



# On the moving pulse solutions in systems with broken parity

Orazio Descalzi<sup>a,c,\*</sup>, Enrique Tirapegui<sup>b,c</sup>

<sup>a</sup>*Facultad de Ingeniería, Universidad de los Andes, Av. San Carlos de Apoquindo, Santiago 2200, Chile*

<sup>b</sup>*Departamento de Física, F.C.F.M. Universidad de Chile, Casilla 487-3, Santiago, Chile*

<sup>c</sup>*Centro de Física No Lineal y Sistemas Complejos de Santiago, Casilla 27122, Santiago, Chile*

Received 3 November 2003; received in revised form 24 December 2003

Available online 12 May 2004

---

## Abstract

We study analytically a system sustaining stable moving localized structures, namely, the one-dimensional quintic complex Ginzburg–Landau (G–L) equation with non-linear gradients. We obtain approximate solutions for the stable moving pulse and its velocity. The results are in excellent agreement with direct numerical simulations.

© 2004 Elsevier B.V. All rights reserved.

*PACS:* 47.20.Ky; 82.40.Bj; 05.70.Ln

*Keywords:* Ginzburg–Landau equation; Moving localized structures

---

## 1. Introduction

In the last decade experimental observation of stable spatially localized structures in systems such as binary fluids mixtures [1,2], nematic liquid crystals [3], chemical reactions [4] and granular media [5], has been reported. From a theoretical point of view stable localized solutions have been found in generic equations near a weakly inverted bifurcation to traveling waves, namely, the Ginzburg–Landau [6,7] and the Swift–Hohenberg equations [8]. Since then a lot effort has been devoted to study properties of these dissipative–dispersive localized structures [9–17]. Mainly this work has been focused on the quintic complex Ginzburg–Landau (G–L) equation without regarding non-linear gradient terms. Some work has been done including these non-linearities

---

\* Corresponding author. Facultad de Ingeniería, Universidad de los Andes, Av. San Carlos de Apoquindo, Santiago 2200, Chile.

*E-mail address:* [oraziod@uandes.cl](mailto:oraziod@uandes.cl) (O. Descalzi).

[18–20], which can be useful in more general models with application in propagation of ultra-short pulses in optical fibers. Very recently we have developed a simple analytical method which enables us to construct approximate expressions of localized solutions and understand their appearance [21–23]. The aim of this article is to generalize this technique to the quintic G–L equation including non-linear gradients.

## 2. Analytical approach

The one-dimensional quintic complex G–L equation including dissipation, dispersion and non-linear gradients can be written as

$$\partial_t A = \mu A + \beta |A|^2 A + \gamma |A|^4 A + D \partial_{xx} A + \lambda \partial_x (|A|^2 A) + \eta A \partial_x (|A|^2). \quad (1)$$

The subscript  $x$  denotes partial derivative with respect to  $x$ ,  $A(x, t) = r \exp i\phi$  is a complex field, and the parameters  $\beta$ ,  $\gamma$ ,  $D$ ,  $\lambda$  and  $\eta$  are in general complex. Nevertheless, Eq. (1) admits stable moving pulses with most parameters being real. In this article we shall consider  $\mu$ ,  $\gamma$  and  $\lambda$  real,  $\beta = \beta_r + i\beta_i$ ,  $D = 1$  and  $\eta = 0$ . The signs of the parameters  $\beta_r > 0$  and  $\gamma < 0$  are chosen in order to guarantee that the bifurcation is subcritical and saturates to quintic order. Because of the parameter  $\beta_i$  the system (without non-linear gradients) is non-variational and admits stable localized structures. The parameter  $\lambda$  breaks the parity symmetry  $x \rightarrow -x$  leading to *moving* pulses. The inclusion of the imaginary parts of the parameters  $\gamma$ ,  $D$ ,  $\lambda$  and  $\eta$  may lead to breathing and chaotic localized structures [13,23].

Making the change of variables:  $y = x - vt$ ;  $\tau = t$ , where  $v$  is the velocity of the pulse, we assume that in the moving frame we can make the following Ansatz:  $r = R(y)$ ;  $\phi = \Omega\tau + \theta(y)$ . Then Eq. (1) reduces to

$$-(v + 3\lambda R^2)R_y = \mu R + \beta_r R^3 + \gamma R^5 + R_{yy} - R\theta_y^2, \quad (2)$$

$$-(v + \lambda R^2)R\theta_y = -\Omega R + \beta_i R^3 + 2R_y\theta_y + R\theta_{yy}. \quad (3)$$

The strategy to calculate approximately  $R(y)$ ,  $\theta(y)$ ,  $\Omega$  and  $v$  consists in considering that  $\theta_y(y)$  (the wave vector) is constant ( $+p_1$  for the left side,  $-p_2$  for the right side) in almost all the domain (*outside the core*) except in a narrow domain around the center of the pulse (*core*), where  $\theta_y(y)$  is considered to be a straight line (see Fig. 1(b)). Because of parity breaking the left and right sides of the pulse must be studied separately.

*Outside the core* and for the left side ( $y < 0$ ), Eqs. (2) and (3) lead to

$$0 = (\mu^{(1)} - p_1^2)R + \beta_r^{(1)}R^3 + \gamma^{(1)}R^5 + R_{yy}, \quad (4)$$

where  $\mu^{(1)} = \mu - \frac{v^2}{2} + \frac{v}{2} \sqrt{v^2 + 4(p_1^2 - \mu)}$ ,  $\beta_r^{(1)} = \beta_r - 2\lambda v - \frac{v\beta_i}{2p_1} + \frac{3}{2} \lambda \sqrt{v^2 + 4(p_1^2 - \mu)}$  and  $\gamma^{(1)} = \gamma - \frac{3}{2} \lambda^2 - \frac{3\lambda\beta_i}{2p_1}$ . Asymptotically, for  $y \rightarrow -\infty$  we obtain  $\Omega = p_1 \sqrt{v^2 + 4(p_1^2 - \mu)}$ , which is a constant in  $(-\infty, 0)$ . For the right side ( $y > 0$ ) one finds the same Eq. (4) with coefficients  $\mu^{(1)} = \mu - \frac{v^2}{2} - \frac{v}{2} \sqrt{v^2 + 4(p_2^2 - \mu)}$ ,  $\beta_r^{(1)} = \beta_r - 2\lambda v + \frac{v\beta_i}{2p_2} - \frac{3}{2} \lambda \sqrt{v^2 + 4(p_2^2 - \mu)}$  and  $\gamma^{(1)} = \gamma - \frac{3}{2} \lambda^2 + \frac{3\lambda\beta_i}{2p_2}$ . For  $y \rightarrow +\infty$  we get  $\Omega =$

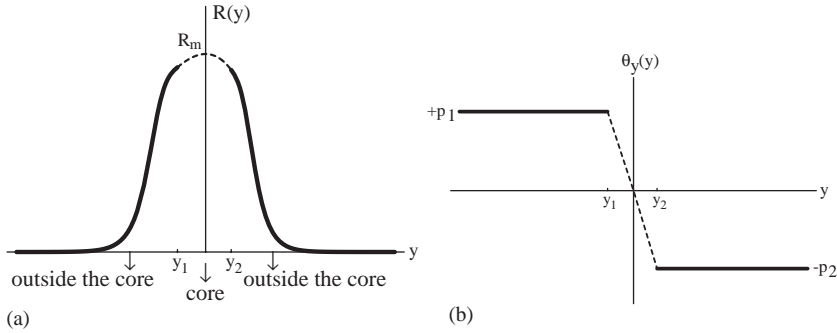


Fig. 1. Analytical approximation for the pulse in the moving frame. For the left ( $y < 0$ ) and right sides ( $y > 0$ ) the space is divided in two regions: *outside the core*, where the wave vector is constant, and *core*, where the wave vector is a straight line. (a) modulus of the pulse, (b) wave vector.

$p_2 \sqrt{v^2 + 4(p_2^2 - \mu)}$ , which is a constant in  $(0, +\infty)$ . Integrating Eq. (4) it is possible to obtain an explicit expression for  $R(y)$ :

$$R(y) = \frac{2b^{\frac{1}{4}} \exp\{\sqrt{-\mu^{(1)} + p^2}(|y| + y_0)\}}{\sqrt{(\exp\{2\sqrt{-\mu^{(1)} + p^2}(|y| + y_0)\} + \frac{a}{\sqrt{b}})^2 - 4}}, \quad (5)$$

where  $a = \frac{-3\beta_r^{(1)}}{2\gamma^{(1)}}$ ,  $b = \frac{-3(-\mu^{(1)} + p^2)}{\gamma^{(1)}}$ ,  $y_0$  is a constant to be determined,  $p = p_1$  for  $y < 0$  and  $p = -p_2$  for  $y > 0$ .

Inside the core and for the left side ( $y < 0$ ) we assume that  $R(y) = R_m^{(left)} - \varepsilon y^2 - \rho y^3$  and  $\theta_y = -\alpha y$ , where  $R_m^{(left)}$  is the highest value of the pulse constructed on the left side. From Eqs. (2) and (3) we can calculate the values of  $\varepsilon$ ,  $\rho$  and  $\alpha$ . Imposing continuity of the amplitude  $R(y)$ , the phase gradient  $\theta_y(y)$ , and the derivative of the amplitude of the analytical expressions calculated inside and outside the core of the pulse at  $y = y_1 = -\frac{p_1}{\alpha}$  we determine  $y_0$  and a relation between  $R_m^{(left)}$  and  $p_1$ :

$$f_1(R_m^{(left)}, p_1) \equiv \sqrt{-\frac{\gamma^{(1)}}{3}} r_c \sqrt{r_c^4 - ar_c^2 + b} + 2\varepsilon y_1 + 3\rho y_1^2 = 0, \quad (6)$$

where  $r_c = R_m^{(left)} - \varepsilon y_1^2 - \rho y_1^3$ .

In order to obtain a second relation between  $R_m^{(left)}$  and  $p_1$  we use a *consistency relation* by multiplying Eq. (3) by  $R(y)$  and integrating from  $-\infty$  to 0.

$$g_1(R_m^{(left)}, p_1) \equiv \Omega - \frac{1}{(I_2^{(0)} + I_2^{(1)})} \{\beta_i(I_4^{(0)} + I_4^{(1)}) + v(p_1 I_2^{(0)} + I_2^{(2)}) + \lambda(p_1 I_4^{(0)} + I_4^{(2)})\} = 0, \quad (7)$$

where  $I_2^{(0)} \equiv \int_{-\infty}^{y_1} R^2 dy$ ;  $I_4^{(0)} \equiv \int_{-\infty}^{y_1} R^4 dy$ ;  $I_2^{(1)} \equiv \int_{y_1}^0 R^2 dy$ ;  $I_4^{(1)} \equiv \int_{y_1}^0 R^4 dy$ ;  $I_2^{(2)} \equiv \int_{y_1}^0 R^2 \theta_y dy$  and  $I_4^{(2)} \equiv \int_{y_1}^0 R^4 \theta_y dy$ , which can be calculated explicitly.

For the right side ( $y > 0$ ) we proceed in an analogous way. We assume  $R(y) = R_m^{(right)} - \varepsilon y^2 - \rho y^3$  and  $\theta_y = -\alpha y$ . Once again  $\varepsilon$ ,  $\rho$  and  $\alpha$  are calculated from Eqs. (2) and (3). The matching between  $R(y)$  outside the core and  $R(y)$  inside the core is carried out at  $y = y_2 = \frac{p_2}{\alpha}$  yielding a relation between  $R_m^{(right)}$  and  $p_2$ :

$$f_2(R_m^{(right)}, p_2) \equiv \sqrt{-\frac{\gamma^{(1)}}{3} r_c \sqrt{r_c^4 - ar_c^2 + b} - 2\varepsilon y_2 - 3\rho y_2^2} = 0, \quad (8)$$

where  $r_c = R_m^{(right)} - \varepsilon y_2^2 - \rho y_2^3$ . The corresponding consistency relation is given by

$$g_2(R_m^{(right)}, p_2) \equiv \Omega - \frac{1}{(I_2^{(0)} + I_2^{(1)})} \{ \beta_i (I_4^{(0)} + I_4^{(1)}) - v(p_2 I_2^{(0)} + I_2^{(2)}) - \lambda(p_2 I_4^{(0)} + I_4^{(2)}) \} = 0, \quad (9)$$

where  $I_2^{(0)} \equiv \int_{y_2}^{+\infty} R^2 dy$ ;  $I_4^{(0)} \equiv \int_{y_2}^{+\infty} R^4 dy$ ;  $I_2^{(1)} \equiv \int_0^{y_2} R^2 dy$ ;  $I_4^{(1)} \equiv \int_0^{y_2} R^4 dy$ ;  $I_2^{(2)} \equiv \int_0^{y_2} R^2 \theta_y dy$  and  $I_4^{(2)} \equiv \int_0^{y_2} R^4 \theta_y dy$ , which can be calculated explicitly.

Thus for fixed values of Eq. (1) and  $v$ , expressions (6)–(9) give us  $R_m^{(left)}$ ,  $R_m^{(right)}$ ,  $p_1$  and  $p_2$ , which enable us to determine the left and right parts of the localized structure. Finally, the continuity of the pulse at  $y = 0$  (or the condition  $R_m^{(left)} = R_m^{(right)}$ ) leads to a unique value of  $v$  (the velocity of the pulse).

### 3. Example

To see how this method works in a concrete example we fix the parameters of Eq. (1):  $\mu = -0.38$ ,  $\beta_r = 3$ ,  $\beta_i = 1$ ,  $\gamma = -2.75$ , and  $\lambda = -0.1$ . Expressions (6) and (7) give us values of  $R_m^{(left)}$  and  $p_1$  for each value of  $v$ . In Fig. 2(a) we draw  $f_1(R_m^{(left)}, p_1) = 0$  (continuous line) and  $g_1(R_m^{(left)}, p_1)$  (dashed line) for  $v = 0.087712$ . Fig. 2(b) shows  $f_2(R_m^{(right)}, p_2) = 0$  (continuous line) and  $g_2(R_m^{(right)}, p_2)$  (dashed line) for the same value of  $v$ . Varying  $v$  we see that there exists a unique value of  $v$  for which  $R_m^{(left)} = R_m^{(right)}$  (In Fig. 3(a) we see that  $R_m^{(left)}(v)$  and  $R_m^{(right)}(v)$  intersect at a unique value  $v = 0.087712$ ). Moreover we can study the relation between  $v$  and  $\lambda$ . We find analytically and from direct numerical simulations that  $v$  varies linearly with  $\lambda$  (see Fig. 3(b)). For  $|\lambda| > 0.2$  our method collapses for the parameters used in this example. The reason may be the fact that for large  $v$  the renormalized parameters  $\mu^{(1)}$ ,  $\beta_r^{(1)}$  and  $\gamma^{(1)}$  lead to a pulse outside the analytical stability tongue. Now we can construct the left and right parts of the moving pulse. Fig. 4(a) shows the analytical approximation for the shape of the pulse (continuous line). Dashed line represents the pulse obtained through a direct numerical simulation. Fig. 4(b) shows a numerical space–time plot for the modulus of the pulse, which leads to a numerical velocity of the pulse  $v = 0.084388$ . This result agrees within 4% with our analytical approach. In Fig. 5 we show a 3-dimensional representation of the analytical expression for the shape  $R(x, t)$  of the pulse (compare with the numerical space–time plot given in Fig. 4(b)).

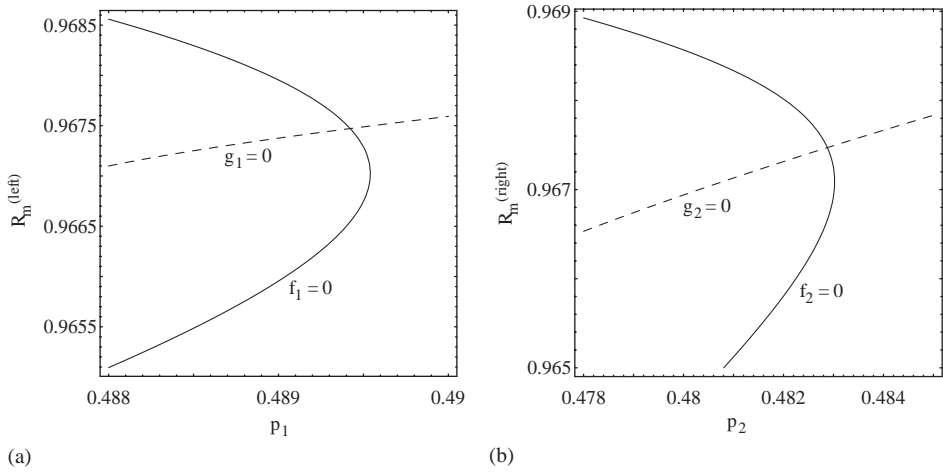


Fig. 2. For the parameters  $\mu = -0.38$ ,  $\beta_r = 3$ ,  $\beta_i = 1$ ,  $\gamma = -2.75$ ,  $\lambda = -0.1$  and  $v = 0.087712$ . (a) left side of the pulse:  $f_1(R_m^{(left)}, p_1) = 0$  is drawn as a continuous line and  $g_1(R_m^{(left)}, p_1)$  as a dashed line. Intersection occurs at  $p_1 = 0.489422$ ,  $R_m^{(left)} = 0.967471$ , (b) right side of the pulse:  $f_2(R_m^{(right)}, p_2) = 0$  is drawn as a continuous line and  $g_2(R_m^{(right)}, p_2)$  as a dashed line. Both curves intersect at  $p_2 = 0.482878$ ,  $R_m^{(right)} = 0.967471$ .

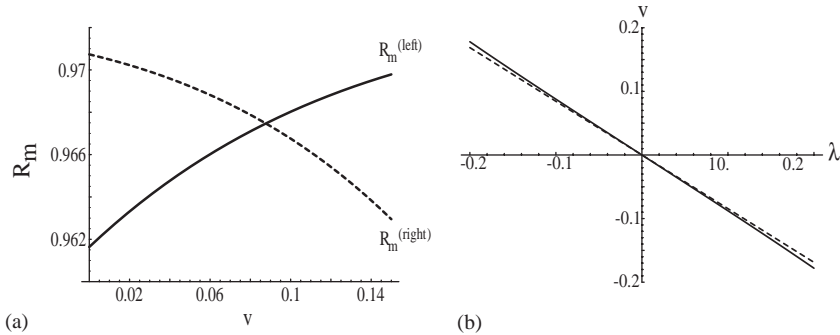


Fig. 3. (a) The intersection between the curves  $R_m^{(left)}(v)$  and  $R_m^{(right)}(v)$  selects the velocity of the pulse. For parameters  $\mu = -0.38$ ,  $\beta_r = 3$ ,  $\beta_i = 1$ ,  $\gamma = -2.75$  and  $\lambda = -0.1$  the selected velocity is  $v = 0.087712$ , (b) Linear relation between  $v$  and  $\lambda$ . The continuous line shows the analytical prediction for the dependence of velocity of the pulse  $v$  with  $\lambda$ . The dashed line stands for the numerical simulation.

### 4. Conclusions

We have studied from an analytical point of view a system with broken parity sustaining moving localized structures, namely, the one-dimensional quintic complex Ginzburg–Landau equation with non-linear gradients. Using a matching approach we have been able to obtain approximate expressions for the stable moving pulse and its velocity. Our results are in good agreement with direct numerical simulations.

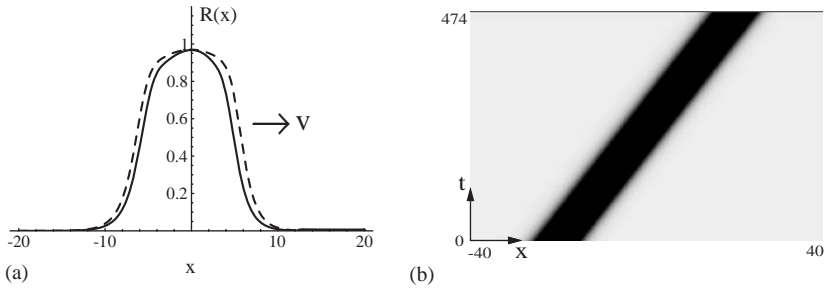


Fig. 4. (a) The analytical shape of the pulse is shown as a continuous line. Direct numerical simulation is represented by a dashed line. (b) Numerical space–time plot for the shape of the pulse.

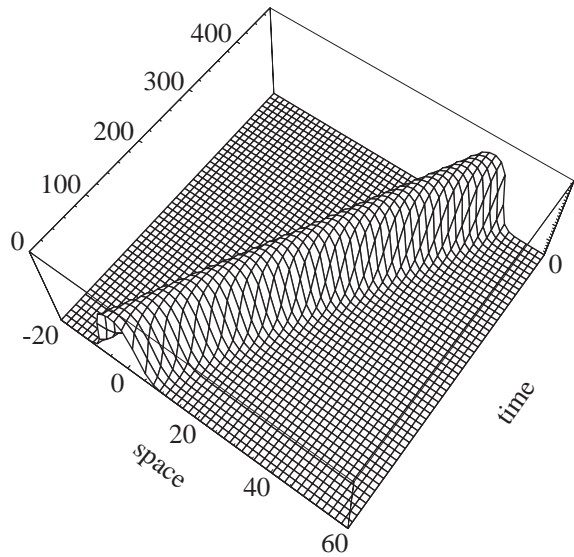


Fig. 5. 3-dimensional plot of the analytical expression for the shape  $R(x,t)$  of the moving pulse in the interval  $(-20, 60) \times (0, 474)$ .

## Acknowledgements

O.D thanks the support of FAI (Universidad de los Andes, Project “Sistemas Reacción-Difusión Lejos del equilibrio”). E.T wish to thank FONDECYT (P.1020374) and FONDAP (P.11980002).

## References

- [1] J.J. Niemela, G. Ahlers, D.S. Cannell, *Phys. Rev. Lett.* 64 (1990) 1365.
- [2] P. Kolodner, *Phys. Rev. Lett.* 69 (1992) 2519.
- [3] M. Dennin, G. Ahlers, D.S. Cannell, *Phys. Rev. Lett.* 77 (1996) 2475.

- [4] K.J. Lee, W.D. McCormick, J.E. Pearson, H.L. Swinney, *Nature* 369 (1994) 215.
- [5] P. Umbanhowar, F. Melo, H.L. Swinney, *Nature* 382 (1996) 793.
- [6] O. Thual, S. Fauve, *J. Phys. France*. 49 (1988) 1829.
- [7] W. van Saarloos, P.C. Hohenberg, *Phys. Rev. Lett.* 64 (1990) 749.
- [8] H. Sakaguchi, H.R. Brand, *Physica D* 117 (1998) 95.
- [9] B.A. Malomed, A.A. Nepomnyashchy, *Phys. Rev. A* 42 (1990) 6009.
- [10] N. Akhmediev, V.V. Afanasjev, *Phys. Rev. Lett.* 75 (1995) 2320.
- [11] J.M. Soto-Crespo, N. Akhmediev, V.V. Afanasjev, S. Wabnitz, *Phys. Rev. E* 55 (1997) 4783.
- [12] N. Akhmediev, J.M. Soto-Crespo, G. Town, *Phys. Rev. E* 63 (2001) 05602.
- [13] R.J. Deissler, H.R. Brand, *Phys. Rev. Lett.* 72 (1994) 478.
- [14] H.R. Brand, R.J. Deissler, *Phys. Rev. Lett.* 63 (1989) 2801.
- [15] R.J. Deissler, H.R. Brand, *Phys. Rev. A* 44 (1991) R3411.
- [16] R.J. Deissler, H.R. Brand, *Phys. Rev. E* 51 (1995) R852.
- [17] R.J. Deissler, H.R. Brand, *Phys. Rev. Lett.* 74 (1995) 4847.
- [18] R.J. Deissler, H.R. Brand, *Phys. Lett. A* 146 (1990) 252.
- [19] R.J. Deissler, H.R. Brand, *Phys. Rev. Lett.* 81 (1998) 3856.
- [20] H. Tian, Z. Li, J. Tian, G. Zhou, *Phys. Rev. E* 66 (2002) 066204.
- [21] O. Descalzi, M. Argentina, E. Tirapegui, *Int. Journal. of Bif. and Chaos* 12 (2002) 2459.
- [22] O. Descalzi, M. Argentina, E. Tirapegui, *Phys. Rev. E* 67 (2003) 015601(R).
- [23] O. Descalzi, *Physica A* 327 (2003) 23.