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Characterization of Lipschitz Continuous Difference of Convex Functions

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Abstract We give a necessary and sufficient condition for a difference of convex (DC, for short) functions, defined on a normed space, to be Lipschitz continuous. Our criterion relies on the intersection of the ε-subdifferentials of the involved functions.

Keywords DC functions · Lipschitz continuity · Integration formulas · ε-subdifferential

1 Introduction

Classical integration formulas [1, 2] have been first established in the Banach spaces setting for proper lower semicontinuous (lsc, for short) convex functions using the usual Fenchel subdifferential. These results have been extended outside the Banach space [3, 4] and the non-convex settings [5] by using the approximate (epsilon) subdifferential mapping.

In this paper, we exploit an idea, recently used in [6], to establish several characterizations for the Lipschitz character of the difference of convex (DC, for short) functions. As a consequence, if the Lipschitz constant is equal to 0, then we obtain an integration formula guaranteeing the coincidence of the involved functions up to an additive constant. The main result is presented in Theorem 2.1 in a slightly more

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general form, valid in the locally convex spaces setting, which characterizes the domination of the variations of DC functions by means of a convex continuous functions. The desired integration formula is obtained in Theorem 2.2.

2 The Main Result

In this paper, we work with a (Hausdorff) real locally convex topological vector space X, whose dual is denoted by X^*. The duality product is denoted by $\langle \cdot, \cdot \rangle : X \times X^* \longrightarrow \mathbb{R}$, and the zero vector (in X and X^*) by θ. Given an extended real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a point x in the domain of f, $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$, the Fenchel subdifferential of f at x is defined as

$$\partial f(x) := \left\{x^* \in X^* : f(y) - f(x) \geq \left\langle y - x, x^* \right\rangle \text{ for all } y \in X\right\}.$$

For $\varepsilon > 0$, the ε-subdifferential of f at x is given by

$$\partial_\varepsilon f(x) := \left\{x^* \in X^* : f(y) - f(x) \geq \left\langle y - x, x^* \right\rangle - \varepsilon \text{ for all } y \in X\right\}.$$

The desired results providing the characterization of Lipschitz DC functions will be given in Theorem 2.2, which is a consequence of the following general Theorem. Hereafter $f, g : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ are two given functions with a common domain

$$D := f^{-1}(\mathbb{R}) = g^{-1}(\mathbb{R}),$$

assumed nonempty and convex.

Theorem 2.1 *Let $h : X \longrightarrow \mathbb{R}$ be a given continuous convex function such that $h(\theta) = 0$. Then, the following statements are equivalent*:

(i) *f and g are convex, lsc on D, and satisfy*

$$f(x) - g(x) \leq f(y) - g(y) + h(x - y) \quad \textit{for all } x, y \in D.$$

(ii) *For each $x \in D$*

$$\emptyset \neq \partial_\varepsilon f(x) \subset \partial_\varepsilon g(x) + \partial_\varepsilon h(\theta) \quad \textit{for all } \varepsilon > 0.$$

(iii) *For each $x \in D$ there exists $\delta > 0$ such that*

$$\emptyset \neq \partial_\varepsilon f(x) \subset \partial_\varepsilon g(x) + \partial_\varepsilon h(\theta) \quad \textit{for all } \varepsilon \in]0, \delta[.$$

(iv) *For each $x \in D$*

$$\partial_\varepsilon f(x) \cap \left(\partial_\varepsilon g(x) + \partial_\varepsilon h(\theta)\right) \neq \emptyset \quad \textit{for all } \varepsilon > 0.$$

(v) *For each $x \in D$ there exists $\delta > 0$ such that*

$$\partial_\varepsilon f(x) \cap \left(\partial_\varepsilon g(x) + \partial_\varepsilon h(\theta)\right) \neq \emptyset \quad \textit{for all } \varepsilon \in]0, \delta[.$$

Proof (i) $\Longrightarrow$ (ii). Since f is proper ($\operatorname{dom} f \neq \emptyset$), convex and lsc on D, for any given $\varepsilon > 0$ the ε-subdifferential operator $\partial_\varepsilon f$ is nonempty on D [7, Proposition 2.4.4(iii)]. For $x \in D$, we define the function $\widetilde{g} : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ as

$$\widetilde{g} := g + f(x) - g(x)$$

so that by (i) the inequality $f \leq \widetilde{g} + h(\cdot - x)$ holds, as well as

$$f(x) = \widetilde{g}(x) + h(\theta) = \widetilde{g}(x).$$

Note that $\operatorname{cl} \widetilde{g} = \operatorname{cl} g + f(x) - g(x)$, where "cl" refers to the corresponding lsc envelope. Hence, as g is lsc on D, $\operatorname{cl} \widetilde{g}$ coincides with $g + f(x) - g(x)$ on D, which implies that it is proper. Therefore, since [8, Lemma 15]

$$\operatorname{cl}\bigl(\widetilde{g} + h(\cdot - x)\bigr) = \operatorname{cl} \widetilde{g} + h(\cdot - x) = \operatorname{cl} g + h(\cdot - x) + f(x) - g(x)$$

and $\partial_\delta(\operatorname{cl} \widetilde{g})(x) = \partial_\delta \widetilde{g}(x) = \partial_\delta g(x)$ (for all $\delta > 0$), by appealing to the sum rule of the ε-subdifferential (e.g., [7]) we get

$$\begin{aligned} \partial_\varepsilon f(x) &\subset \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \bigl(\partial_{\varepsilon_1}(\operatorname{cl} \widetilde{g})(x) + \partial_{\varepsilon_2} h(\theta)\bigr) \\ &= \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \bigl(\partial_{\varepsilon_1} g(x) + \partial_{\varepsilon_2} h(\theta)\bigr) \subset \partial_\varepsilon g(x) + \partial_\varepsilon h(\theta); \end{aligned}$$

showing that (ii) holds.

The implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (v) and (ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) are obvious.

(v) $\Longrightarrow$ (i). We fix $x, y \in D$ and take an arbitrary number $\varepsilon > 0$. For $m = 1, 2, \ldots$ we denote

$$x_{m,i} := x + \frac{i}{m}(y - x) \quad \text{for } i = 0, 1, \ldots, m.$$

Then, by invoking the current assumption (v) for each i and m, there exists some $\gamma_{m,i} \in]0, m^{-1}[$ such that

$$\partial_{m^{-1}\gamma\varepsilon} f(x_{m,i}) \cap \bigl[\partial_{m^{-1}\gamma\varepsilon} g(x_{m,i}) + \partial_{m^{-1}\gamma\varepsilon} h(\theta)\bigr] \neq \emptyset \quad \text{for all } \gamma \in]0, \gamma_{m,i}[.$$

Set

$$\gamma_m := \min_{i \in \{1, \ldots, m\}} \gamma_{m,i},$$

so that $\gamma_m > 0$, and choose $u^*_{m,i} \in \partial_{m^{-1}\gamma_m\varepsilon} f(x_{m,i})$, $v^*_{m,i} \in \partial_{m^{-1}\gamma_m\varepsilon} g(x_{m,i})$ and $w^*_{m,i} \in \partial_{m^{-1}\gamma\varepsilon} h(\theta)$ such that $u^*_{m,i} = v^*_{m,i} + w^*_{m,i}$ for $i = 1, \ldots, m - 1$. In this way, if $u^* \in \partial_\varepsilon f(x)$ and $v^* \in \partial_\varepsilon g(y)$ are given, we write

$$f(x_{m,1}) - f(x) \geq \frac{1}{m}\bigl\langle y - x, u^*\bigr\rangle - \varepsilon$$

$$f(x_{m,i+1}) - f(x_{m,i}) \geq \frac{1}{m}\bigl\langle y - x, u^*_{m,i}\bigr\rangle - m^{-1}\gamma_m\varepsilon \quad (i = 1, \ldots, m - 1)$$

$$g(x_{m,i-1}) - g(x_{m,i}) \geq -\frac{1}{m}\langle y - x, v^*_{m,i}\rangle - m^{-1}\gamma_m \varepsilon \quad (i = 1, \ldots, m-1)$$
$$g(x_{m,m-1}) - g(y) \geq -\frac{1}{m}\langle y - x, v^*\rangle - \varepsilon.$$

Adding up these inequalities and using the facts that $x_{m,m} = y$ and $x_{m,0} = x$, together with $u^*_{m,i} = v^*_{m,i} + w^*_{m,i}$, we obtain

$$f(y) - f(x) + g(x) - g(y) \geq \frac{1}{m}\langle y - x, u^* - v^*\rangle + \frac{1}{m}\sum_{i=1}^{m-1}\langle y - x, w^*_{m,i}\rangle$$
$$- 2(m-1)m^{-1}\gamma_m\varepsilon - 2\varepsilon.$$

Thus, since $w^*_{m,i} \in \partial_{m^{-1}\gamma\varepsilon} h(\theta)$, we deduce that

$$f(y) - f(x) + g(x) - g(y) \geq \frac{1}{m}\langle y - x, u^* - v^*\rangle - \frac{m-1}{m}h(x - y)$$
$$- 2(m-1)m^{-1}\gamma_m\varepsilon - 2\varepsilon,$$

which gives us, as m goes to ∞ (recall that $0 < \gamma_m \leq m^{-1}$),

$$f(y) - f(x) + g(x) - g(y) \geq -h(x - y) - 2\varepsilon.$$

Hence, by letting ε go to 0 we get

$$f(x) - g(x) \leq f(y) - g(y) + h(x - y);$$

that is, (i) follows. □

The particular case $h \equiv 0$ in Theorem 2.1 yields a new integration result, which relies on the intersection of the ε-subdifferentials of the nominal functions. We will denote by f_D and g_D the restrictions of f and g to D, respectively.

Corollary 2.1 (Cf. [9, Corollary 2.5]) *The following statements are equivalent*:

(i) *f and g are convex, lsc on D, and $f_D - g_D$ is constant.*
(ii) *For each $x \in D$*

$$\emptyset \neq \partial_\varepsilon f(x) \subset \partial_\varepsilon g(x) \quad \textit{for all } \varepsilon > 0.$$

(iii) *For each $x \in D$ there exists $\delta > 0$ such that*

$$\emptyset \neq \partial_\varepsilon f(x) \subset \partial_\varepsilon g(x) \quad \textit{for all } \varepsilon \in]0, \delta[.$$

(iv) *For each $x \in D$*

$$\partial_\varepsilon f(x) \cap \partial_\varepsilon g(x) \neq \emptyset \quad \textit{for all } \varepsilon > 0.$$

(v) *For each $x \in D$ there exists $\delta > 0$ such that*

$$\partial_\varepsilon f(x) \cap \partial_\varepsilon g(x) \neq \emptyset \quad \textit{for all } \varepsilon \in]0, \delta[.$$

The following corollary, giving a criterion for integrating the Fenchel subdifferential, is an immediate consequence of Corollary 2.1 in view of the straightforward relationships $\partial f(x) \subset \partial_\varepsilon f(x)$ and $\partial g(x) \subset \partial_\varepsilon g(x)$ for every $x \in D$ and every $\varepsilon > 0$.

Corollary 2.2 (Cf. [6, Theorem 1]) *The following statements are equivalent*:

(i) *For each* $x \in D$

$$\emptyset \neq \partial f(x) \subset \partial g(x).$$

(ii) *For each* $x \in D$

$$\partial f(x) \cap \partial g(x) \neq \emptyset.$$

(iii) *For each* $x \in D$

$$\emptyset \neq \partial f(x) = \partial g(x).$$

If these statements hold, then f *and* g *are convex, lsc on* D, *and* $f_D - g_D$ *is constant.*

Remark 2.1

(a) The preceding results remain true if X is an arbitrary locally convex real topological vector space (not necessarily Hausdorff). Indeed, the equivalence between the convex and the lsc character of a function and the nonemptiness of its ε-subdifferentials is a reformulation of the Fenchel–Moreau Theorem, the validity of which in non-Hausdorff spaces has been proved by S. Simons [10, Theorem 10.1].
(b) The equivalence between (i) and (ii) in Corollary 2.1 also follows from a well-known characterization of global minima of DC functions due to J.-B. Hiriart-Urruty [11, Theorem 4.4]. Indeed, according to this characterization, if f and g are convex, then one has $\partial_\varepsilon f(x) \subset \partial_\varepsilon g(x)$ for all $\varepsilon > 0$ if and only if x is a global minimum of $f_D - g_D$. Hence, that condition holds for every $x \in D$ if and only if every $x \in D$ is a global minimum of $f_D - g_D$, which is obviously equivalent to $f_D - g_D$ being constant on D.

From now on, we suppose that X is a normed space with a norm $\|\cdot\|$; the dual norm is denoted by $\|\cdot\|_*$. We use $B_*(\theta, K)$ to denote the closed ball in $(X^*, \|\cdot\|_*)$ with center θ and radius $K \geq 0$, and for $A, B \subset X^*$, we set

$$d(A, B) := \inf\big\{\|a - b\|_* : a \in A,\ b \in B\big\},$$

with the convention that $d(A, B) := +\infty$ if A or B are empty.

At this moment, we easily get the main result of the paper from Theorem 2.1 when h is a multiple of the norm function.

Theorem 2.2 *Let $K \geq 0$. Then, the following statements are equivalent*:

(i) *f and g are convex, lsc on D, and $f_D - g_D$ is Lipschitz with constant K.*
(ii) *For each $x \in D$*

$$\emptyset \neq \partial_\varepsilon f(x) \subset \partial_\varepsilon g(x) + B_*(\theta, K) \quad \textit{for all } \varepsilon > 0.$$

(iii) *For each $x \in D$ there exists $\delta > 0$ such that*

$$\emptyset \neq \partial_\varepsilon f(x) \subset \partial_\varepsilon g(x) + B_*(\theta, K) \quad \textit{for all } \varepsilon \in]0, \delta[.$$

(iv) *For each $x \in D$*

$$\partial_\varepsilon f(x) \cap \big[\partial_\varepsilon g(x) + B_*(\theta, K)\big] \neq \emptyset \quad \textit{for all } \varepsilon > 0.$$

(v) *For each $x \in D$ there exists $\delta > 0$ such that*

$$\partial_\varepsilon f(x) \cap \big[\partial_\varepsilon g(x) + B_*(\theta, K)\big] \neq \emptyset \quad \textit{for all } \varepsilon \in]0, \delta[.$$

(vi) *For each $x \in D$*

$$d\big(\partial_\varepsilon f(x), \partial_\varepsilon g(x)\big) \leq K \quad \textit{for all } \varepsilon > 0.$$

(vii) *For each $x \in D$ there exists $\delta > 0$ such that*

$$d\big(\partial_\varepsilon f(x), \partial_\varepsilon g(x)\big) \leq K \quad \textit{for all } \varepsilon \in]0, \delta[.$$

Proof The proofs of the equivalences

$$\text{(i)} \Longleftrightarrow \text{(ii)} \Longleftrightarrow \text{(iii)} \Longleftrightarrow \text{(iv)} \Longleftrightarrow \text{(v)}$$

follow from Theorem 2.1 by observing that $\partial_\varepsilon(K\|\cdot\|)(\theta) = B_*(\theta, K)$. Since the implications (iv) $\Longrightarrow$ (vi) $\Longrightarrow$ (vii) are obvious, we need only to prove that (vii) $\Longrightarrow$ (i). Given $x \in D$, we note that (vii) implies the existence of $\delta > 0$ such that, for all $\gamma > 0$,

$$\partial_\varepsilon f(x) \cap \big[\partial_\varepsilon g(x) + B_*(\theta, K + \gamma)\big] \neq \emptyset \quad \text{for all } \varepsilon \in]0, \delta[.$$

Hence, by the equivalence between (v) and (i), f and g are convex, lsc on D, and $f_D - g_D$ is Lipschitz with constant $K + \gamma$. Therefore, since γ is arbitrary, $f_D - g_D$ is Lipschitz with constant K. □

Observing that statements (i), (iv), (v), (vi) and (vii) in Theorem 2.2 are symmetric in f and g, it turns out that, under the assumptions of this theorem, statements (ii) and (iii) are also symmetric; therefore, if one has

$$\emptyset \neq \partial_\varepsilon f(x) \subset \partial_\varepsilon g(x) + B^*(\theta, K) \quad \text{for all } \varepsilon > 0$$

for each $x \in D$, then one also has

$$\emptyset \neq \partial_\varepsilon g(x) \subset \partial_\varepsilon f(x) + B^*(\theta, K) \quad \text{for all } \varepsilon > 0$$

for each $x \in D$. We thus obtain the following corollary:

Corollary 2.3 *Let* $K \geq 0$. *If some* (*hence all*) *of the statements* (i)–(vii) *of Theorem* 2.2 *holds*, *then*, *for every* $x \in D$ *and every* $\varepsilon > 0$, *the Hausdorff distance between* $\partial_\varepsilon f(x)$ *and* $\partial_\varepsilon g(x)$ *does not exceed the constant* K.

Corollary 2.4 *The following statements are equivalent*:

(i) f *and* g *are convex*, *lsc on* D, *and* $f_D - g_D$ *is constant.*
(ii) *For each* $x \in D$

$$d\big(\partial_\varepsilon f(x), \partial_\varepsilon g(x)\big) = 0 \quad \textit{for all } \varepsilon > 0.$$

(iii) *For each* $x \in D$ *there exists* $\delta > 0$ *such that*

$$d\big(\partial_\varepsilon f(x), \partial_\varepsilon g(x)\big) = 0 \quad \textit{for all } \varepsilon \in]0, \delta[.$$

From the previous result we obtain a complement to Corollary 2.2:

Corollary 2.5 *The following statements are equivalent*:

(i) *For each* $x \in D$

$$\emptyset \neq \partial f(x) = \partial g(x).$$

(ii) *For each* $x \in D$

$$d\big(\partial f(x), \partial g(x)\big) = 0.$$

3 Conclusion

This paper provides criteria for Lipschitz continuity of DC functions defined in Banach spaces, involving ε-subdifferentials. The key of the proof of the main result (Theorem 2.1), traced out from [6], allowed us to work in the more general setting of locally convex spaces. The use of ε-subdifferentials rather than (exact) subdifferentials is suitable in the current paper, since it deals with (proper lsc) convex functions which are not necessarily continuous.

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