_{ik} = \mathbb{P}_i\{X_{T_{B^{n+1}}} \in S_k\}.$$

Let W^k be the k column of W with $k \in \widetilde{B}^n$. We notice that $h^k = W^k$; then $(W^k : k \in \widetilde{B}^n)$ is a basis of $\mathcal{H}^n$. In particular $\bar{Q}^{(n)} W^k = 0$.

From definition $D_{ik} = \sum_{\ell \in S_k} E_{i\ell}$. This equality can be written as $D = EM^t$, where M^t is the transpose of the incidence matrix $M = (M_{k\ell} : k \in \widetilde{B}^n, \ell \in B^{n+1})$ with $M_{k\ell} = 1$ if $\ell \in S_k$ and $M_{k\ell} = 0$ otherwise.

Let $i \in I^n$ and $k \in \widetilde{B}^n$. Since

$$\mathbb{P}_i\{T_k < \infty\} = \sum_{j \in \widetilde{B}^n} \mathbb{P}_i\{X_{T_{\widetilde{B}^n}} = j\}\mathbb{P}_j\{T_k < \infty\} \quad \text{and} \quad U_{ik} = \mathbb{P}_i\{T_k < \infty\}U_{kk},$$

we find $U_{ik} = \sum_{j \in \widetilde{B}^n} \mathbb{P}_i\{X_{T_{\widetilde{B}^n}} = j\}U_{jk}$. Hence we obtain

$$U_{I^n\, \widetilde{B}^n} = W U_{\widetilde{B}^n\, \widetilde{B}^n} \quad \text{and so} \quad W = U_{I^n \widetilde{B}^n}(U_{\widetilde{B}^n \widetilde{B}^n})^{-1}. \tag{2.3}$$

Analogously we get $E = U_{I^n B^{n+1}}(U_{B^{n+1}\, B^{n+1}})^{-1}$. From the equality $D = EM^t$ we find $D = U_{I^n\, B^{n+1}}(U_{B^{n+1}\, B^{n+1}})^{-1}M^t$. Since $U_{i\ell} = w_{|i|} = U_{ik}$ when $k \in \widetilde{B}^n$ and $\ell \in S_k$, we obtain

$$U_{I^n\, B^{n+1}} = U_{I^n \widetilde{B}^n} M. \tag{2.4}$$

Then

$$D = U_{I^n\, \widetilde{B}^n} M (U_{B^{n+1}\, B^{n+1}})^{-1} M^t. \tag{2.5}$$

Let us now show that

$$H = U_{I^n \widetilde{B}^n} M (U_{B^{n+1} B^{n+1}})^{-1} M^t U_{\widetilde{B}^n\, I^n} \tag{2.6}$$

or equivalently $H = DU_{\widetilde{B}^n\, I^n}$. For $i, j \in I^n$, we have

$$\begin{aligned} U_{ij} &= \mathbb{E}_i\left(\int_0^\zeta \mathbf{1}_{\{X_t = j\}}\, dt\right) \\ &= \mathbb{E}_i\left(\int_0^{T_{B^{n+1}}} \mathbf{1}_{\{X_t = j\}}\, dt\right) + \mathbb{E}_i\left(\int_{T_{B^{n+1}}}^\zeta \mathbf{1}_{\{X_t = j\}}\, dt, T_{B^{n+1}} < \zeta\right) \\ &= \mathbb{E}_i\left(\int_0^{T_{B^{n+1}}} \mathbf{1}_{\{X_t = j\}}\, dt\right) + \mathbb{E}_i\left(\mathbb{E}_{X_{T_{B^{n+1}}}}\left(\int_0^\zeta \mathbf{1}_{\{X_t = j\}} dt\right), T_{B^{n+1}} < \zeta\right), \end{aligned}$$

where in the last equality we have used the strong Markov property. Hence $U_{ij} = V_{ij}^{(n)} + \sum_{\ell \in B^{n+1}} \mathbb{P}_i\{X_{T_{B^{n+1}}} = \ell\}U_{\ell j}$, or equivalently

$$U_{ij} = V_{ij}^{(n)} + \mathbb{E}_i(U_{X_{T_{B^{n+1}}}\, j},\ T_{B^{n+1}} < \infty) = \mathbb{E}_i(U_{X_{T_{B^{n+1}}}\, j}). \tag{2.7}$$

The last equality follows from the fact that on the set $T_{B^{n+1}} = \infty$ one has $X_{T_{B^{n+1}}} = \partial_r$ $\mathbb{P}_i$-a.s. and that by definition $U_{j\partial_r} = U_{\partial_r j} = 0$. Thus $H_{ij} = \mathbb{E}_i(U_{X_{T_{B^{n+1}}}\, j})$, and by using (2.4) we find

$$H_{ij} = \sum_{\ell \in B^{n+1}} \mathbb{P}_i\{X_{T_{B^{n+1}}} = \ell\}U_{\ell j} = \sum_{k \in \widetilde{B}^n} \mathbb{P}_i\{X_{T_{B^{n+1}}} \in S_k\}U_{kj} \quad \text{for } i, j \in I^n.$$

This gives $H = DU_{\widetilde{B}^n\, I^n}$, which together with (2.5) shows that (2.6) holds. From (2.6) we deduce $\operatorname{rank} H = \operatorname{rank} U_{\widetilde{B}^n\, I^n} = |\widetilde{B}^n| = \dim \mathcal{H}^n$. On the other hand, from (2.3) and (2.6) we get

$$H = W U_{\widetilde{B}^n \widetilde{B}^n} M (U_{B^{n+1} B^{n+1}})^{-1} M^t U_{\widetilde{B}^n \widetilde{B}^n} W^t. \tag{2.8}$$

From $\bar{Q}^{(n)}W = 0$ we obtain $\bar{Q}^{(n)}H = 0$. Therefore, the columns of H belong to the space $\mathcal{H}^n$. Given that $\text{rank}(H) = \dim(\mathcal{H}^n)$, the columns of H generate this space. On the other hand, by (2.6) the columns of $U_{I^n \widetilde{B}^n}$ generate $\mathcal{H}^n$. Since the rank of this matrix is equal to the dimension of $\mathcal{H}^n$, the proposition is shown. □

3 Harmonic Functions and the Martin Kernel

From now on we assume that $(I, \mathcal{T})$ is an infinite rooted tree and that all its branches are infinite.

We denote $\widehat{Q}$ the extension of Q to $I \cup \{\partial_r\}$ given by

$$\begin{aligned}
&\widehat{Q}_{r\partial_r} = w_0^{-1},\\
&\widehat{Q}_{i\partial_r} = \widehat{Q}_{\partial_r\partial_r} = 0 \quad \text{for } i \neq r, i \in I,\\
&\widehat{Q}_{\partial_r i} = 0 \quad \text{for all } i \in I.
\end{aligned} \tag{3.1}$$

This extension is a nonsymmetric conservative q-matrix in $I \cup \{\partial_r\}$, having ∂_r as an absorbing state.

We consider the minimal transition semigroup $\widehat{P}_t$ associated to $\widehat{Q}$ (this semigroup can be constructed by a truncation method as in [4] Proposition 2.14). Let $\widehat{X} = (\widehat{X}_t : 0 \le t < \widehat{\zeta})$ be a time-continuous Markov process with infinitesimal generator $\widehat{Q}$ and lifetime $\widehat{\zeta}$. If we stop $\widehat{X}$ when it hits ∂_r, we get a Markov process $X = (\widehat{X}_t : 0 \le t < \zeta)$ with generator Q, state space I, and lifetime $\zeta = T_{\partial_r} \wedge \widehat{\zeta}$. Let (P_t) be the semigroup associated to X and $V = \int_0^\infty P_t\,dt$ be the induced potential on I.

Remark 3.1 We mention that the discrete skeleton of $\widehat{X}$ whose transition probabilities are

$$\begin{aligned}
&p_{ij} = \frac{\widehat{Q}_{ij}}{\sum_{k \in S_i \cup \{i^-\}} \widehat{Q}_{ik}} \quad \text{for } i \in I, j \in S_i \cup \{i^-\},\\
&p_{\partial_r\partial_r} = 1,\ p_{\partial_r i} = 0 \quad \text{for } i \in I,
\end{aligned}$$

has electrical circuits interpretation. It is the Markov chain with conductances $C_{ii^-} := Q_{ii-}$ (see [17], Sect. 9, and [20], Sect. 2).

Let $I^n = \{i \in I : |i| \le n\}$. As in the previous section, $V^{(n)}$ is the potential associated to $Q_{I_n I_n}$, and $\mathcal{H}^n$ is the set of $\bar{Q}^{(n)}$-harmonic functions in I^n. Let $X^{(n)} := (X_t : t < T_{\partial_r} \wedge T_{B^{n+1}})$ be the chain killed at $B^{n+1} \cup \{\partial_r\}$, with generator $Q_{I^n I^n}$, Markov semigroup $P_t^{(n)} = e^{tQ_{I^n I^n}}$, and potential $V^{(n)} = \int_0^\infty P_t^{(n)} dt = -Q_{I^n I^n}^{-1}$. Clearly $(P_t^{(n)})_{ij} \le (P_t^{(n+1)})_{ij}$ and $V_{ij}^{(n)} \le V_{ij}^{(n+1)}$ for $i, j \in I^n$. By the Monotone Convergence Theorem their limits are $(P_t)_{ij}$ and V_{ij}, respectively. From (2.7) we get $V_{ij}^{(n)} \le U_{ij}$, so $V \le U$.

A classical procedure (for instance, see [8]) shows that X_ζ is a well-defined variable in $I \cup \partial_\infty \cup \partial_r$. Let us briefly do this. If $T_{\partial_r} < \infty$, this is obvious because $T_{\partial_r} = \zeta$ and $X_\zeta = \partial_r$. In the set $T_{\partial_r} = \infty$, the Borel–Cantelli Lemma implies that the trajectories must visit each site of I only a finite number of times. Since they are not absorbed at ∂_r, they must converge to a point in the boundary ∂_∞ because (1.5) is fulfilled. To describe this phenomenon we need the stopping times $R_n = \inf\{t \geq 0 : |X_t| \geq n\}$ and $R_\infty := \lim_{n\to\infty} R_n$. The above discussion is summarized in

$$X_\zeta = \partial_r \quad \text{if } T_{\partial_r} < R_\infty \quad \text{and} \quad X_\zeta = \lim_{n\to\infty} X_{R_n} = \lim_{n\to\infty} X_\zeta(n) \in \partial_\infty \quad \text{if } R_\infty \leq T_{\partial_r}. \tag{3.2}$$

Here, as already introduced in Sect. 1.2, $X_\zeta(n)$ is the point at level n in $geod(r, X_\zeta)$.

The tree matrix is said to be *transient* whenever $\mathbb{P}_r\{T_{\partial_r} < \infty\} < 1$ or equivalently $\mathbb{P}_r\{X_\zeta \in \partial_\infty\} > 0$. Otherwise, the tree matrix is said to be *recurrent*. This classification corresponds to the recurrence or transient property for the chain reflected at r. For a simple criterion on transience for reversible Markov chains, see [22].

Now, consider $|i| \leq n$. Equality (2.7) gives $U_{ij} = V_{ij}^{(n)} + \mathbb{E}_i(U_{X_{T_{B^{n+1}}}\,j})$, which, together with the fact that $U_{X_{T_{B^{n+1}}}\,j} \leq U_{jj}$ and $\lim_{n\to\infty} U_{X_{T_{B^{n+1}}}\,j} = U_{X_\zeta\,j}$ $\mathbb{P}_i$-a.e., yields

$$\lim_{n\to\infty} \mathbb{E}_i(U_{X_{T_{B^{n+1}}}\,j}) = \mathbb{E}_i(U_{X_\zeta\,j}).$$

Combining these relations with $\lim_{n\to\infty} V_{ij}^{(n)} = V_{ij}$ allows us to get

$$U_{ij} = V_{ij} + \mathbb{E}_i(U_{X_\zeta\,j}) = V_{ij} + \int_{\partial_\infty} U_{\eta j}\, \mathbb{P}_i\{X_\zeta \in d\eta\}. \tag{3.3}$$

Given that $V_{ij} = V_{ji} = \mathbb{P}_j\{T_i < \infty\} V_{ii}$, the following limit exists:

$$V_{i\xi} := \lim_{j\to\xi} V_{ij} = V_{ii} \cdot \lim_{j\to\xi} \mathbb{P}_j\{T_i < \infty\} \geq 0, \quad \text{for } i \in I, \xi \in \partial_\infty. \tag{3.4}$$

Therefore, passing to the limit as $j \to \xi \in \partial_\infty$ in relation (3.3) and using the Monotone Convergence Theorem lead to

$$U_{i\xi} = V_{i\xi} + \int_{\partial_\infty} U_{\eta\xi} \mathbb{P}_i\{X_\zeta \in d\eta\}. \tag{3.5}$$

A conclusion derived from (3.3) is that the recurrent case $\mathbb{P}_r\{X_\zeta \in \partial_\infty\} = 0$ is completely characterized by the equality $V = U$. In particular the tree matrix U is the potential of (X_t).

In the transient case we denote by μ the exit measure on the boundary ∂_∞, that is, the probability measure defined on ∂_∞ by

$$\mu(\bullet) = \mathbb{P}_r\{X_\zeta \in \bullet \mid X_\zeta \in \partial_\infty\}. \tag{3.6}$$

Remark 3.2 If U is unbounded, that is, $w_\infty = \infty$, the measure μ is atomless. In fact, from (3.5) we get

$$\infty > w_0 = U_{r\xi} \geq \int_{\partial_\infty \setminus \{\xi\}} U_{\eta\xi} \mathbb{P}_r\{X_\zeta \in d\eta\} + w_\infty \mathbb{P}_r\{X_\zeta = \xi\}.$$

In what follows we concentrate ourselves on the transient case. Nevertheless, at some points we shall state the corresponding results for the recurrent case.

3.1 Harmonic Functions

In this subsection we study basic properties of the harmonic functions on I. We notice that the restriction of a $\widehat{Q}$-harmonic function to I is not necessarily Q-harmonic. An example of this is the constant **1** function. In fact, the unique $\widehat{Q}$-harmonic functions whose restrictions are Q-harmonic are those vanishing at ∂_r. Obviously the reciprocal also holds, that is, the unique $\widehat{Q}$-harmonic extension of a Q-harmonic function is the one extended by 0 at ∂_r. In the sequel a harmonic function is to be understood as a Q-harmonic function, and for a function defined on a subset of $I \cup \partial_\infty$, we assume implicitly that it takes the value 0 at ∂_r, unless otherwise is specified.

In the sequel an important role will be played by the function

$$\bar{g}(j) = \mathbb{P}_j\{T_{\partial_r} < \infty\}, \quad j \in I \cup \{\partial_r\}, \tag{3.7}$$

which is the Martin kernel for $\widehat{Q}$ at ∂_r. We point out that both $\bar{g}$ and $1 - \bar{g}$ are $\widehat{Q}$-harmonic, but only $1 - \bar{g}$ is Q-harmonic. We also note that $\bar{g}$ is nonnegative and decreasing on each branch, which allows to define it on ∂_∞ by

$$\bar{g}(\eta) := \lim_{j \to \eta} \mathbb{P}_j\{T_{\partial_r} < \infty\}, \quad \eta \in \partial_\infty.$$

The following notion will enable us to study limiting properties on the boundary for functions defined on the extended tree.

Given $g : I \to \overline{\mathbb{R}}$ an extended real function defined on the tree, we consider the sequence of functions (g_n) defined on the boundary by

$$g_n(\xi) = g\big(\xi(n)\big) \quad \text{for } n \in \mathbb{N} \quad \text{and} \quad \xi \in \partial_\infty.$$

Definition 3.1 Let $g : I \to \overline{\mathbb{R}}$ and $\varphi : \partial_\infty \to \overline{\mathbb{R}}$. We put $\lim g = \varphi$ pointwise (respectively μ-a.e.) if $\lim_{n\to\infty} g_n = \varphi$ pointwise (respectively μ-a.e.).

Let $\bar{R}_n := \inf\{t \geq 0 : |X_t| \geq n \text{ or } X_t = \partial_r\}$. A standard argument gives

$$\begin{aligned} &h : I \to \mathbb{R} \text{ is harmonic} \\ &\quad \Longleftrightarrow \quad \big[\forall n \geq 1,\ \forall \tau \text{ stopping time: } \forall i \in I, h(i) = \mathbb{E}_i\big(h(X_{\tau \wedge \bar{R}_n})\big)\big]. \end{aligned}$$

In the transient case, an application of the Dominated Convergence Theorem and Fatou's Theorem give that for any bounded harmonic function $h : I \to \mathbb{R}$, the limit $\varphi = \lim h$ μ-a.e. exists and moreover

$$h(i) = \mathbb{E}_i\big(\varphi(X_\zeta)\big).$$

Indeed, this is a consequence of Theorem 2.6 in [8], because h is bounded if and only if $h/(1 - \bar{g})$ is bounded. Thus, if h_1, h_2 are bounded harmonic functions such that $\lim h_1 = \lim h_2$ μ-a.e., then $h_1 \equiv h_2$.

Obviously in the recurrent case the unique bounded harmonic function is $h \equiv 0$.

Proposition 3.1 *If U is bounded, then the tree matrix is transient.*

Proof The function $h(i) = U_{i\eta}$ is harmonic, bounded, and nonzero, which implies that the tree matrix must be transient. □

A distinguished class of harmonic functions is given by the Martin kernel at ∞, see [8, 17], or [28].

Definition 3.2 The Martin kernel (at ∞) $\kappa : I \times \partial_\infty \to \mathbb{R}$ is given by

$$\kappa(i,\eta) := \lim_{j\to\eta} \frac{V_{ij}}{V_{rj}} \quad \text{for } i \in I, \eta \in \partial_\infty.$$

It is a well-known fact that $\kappa(\bullet,\eta)$ is a harmonic function on I (see [8] or [28]). Consider $i \in I$, $\xi \in \partial_\infty$, and $n > |i \wedge \xi|$. Take $j = \xi(n)$ and recall that $C^n(\xi) = \partial_\infty(\xi(n))$ (see (1.4)). From $V_{ij} = \mathbb{P}_i\{T_j < \infty\}V_{jj}$ and the strong Markov property we obtain

$$\mathbb{P}_i\big\{X_\zeta \in C^n(\xi)\big\} = \mathbb{P}_i\{T_j < \infty\}\mathbb{P}_j\big\{X_\zeta \in C^n(\xi)\big\} = \frac{V_{ij}}{V_{jj}}\mathbb{P}_{\xi(n)}\big\{X_\zeta \in C^n(\xi)\big\}.$$

Thus we get

$$\frac{V_{ij}}{V_{rj}} = \frac{\mathbb{P}_i\{X_\zeta \in C^n(\xi)\}}{\mathbb{P}_r\{X_\zeta \in C^n(\xi)\}},$$

and passing to the limit, we find that

$$\kappa(i,\xi) = \lim_{j\to\xi} \frac{\mathbb{P}_i\{X_\zeta \in \partial_\infty(j)\}}{\mathbb{P}_r\{X_\zeta \in \partial_\infty(j)\}} = \frac{\mathbb{P}_i\{X_\zeta \in C^n(\xi)\}}{\mathbb{P}_r\{X_\zeta \in C^n(\xi)\}}.$$

On the other hand, $\mathbb{P}_i\{X_\zeta \in C^n(\xi)\} = \mathbb{P}_i\{T_{i\wedge\xi} < \infty\}\mathbb{P}_{i\wedge\xi}\{X_\zeta \in C^n(\xi)\}$. Also, since every trajectory starting at r and reaching $C^n(\xi)$ must cross $i \wedge \xi$, we obtain $\mathbb{P}_r\{X_\zeta \in C^n(\xi)\} = \mathbb{P}_r\{T_{i\wedge\xi} < \infty\}\mathbb{P}_{i\wedge\xi}\{X_\zeta \in C^n(\xi)\}$.

Thus for $i \in I$, $\xi \in \partial_\infty$, and $n > |i \wedge \xi|$, we get the formula

$$\kappa(i,\xi) = \lim_{j\to\xi} \frac{\mathbb{P}_i\{X_\zeta \in \partial_\infty(j)\}}{\mathbb{P}_r\{X_\zeta \in \partial_\infty(j)\}} = \frac{\mathbb{P}_i\{X_\zeta \in C^n(\xi)\}}{\mathbb{P}_r\{X_\zeta \in C^n(\xi)\}} = \frac{\mathbb{P}_i\{T_{i\wedge\xi} < \infty\}}{\mathbb{P}_r\{T_{i\wedge\xi} < \infty\}}. \tag{3.8}$$

In particular $\kappa(i,\bullet)$ is the Radon–Nikodým derivative of $\mathbb{P}_i\{X_\zeta \in \bullet\}$ with respect to $\mathbb{P}_r\{X_\zeta \in \bullet\}$ (see [8]), so

$$U_{i\xi} = V_{i\xi} + \int_{\partial_\infty} U_{\xi\eta}\, \kappa(i,\eta)\, \mathbb{P}_r\{X_\zeta \in d\eta\}.$$

Remark 3.3 When the tree is recurrent, that is, $V = U$, the Martin kernel is easily computed as

$$\kappa(i,\eta) = \lim_{j\to\eta} \frac{V_{ij}}{V_{rj}} = \frac{U_{i\eta}}{w_0}.$$

Therefore, $\{U_{\bullet\eta}/w_0 : \eta \in \partial_\infty\}$ is the Martin kernel.

3.2 Regular Points

To study the potential theory in the transient case one needs the description of the regular points on ∂_∞. In the classical setting regularity is needed for the continuity up to the boundary for solutions to the Dirichlet boundary problem (see, for example, [10], Theorem 1.23). In our context this is stated in Lemma 3.1 (ii).

Definition 3.3 A point $\eta \in \partial_\infty$ is regular if $\bar{g}(\eta) = 0$, that is, $\lim_{j\to\eta} \mathbb{P}_j\{T_{\partial_r} < \infty\} = 0$. We denote by $\partial_\infty^{\text{reg}}$ the set of regular points.

The classification on regular points is the same if instead of T_{∂_r}, we use T_i for any $i \in I$. Indeed η is regular if and only if $\lim_{j\to\eta} \mathbb{P}_j\{T_i < \infty\} = 0$ for all $i \in I$. By (3.4) this is exactly the case where $V_{i\eta} = 0$.

Lemma 3.1

(i) *The measure μ is concentrated on the set of regular points*: $\mu(\partial_\infty^{\text{reg}}) = 1$.

(ii) *A point $\eta \in \partial_\infty$ is regular if and only if any bounded continuous real function f defined in $\partial_\infty \cup \{\partial_r\}$ with $f(\partial_r) = 0$ verifies*

$$\lim_{j\to\eta} \mathbb{E}_j\big(f(X_\zeta)\big) = f(\eta). \tag{3.9}$$

(iii) *Every regular point $\eta \in \partial_\infty$ belongs to the closed support of μ, that is,*

$$\mathbb{P}_r\big\{X_\zeta \in [\eta(n), \infty]\big\} > 0 \quad \textit{for all } n.$$

Proof (i) The function $\bar{g}(j) = \mathbb{P}_j\{T_{\partial_r} < \infty\}$ is bounded and $\widehat{Q}$-harmonic and verifies $\bar{g}(\partial_r) = 1$. For any $n \geq 1$, it holds $\bar{g}(r) = \mathbb{E}_r(\bar{g}(X_{T_{B^n} \wedge T_{\partial_r}}))$. Hence, the Dominated Convergence Theorem gives

$$\bar{g}(r) = \mathbb{E}_r\big(\bar{g}(X_\zeta)\big) = \mathbb{P}_r\{T_{\partial_r} < \infty\} + \int \bar{g}(\xi)\mathbb{P}_r\{X_\zeta \in d\xi\}.$$

From this relation we conclude that $\bar{g} = 0$ μ-a.e. Therefore $\mu(\partial_\infty^{\text{reg}}) = 1$.

(ii) Since f is continuous and bounded, for every fixed $\varepsilon > 0$, there exists n such that $|f(\xi) - f(\eta)| \leq \varepsilon$ if $\xi(n) = \eta(n)$. Then for $j \in [\eta(n), \infty)$, we have

$$\big|\mathbb{E}_j\big(f(X_\zeta)\big) - f(\eta)\big| \leq 2M\mathbb{P}_j\{T_{\eta(n)} < \zeta\} + 2\varepsilon\mathbb{P}_j\{\zeta \leq T_{\eta(n)}\},$$

where M is a bound for f. From this inequality we conclude that

$$\limsup_{j\to\eta} \big|\mathbb{E}_j\big(f(X_\zeta)\big) - f(\eta)\big| \leq 2\varepsilon$$

and then obtain the desired limit in (3.9).

Conversely, assume now that (3.9) holds for $f = \mathbf{1}_{\partial_\infty}$ (so $f(\partial_r) = 0$). Then

$$\mathbb{E}_j\big(f(X_\zeta)\big) = \mathbb{P}_j\{R_\infty \leq T_{\partial_r}\} = 1 - \mathbb{P}_j\{T_{\partial_r} < \infty\} \underset{j\to\eta}{\longrightarrow} 1 = f(\eta),$$

proving that η is regular.

(iii) Fix a regular point η and $n \in \mathbb{N}$. Consider the indicator function $f(\bullet) = \mathbf{1}_{C^n(\eta)}(\bullet)$ of $C^n(\eta)$. For $j \in I$ close enough to η, we have $\mathbb{P}_j\{X_\zeta \in C^n(\eta)\} > 0$, which implies $\mathbb{P}_r\{X_\zeta \in C^n(\eta)\} > 0$. Thus η is in the closed support of μ. $\square$

3.3 The Kernel at the Boundary Is a Filtered Operator

Let us introduce the operator W, with kernel U, acting on $L^p(\mu)$. U and W acting on ∂_∞ were introduced in [19], Sect. 4, and used in [20], Sect. 2.3, to study the capacity function on the boundary.

Definition 3.4 For any (positive) bounded, real, and measurable function f with domain in ∂_∞, we define

$$Wf(\eta) = \int_{\partial_\infty} U_{\eta\xi} f(\xi)\mu(d\xi),$$

which is also a (positive) real and measurable function.

We notice that the integral defining W can be made over ∂_∞ or $\partial_\infty^{\mathrm{reg}}$, because this last set is of full measure μ. We have from (3.5) and $w_0 = U_{r\eta}$ that

$$W\mathbf{1}(\eta) = \int_{\partial_\infty} U_{\eta\xi}\mu(d\xi) = \frac{w_0 - V_{r\eta}}{\mathbb{P}_r\{X_\zeta \in \partial_\infty\}}.$$

Then Wf is bounded for any bounded f. Since $V_{r\eta} = 0$ for any regular point η, we conclude that $W\mathbf{1}$ is constant μ-a.e., where this constant, denoted by α, is given by $\alpha = w_0/\mathbb{P}_r\{X_\zeta \in \partial_\infty\}$. In general we have $W\mathbf{1} \le \alpha$ in ∂_∞.

The action of W on measures is given by $\nu W(A) = \int W\mathbf{1}_A(\xi)\nu(d\xi)$. It is direct to see that $\mu W = \alpha\mu$. Then $\alpha^{-1}W$ is a Markov operator preserving μ. Hence, for every $p \ge 1$, the operator $W : L^p(\mu) \to L^p(\mu)$ is well defined, $\|W\|_p = \alpha$, and W is self-adjoint in $L^2(\mu)$.

For $f \in L^1(\mu)$, it is verified

$$Wf(\eta) = \sum_{k\in\mathbb{N}} w_k \int_{C^k(\eta)\setminus C^{k+1}(\eta)} f\,d\mu = \sum_{k\in\mathbb{N}} (w_k - w_{k-1}) \int_{C^k(\eta)} f\,d\mu. \tag{3.10}$$

Recall that $\mathcal{F} = (\mathcal{F}_k : k \in \mathbb{N})$ is a generating filtration in ∂_∞, where for each k, the σ-field $\mathcal{F}_k$ is generated by the finite family of sets $\{C^k(\xi) : \xi \in \partial_\infty\}$. Using the associated conditional expectations, (3.10) can be written as

$$Wf(\bullet) = \sum_{k\in\mathbb{N}} (w_k - w_{k-1})\mu\big(C^k(\bullet)\big)\mathbb{E}_\mu(f|\mathcal{F}_k)(\bullet). \tag{3.11}$$

Now, consider the following process defined on ∂_∞:

$$G = (G_n : n \in \mathbb{N}) \quad \text{where } G_n(\eta) = \sum_{k\ge n} (w_k - w_{k-1})\mu\big(C^k(\eta)\big). \tag{3.12}$$

Since $G_0 = W\mathbf{1} \leq \alpha$, we obtain that G_0 is a convergent series. On the other hand, since every regular point is in the closed support of μ, we conclude that $\mu(C^k(\xi)) > 0$ for all $k \in \mathbb{N}$ and $\xi \in \partial_\infty^{\mathrm{reg}}$. In particular, $G_n > 0$ μ-a.e. for every $n \in \mathbb{N}$. We also have for $n \geq 1$,

$$G_n(\eta) = G_0 - \sum_{k=0}^{n-1} (w_k - w_{k-1})\mu\big(C^k(\eta)\big) \quad \text{is } \mathcal{F}_{n-1}\text{-measurable.}$$

Therefore if $|\xi \wedge \eta| \geq n$, we have $G_i(\eta) - G_{i+1}(\eta) = G_i(\xi) - G_{i+1}(\xi)$, $i = 0, \dots, n$. Moreover, if ξ, η are regular points, then $G_0(\eta) = G_0(\xi) = \alpha$ and

$$G_i(\xi) = G_i(\eta) \quad \text{for all } i \leq |\xi \wedge \eta|. \tag{3.13}$$

The process (G_n) is $\mathcal{F}$-predictable, positive, bounded by α, and decreasing to 0 as $n \to \infty$. Then $G_n \mathbb{E}_\mu(\,|\mathcal{F}_n)$ converges to 0 in $L^p(\mu)$ for every $p \in [1, \infty]$. Therefore, integration by parts on (3.11) gives

$$W = \sum_{n \in \mathbb{N}} (G_n - G_{n+1})\mathbb{E}_\mu(\,|\mathcal{F}_n) = \sum_{n \in \mathbb{N}} G_n\big(\mathbb{E}_\mu(\,|\mathcal{F}_n) - \mathbb{E}_\mu(\,|\mathcal{F}_{n-1})\big), \tag{3.14}$$

the equality being in the sense of operators. Thus, we have shown the following result.

Proposition 3.2 *The self adjoint operator W acting on $L^2(\mu)$ is a stochastic integral operator* (*or a filtered operator*), *that is, there exist a filtration $\mathcal{F} = (\mathcal{F}_n)$ and an $\mathcal{F}$-predictable process $G = (G_n)$ such that $W = \sum_{n \in \mathbb{N}} G_n(\mathbb{E}_\mu(\,|\mathcal{F}_n) - \mathbb{E}_\mu(\,|\mathcal{F}_{n-1}))$.*

For definitions and properties of stochastic integral operators, see [14], and for its characterization in the countable case, see [11].

Let us consider $\mathcal{D} = \bigcup_{n \in \mathbb{N}} L^2(\mathcal{F}_n, \mu)$, the set of simple functions over the algebra $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n$. Clearly $\mathcal{D}$ is a dense subset in $L^2(\mu)$. The operator $L = \sum_{n \in \mathbb{N}} G_n^{-1} \times (\mathbb{E}_\mu(\,|\mathcal{F}_n) - \mathbb{E}_\mu(\,|\mathcal{F}_{n-1}))$ is well defined in $\mathcal{D}$. Since G_n is $\mathcal{F}_{n-1}$-measurable, the following equalities hold on $\mathcal{D}$:

$$LW = WL = \sum_{n \in \mathbb{N}} \mathbb{E}_\mu(\,|\mathcal{F}_n) - \mathbb{E}_\mu(\,|\mathcal{F}_{n-1}) = \mathbb{I}_{\mathcal{D}}.$$

Here $\mathbb{I}_{\mathcal{D}}$ is the identity on $\mathcal{D}$. In particular, $\mathrm{Im}(W) = W(L^2(\mu))$ contains $\mathcal{D}$, so $\mathrm{Im}(W)$ is dense in $L^2(\mu)$. Since W is a self-adjoint operator, we get that W is one-to-one. Hence we can extend L to $\mathrm{Im}(W)$ by $Lg = f$ for $g \in \mathrm{Im}(W)$, $g = Wf$. Therefore $WL = \mathbb{I}_{\mathrm{Im}(W)}$ and $LW = \mathbb{I}_{L^2(\mu)}$. We put $L = W^{-1}$ and assume implicitly that its domain is $\mathrm{Im}(W)$, so

$$W^{-1} = \sum_{n \in \mathbb{N}} G_n^{-1}\big(\mathbb{E}_\mu(\,|\mathcal{F}_n) - \mathbb{E}_\mu(\,|\mathcal{F}_{n-1})\big). \tag{3.15}$$

Observe that $W^{-1}\mathbf{1} = \alpha^{-1}$ μ-a.e.

Let us compute W^{-1} in $\mathcal{D}$. For that purpose, fix a set $C^n \in \mathcal{F}_n$. For $k \le n$, we denote by C^k the element in $\mathcal{F}_k$ such that $C^n \subseteq C^k$. We also put $C^{-1} = \emptyset$. From (3.15) we obtain

$$\begin{aligned} W^{-1}\mathbf{1}_{C^n} &= \sum_{k=0}^{n} G_k^{-1}\big(\mathbb{E}_\mu(\mathbf{1}_{C^n}|\mathcal{F}_k) - \mathbb{E}_\mu(\mathbf{1}_{C^n}|\mathcal{F}_{k-1})\big) \\ &= \sum_{k=0}^{n} G_k^{-1}\left(\frac{\mu(C^n)}{\mu(C^k)}\mathbf{1}_{C^k} - \frac{\mu(C^n)}{\mu(C^{k-1})}\mathbf{1}_{C^{k-1}}\right) \\ &= G_n^{-1}\mathbf{1}_{C^n} + \sum_{k=0}^{n-1}\big(G_k^{-1} - G_{k+1}^{-1}\big)\frac{\mu(C^n)}{\mu(C^k)}\mathbf{1}_{C^k}. \end{aligned} \tag{3.16}$$

This formula shows that $W^{-1}\mathbf{1}_{C^n}$ is a bounded function.

As a particular case, consider $\eta, \xi \in \partial_\infty$, $\eta \ne \xi$, such that $n > |\eta \wedge \xi|$. We consider in the previous formulae the sets $C^k = C^k(\eta)$ for $k = 0, \dots, n$ and get

$$W^{-1}\mathbf{1}_{C^n(\eta)}(\xi) = \sum_{k=0}^{|\eta\wedge\xi|} \big(G_k^{-1}(\eta) - G_{k+1}^{-1}(\eta)\big)\frac{\mu(C^n(\eta))}{\mu(C^k(\eta))}, \tag{3.17}$$

since $C^k(\xi) = C^k(\eta)$ for $k \le |\eta \wedge \xi|$ and $C^k(\xi) \cap C^k(\eta) = \emptyset$ for $k > |\eta \wedge \xi|$.

3.4 The Martin Kernel

We note that by (3.2) the Martin kernel for a nonregular point ξ is given by

$$\kappa(i,\xi) = \frac{U_{i\xi} - \int_{\partial_\infty} U_{\eta\xi}\mathbb{P}_i\{X_\zeta \in d\eta\}}{w_0 - \int_{\partial_\infty} U_{\eta\xi}\mathbb{P}_r\{X_\zeta \in d\eta\}}. \tag{3.18}$$

For regular points, the numerator and denominator in the previous expression vanish. Hence, for obtaining an expression for the Martin kernel, we need new elements in the study of regular points. In the next formulae we get relations between the operator W and the exit measure μ.

Proposition 3.3 *For any $i, j \in I$, we have*

$$\mathbb{P}_i\big\{X_\zeta \in \partial_\infty(j)\big\} = \int_{\partial_\infty} U_{i\xi}\big(W^{-1}\mathbf{1}_{\partial_\infty(j)}\big)(\xi)\mu(d\xi). \tag{3.19}$$

Proof The function $h_1(i) = \mathbb{P}_i\{X_\zeta \in \partial_\infty(j)\}$ is harmonic and bounded. Moreover for any regular point η, we have

$$\lim_{i\to\eta} \mathbb{P}_i\big\{X_\zeta \in \partial_\infty(j)\big\} = \mathbf{1}_{\partial_\infty(j)}(\eta),$$

which implies that $\lim h_1 = \mathbf{1}_{\partial_\infty(j)}$ μ-a.e.

On the other hand, consider $h_2(i) := \int_{\partial_\infty} U_{i\xi}(W^{-1}\mathbf{1}_{\partial_\infty(j)})(\xi)\mu(d\xi)$. This function is also harmonic because for every $\xi \in \partial_\infty$, the function $U_{i\xi}$ is harmonic on I.

Let us show that h_2 is a bounded function. From (3.16) one checks that $\|W^{-1}\mathbf{1}_{\partial_\infty(j)}\|_\infty < \infty$. Then for $\eta \in \partial_\infty(i)$, we find

$$\big|h_2(i)\big| \le \big\|W^{-1}\mathbf{1}_{\partial_\infty(j)}\big\|_\infty \int_{\partial_\infty} U_{\eta\xi}\,\mu(d\xi) = \big\|W^{-1}\mathbf{1}_{\partial_\infty(j)}\big\|_\infty W\mathbf{1}(\eta) < \infty.$$

Hence h_2 is bounded. Finally, by the Dominated Convergence Theorem we conclude the pointwise convergence

$$\lim_{i\to\eta} h_2(i) = \int_{\partial_\infty} U_{\eta\xi}\big(W^{-1}\mathbf{1}_{\partial_\infty(j)}\big)(\xi)\mu(d\xi).$$

The result follows from the equality $\int_{\partial_\infty} U_{\eta\xi}(W^{-1}\mathbf{1}_{\partial_\infty(j)})(\xi)\mu(d\xi) = \mathbf{1}_{\partial_\infty(j)}(\eta)$ μ-a.e. in $\eta \in \partial_\infty$. □

Corollary 3.1 *Let* $h : I \to \mathbb{R}$ *be a harmonic function such that* $\lim h = \varphi$ μ*-a.e.* (*for example*, *if h is bounded*). *Assume that* φ *is a simple function*, *that is*, $\varphi \in \mathcal{D}$ (*in particular* φ *is in the domain of* W^{-1}). *Then for all* $i \in I$,

$$h(i) = \int U_{i\xi}\big(W^{-1}\varphi\big)(\xi)\mu(d\xi). \tag{3.20}$$

Proof It is straightforward from (3.19) by decomposing φ as a finite linear combination of indicator functions based on the sets $C^{n_1}(\eta_1), \dots, C^{n_k}(\eta_k)$. □

Remark 3.4 Then, in a dense class of harmonic functions we have the representation $h(i) = \int U_{i\xi}\,d\nu(\xi)$ with $d\nu(\xi) = W^{-1}\varphi(\xi)\mu(d\xi)$. This representation is similar to the one using the Martin kernel as in [8]. Nevertheless, there are some differences. Even if h is positive, ν may be a signed measure. On the other hand, the characterization $d\nu = W^{-1}\varphi\,d\mu$ gives additional information on this signed measure. We recall that in the Martin representation, φ is the Radon–Nikodým derivative of the absolute continuous part, with respect to μ, of the representing measure (see, for example, [30]).

Let us characterize the increasing harmonic functions. A real function $f : I \to \mathbb{R}$ is increasing in the rooted tree if $i \preceq j$ implies $f(i) \le f(j)$.

Theorem 3.1 *A function* $h : I \to \mathbb{R}_+$ *is harmonic and increasing if and only there exists a finite* (*nonnegative*) *measure* ν *in* ∂_∞ *such that*

$$h(i) = \int_{\partial_\infty} U_{i\xi}\,d\nu(\xi) \quad \textit{for every } i \in I. \tag{3.21}$$

Proof If h verifies (3.21), then it is harmonic and increasing, since so is $U_{\bullet\xi}$.

Let us now prove the converse. Assume that h is a nonnegative harmonic and increasing function. From Proposition 2.4 proven for finite matrices we get

$$\forall n\ \exists!\, \alpha^{(n)} : B^n \to \mathbb{R} \quad \text{such that if } |i| \le n : h(i) = \sum_{j\in B^n} U_{ij}\alpha^{(n)}(j).$$

In particular if $|i| = n-1$, we find

$$\begin{aligned} h\big(i^+\big) &= \sum_{j\in B^n} U_{i^+j}\,\alpha^{(n)}(j) = \sum_{j\in B^n, j\neq i^+} U_{ij}\alpha^{(n)}(j) + U_{i^+i^+}\alpha^{(n)}\big(i^+\big) \\ &= h(i) + (U_{i^+i^+} - U_{ii^+})\alpha^{(n)}\big(i^+\big). \end{aligned}$$

Therefore

$$\alpha^{(n)}\big(i^+\big) = \frac{h(i^+) - h(i)}{U_{i^+i^+} - U_{ii^+}},$$

and $\alpha^{(n)}$ is a measure in B^n. Let us show that these measures verify the consistence property. We have

$$\begin{aligned} \text{for } |i| \le n: \quad h(i) &= \sum_{j\in \widetilde{B}^{n+1}} U_{ij}\alpha^{(n+1)}(j) = \sum_{k\in B^n} U_{ik}\left(\sum_{j\in S_k}\alpha^{(n+1)}(j)\right) \\ &= \sum_{k\in B^n} U_{ik}\alpha^{(n)}(k). \end{aligned}$$

From the uniqueness of $\alpha^{(n)}$ we deduce $\alpha^{(n)}(k) = \sum_{j\in S_k}\alpha^{(n+1)}(j)$. Then the consistence property is verified. The total mass of $\alpha^{(n)}$ is obtained from $h(r) = w_0\sum_{j\in B^n}\alpha^{(n)}(j)$. Then there exists a finite measure in the boundary such that $h(i) = \int_{\partial_\infty} U_{i\xi}\,d\nu(\xi)$, for $i \in I$. □

Remark 3.5 The measure ν in the previous result can be singular with respect to μ. For example, when ξ is a point outside the closed support of μ, the function $h(i) = U_{i\xi}$ is represented by the measure $\nu = \delta_\xi$, which is clearly singular with respect to μ.

The next result is a representation, as an integral of U, of all harmonic functions that satisfy a certain finite-variation condition.

Theorem 3.2 *Assume that $h : I \to \mathbb{R}$ is a bounded harmonic function. Then, there exists a finite signed measure ν such that*

$$h(i) = \int_{\partial_\infty} U_{i\xi}\,d\nu(\xi) \quad \textit{for every } i \in I \tag{3.22}$$

if and only if the following condition holds:

$$\sup_{n\ge 1} \frac{1}{w_n - w_{n-1}} \sum_{j\in B^n} \big|h(j) - h\big(j^-\big)\big| < \infty. \tag{3.23}$$

In particular, if this condition holds, then $h = h^+ - h^-$ is the difference of two increasing nonnegative harmonic functions h^+, h^- given by the positive and negative parts of ν.

Proof Let us first assume that h is strictly positive. If (3.22) holds, then

$$h(i) - h(i^-) = \int (U_{i\xi} - U_{i^-\xi})\, d\nu(\xi) = (U_{ii} - U_{i^-i})\nu\big(\partial_\infty(i)\big),$$

so $|h(i) - h(i^-)| \le (w_n - w_{n-1})|\nu|(\partial_\infty(i))$ for $i \in B^n$. Summing over B^n, this inequality yields

$$\frac{1}{w_n - w_{n-1}} \sum_{i\in B^n} \big|h(i) - h\big(i^-\big)\big| \le |\nu|(\partial_\infty) < \infty.$$

Let us now assume that (3.23) holds. As in the proof of Theorem 3.1, we have that, for all n and all $i \in I$ such that $|i| \le n$,

$$h(i) = \sum_{j\in B^n} U_{ij}\, \alpha^{(n)}(j),$$

where

$$\alpha^{(n)}(j) = \frac{h(j) - h(j^-)}{U_{jj} - U_{jj^-}} = \frac{h(j) - h(j^-)}{w_n - w_{n-1}}.$$

Let us define the signed measure ν_n by $\nu_n(\partial_\infty(j)) = \alpha^n(j)$. Then we obtain that $\nu_n(\partial_\infty) = h(r)/w_0 > 0$ and

$$\sup_{n\ge 1} |\nu_n|(\partial_\infty) < \infty.$$

Therefore, there exists a subsequence (ν_{n_k}) converging weakly to a finite signed measure $\nu \ne 0$. Moreover, $\nu(\partial_\infty) = h(r)/w_0$, and since $U_{i\bullet}$ is a bounded continuous function, we get

$$h(i) = \lim_k \int U_{i\xi}\, d\nu_{n_k}(\xi) = \int U_{i\xi}\, d\nu(\xi).$$

Let us prove the general case. Recall that the function $\ell(i) =: 1 - \bar{g}(i) = \mathbb{P}_i(X_\zeta \in \partial_\infty)$ is nonnegative and harmonic, which is also increasing with limit 1 at the boundary. Then

$$\ell(i) = \int U_{i\xi}\big(W^{-1}\mathbf{1}\big)(\xi)\, d\mu(\xi) = \frac{\ell(r)}{w_0} \int U_{i\xi}\, d\mu(\xi) = \int U_{i\xi}\, d\nu(\xi),$$

where ν is the finite measure $\frac{\ell(r)}{w_0}\mu$. Since $\ell(i) \ge \ell(r) > 0$, we can take a large constant C such that the function $\bar{h} = h + C\ell$ is a nonnegative bounded harmonic function. It is straightforward to check that h satisfies (3.23) if and only if $\bar{h}$ satisfies it, whence the result follows. □

Let have a close look at (3.23). Since h is harmonic, we have for $n = |j|$,

$$\frac{1}{w_n - w_{n-1}}\big(h(j) - h\big(j^-\big)\big) = Q_{jj^-}\big(h(j) - h\big(j^-\big)\big) = \sum_{j^+} Q_{jj^+}\big(h\big(j^+\big) - h(j)\big).$$

Then,

$$\frac{1}{w_n - w_{n-1}}\big|h(j) - h\big(j^-\big)\big| \le \sum_{j^+} \frac{1}{w_{n+1} - w_n}\big|h\big(j^+\big) - h(j)\big|,$$

implying that $\frac{1}{w_n - w_{n-1}} \sum_{j\in B^n} |h(j) - h(j^-)|$ increases with n.

Now, we go back to the problem of a formula for the Martin kernel in terms of U and μ. For this reason, we prove the following result.

Lemma 3.2 *For $\eta \in \partial_\infty$, $i \in I$, $n \ge 1$, we have*

$$\begin{aligned}\mathbb{P}_i\big\{X_\zeta \in C^n(\eta)\big\} = \mu\big(C^n(\eta)\big)\Bigg[&\frac{U_{i\eta}}{G_{|i\wedge\eta|+1}(\eta)}\,\mathbf{1}_{I\setminus[\eta(n),\infty)}(i)\\ &+\frac{1}{G_n(\eta)}\mathbb{E}(U_{i\bullet}|\mathcal{F}_n)(\eta)\,\mathbf{1}_{[\eta(n),\infty)}(i)\\ &+\sum_{k=0}^{n-1}\bigg(\frac{1}{G_k(\eta)} - \frac{1}{G_{k+1}(\eta)}\bigg)\mathbb{E}(U_{i\bullet}|\mathcal{F}_k)(\eta)\mathbf{1}_{[\eta(k),\infty)}(i)\Bigg].\end{aligned}$$

In particular, if η is in the closed support of μ and $n > |i \wedge \eta|$, we get

$$\frac{\mathbb{P}_i\{X_\zeta \in C^n(\eta)\}}{\mu(C^n(\eta))} = \frac{U_{i\eta}}{G_{|i\wedge\eta|+1}(\eta)} + \sum_{k=0}^{|i\wedge\eta|}\bigg(\frac{1}{G_k(\eta)} - \frac{1}{G_{k+1}(\eta)}\bigg)\mathbb{E}(U_{i\bullet}|\mathcal{F}_k)(\eta). \quad (3.24)$$

Proof Let η and n be fixed. We denote $C^k = C^k(\eta)$ and $A^k = [\eta(k), \infty)$ for $k \in \mathbb{N}$. From (3.19) we have $h_\eta(i) := \mathbb{P}_i\{X_\zeta \in C^n\} = \int_{\partial_\infty} U_{i\xi}(W^{-1}\mathbf{1}_{C^n})(\xi)\mu(d\xi)$. Now, let us compute $\rho^k(i) := \int_{\partial_\infty} U_{i\xi}\mathbf{1}_{C^k}(\xi)\mu(d\xi)$.

We examine two different cases. If $i \notin A^k$, then $U_{i\xi} = U_{i\eta(k)}$ for every $\xi \in C^k$, and so $\rho^k(i) = U_{i\eta(k)}\mu(C^k)$. If $i \in A^k$, then $\rho^k(i) = \sum_{\substack{j\in A^k\\ |j|=|i|}} U_{ij}\mu(\partial_\infty(j))$. We summarize these relations in

$$\rho^k(i) = U_{i\eta(k)}\mu\big(C^k\big)\mathbf{1}_{I\setminus A^k}(i) + \sum_{\substack{j\in A^n\\ |j|=|i|}} U_{ij}\mu\big(\partial_\infty(j)\big)\mathbf{1}_{A^k}(i). \quad (3.25)$$

Now we use (3.16) to get

$$
\begin{aligned}
&\int U_{i\xi}\big(W^{-1}\mathbf{1}_{C^n}\big)(\xi)\mu(d\xi)\\
&\quad=\frac{1}{G_n}\Bigg[U_{i\eta(n)}\mu\big(C^n\big)\mathbf{1}_{I\setminus A^n}(i)+\sum_{\substack{j\in A^n\\ |j|=|i|}}U_{ij}\mu\big(\partial_\infty(j)\big)\mathbf{1}_{A^n}(i)\Bigg]\\
&\qquad+\sum_{k=0}^{n-1}\bigg(\frac{1}{G_k}-\frac{1}{G_{k+1}}\bigg)\frac{\mu(C^n)}{\mu\big(C^k\big)}\bigg[U_{i\eta(k)}\mu\big(C^k\big)\mathbf{1}_{I\setminus A^k}(i)\\
&\qquad+\sum_{\substack{j\in A^k\\ |j|=|i|}}U_{ij}\mu\big(\partial_\infty(j)\big)\mathbf{1}_{A^k}(i)\bigg].
\end{aligned}
$$

Using that

$$
\mathbb{E}(U_{i\bullet}|\mathcal{F}_n)(\eta)=\begin{cases}U_{i\eta(k)} & \text{if } i\notin A^k,\\ \sum_{\substack{j\in A^n\\ |j|=|i|}}U_{ij}\mu(\partial_\infty(j)) & \text{if } i\in A^k,\end{cases}
$$

yields

$$
\begin{aligned}
&\int U_{i\xi}\big(W^{-1}\mathbf{1}_{C^n}\big)(\xi)\mu(d\xi)\\
&\quad=\mu\big(C^n\big)\Bigg[\sum_{k=0}^{n-1}\bigg(\frac{1}{G_k}-\frac{1}{G_{k+1}}\bigg)\mathbb{E}(U_{i\bullet}|\mathcal{F}_k)(\eta)\mathbf{1}_{A^k}(i)+\frac{1}{G_n}\mathbb{E}(U_{i\bullet}|\mathcal{F}_n)(\eta)\mathbf{1}_{A^n}(i)\Bigg]\\
&\qquad+\mu\big(C^n\big)\Bigg[\sum_{k=0}^{n-1}\bigg(\frac{1}{G_k}-\frac{1}{G_{k+1}}\bigg)U_{i\eta(k)}\,\mathbf{1}_{I\setminus A^k}(i)+\frac{1}{G_n}U_{i\eta(n)}\,\mathbf{1}_{I\setminus A^n}(i)\Bigg].
\end{aligned}
$$

Now $i\in I\setminus A^k$ implies $k>|i\wedge\eta|$. Since $U_{i\eta(k)}=U_{i\eta}$ for $k\geq|i\wedge\eta|$, we can simplify the last term in the previous equation to $\frac{\mu(C^n)}{G_{|i\wedge\eta|+1}}U_{i\eta}\mathbf{1}_{I\setminus A^n}(i)$. Then we get

$$
\begin{aligned}
\mathbb{P}_i\big\{X_\zeta\in C^n(\eta)\big\}=\mu\big(C^n(\eta)\big)\Bigg[&\frac{U_{i\eta}}{G_{|i\wedge\eta|+1}}\,\mathbf{1}_{I\setminus A^n}(i)+\frac{1}{G_n}\mathbb{E}(U_{i\bullet}|\mathcal{F}_n)(\eta)\,\mathbf{1}_{A^n}(i)\\
&+\sum_{k=0}^{n-1}\bigg(\frac{1}{G_k(\eta)}-\frac{1}{G_{k+1}(\eta)}\bigg)\mathbb{E}(U_{i\bullet}|\mathcal{F}_k)(\eta)\mathbf{1}_{A^k}(i)\Bigg].
\end{aligned}
$$

□

Theorem 3.3 *Let $i\in I$, and let η be a point in the closed support of μ. Then the Martin kernel has the representation*

$$
\kappa(i,\eta)=\frac{1}{\mathbb{P}_r\{X_\zeta\in\partial_\infty\}}\sum_{k=0}^{|i\wedge\eta|+1}\frac{1}{G_k(\eta)}\big(\mathbb{E}(U_{i\bullet}|\mathcal{F}_k)(\eta)-\mathbb{E}(U_{i\bullet}|\mathcal{F}_{k-1})(\eta)\big),\tag{3.26}
$$

where by convention $\mathbb{E}(\,|\mathcal{F}_{-1})=0$.

Proof We use Lemma 3.2 and the equality $\frac{U_{r\eta}}{G_0} = \frac{w_0}{G_0} = \mathbb{P}_r\{X_\zeta \in \partial_\infty\}$ to get

$$\kappa(i,\eta) = \frac{1}{\mathbb{P}_r\{X_\zeta \in \partial_\infty\}}\left[\frac{U_{i\eta}}{G_{|i\wedge\eta|+1}(\eta)} + \sum_{k=0}^{|i\wedge\eta|}\left(\frac{1}{G_k(\eta)} - \frac{1}{G_{k+1}(\eta)}\right)\mathbb{E}(U_{i\bullet}|\mathcal{F}_k)(\eta)\right],$$

and the result follows. □

Corollary 3.2 *For $i \in I$ fixed, the Martin kernel $\kappa(i,\bullet)$ is the image of $U_{i\bullet}$ by a stochastic integral operator, in fact,*

$$\kappa(i,\eta) = \sum_{k=0}^{\infty} \widetilde{G}_k^{(i)}(\eta)\big(\mathbb{E}(U_{i\bullet}|\mathcal{F}_k) - \mathbb{E}(U_{i\bullet}|\mathcal{F}_{k-1})\big)(\eta),$$

where $\widetilde{G}^{(i)} = (\widetilde{G}_k^{(i)} : k \in \mathbb{N})$ is an $\mathcal{F}$-predictable process.

Proof It suffices to take $\widetilde{G}_k^{(i)} = \mathbf{1}_{D_k^{(i)}}\mathbb{P}_r\{X_\zeta \in \partial_\infty\}^{-1}G_k{}^{-1}$, where $D_k^{(i)} = \{\xi \in \partial_\infty : \xi \wedge i \geq k-1\}$ is an $\mathcal{F}_{k-1}$-measurable set. □

4 Trees Potential without Absorption

4.1 Reflecting at the Root

Let $(I,\mathcal{T})$ be a tree rooted at r. In this section we consider the case where r is a reflecting barrier. As before, we take a strictly positive and strictly increasing sequence $(w_n : n \in \mathbf{N})$ and consider a symmetric q-matrix Q on $I \times I$, supported on the tree, and the diagonal defined as in (2.2) except at the pair (r,r), where we put $Q_{rr} = -\frac{|S_r|}{w_1 - w_0}$. It is straightforward to check that Q is conservative: $\sum_{j\in I} Q_{ij} = 0$ for every $i \in I$. We assume that the Markov process (X_t) associated to Q is transient, so $\mathbb{P}_r\{X_\zeta \in \partial_\infty\} = 1$, and that all points in ∂_∞ are regular.

The aim is to obtain a representation of the potential V and the Martin kernel in terms of the tree matrix $U = (U_{ij} = w_{|i\wedge j|} : i,j \in I)$. For this purpose, consider the translated matrix $U^{(a)} := U + a$ for $a > 0$, which is the tree matrix associated to the level function $w_n^{(a)} = w_n + a$. Define the matrix $Q^{(a)}$ on $I \times I$ as in (2.2) with respect to this level function. At (r,r) it takes the value $Q_{rr}^{(a)} = Q_{rr} - \frac{1}{w_0 + a}$. The matrices $Q^{(a)}$ and Q in $I \times I$ only differ at (r,r). We note that $Q^{(a)}$ is not conservative at r.

As a tends to infinity, $Q^{(a)}$ converges to Q, and the associated processes also converge. In fact, a coupling argument allows us to construct an increasing sequence of stopping times $T^{(a)} \uparrow_{a\to\infty} \infty$ such that

$$X_t^{(a)} = X_t \quad \text{if } t < T^{(a)} \quad \text{and} \quad X_t^{(a)} = \partial_r \quad \text{if } t \geq T^{(a)}$$

is a Markov process with generator $Q^{(a)}$. The lifetime variables $\zeta^{(a)}$ and ζ associated respectively to $X^{(a)}$ and X verify $\zeta^{(a)} = \zeta \wedge T^{(a)}$. From this representation it also follows immediately that the potentials $V^{(a)}$ and V associated to $Q^{(a)}$ and Q, respectively, verify $\forall i, j$, $V_{ij}^{(a)} \uparrow_{a\to\infty} V_{ij}$. Therefore, the representation (3.3) reads

$$U_{ij}^{(a)} - V_{ij}^{(a)} = \int_{\partial_\infty} U_{\eta j}^{(a)} \mathbb{P}_i\big\{X_\zeta \in d\eta, \zeta \le T^{(a)}\big\}$$

or equivalently $U_{ij} - V_{ij}^{(a)} = \int_{\partial_\infty} U_{\eta j}\, \mathbb{P}_i\{X_\zeta \in d\eta, \zeta \le T^{(a)}\} - a\mathbb{P}_i\{T^{(a)} < \zeta\}$. Passing to the limit as $a \to \infty$, we obtain that $\lim_{a\to\infty} a\mathbb{P}_i\{T^{(a)} < \zeta\}$ exists and moreover

$$U_{ij} - V_{ij} = \int_{\partial_\infty} U_{\eta j}\mathbb{P}_i\{X_\zeta \in d\eta\} - \lim_{a\to\infty} a\mathbb{P}_i\big\{T^{(a)} < \zeta\big\}.$$

Substituting j by r in the last equality and using that $U_{ir} = U_{\eta r} = w_0$, we find $\lim_{a\to\infty} a\mathbb{P}_i\{T^{(a)} < \zeta\} = V_{ir}$, and therefore we get

$$U_{ij} - V_{ij} = \int_{\partial_\infty} U_{\eta j}\, \mathbb{P}_i\{X_\zeta \in d\eta\} - V_{ir}. \tag{4.1}$$

Now, taking $j \to \xi \in \partial_\infty^{\mathrm{reg}}$, we obtain

$$V_{ir} = \int_{\partial_\infty} U_{\eta\xi}\, \mathbb{P}_i\{X_\zeta \in d\eta\} - U_{i\xi} = \int_{\partial_\infty} (U_{\eta\xi} - U_{i\xi})\, \mathbb{P}_i\{X_\zeta \in d\eta\}.$$

Thus, we have proven the equality

$$U_{ij} - V_{ij} = \int_{\partial_\infty} (U_{\eta j} + U_{i\xi} - U_{\eta\xi})\, \mathbb{P}_i\{X_\zeta \in d\eta\}, \tag{4.2}$$

which is independent of $\xi \in \partial_\infty^{\mathrm{reg}}$. Integrating (4.2) with respect to $\mathbb{P}_j\{X_\zeta \in d\xi\}$ gives

$$\begin{aligned} U_{ij} - V_{ij} = {} & \int_{\partial_\infty} U_{\eta j}\, \mathbb{P}_i\{X_\zeta \in d\eta\} + \int_{\partial_\infty} U_{i\xi}\mathbb{P}_j\{X_\zeta \in d\xi\} \\ & - \int_{\partial_\infty}\int_{\partial_\infty} U_{\eta\xi}\, \mathbb{P}_j\{X_\zeta \in d\xi\}\mathbb{P}_i\{X_\zeta \in d\eta\}. \end{aligned}$$

The Martin kernel $\kappa^{(a)}$ associated to $Q^{(a)}$ can be computed as in Theorem 3.3. Take $i \in I$, $\eta \in \partial_\infty$, and $n > |i \wedge \eta|$; then

$$\kappa^{(a)}(i, \eta) = \frac{\mathbb{P}_i\{X_\zeta \in C^n(\eta), \zeta \le T^{(a)}\}}{\mathbb{P}_r\{X_\zeta \in C^n(\eta), \zeta \le T^{(a)}\}}.$$

Therefore, the Martin kernel is continuous with respect to a. Passing to the limit as $a \to \infty$ and using the representation (3.26), we obtain

$$\kappa(i, \eta) = \lim_{a\to\infty} \sum_{k=0}^{|i\wedge\eta|+1} \frac{1}{G_k^{(a)}} \big(\mathbb{E}_{\mu^{(a)}}\big(U_{i\bullet}^{(a)} | \mathcal{F}_k\big)(\eta) - \mathbb{E}_{\mu^{(a)}}\big(U_{i\bullet}^{(a)} | \mathcal{F}_{k-1}\big)(\eta)\big), \tag{4.3}$$

where $G_k^{(a)} = G_k^{(a)}(\eta) = \sum_{n\geq k}(w_n^{(a)} - w_{n-1}^{(a)})\ \mu^{(a)}(C^n(\eta))$ and $\mu^{(a)}(\bullet) = \frac{\mathbb{P}_r\{X_\zeta\in\bullet,\zeta\leq T^{(a)}\}}{\mathbb{P}_r\{\zeta\leq T^{(a)}\}}$. We notice that $G_0^{(a)} = \frac{w_0+a}{\mathbb{P}_r\{\zeta\leq T^{(a)}\}}$, and for $k\geq 1$, it holds

$$
\begin{aligned}
G_k^{(a)} &= G_0^{(a)} - \sum_{n=0}^{k-1}\big(w_n^{(a)} - w_{n-1}^{(a)}\big)\mu^{(a)}\big(C^n(\eta)\big)\\
&= (w_0+a)\frac{\mathbb{P}_r\{T^{(a)}<\zeta\}}{\mathbb{P}_r\{\zeta\leq T^{(a)}\}} - \sum_{n=1}^{k-1}(w_n - w_{n-1})\ \mu^{(a)}\big(C^n(\eta)\big).
\end{aligned}
$$

The previous computations are summarized in the following result.

Theorem 4.1 *Let $\mu(\bullet) = \mathbb{P}_r\{X_\zeta\in\bullet\}$. Consider $G_0(\eta) := \int U_{\eta\xi}\mathbb{P}_r\{X_\zeta\in d\xi\}$ and $G_k := \lim_{a\to\infty} G_k^{(a)}$. Then, $G_0(\eta) = V_{rr} + w_0$ is a constant, and $(G_k : k\geq 1)$ is a positive decreasing predictable process that verifies*

$$
G_k(\eta) = G_0 - \sum_{n=0}^{k-1}(w_n - w_{n-1})\mu\big(C^n(\eta)\big) = \sum_{n\geq k}(w_n - w_{n-1})\mu\big(C^n(\eta)\big) \quad \textit{for } k\geq 1;
$$

moreover, the following representation holds:

$$
\kappa(i,\eta) = 1 + \sum_{k=1}^{|i\wedge\eta|+1}\frac{1}{G_k(\eta)}\big(\mathbb{E}_\mu(U_{i\bullet}|\mathcal{F}_k)(\eta) - \mathbb{E}_\mu(U_{i\bullet}|\mathcal{F}_{k-1})(\eta)\big). \tag{4.4}
$$

Remark 4.1 It can be shown that $\mu^{(a)}$ does not depend on $a\geq 0$ (recall that $\mu^{(0)} = \mu$ is the exit measure defined in (3.6) for the chain absorbed at ∂_r). Indeed this follows from the independence relation

$$
\mathbb{P}_r\big\{X_\zeta\in\bullet,\zeta\leq T^{(a)}\big\} = \mathbb{P}_r\{X_\zeta\in\bullet\}\mathbb{P}_r\big\{\zeta\leq T^{(a)}\big\};
$$

then $\mu^{(a)} = \mu$ for $a\geq 0$. Further, if N_r^* is the number of visits in the strict future to r of the discrete skeleton of (X_t), then a simple argument shows that $\mu(\bullet) = \mathbb{P}_r\{X_\zeta\in\bullet|N_r^* = 0\}$.

Remark 4.2 Let $W^{(a)} = \sum_{n\in\mathbb{N}} G_n^{(a)}(\mathbb{E}_{\mu^{(a)}}(\,|\mathcal{F}_n) - \mathbb{E}_{\mu^{(a)}}(\,|\mathcal{F}_{n-1}))$. Then, $\lim_{a\to\infty}(W^{(a)})^{-1} = \sum_{n\geq 1} G_n^{-1}(\mathbb{E}_\mu(\,|\mathcal{F}_n) - \mathbb{E}_\mu(\,|\mathcal{F}_{n-1}))$ verifies $\lim_{a\to\infty}(W^{(a)})^{-1}\mathbf{1} = 0$, and it is the generator of a Markov process defined on ∂_∞. This process has been partially studied in [24].

4.2 Potential for Homogeneous Trees

In this section we consider a random walk on a homogeneous tree of degree $p+1\geq 3$. In this case the previous calculations supply a closed form of the Martin kernel. Most of the following formulas are well known, see, for instance, [30]. We assume that $\mathcal{T}$ is an infinite rooted tree with $|S_r| = p+1$ and $|S_i| = p$ for $i \neq r$. As a weight function, we take $w_n = n+1$ and assume that r is reflecting. In this way we have

$$Q_{ii^+} = 1, \quad Q_{ii} = -(p+1) \quad \text{for } i \in I, \quad \text{and} \quad Q_{ii^-} = 1 \quad \text{for } i \neq r.$$

It is well known that this tree matrix is transient for all $p \geq 2$. Also it is clear that μ is the uniform measure on ∂_∞ and that all points in ∂_∞ are regular. Let us now compute the quantities involved on (4.4) to get the Martin kernel.

We fix $i \in I$, $\eta \in \partial_\infty$ and put $m = |i|$, $n = |i \wedge \eta|$. For $m = 0$, we have $i = r$ and $\kappa(r, \eta) = 1$. For the rest, we assume $m \geq 1$. We set $C^k = C^k(\eta)$ for all $k \in \mathbb{N}$. Therefore, $\mu(C^k) = ((p+1)p^{k-1})^{-1}$ for all $k \geq 1$, and $\mu(C^0) = 1$. Then, for $k \geq 1$,

$$G_k(\eta) = \sum_{l \geq k} (w_l - w_{l-1}) \mu\big(C^l(\eta)\big) = \sum_{l \geq k} \frac{1}{(p+1)p^{l-1}} = \frac{1}{(p^2-1)p^{k-2}}.$$

On the other hand, an explicit computation of $\mathbb{E}_\mu(U_{i\bullet}|\mathcal{F}_k)(\eta) = \frac{1}{\mu(C^k)} \int_{C^k} U_{i\xi}\, \mu(d\xi)$ gives

$$\mathbb{E}_\mu(U_{i\bullet}|\mathcal{F}_0)(\eta) = \frac{1}{(p+1)p^{m-1}} \left(m + 1 + p^m + (p-1) \sum_{t=2}^{m} t p^{m-t} \right),$$

$$\mathbb{E}_\mu(U_{i\bullet}|\mathcal{F}_k)(\eta) = p^{k-m} \left(m + 1 + (p-1) \sum_{t=k+1}^{m} t p^{m-t} \right) \quad \text{for } 1 \leq k \leq n.$$

Then we obtain (see, for example, [30], Theorem 8.1) $\kappa(i, \eta) = p^{2n-m}$, where $m = |i|$, $n = |i \wedge \eta|$. In particular, if $|i \wedge \eta| = 0$, we get $p^{-m} = \kappa(i, \eta) = \frac{\mathbb{P}_i\{X_\zeta \in C^k(\eta)\}}{\mathbb{P}_r\{X_\zeta \in C^k(\eta)\}} = \mathbb{P}_i\{T_r < \infty\}$, where $k \geq 1$. In a similar way we obtain $\mathbb{P}_i\{T_{i^-} < \infty\} = p^{-1}$.

With respect to the potential V we have from (4.1)

$$V_{rj} - V_{rr} = V_{rr}\big(\mathbb{P}_j\{T_r < \infty\} - 1\big) = 1 - \int U_{j\eta} \mathbb{P}_r\{X_\zeta \in d\eta\}.$$

We get

$$V_{rr} = \frac{p}{(p+1)(p-1)} \quad \text{and more generally} \quad V_{jr} = \frac{p^{1-|j|}}{(p+1)(p-1)}.$$

A simple argument based on time reversal shows that $\mathbb{P}_r\{T_j < \infty\} = \mathbb{P}_j\{T_r < \infty\} = p^{-|j|}$, and in general $\mathbb{P}_i\{T_j < \infty\} = p^{-|geod(i,j)|}$. Since $V_{jr} = \mathbb{P}_r\{T_j < \infty\} V_{jj}$, one deduces $V_{jj} = \frac{p}{(p+1)(p-1)}$. Analogously one finds $V_{ij} = \frac{p^{1-|geod(i,j)|}}{(p+1)(p-1)}$. Finally, from (4.1) we get that

$$\int U_{j\eta}\, \mathbb{P}_i\{X_\zeta \in d\eta\} = |i \wedge j| + 1 + \frac{p^{1-|i|} - p^{1-|geod(i,j)|}}{(p+1)(p-1)}.$$

5 Ultrametricity

There is a wide literature concerning ultrametricity, but it is not a common notion in potential theory. So, we supply some basic properties which are a consequence of the ultrametric inequality. The core of this section is Sect. 5.3 , where we construct the Markov semigroup associated to an ultrametric matrix, in terms of a minimal tree matrix extension. We also study the representation of harmonic functions in Sect. 5.4.

5.1 Basic Notions and the Minimal Rooted Tree Extension

We give conditions in order that an ultrametric matrix can be immersed in a countable and locally finite tree. It is known that a tree structure is behind an ultrametric (for a deep study of this relation, see [16]), but we prefer here to give an explicit construction because it allows a better understanding of the main results of this section.

Most of the properties we present are easily deduced from the ultrametric inequality, so they are established without a proof. We note that up to Lemma 5.2 the set I will have no restriction.

Definition 5.1 $U=(U_{ij}:i,j\in I)$ is an ultrametric arrangement if it is symmetric, that is, $U_{ij}=U_{ji}$ for any couple $i,j\in I$ and verifies the ultrametric inequality

$$U_{ij}\ge \min\{U_{ik},U_{kj}\}\quad \text{for any } i,j,k\in I.$$

In particular, $U_{ii}\ge U_{ij}$ for any $i,j\in I$, so $U_{ij}=U_{ii}\Rightarrow U_{jj}\ge U_{ii}$. Observe that for any triple $i_1,i_2,i_3\in I$, there exists a permutation φ of $\{1,2,3\}$ such that

$$U_{i_{\varphi(1)}i_{\varphi(2)}}=\min\{U_{i_{\varphi(2)}i_{\varphi(3)}},U_{i_{\varphi(3)}i_{\varphi(1)}}\}.$$

Hence, $U_{ik}>U_{kj}\Rightarrow U_{ik}>U_{kj}=U_{ij}$ and $U_{ik}=U_{kj}\Rightarrow U_{ij}\ge U_{ik}=U_{kj}$.

Consider the equivalence relation

$$i\sim j\quad\Longleftrightarrow\quad (\forall k\in I:\ U_{ik}=U_{jk}).$$

Notice that $i\sim j\quad\Longleftrightarrow\quad U_{ii}=U_{ij}=U_{jj}$. Let us introduce the relation

$$i\preceq j\quad\Longleftrightarrow\quad U_{ij}=U_{ii}.$$

From $U_{jk}\ge\min\{U_{ji},U_{ik}\}=\min\{U_{ii},U_{ik}\}=U_{ik}$ we get

$$i\preceq j\quad\Longleftrightarrow\quad U_{i\bullet}\le U_{j\bullet}\quad(\text{that is, }\forall k\in I: U_{ik}\le U_{jk}),$$

so the relation $\preceq$ is a preorder, and its associated equivalence relation is $\sim$. On the other hand, $i\preceq j\Rightarrow U_{ii}\le U_{jj}$.

The left and right intervals defined by $i\in I$ are respectively

$$[i,\infty)^U=\{j\in I:i\preceq j\}\quad\text{and}\quad(-\infty,i]^U=\{j\in I:j\preceq i\}.$$

Notice that for any $i\in I$, the set $(-\infty,\ i]^U$ is totally preordered by $\preceq$, that is, for all $j,k\in(-\infty,\ i]^U$, it holds $j\preceq k$ or $k\preceq j$.

In the sequel we will assume the following separation condition:

$$i \sim j \quad \Longleftrightarrow \quad i = j. \tag{$H1$}$$

Property $(H1)$ is equivalent to the fact that $\preceq$ is an order or equivalently to the relation $i \neq j \Rightarrow U_{ij} < \max\{U_{ii}, U_{jj}\}$. We point out that if I is finite and $U > 0$, condition $(H1)$ is equivalent to the nonsingularity of U (see [12, 25], or [27]).

We denote $\mathcal{W} = \{U_{ij} : i, j \in I\}$ the set of values of U. To every $w \in \mathcal{W}$, we associate the nonempty set $J(w) = \{i \in I : U_{ii} \geq w\}$ and the relation

$$i \equiv_w j \quad \Longleftrightarrow \quad U_{ij} \geq w.$$

The ultrametric inequality implies that $\equiv_w$ is an equivalence relation in $J(w)$. By E^w we mean the equivalence class of $\equiv_w$, and E_i^w denotes the equivalence class containing $i \in J(w)$. In the case $U_{ii} < w$, that is, $i \notin J(w)$, we put $E_i^w = \emptyset$. As usual, $J(w)/\equiv_w$ denotes the set of equivalence classes of elements of $J(w)$.

Let us introduce the following set

$$\widetilde{I} = \big\{\big(E^w, w\big) : E^w \in J(w)/\equiv_w, w \in W\big\}.$$

The function

$$\mathbf{i}^U : I \to \widetilde{I}, \qquad \mathbf{i}^U(i) = \big(E_i^{U_{ii}}, U_{ii}\big)$$

is one-to-one. In fact, if $\mathbf{i}^U(i) = \mathbf{i}^U(j)$, then $U_{ij} \geq U_{ii} = U_{jj}$. From condition $(H1)$ we deduce $i = j$. In this way we identify $i \in I$ with $\mathbf{i}^U(i) = (E_i^{U_{ii}}, U_{ii}) \in \widetilde{I}$.

Observe that $E_i^{U_{ii}} = [i, \infty)^U$ for every $i \in I$. Also it holds

$$\big[w \leq w' \Rightarrow E^{w'} \subseteq E^w\big] \quad \text{and} \quad \big[\big(w \leq w', E^{w'} \neq E^w\big) \Rightarrow w < w'\big].$$

Lemma 5.1 *If $E^{w'} \nsubseteq E^w$ and $E^w \nsubseteq E^{w'}$, then*

$$\forall k, k' \in E^w,\ \forall \ell, \ell' \in E^{w'} : \quad U_{k\ell} = U_{k'\ell'} < \min\{w, w'\} \quad \textit{and} \quad E^w \cap E^{w'} = \emptyset.$$

Proof Let $k \in E^w \setminus E^{w'}$ and $l \in E^{w'} \setminus E^w$. Also take $k' \in E^w$, $l' \in E^{w'}$. Since $\equiv_w, \equiv_{w'}$ are equivalent relations on their respective domains, we get $U_{kl'} < w'$ and $U_{k'l} < w$. In particular, $U_{kl} < \min\{w, w'\}$. On the other hand, the definition of E^w implies $U_{kk'} \geq w$. Using the ultrametric property, we get $U_{k'l} \geq \min\{U_{k'k}, U_{kl}\} = U_{kl}$ and similarly $U_{kl} \geq U_{k'l}$, from which the equality $U_{kl} = U_{k'l}$ holds. In an analogous way the equality $U_{kl} = U_{kl'}$ is deduced, and we get that

$$k' \in E^w \setminus E^{w'} \quad \text{and} \quad l' \in E^{w'} \setminus E^w.$$

This implies that $E^w \cap E^{w'} = \emptyset$. By using again the ultrametricity we find

$$U_{k'l'} \geq \min\{U_{k'l}, U_{ll'}\} = U_{k'l} = U_{kl}.$$

By exchanging the roles of k with k' and l with l' we deduce the result. □

The previous result implies that either two classes E^w and $E^{w'}$ are disjoint or one is included in the other. Now we define $\widetilde{U}$, an extension of U to $\widetilde{I}$.

Definition 5.2 Let $\tilde{\imath} = (E^w, w) \in \widetilde{I}$, $\tilde{\jmath} = (E^{w'}, w') \in \widetilde{I}$. If $E^{w'} \subseteq E^w$ or $E^w \subseteq E^{w'}$, we put $\tilde{U}_{\tilde{\imath}\tilde{\jmath}} = \min\{w, w'\}$. If, on the contrary, $E^w \cap E^{w'} = \emptyset$, we put $\tilde{U}_{\tilde{\imath}\tilde{\jmath}} = U_{k\ell}$, where $k \in E^w$ and $\ell \in E^{w'}$.

By Lemma 5.1, $\widetilde{U}$ is well defined. On the other hand, it is straightforward to prove that for any $i, j \in I$, it holds $U_{ij} = \widetilde{U}_{\mathbf{i}^U(i)\,\mathbf{i}^U(j)}$. In this way $\widetilde{U}$ is an extension of U. Also, if $i \in E^w$, $j \in E^{w'}$, then $U_{ij} \geq \widetilde{U}_{\alpha\beta}$, where $\alpha = (E^w, w)$, $\beta = (E^{w'}, w')$.

Lemma 5.2 $\widetilde{U} = (\widetilde{U}_{\tilde{\imath}\tilde{\jmath}} : \tilde{\imath}, \tilde{\jmath} \in \widetilde{I})$ *is ultrametric.*

Proof For $u, v, w \in W$, consider the following elements of $\widetilde{I}$: $\tilde{\imath} = (E^u, u)$, $\tilde{\jmath} = (E^v, v)$, and $\tilde{k} = (E^w, w)$. Take $i \in E^u, j \in E^v, k \in E^w$. The proof is divided into two cases.

Case 1. We assume that $E^u \cap E^v = \emptyset$. The ultrametric property satisfied by U and the definition of $\widetilde{U}$ imply $\widetilde{U}_{\tilde{\imath}\tilde{\jmath}} = U_{ij} \geq \min\{U_{ik}, U_{kj}\} \geq \min\{\widetilde{U}_{\tilde{\imath}\tilde{k}}, \widetilde{U}_{\tilde{k}\tilde{\jmath}}\}$. Then the property holds.

Case 2. We assume, without loss of generality, that $E^u \subseteq E^v$ and $v \leq u$. If $E^w \cap E^v = \emptyset$, one gets that $\widetilde{U}_{\tilde{\jmath}\tilde{k}} = U_{jk} < v = \widetilde{U}_{\tilde{\imath}\tilde{\jmath}}$, and the property is verified. Finally, if $E^w \cap E^v \neq \emptyset$, then $\widetilde{U}_{\tilde{\jmath}\tilde{k}} = \min\{v, w\} \leq v = \widetilde{U}_{\tilde{\imath}\tilde{\jmath}}$. □

In the sequel we shall assume that I is countable and the following hypothesis holds:

$$\mathcal{W} = \{U_{ij} : i, j \in I\} \subset \mathbb{R}_+^* \text{ has no finite accumulation point.} \qquad (H2)$$

We put $\mathcal{W} = \{w_n : n \in \mathbb{N}\}$, where (w_n) increases with $n \in \mathbb{N}$, $w_0 > 0$. Under $(H2)$, we are able to define in $\widetilde{I}$ the following binary relation $\widetilde{\mathcal{T}}$. For $u, v \in \mathcal{W}$, we set

$$\big((E^u, u), (E^v, v)\big) \in \widetilde{\mathcal{T}} \quad \Longleftrightarrow \quad \exists n \in \mathbb{N} : \{u, v\} = \{w_n, w_{n+1}\} \quad \text{and} \quad E^u \cap E^v \neq \emptyset.$$

Two points $\tilde{\imath}, \tilde{\jmath} \in \widetilde{I}$ are said to be neighbors in $\widetilde{\mathcal{T}}$ if $(\tilde{\imath}, \tilde{\jmath}) \in \widetilde{\mathcal{T}}$.

Observe that if $((E^{w_n}, w_n), (E^{w_{n+1}}, w_{n+1})) \in \widetilde{\mathcal{T}}$, then $E^{w_{n+1}} \subseteq E^{w_n}$. The strict inclusion $E^{w_{n+1}} \neq E^{w_n}$ holds if and only if there exists a unique $i \in E^{w_n}$ such that $w_n = U_{ii}$. Indeed, it suffices to show the uniqueness. Let $i \in E^{w_n} \setminus E^{w_{n+1}}$; then $w_n \leq U_{ii} < w_{n+1}$. For any other $k \in E^{w_n}$ for which $U_{kk} = w_n$, it holds $U_{ik} \geq w_n$. We get $i \sim k$, and from $(H1)$ we conclude that $i = k$.

It is easy to see that $(\widetilde{I}, \widetilde{\mathcal{T}})$ is a tree rooted at $\tilde{r}$, where $\widetilde{U}_{\tilde{r}\tilde{r}} = w_0$. This point $\tilde{r}$ exists (and is unique) because either there exists $i_0 \in I$ verifying $U_{i_0 i_0} = w_0$, in which case $\tilde{r} = \tilde{\imath}_0$, or in the contrary, our construction adds a point $\tilde{r} \in \widetilde{I} \setminus I$ such that $\widetilde{U}_{\tilde{r}\tilde{r}} = w_0$.

By construction $\widetilde{U}$ is the minimal tree matrix extending U, that is, we can immerse $\widetilde{U}$ in any other tree extension of U. The tree $(\widetilde{I}, \widetilde{\mathcal{T}})$ supporting this minimal extension is locally finite if and only if the following assumption is verified:

$$\forall w \in \mathcal{W}, \quad \text{it holds } \big|J(w)/\equiv_w\big| < \infty. \qquad (H3)$$

Since $(\widetilde{I}, \widetilde{\mathcal{T}})$ is a rooted tree, all the concepts defined in the Introduction apply to it. In particular we denote by $\widetilde{\preceq}$ the order relation introduced in (1.1); by $\widetilde{\curlywedge}$ the associated minimum, by $[\tilde{\imath}, \infty)$ the branch born at $\tilde{\imath}$, and by $geod(\tilde{\imath}, \tilde{\jmath})$ the geodesic between two points in $\widetilde{I}$. Since we have identified $i \in I$ with $\mathbf{i}^U(i) \in \widetilde{I}$, all these concepts have a meaning for elements in I. In particular $\widetilde{\preceq}$ is an extension of the order relation $\preceq$, and we have the equality $[i, \infty)^U = [i, \infty) \cap I$.

Observe that the $\widetilde{\preceq}$−minimum in $(\widetilde{I}, \widetilde{\mathcal{T}})$ is characterized as follows. Take $(E^u, u), (E^v, v) \in \widetilde{I}$, and any $i \in E^u$; then $(E^u, u)\widetilde{\curlywedge}(E^v, v) = E_i^w$, where $w = \sup\{z \in \mathcal{W} : z \le u,\ E_i^z \supseteq E^v\}$. Notice that $E_i^{w_0} = I$.

5.2 Neighbor Relation

We will assume that hypotheses $(H1)$–$(H3)$ are fulfilled. The next definition is a notion of neighbor on I giving a better understanding of the embedding I in $\widetilde{I}$ and, in particular, describing how the elements in $\widetilde{I} \setminus I$ are surrounded by I.

Definition 5.3 Let $i \in I$.

- (i) The set $\mathcal{V}(i) = \{j \in I : j \neq i, geod(i, j) \cap I = \{i, j\}\}$ is called the set of U-neighbors of i. We also put $\mathcal{V}^*(i) = \mathcal{V}(i) \cup \{i\}$.
- (ii) The set $\mathcal{B}(i) = \{\tilde{\jmath} \in \widetilde{I} : geod(i, \tilde{\jmath}) \cap I \subseteq \{i, \tilde{\jmath}\}\}$ is called the attraction basin of i.

Notice that $\mathcal{V}^*(i) \subseteq \mathcal{B}(i)$. In the next result we summarize some useful properties of $\mathcal{B}(i)$, $\mathcal{V}(i)$, and $\mathcal{V}^*(i)$.

Lemma 5.3

- (i) *$\tilde{\jmath} \in \mathcal{B}(i) \setminus \mathcal{V}(i)$ if and only if $geod(\tilde{\jmath}, i) \cap I = \{i\}$. Moreover, $\mathcal{V}^*(i) = \mathcal{B}(i) \cap I$ and $\mathcal{B}(i) \setminus \mathcal{V}^*(i) = \mathcal{B}(i) \setminus I$.*
- (ii) *If $\tilde{\jmath} \in \mathcal{B}(i) \setminus \mathcal{V}^*(i)$, then all its neighbors in $(\widetilde{I}, \widetilde{\mathcal{T}})$ belong to $\mathcal{B}(i)$. Thus, $(\mathcal{B}(i), \widetilde{\mathcal{T}}|_{\mathcal{B}(i)\times\mathcal{B}(i)})$ is a tree. If we fix the root of this tree at i, then the set of leaves is $\mathcal{V}(i)$.*
- (iii) *For every $\tilde{\jmath} \notin \mathcal{B}(i)$, there exists a unique $k = k(i) \in \mathcal{V}(i)$ such that $geod(i, \tilde{\jmath}) \cap \mathcal{V}^*(i) = \{i, k\}$. This unique k also verifies that $k \in geod(\tilde{l}, \tilde{\jmath}) \cap \mathcal{V}^*(i)$ for every $\tilde{l} \in \mathcal{B}(i)$.*
- (iv) *For every $\tilde{\jmath} \in \widetilde{I}$, there exists $i \in I$ such that $\tilde{\jmath} \in \mathcal{B}(i)$.*
- (v) *For $j \in \mathcal{V}(i)$, either $(i, j) \in \widetilde{\mathcal{T}}$, that is, i, j are neighbors on $\widetilde{\mathcal{T}}$, or there is a unique $\tilde{k} \in \widetilde{I} \setminus I$ such that $(\tilde{k}, i) \in \widetilde{\mathcal{T}}$ and $\tilde{k} \in \mathcal{B}(j) \cap geod(i, j)$.*

Proof (i) and (ii) are straightforward from the definitions.

(iii) Take $\tilde{\jmath} \notin \mathcal{B}(i)$. If $geod(\tilde{\jmath}, i) \cap \mathcal{V}^*(i) = \{i\}$, then $geod(\tilde{\jmath}, i) \cap I = \{i\}$. In fact, if this intersection contains another point $\ell \in I$ and if we take $m \in (geod(\ell, i) \cap I) \setminus \{i\}$, the closest point to i, we obtain $m \in \mathcal{V}^*(i)$, which is a contradiction. Therefore, $geod(\tilde{\jmath}, i) \cap I = \{i\}$ and then $\tilde{\jmath} \in \mathcal{B}(i)$, which is also a contradiction.

Thus we can assume that $|geod(\tilde{\jmath}, i) \cap \mathcal{V}^*(i)| \geq 2$. If this intersection has at least three points, from the inclusion $geod(\tilde{\jmath}, i) \subseteq geod(\tilde{\ell}, \tilde{\jmath}) \cup geod(\tilde{\ell}, i)$ for any $\tilde{\ell} \in \widetilde{I}$ we would find a point $k \in \mathcal{V}^*(i)$ for which $geod(k, i) \cap I$ contains at least three points. This is a contradiction, and the result follows.

(iv) For $\tilde{j}$ and $k \in I$, we consider $geod(\tilde{j}, k)$. The first point in this geodesics (when starting from $\tilde{j}$) belonging to I makes the job.

(v) If i, j are not neighbors in $\widetilde{\mathcal{T}}$, then $geod(i, j)$ contains strictly $\{i, j\}$. Take $\tilde{k} \neq i$ the closest point to i in $geod(i, j)$. Clearly $\tilde{k} \in \widetilde{I} \setminus I$, otherwise $j \notin \mathcal{V}^*(i)$. By the same reason $geod(\tilde{k}, j) \cap I = \{j\}$, and therefore $\tilde{k} \in \mathcal{B}(j)$. □

Let us fix some $\tilde{j} \in \widetilde{I} \setminus I$. By Lemma 5.3 there exists $i \in I$ such that $\tilde{j} \in \mathcal{B}(i)$. Then the following set is well defined, and the following equality holds:

$$\widetilde{I}(\tilde{j}) := \bigcap_{i \in I : \tilde{j} \in \mathcal{B}(i)} \mathcal{B}(i) = \left\{\tilde{k} \in \widetilde{I} : geod(\tilde{j}, \tilde{k}) \cap \left(I \setminus \{\tilde{k}\}\right) = \emptyset\right\}. \tag{5.1}$$

The set $\widetilde{I}(\tilde{j})$ endowed with the set of edges $\widetilde{\mathcal{T}} \cap (\widetilde{I}(\tilde{j}) \times \widetilde{I}(\tilde{j}))$ is the smallest subtree containing $\tilde{j}$, and the set of extremal points $\mathcal{E}(\tilde{j}) = \{\tilde{k} \in \widetilde{I}(\tilde{j}) : \tilde{k}$ has a unique neighbor in $\widetilde{I}(\tilde{j})\}$ is contained in I.

The property that every point in I has a finite number of U-neighbors supplies a good example for the next section. Observe that the sets $\mathcal{V}(i)$ are finite for $i \in I$ if and only if $\mathcal{B}(i)$ are finite for $i \in I$. This property can be easily expressed in terms of U.

Lemma 5.4 *The sets $\mathcal{B}(i)$ are finite for all $i \in I$ if and only if*

$$\begin{aligned} &\forall w \in \mathcal{W}\ \exists I^w \subset I \quad \textit{finite such that}: \\ &\forall i \in I \setminus I^w, \quad \max\left\{U_{ij} : j \in I^w, U_{ij} = U_{jj}\right\} > w. \end{aligned} \tag{5.2}$$

Proof Assume that $\mathcal{B}(i)$ are finite. Clearly, it is enough to prove (5.2) for large $w \in W$. We shall assume that the finite set $L = \{j \in I : U_{jj} \leq w\}$ is nonempty, and we define $I^w = \bigcup_{j \in L} \mathcal{V}^*(j)$.

Fix $i_0 \in L$ as one of the closest points in I to the root $\tilde{r}$. For $i \in I \setminus I^w$, the geodesic $geod(i, \tilde{r})$ must contain points on I^w, otherwise $geod(i, i_0) = \{i, i_0\}$, which implies $i \in \mathcal{V}^*(i_0)$ given a contradiction. Take $k \in geod(i, \tilde{r}) \cap I^w$ the farthest point from $\tilde{r}$. It is clear that $U_{ik} = U_{kk}$. Assume that $U_{kk} \leq w$, so $k \in L$. If $geod(k, i) \cap I = \{k, i\}$, then $i \in I^w$, which is a contradiction. Therefore, there is at least one $m \in (geod(k, i) \cap I) \setminus \{i, k\}$. Take m the closest of such points to k. Clearly $m \in \mathcal{V}^*(k) \subseteq I^w$, contradicting the maximality of k. Then $U_{kk} > w$, proving the desired property.

Conversely, take $i \in I$ and consider $w = U_{ii}$. We shall prove that $\mathcal{V}^*(i) \subseteq I^w$. In fact, take $j \in \mathcal{V}^*(i) \setminus I^w$. By hypothesis there is $k \in I^w$ such that $U_{kk} = U_{jk} > w$. Since $U_{jj} \geq U_{jk} = U_{kk} > w = U_{ii}$ and $j \in \mathcal{V}^*(i)$, we conclude $k \in geod(i, j)$ and $k \neq i$. Since $k \neq j$, because $k \in I^w$, we arrive to a contradiction with the definition of $\mathcal{V}^*(i)$, proving the result. □

5.3 Generator and Harmonic Functions of an Ultrametric Matrix

In this section we associate to an ultrametric matrix U a q-matrix through its extension $\widetilde{U}$. Consider the q-matrix $\widetilde{Q}$ given by (2.2), which satisfies $\widetilde{Q}\widetilde{U} = \widetilde{U}\widetilde{Q} = -\mathbb{I}_{\widetilde{I}}$.

We can also assume that $\widetilde{Q}$ is defined in $\widetilde{I} \cup \partial_{\tilde{r}}$ as in (3.1). Further, we consider the Markov process $\widetilde{X}$ associated to $\widetilde{Q}$ with lifetime $\widetilde{\zeta}$.

We assume that $\widetilde{X}$ is transient. We denote by $\tilde{\mu}$ the probability measure defined on $\widetilde{\partial}_\infty$, the boundary of $(\widetilde{I}, \widetilde{\mathcal{T}})$, that is proportional to the exit distribution of $\widetilde{X}$.

In the sequel we consider

$$\tau := \inf\{t > 0 : \widetilde{X}_t \in I \cup \partial_{\tilde{r}}\} \wedge \widetilde{\zeta}\,. \tag{5.3}$$

We point out that $\widetilde{X}_\tau$ belongs to $I \cup \partial_{\tilde{r}} \cup \widetilde{\partial}_\infty$ with probability one. Notice that if $\widetilde{X}_0 = \tilde{j} \in \widetilde{I} \setminus I$, then $\tau = \inf\{t > 0 : \widetilde{X}_t \in \mathcal{E}(\tilde{j}) \cup \partial_{\tilde{r}}\} \wedge \widetilde{\zeta}$.

Our main assumption is

$$\forall \tilde{j} \in \widetilde{I} \setminus I : \quad \mathbb{P}_{\tilde{j}}\{\widetilde{X}_\tau \in I \cup \partial_{\tilde{r}}\} = 1. \tag{$H4$}$$

We can also write $(H4)$ as $\mathbb{P}_{\tilde{j}}\{\tau < \widetilde{\zeta}\} = 1$ for every $\tilde{j} \in \widetilde{I} \setminus I$. This is also equivalent to $\mathbb{P}_{\tilde{j}}\{\widetilde{X}_\tau \in \widetilde{\partial}_\infty\} = 0$ for every $\tilde{j} \in \widetilde{I} \setminus I$.

In the next theorem we associate a q-matrix to a general ultrametric matrix verifying $(H1)$–$(H4)$.

Theorem 5.1 *Assume that U satisfies $(H1)$–$(H4)$. Then there exists a matrix $Q : I \times I \to \mathbb{R}$ such that $QU = UQ = -\mathbb{I}_I$. Moreover, $Q_{ij} \neq 0$ if and only if $j \in \mathcal{V}^*(i)$, and we have*

$$Q_{ij} = \widetilde{Q}_{ij} + \sum_{\tilde{k} \in \widetilde{I} \setminus I} \widetilde{Q}_{i\tilde{k}} \mathbb{P}_{\tilde{k}}(\widetilde{X}_\tau = j). \tag{5.4}$$

For $i \neq j$, this formula takes the form

$$Q_{ij} = \widetilde{Q}_{ij} \quad \text{if } (i,j) \in \widetilde{\mathcal{T}} \quad \text{and} \quad Q_{ij} = \widetilde{Q}_{i\tilde{k}} \mathbb{P}_{\tilde{k}}(\widetilde{X}_\tau = j) \quad \text{if } (i,j) \notin \widetilde{\mathcal{T}},\ j \in \mathcal{V}^*(i),$$

where $\tilde{k} \in \widetilde{I} \setminus I$ is the unique neighbor of i in $\widetilde{\mathcal{T}}$ that belongs to $geod(i,j)$.

Proof We set $A = \tilde{Q}_{II}$, $B = \widetilde{Q}_{I,\widetilde{I}\setminus I}$, and $V = \widetilde{U}_{\widetilde{I}\setminus I, I}$. Since $\widetilde{U}_{II} = U$, we get $AU + BV = -\mathbb{I}_I$.

The crucial step in the proof is to get a $(\widetilde{I} \setminus I) \times I$ matrix Z with summable rows and verifying $ZU = V$, which means

$$\widetilde{U}_{\tilde{j}i} = \sum_{k \in I} Z_{\tilde{j}k} U_{ki} \quad \text{for all } \tilde{j} \in \widetilde{I} \setminus I,\ i \in I.$$

For any $\tilde{j} \in \widetilde{I} \setminus I$, consider the subtree $\widetilde{J} := \widetilde{I}(\tilde{j})$ given by (5.1). We denote by $\mathcal{E} \subset I$ the set of extremal points of $\widetilde{J}$. Note that $\widetilde{J} \setminus \mathcal{E} \subseteq \widetilde{I}$. We consider the following q-matrix on $\widetilde{J} \times \widetilde{J}$:

$$C_{\tilde{l}\tilde{k}} = \widetilde{Q}_{\tilde{l}\tilde{k}} \quad \text{if } \tilde{l} \in \widetilde{J} \setminus \mathcal{E} \quad \text{and} \quad C_{\tilde{l}\tilde{k}} = 0 \quad \text{otherwise.}$$

By definition of the random time τ in (5.3), the Markov process induced by C is just the stopped process $\widetilde{X}^\tau$. From the property $\widetilde{Q}\widetilde{U} = -\mathbb{I}_{\widetilde{I}}$ we deduce that for each $i \in I$, the restriction of $U_{\bullet i}$ to $\widetilde{J}$ is a C-harmonic function. Therefore,

$$U_{\tilde{j}i} = \mathbb{E}_{\tilde{j}}(U_{\widetilde{X}_\tau i}) = \sum_{k\in\mathcal{E}} \mathbb{P}_{\tilde{j}}(\widetilde{X}_\tau = k)U_{ki},$$

which gives the desired matrix Z with $Z_{\tilde{j}k} = \mathbb{P}_{\tilde{j}}(\widetilde{X}_\tau = k)\mathbf{1}_{\mathcal{E}}(k)$ for $\tilde{j} \in \widetilde{I} \setminus I$, $k \in I$. Since B is finitely supported and the rows of Z are summable, we get

$$(A + BZ)U = -\mathbb{I}_I; \tag{5.5}$$

then $Q = A + BZ$ should be the desired q-matrix. The explicit formula for Q is

$$Q_{ij} = \widetilde{Q}_{ij} + \sum_{\tilde{k}\in\widetilde{I}\setminus I} \widetilde{Q}_{i\tilde{k}} Z_{\tilde{k}j} = \widetilde{Q}_{ij} + \sum_{\tilde{k}\in\widetilde{I}\setminus I} \widetilde{Q}_{i\tilde{k}} \mathbb{P}_{\tilde{k}}(\widetilde{X}_\tau = j). \tag{5.6}$$

From the structure of $\widetilde{Q}$ the last sum in (5.6) runs over $\tilde{k} \in \widetilde{I} \setminus I$ which are neighbors of i with respect to $\widetilde{\mathcal{T}}$. From the shape of Z these values of $\tilde{k}$ are further restricted to the set $\mathcal{V}^*(j)$. According to the Lemma 5.3, part (v), the set of such points is not empty when $(i,j) \notin \widetilde{\mathcal{T}}$, and moreover this set contains exactly one point $\tilde{k} \in \widetilde{I}$. In summary, we have for $i \neq j$,

$$Q_{ij} = \widetilde{Q}_{ij} \quad \text{if } (i,j) \in \widetilde{\mathcal{T}} \quad \text{and} \quad Q_{ij} = \widetilde{Q}_{i\tilde{k}} \mathbb{P}_{\tilde{k}}(\widetilde{X}_\tau = j) \quad \text{if } (i,j) \notin \widetilde{\mathcal{T}},\, j \in \mathcal{V}(i),$$

where in the last case, $\tilde{k}$ is the unique neighbor of i in $\widetilde{\mathcal{T}}$ belonging to $geod(i,j)$. From this formula we deduce that for $i \neq j$, we have $Q_{ij} > 0$ if and only if $j \in \mathcal{V}(i)$. From (5.5) we deduce that $Q_{ii} < 0$. Also, we get

$$Q_{ii} = \widetilde{Q}_{ii} + \sum_{\tilde{k}\in\widetilde{I}:\, (\tilde{k},i)\in\widetilde{\mathcal{T}}} \widetilde{Q}_{i\tilde{k}} \mathbb{P}_{\tilde{k}}(\widetilde{X}_\tau = i).$$

Now, let us prove that Q is a q-matrix. Let $k \in \mathcal{V}(i)$ be such that $U_{ki} = \min\{U_{ji} : j \in \mathcal{V}(i)\}$. This minimum is attained because the set $\{w \in \mathcal{W} : w \le U_{ii}\}$ is finite. From the ultrametric property of U we have $U_{jk} \ge \min\{U_{ji}, U_{ik}\} = U_{ik}$ for $j \in \mathcal{V}^*(i)$. Then, by using (5.5) we deduce that

$$0 \ge Q_{ii}U_{ik} + \sum_{j\in\mathcal{V}(i)} Q_{ij}U_{jk} \ge U_{ik}\left(\sum_{j\in I} Q_{ij}\right).$$

Hence Q is a q-matrix.

To finish the proof it is enough to show that Q is a symmetric matrix. This is equivalent to prove that

$$\widetilde{Q}_{i\tilde{k}} \mathbb{P}_{\tilde{k}}(\widetilde{X}_\tau = j) = \widetilde{Q}_{j\tilde{l}} \mathbb{P}_{\tilde{l}}(\widetilde{X}_\tau = i) \quad \text{for } j \in \mathcal{V}(i),\ (j,i) \notin \widetilde{\mathcal{T}}, \tag{5.7}$$

where $\tilde{k}$ (respectively $\tilde{l}$) is the unique neighbor in $\widetilde{\mathcal{T}}$ of i (of j, respectively) given by Lemma 5.3, part (v). The probabilities appearing in (5.7) can be computed in terms of $\widetilde{Y}=(\widetilde{Y}_n : n\in\mathbb{N})$, the discrete skeleton of the Markov chain on $\widetilde{X}$ taking values on $\widetilde{I}$. The transition probabilities for this discrete time chain are

$$\mathbb{P}(\widetilde{Y}_1=y_1|\widetilde{Y}_0=y_0)=\frac{\widetilde{Q}_{y_0y_1}}{(-\widetilde{Q}_{y_0y_0})}.$$

If we define $N=\min\{n\in\mathbb{N}:\ \widetilde{Y}_n\in I\cup\{\partial_{\tilde r}\}\}$, then $\mathbb{P}_{\tilde k}(\widetilde{X}_\tau=j)=\mathbb{P}_{\tilde k}(\widetilde{Y}_N=j)$. This last probability can be computed by summing up all possible trajectories $\widetilde{Y}_0=\tilde k$, $\widetilde{Y}_1=y_1,\dots,\widetilde{Y}_{n-2}=y_{n-2}$, $\widetilde{Y}_{n-1}=\tilde\ell$, $\widetilde{Y}_n=j$, which do not visit I at any intermediate state. The probability of such a trajectory is

$$\frac{\widetilde{Q}_{\tilde k y_1}}{(-\widetilde{Q}_{\tilde k\tilde k})}\frac{\widetilde{Q}_{y_1y_2}}{(-\widetilde{Q}_{y_1y_1})}\cdots\frac{\widetilde{Q}_{y_{n-2}\tilde\ell}}{(-\widetilde{Q}_{y_{n-2}y_{n-2}})}\frac{\widetilde{Q}_{\tilde\ell j}}{(-\widetilde{Q}_{\tilde\ell\tilde\ell})}.$$

The probability of the reverse trajectory $\widetilde{Y}_0=\tilde\ell$, $\widetilde{Y}_1=y_{n-2},\dots,\widetilde{Y}_{n-2}=y_1$, $\widetilde{Y}_{n-1}=\tilde k$, $\widetilde{Y}_n=i$, is

$$\frac{\widetilde{Q}_{\tilde\ell y_{n-2}}}{(-\widetilde{Q}_{\tilde\ell\tilde\ell})}\frac{\widetilde{Q}_{y_{n-2}y_{n-3}}}{(-\widetilde{Q}_{y_{n-2}y_{n-2}})}\cdots\frac{\widetilde{Q}_{y_1\tilde k}}{(-\widetilde{Q}_{y_1y_1})}\frac{\widetilde{Q}_{\tilde k i}}{(-\widetilde{Q}_{\tilde k\tilde k})}.$$

The symmetry of $\widetilde{Q}$ implies that (5.7) holds. Therefore, Q is symmetric, and we deduce that $UQ=-\mathbb{I}_I$. This finishes the proof. □

As usual, we say that a function $h:I\to\mathbb{R}$ is Q-harmonic if $Qh=0$. Our main result in relation with harmonic functions for ultrametric matrices is the following one.

Theorem 5.2 *Assume that U satisfies $(H1)$–$(H4)$. Given a bounded Q-harmonic function h defined on I, there exists a unique $\widetilde{Q}$-harmonic function $\tilde h$ defined on $\widetilde{I}$ which is an extension of h.*

Proof Consider the function $\tilde h(\tilde\imath)=\mathbb{E}_{\tilde\imath}(h(\widetilde{X}_\tau))$, $\tilde\imath\in\widetilde{I}$, where τ is given by (5.3). Clearly $\tilde h$ is an extension of h. Using the strong Markov property for the time of first jump of $\widetilde{X}$, we deduce that $\tilde h$ is $\widetilde{Q}$-harmonic at every $\tilde\jmath\in\widetilde{I}\setminus I$. Now, for $i\in I$, we have

$$\begin{aligned}\sum_{\tilde\jmath\in\widetilde{I}}\widetilde{Q}_{i\tilde\jmath}\tilde h(\tilde\jmath)&=\sum_{j\in I}\widetilde{Q}_{ij}h(j)+\sum_{\tilde\jmath\in\widetilde{I}\setminus I}\widetilde{Q}_{i\tilde\jmath}\tilde h(\tilde\jmath)=\sum_{j\in I}\widetilde{Q}_{ij}h(j)+\sum_{\tilde\jmath\in\widetilde{I}\setminus I}\widetilde{Q}_{i\tilde\jmath}\mathbb{E}_{\tilde\jmath}\big(h(\widetilde{X}_\tau)\big)\\&=\sum_{j\in I}\widetilde{Q}_{ij}h(j)+\sum_{\tilde\jmath\in\widetilde{I}\setminus I}\widetilde{Q}_{i\tilde\jmath}\Big(\sum_{k\in I}\mathbb{P}_{\tilde\jmath}(\widetilde{X}_\tau=k)h(k)\Big)\\&=\sum_{j\in I}\Big(\widetilde{Q}_{ij}+\sum_{\tilde k\in\widetilde{I}\setminus I}\widetilde{Q}_{i\tilde k}\mathbb{P}_{\tilde k}(\widetilde{X}_\tau=j)\Big)h(j)=\sum_{j\in I}Q_{ij}h(j),\end{aligned}$$

where the last equality follows from (5.4). Since h is Q-harmonic, we get $\sum_{\tilde{j}\in\tilde{I}} \widetilde{Q}_{i\tilde{j}} \times \tilde{h}(\tilde{j})=0$. Then $\tilde{h}$ is $\widetilde{Q}$-harmonic at $i\in I$. □

5.4 The Boundary of an Ultrametric Matrix

The boundary $\widetilde{\partial}_\infty$ can be identified with

$$\widetilde{\partial}_\infty=\big\{(\tilde{\imath}_n:n\in\mathbb{N}):\tilde{\imath}_0=\tilde{r},\forall n\in\mathbb{N},|\tilde{\imath}_n|=n \text{ and } (\tilde{\imath}_n,\tilde{\imath}_{n+1})\in\widetilde{\mathcal{T}}\big\},$$

and it is endowed with the topology generated by the sets $\widetilde{\mathfrak{C}}=\{\widetilde{\partial}_\infty(\tilde{\imath})=[\tilde{\imath},\infty]\cap\widetilde{\partial}_\infty : \tilde{\imath}\in\widetilde{I}\}$. We denote by $\widetilde{\mathcal{F}}_\infty$ the associated σ-field. We denote by $\mathcal{F}_\infty$ the σ-field on $\widetilde{\partial}_\infty$ generated by the sets $\{\widetilde{\partial}_\infty(i):i\in I\}$. We have $\mathcal{F}_\infty\subseteq\widetilde{\mathcal{F}}_\infty$, and this inclusion can be strict.

The following definition of the boundary ∂^U_∞ associated to an ultrametric matrix extends the one for a tree. An infinite path $(i_n:n\in\mathbb{N})$ in I is called a $\preceq$-chain if $i_n\prec i_{n+1}$ for every $n\in\mathbb{N}$, and the $\preceq$-chain is maximal if we cannot add any element of I to it in such a way that it continues to be a $\preceq$-chain. In a tree a $\preceq$-chain $(i_n:n\in\mathbb{N})$ is maximal if and only if $i_0=r$ and $|i_n|=n$ for every $n\in\mathbb{N}$. The boundary of I with respect to the ultrametric matrix U is defined as $\partial^U_\infty=\{(i_n:n\in\mathbb{N})$ is a maximal $\preceq$-chain $\}$. We endowed ∂^U_∞ with the trace topology from $\tilde{\partial}_\infty$. The equality

$$\partial^U_\infty=\bigcap_{n\in\mathbb{N}}\bigg(\bigcup_{m\geq n}\bigcup_{i\in I:|i|=m}\big\{\xi\in\widetilde{\partial}_\infty:\xi(m)=i\big\}\bigg),$$

shows that $\partial^U_\infty\in\widetilde{\mathcal{F}}_\infty$.

The function

$$\begin{aligned}&\mathbf{i}^U_\infty:\partial^U_\infty\to\widetilde{\partial}_\infty,\quad \mathbf{i}^U_\infty\big((i_n:n\in\mathbb{N})\big)=(\tilde{\imath}_n:n\in\mathbb{N})\\&\quad\Longleftrightarrow\quad \{i_n:n\in\mathbb{N}\}\subseteq\{\tilde{\imath}_n:n\in\mathbb{N}\},\end{aligned}\tag{5.8}$$

is well defined and is one-to-one. We will identify ∂^U_∞ and $\mathbf{i}^U_\infty(\partial^U_\infty)$. In general, $\mathbf{i}^U_\infty$ is not onto, and ∂^U_∞ is small compared to $\widetilde{\partial}_\infty$. But, as the following result shows, under $(H4)$, it has full $\tilde{\mu}$-measure.

Lemma 5.5 *Property* $(H4)$ *is equivalent to* $\tilde{\mu}(\partial^U_\infty)=1$.

Proof First notice that if for some $\tilde{\jmath}\in\widetilde{I}\setminus I$, it holds $\mathbb{P}_{\tilde{\jmath}}\{\widetilde{X}_\tau\in I\cup\partial_{\tilde{r}}\}<1$, then $\mathbb{P}_{\tilde{\jmath}}\{\widetilde{X}_{\tilde{\zeta}}\in\widetilde{\partial}_\infty\setminus\partial^U_\infty\}>0$. Hence, the condition is necessary for $(H4)$. For the reciprocal, assume that $\tilde{\mu}(\widetilde{\partial}_\infty\setminus\partial^U_\infty)>0$. Therefore there exists $n\in\mathbb{N}$ such that $\mathbb{P}_{\tilde{r}}\{A_n\}>0$, where $A_n=\bigcap_{m\geq n}\bigcap_{i\in I:|i|=m}\{\xi\in\widetilde{\partial}_\infty:\xi(m)\neq i\}$. Take any $\tilde{\imath}\in\widetilde{I}\setminus I$, $|\tilde{\imath}|=n$, such that $\widetilde{\partial}_{\tilde{\imath}}(\infty)\cap A_n$ has positive $\mathbb{P}_{\tilde{r}}$-measure. Then we have $\mathbb{P}_{\tilde{\imath}}\{\widetilde{X}_{\tilde{\zeta}}\in\widetilde{\partial}_\infty,\ \widetilde{X}_{\tilde{\zeta}}(\ell)\notin I, \forall\ell\geq 0\}>0$, which contradicts hypothesis $(H4)$. □

Theorem 5.3 *Assume that U satisfies $(H1)$–$(H4)$. Let h be a bounded Q-harmonic function such that $\lim_{i\to\xi}h(i)=\varphi(\xi)$ for every $\xi\in\partial^U_\infty$. Then, there exists $\tilde{\varphi}=$*

$\lim \tilde{h}$ $\tilde{\mu}$*-a.e., where* $\tilde{h}$ *is the harmonic function associated to* h *in Theorem* 5.2. *Moreover, if* $\tilde{\varphi}$ *is in the domain of* $\widetilde{W}^{-1}$, *then* h *has the representation*

$$h(i) = \int_{\widetilde{\partial}_\infty} U_{i,\eta}\big(\widetilde{W}^{-1}\tilde{\varphi}\big)(\eta)\tilde{\mu}(d\eta). \tag{5.9}$$

Proof From Lemma 5.5 we have $\partial_\infty^U = \widetilde{\partial}_\infty$ $\tilde{\mu}$-a.e., and therefore (almost) every point $\xi \in \widetilde{\partial}_\infty$ verifies $|\{n \in \mathbb{N} : \xi(n) \in I\}| = \infty$. Also, by the hypothesis there exists $a = \lim_{\substack{n\to\infty \\ \xi(n)\in I}} h(\xi(n))$. For the first part of the statement, it suffices to show that $a = \lim_{n\to\infty} h(\xi(n))$. Let us consider the subsequence $k(n) = \max\{m \le n : \xi(m) \in I\}$. We have $\lim_{n\to\infty} k(n) = \infty$. On the other hand, for large n, $\mathcal{V}(\xi(n)) \subset [\xi(k(n)), \infty)$; then $\tilde{h}(\xi(n)) = \mathbb{E}_{\xi(n)}(h(\widetilde{X}_\tau))$ belongs to the convex closure of the set $\{h(\xi(m)) : \xi(m) \in I, m \ge k(n)\}$. Hence the result follows.

Now we are able to show relation (5.9). It suffices to notice that, for every $i \in I \cup \widetilde{\partial}_\infty$ and $\tilde{\mu}$-a.e. $\eta \in \widetilde{\partial}_\infty$, it holds $\widetilde{U}_{i\eta} = U_{i\eta}$. Then the proof follows from Corollary 3.1. □

Remark 5.1 From a topological point of view, ∂_∞^U is dense in $\widetilde{\partial}_\infty$ if for all $i \in I$, there exists $j \in I$, $j \neq i$, such that $U_{ij} = U_{ii}$ (that is, if for all $i \in I$, the set $[i,\infty)^U$ is infinite). Indeed, by the definition of the minimal tree, for all $\xi \in \widetilde{\partial}_\infty$ and $n \ge 1$, there exists some $i \in I$ such that $i \in \widetilde{\partial}_\infty(\xi(n))$. The desired density follows by taking any $\eta \in \partial_\infty^U$ hanging from i.

Remark 5.2 The referee has put the following problem. Consider the tree $(I, \mathcal{T})$ as in Sect. 2. What happens with the results of the work if the strictly increasing function w is defined on the set I of tree vertices rather than on $\mathbf{N}$? To give a partial answer to it, let us consider a strictly increasing function $w^* : I \to \mathbb{R}_+$ (that is, $i \prec j$ implies $w^*(i) < w^*(j)$) satisfying $w^*(r) = 0$. Define $U_{ij} = w^*(i \wedge j)$. Then, the tree structure implies that $U = (U_{ij} : i, j \in I)$ is an ultrametric matrix. Hence, the results of this section can be applied. Condition $(H1)$ holds, and condition $(H2)$ reads: the set $\{w^*(i) : i \in I\}$ has no finite accumulation points, and it implies $(H3)$, while $(H4)$ holds anyways.

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