A NEW CLASS OF INVERSE M-MATRICES OF TREE-LIKE TYPE*

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Abstract. In this paper, we use weighted dyadic trees to introduce a new class of nonnegative matrices whose inverses are column diagonally dominant M-matrices.

1. Introduction. It is a longstanding and difficult problem to characterize all nonnegative matrices whose inverses are M-matrices, although inverses of all non-singular M-matrices are always nonnegative matrices. In 1977, Willoughby [16] called the problem of finding or characterizing nonnegative matrices whose inverses are M-matrices the *inverse M-matrix problem*. Johnson [7], Fiedler, Johnson, and Markham [6], and Fiedler [4] devoted much effort to general properties of inverse M-matrices. For definitions, references, and background on M-matrices and the inverse M-matrix problem, the reader is referred to Berman and Plemmons [1] and Johnson [7]. However, until now there have been just a few known classes of inverse M-matrices. The oldest class of symmetric inverse M-matrices is the class of positive type D matrices defined by Markham [8]. In 1994, Martínez, Michon, and San Martín introduced a strictly symmetric ultrametric matrix $A = (a_{ij})$ whose entries satisfy

$$a_{ij} \geq \min\{a_{ik}, a_{kj}\} \quad \text{for all} \quad i, j, k,$$

$$a_{ii} > \max_{j \neq i} a_{ij} \quad \text{for all} \quad i$$

and proved that inverses of strictly symmetric ultrametric matrices are row and column diagonally dominant M-matrices (see [9] and also [13]). Later, nonsymmetric ultrametric matrices were independently introduced by McDonald et al. [11] and Nabben and Varga [14], i.e., nested block form and generalized ultrametric matrices. After a suitable permutation, every generalized ultrametric matrix can be put into nested block form, which contains type D matrices. Recently, Fiedler [5] introduced a new class of inverse M-matrices. Furthermore, Nabben [12] was motivated by Fiedler's result and introduced a new class of inverse M-matrices.

We have been motivated by the results in [3], [5], [10], [11], [14], and [12] to introduce in section 2 a new class of nonnegative matrices by using weighted dyadic trees. We state the following condition under which our main result holds: their

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inverses are column diagonally dominant M-matrices. In section 3, some preliminary properties and lemmas are presented. In particular, it is shown that these weighted tree matrices admit a representation that we call the quasi-nested block form. The proof of the main result is supplied in section 4. Finally, in section 5, we study the class of all the permutations, which leads to the matrix being presented in a quasi-nested block form.

2. Definitions and main result. Let $T = (V, E)$ be a tree on n vertices and edge set E. Sometimes we also write $V = V(T)$, $E = E(T)$. For any two vertices s and t, there is a unique path $geod(s, t)$ from vertex s to vertex t. In particular $geod(s, s) = \{s\}$. Let vertex $r \in V$ be a root of the tree T. We may define a partial order relation "$\preceq$" on T: $s \preceq t$ if and only if $s \in geod(r, t)$. Moreover, for $s, t \in V$, $s \wedge t = \sup\{v : v \in geod(r, s) \cap geod(r, t)\}$ denotes the closest common ancestor of s and t. Thus $\mathbf{s}(t) = \{v \in V : t \preceq v, (t, v) \in E\}$ is the set of successors of t, and $I = \{i \in T : \mathbf{s}(i) = \emptyset\}$ is the set of leaves of the tree T. A tree is called *dyadic* if the cardinality of set $\mathbf{s}(t)$ is $|\mathbf{s}(t)| = 2$ for $t \notin I$. For vertex $t \notin I$ of a dyadic tree T, its successors are signed and denoted by t^- and t^+ (the signs $-$ or $+$ of the successors are fixed). In addition, since vertex $t \in T$ and the set $L(t) = \{i \in I : t \in geod(r, i)\}$ are in one-to-one correspondence relations, we may identify $L(t)$ with t. Thus, the root r is identified with I. The distinction between the roles of $L \in V$ and $L \subseteq I$ will be clear in the context when we use them. We usually say "element L" when referring to $L \in V$ and "set L" to mean $L \subseteq I$.

For $L \in T$, we denote by $T_L = (V_L, E_L)$ the dyadic subtree rooted by L, that is, $V_L = \{v \in V : L \preceq v\}$, $E_L = E \cap (V_L \times V_L)$. Its leaves are the elements of L. For $v \in V_L$, its signed successors in T_L coincide with its signed successors in T.

For a dyadic tree T, its set I of leaves can be totally ordered as follows: $i \leq j$ if $i \in t^-$, $j \in t^+$, where $t = i \wedge j$. We denote by $P^\phi : I \to \{1, \dots, n\}$ the permutation which assigns i to its rank in the total order and we call it the *canonical permutation*.

DEFINITION 2.1. *A matrix $U = (u_{ij} : i, j \in I)$ is called a $\mathcal{W}$ matrix if there exists a dyadic tree $T = (V, E)$ with set I of leaves and nonnegative vectors $\overrightarrow{\alpha} = (\alpha_i : i \in V)$, $\overrightarrow{\beta} = (\beta_i : i \in V)$ satisfying that*

(i) $\alpha_i = \beta_i > 0$ *for* $i \in I$;

(ii) $0 \leq \alpha_i \leq 1$ *and* $0 \leq \beta_i \leq 1$ *for* $i \in V \setminus I$;

(iii) β *is* $\preceq$-*increasing in* $V \setminus I$, *that is,* $s \preceq t \in V \setminus I$ *implies* $\beta_s \leq \beta_t$;

(iv) $u_{ij} = \alpha_i \Pi_{(l, l^-) \in geod(t, i)} \alpha_l$ *if* $(i, j) \in (t^-, t^+)$, *and* $u_{ij} = \beta_t \alpha_i \Pi_{(l, l^-) \in geod(t, i)} \alpha_l$ *if* $(i, j) \in (t^+, t^-)$, *where* $t = i \wedge j$;

(v) $u_{ii} = \alpha_i$ *for* $i \in I$.

The matrix U is said to be supported by the dyadic tree T and defined by $\overrightarrow{\alpha}, \overrightarrow{\beta}$ on T.

For $J, K \subseteq I$, denote $U_{JK} = (u_{ij} : i \in J, j \in K)$. It is easy to see that if U is a $\mathcal{W}$ matrix supported by $T = (V, E)$ and $L \in V$, then U_{LL} is also a $\mathcal{W}$ matrix supported by T_L and defined by the restricted vectors $\overrightarrow{\alpha}|_{V_L}$ and $\overrightarrow{\beta}|_{V_L}$ on V_L.

The main result of this paper is the following.

THEOREM 2.2. *Let U be a $\mathcal{W}$ matrix. If U does not contain a row of zeros and no two columns in U are the same, then U is nonsingular and its inverse is a column diagonally dominant M-matrix.*

3. Preliminaries and lemmas. In this section, we first present an equivalent condition for $U \in \mathcal{W}$.

DEFINITION 3.1. *Let $C = (c_{ij})$ be a nonnegative matrix of order n with positive main diagonal elements. We define inductively as follows what it means for C to be in quasi-nested block form:*

(i) *If $n = 1$, then C is in quasi-nested block form.*

(ii) *If $n > 1$, and quasi-nested block form has been defined for all $k \times k$ nonnegative matrices with $k < n$, then C is in quasi-nested block form if*

$$C = \begin{pmatrix} C_{11} & b_{12}b_J e_K{}^T \\ b_{21}b_K e_J{}^T & C_{22} \end{pmatrix},$$

where C_{11} and C_{22} are $n_1 \times n_1$ and $n_2 \times n_2$ square matrices in quasi-nested block form with $n_1 \geq 1$, $n_2 \geq 1$, $n = n_1 + n_2$; b_J and b_K are the last columns of C_{11} and C_{22}, respectively; e is a vector of all ones with suitable dimension; $0 \leq b_{12} \leq 1$, $0 \leq b_{21} \leq 1$; and $c_{ij} \geq c_{ik}$ for all $k \geq j \geq i$, $c_{ij} \geq c_{ik}$ for all $i \geq j \geq k$.

THEOREM 3.2. *U is a $\mathcal{W}$ matrix if and only if there exists a permutation matrix P such that PUP^T is a matrix in quasi-nested block form. Moreover, P can be taken to be the matrix associated with the canonical permutation P^ϕ.*

Proof. Necessity. We prove the assertion by induction on n, the dimension of U. It is clear for $n = 1, 2$. Assume that the assertion holds for less than n. Let us consider the total order $\leq$ on I defined by the dyadic tree T supporting U. The successors of the root I are denoted by $J = I^-$ and $K = I^+$. Then there exists a permutation matrix P such that

$$PUP^T = \begin{pmatrix} U_{JJ} & U_{JK} \\ U_{KJ} & U_{KK} \end{pmatrix},$$

where the matrices U_{JJ} and U_{KK} are $\mathcal{W}$ matrices. We denote by n_1 and n_2 the orders of U_{JJ} and U_{KK}, respectively. Clearly $n_1 > 0$, $n_2 > 0$, and $n_1 + n_2 = n$. Hence by the induction hypothesis, there exist permutation matrices Q_J and Q_K such that $Q_J U_{JJ} Q_J^T = C_{11}$ and $Q_K U_{KK} Q_K^T = C_{22}$ are matrices in quasi-nested block form. Moreover, Q_J and Q_K can be taken to be the matrices associated with permutations $Q_J^{\phi_1}$ and $Q_K^{\phi_2}$, respectively.

Let $P_1 = \text{diag}(Q_J, Q_K)P$. Then

$$P_1 U P_1^T = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} := C.$$

For $i \leq n_1 < j$, since $i \wedge j = I$ and $i \in I^-$, we get $c_{ij} = \alpha_I(\alpha_i \prod_{(l,l^-) \in geod(I^-,i)} \alpha_l) = \alpha_I c_{i,n_1}$. Hence $C_{12} = \alpha_I b_J e_K{}^T$, where b_J is the last column of C_{11}. By a similar argument, we may show that $C_{21} = \beta_I b_K e_J^T$, where b_K is the last column of C_{22}.

Let $i \leq j \leq k$. If $i \leq j \leq k \leq n_1$ or $n_1 < i \leq j \leq k$, then by the induction hypothesis, $c_{ij} \geq c_{ik}$; if $i \leq j \leq n_1 < k$, also by the induction hypothesis we get $c_{ij} \geq c_{i,n_1} \geq c_{i,n_1}\alpha_I = c_{ik}$; and in the case $i \leq n_1 < j \leq k$, we find directly $c_{ij} = c_{ik}$. Let $i \geq j \geq k$. If $i \geq j \geq k \geq n_1$ or $n_1 > i \geq j \geq k$, then by the induction hypothesis, $c_{ij} \geq c_{ik}$; if $i > n_1 \geq j \geq k$, then $c_{ij} = c_{ik}$; and if $i \geq j > n_1 \geq k$, then $i \wedge k = I$, $i \wedge j = t$, and

$$c_{ij} = \alpha_i \beta_t \Pi_{(l,l^-) \in geod(t,i)} \alpha_l \quad \text{and} \quad c_{ik} = \alpha_i \beta_I \Pi_{(l,l^-) \in geod(t,i)} \alpha_l \Pi_{(l,l^-) \in geod(I,t)} \alpha_l$$

since $0 \leq \alpha_l \leq 1$ and $\beta_I \leq \beta_t$, we have $c_{ij} \geq c_{ik}$. Hence C is a matrix in quasi-nested block form. Moreover, with this construction, an induction argument shows that the final P_1 will correspond to the canonical permutation P^ϕ.

Sufficiency. We proceed as before by induction on the size of the matrix. For $n = 2$,

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

where $c_{12} \leq c_{11}$ and $c_{21} \leq c_{22}$. Let T be a dyadic tree with tree elements $V = \{I, I^-, I^+\}$, $\alpha_{I^-} = \beta_{I^-} = c_{11}$, $\alpha_{I^+} = \beta_{I^+} = c_{22}$, $\alpha_I = \frac{c_{12}}{c_{11}}$, and $\beta_I = \frac{c_{21}}{c_{22}}$. The matrix U with support tree T is just C. Hence the assertion holds for $n = 2$. Assume that the assertion holds when the dimension is less than n. By the definition of matrix C in quasi-nested block form,

$$C = \begin{pmatrix} C_{11} & b_{12}c_J e_K^T \\ b_{21}c_K e_J^T & C_{22} \end{pmatrix},$$

where C_{ii} is a matrix of order n_i in quasi-nested block form for $i = 1, 2$ and both c_J and c_K are the last columns of C_{11} and C_{22}, respectively. By the induction hypothesis, there exist two dyadic trees T_1 and T_2 with roots J and K and $\overrightarrow{\alpha} = (\alpha_t : t \in V(T_i))$, $\overrightarrow{\beta} = (\beta_t : t \in V(T_i))$ for $i = 1, 2$. Now we define a new tree T obtained from $T_1 \cup T_2$ by adding a new root vertex I associated with $\alpha_I = b_{12}$ and $\beta_I = b_{21}$ and two edges (I, J) and (I, K), where $J = I^-$ and $K = I^+$. Then the matrix associated with T has the following form:

$$U = \begin{pmatrix} U_{JJ} & U_{12} \\ U_{21} & U_{KK} \end{pmatrix} = \begin{pmatrix} C_{11} & U_{12} \\ U_{21} & C_{22} \end{pmatrix}.$$

For $i \leq n_1 < j$, $u_{ij} = \alpha_i \Pi_{(l,l^-)\in geod(I,i)}\alpha_l = \alpha_I \alpha_i \Pi_{(l,l^-)\in geod(I^-,i)}\alpha_l = \alpha_I u_{i,n_1}$. Hence $U_{12} = b_{12}c_J e_K^T$, where c_J is the last column of $U_{JJ} = C_{11}$. Similarly, $U_{21} = b_{21}c_K e_J^T$, where c_K is the last column of $U_{KK} = C_{22}$. Therefore $U = C$ and C is a $\mathcal{W}$ matrix. Since the permutation matrix P corresponds to renumbering of the vertices, PCP^T is still a $\mathcal{W}$ matrix.   □

LEMMA 3.3. *Let $U = (u_{ij} : i, j \in I)$ be a $\mathcal{W}$ matrix associated with tree T in quasi-nested block form and $\overrightarrow{\alpha}, \overrightarrow{\beta}$. If $0 \leq \delta \leq \beta_I$ and $\delta < 1$, then $\widetilde{U} = U - \delta b_I e^T$ is still a $\mathcal{W}$ matrix associated with T and $\widetilde{\alpha_I} = \frac{(1-\delta)\alpha_I}{1-\delta\alpha_I}, \widetilde{\beta_I} = \frac{\beta_I - \delta}{1-\delta}$, where b_I is the last column of U.*

Proof. We assume $I = \{1, \ldots, n\}$ is totally ordered by the tree T. We proceed on n, the dimension of matrix U. If U is a 2×2 matrix with the root I of the tree T and the set $\{1, 2\}$ of leaves, then we assume $1 = I^-, 2 = I^+$. Hence

$$U = \begin{pmatrix} \alpha_1 & \alpha_1 \alpha_I \\ \beta_I \alpha_2 & \alpha_2 \end{pmatrix},$$

where $0 \leq \alpha_I, \beta_I \leq 1$. Then

$$\widetilde{U} = \begin{pmatrix} (1-\delta\alpha_I)\alpha_1 & (1-\delta)\alpha_I\alpha_1 \\ (\beta_I - \delta)\alpha_2 & (1-\delta)\alpha_2 \end{pmatrix}.$$

We take the same tree T with vectors $\overrightarrow{\widetilde{\alpha}}, \overrightarrow{\widetilde{\beta}}$ given by $\widetilde{\alpha_1} = (1-\delta\alpha_I)\alpha_1, \widetilde{\alpha_2} = (1-\delta)\alpha_2$, and

$$\widetilde{\alpha_I} = \frac{(1-\delta)\alpha_I}{1-\delta\alpha_I}, \quad \widetilde{\beta_I} = \frac{\beta_I - \delta}{1-\delta}.$$

It is clear that $0 \leq \widetilde{\alpha}_I, \widetilde{\beta}_I \leq 1$ and that $\widetilde{U}$ is just the matrix defined by vectors $\overrightarrow{\widetilde{\alpha}}$, $\overrightarrow{\widetilde{\beta}}$ on the tree T. Hence the assertion holds for $n = 2$. Assume that the assertion holds when the dimension of a matrix is less than n. Let U be an $n \times n$ matrix. By Theorem 3.2, we may assume that

$$U = \begin{pmatrix} U_{JJ} & \alpha_I b_J e_K^T \\ \beta_I b_K e_J^T & U_{KK} \end{pmatrix}$$

is associated with tree T, U_{JJ} with subtree T_1, and tree U_{KK} with subtree T_2. Then $b_I = \binom{\alpha_I b_J}{b_K}$ and

$$\widetilde{U} = U - \delta b_I e^T = \begin{pmatrix} U_{JJ} - \delta\alpha_I b_J e_J^T & (1-\delta)\alpha_I b_J e_K^T \\ (\beta_I - \delta)b_K e_J^T & U_{KK} - \delta b_K e_K^T \end{pmatrix} := \begin{pmatrix} \overline{U}_{JJ} & \overline{U}_{12} \\ \overline{U}_{21} & \overline{U}_{KK} \end{pmatrix},$$

where b_J and b_K are the last columns of U_{JJ} and U_{KK}, respectively. Since $\overrightarrow{\beta}$ is increasing and $0 \leq \alpha_I \leq 1$, we have $0 \leq \delta\alpha_I \leq \beta_I \alpha_I \leq \beta_J$ and $\delta\alpha_I < 1$. Hence by the induction hypothesis, $U_{JJ} - \delta\alpha_I b_J e_J^T = \overline{U}_{JJ}$ is a $\mathcal{W}$ matrix defined by vectors $(\widetilde{\alpha}_t : t \in V(T_1))$ and $(\widetilde{\beta}_t : t \in V(T_1))$ on the subtree T_1. Moreover,

$$\widetilde{\alpha}_J = \frac{(1-\delta\alpha_I)\alpha_J}{1-\delta\alpha_I\alpha_J}, \quad \widetilde{\beta}_J = \frac{\beta_J - \delta\alpha_I}{1-\delta\alpha_I}.$$

By a similar argument, $U_{KK} - \delta b_K e_K^T = \overline{U}_K$ is a $\mathcal{W}$ matrix associated with subtree T_2 and vectors $(\widetilde{\alpha}_t : t \in V(T_2))$ and $(\widetilde{\beta}_t : t \in V(T_2))$. Moreover,

$$\widetilde{\alpha}_K = \frac{(1-\delta)\alpha_K}{1-\delta\alpha_K}, \quad \widetilde{\beta}_K = \frac{\beta_K - \delta}{1-\delta}.$$

Define $\widetilde{\alpha}_I = \frac{(1-\delta)\alpha_I}{1-\delta\alpha_I}$, $\widetilde{\beta}_I = \frac{\beta_I - \delta}{1-\delta}$. We have $0 \leq \widetilde{\alpha}_I, \widetilde{\beta}_I \leq 1$ and

$$\widetilde{\beta}_I = \frac{\beta_I - \delta}{1-\delta} \leq \frac{\beta_K - \delta}{1-\delta} = \widetilde{\beta}_K,$$

$$\widetilde{\beta}_I = \frac{\beta_I - \delta}{1-\delta} \leq \frac{\beta_I - \delta\alpha_I}{1-\delta\alpha_I} \leq \frac{\beta_J - \delta\alpha_I}{1-\delta\alpha_I} = \widetilde{\beta}_J.$$

Then the matrix X associated with the tree T and vectors $(\widetilde{\alpha}_t : t \in V(T))$, $(\widetilde{\beta}_t : t \in V(T))$ is just $\widetilde{U}$. In fact, $0 \leq \widetilde{\alpha}_t, \widetilde{\beta}_t \leq 1$ for $t \in V \setminus I$ and $\widetilde{\beta}$ is increasing in $V \setminus I$. For $i, j \in I^- = J$ or $i, j \in I^+ = K$, $X_{ij} = (\overline{U}_{JJ})_{ij} = \widetilde{U}_{ij}$ or $X_{ij} = (\overline{U}_{KK})_{ij} = \widetilde{U}_{ij}$; for $i \in J$, $j \in K$, and $|J| = n_1$, $X_{ij} = \widetilde{\alpha}_i \Pi_{(l,l^-)\in geod(I,i)}\widetilde{\alpha}_l = \widetilde{\alpha}_I X_{i,n_1} = \widetilde{\alpha}_I(\overline{U}_{JJ})_{i,n_1} = (1-\delta)\alpha_I(U_{JJ})_{i,n_1} = (\widetilde{U})_{ij}$; for $i \in K$, $j \in J$, $X_{ij} = \widetilde{\alpha}_i\widetilde{\beta}_I \Pi_{(l,l^-)\in geod(I,i)}\widetilde{\alpha}_l = \widetilde{\beta}_I\widetilde{\alpha}_i \Pi_{(l,l^-)\in geod(s,i)}\widetilde{\alpha}_l = \widetilde{\beta}_I X_{in} = (\widetilde{U})_{ij}$, where $i \wedge n = s$, since each edge from vertex I to vertex s is (t, t^+). This completes our proof. $\square$

4. Proof of Theorem 2.2.

LEMMA 4.1. *Let U be a $\mathcal{W}$ matrix defined by vectors $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$ on tree T. Then U does not contain a row of zeros and no two columns in U are the same if and only if $\alpha_t\beta_t < 1$ for $t \in V(T) \setminus I$ and $\alpha_i > 0$ for $i \in I$, where I is the set of leaves of T.*

Proof. Necessity. We use the induction on the size of matrix U. It is clear that the assertion holds for $|I| = 1, 2$. Since U does not contain a row of zeros, $U_{ii} = \alpha_i > 0$

for $i \in I$. Let $J = I^-$ and $K = I^+$. It is easy to see that no two columns in U_{JJ} and U_{KK} are the same. By the induction hypothesis, it suffices to verify that $\alpha_I \beta_I < 1$. Assume that

$$U = \begin{pmatrix} U_{JJ} & \alpha_I b_J e_K{}^T \\ \beta_I b_K e_J{}^T & U_{KK} \end{pmatrix}.$$

If $\alpha_I \beta_I = 1$, then $\alpha_I = \beta_I = 1$. Hence the $|I^-|$th and nth columns are the same, which is a contradiction. Thus $\alpha_I \beta_I < 1$.

Conversely, since $\alpha_i > 0$ it is clear that the assertion holds for $n = 1, 2$. We may assume that

$$U = \begin{pmatrix} U_{JJ} & \alpha_I b_J e_K^T \\ \beta_I b_K e_J^T & U_{KK} \end{pmatrix},$$

where U_{JJ} is an $n_1 \times n_1$ matrix. By the induction hypothesis, no two columns in U_{JJ} and U_{KK} are the same. Suppose that the ith and jth columns in U are the same with $i < j$. Then $i \le n_1 < j$ and $U_{ii} = U_{ij}$, $U_{ji} = U_{jj}$. On the other hand, $U_{ij} = \alpha_I U_{i,n_1} \le U_{ii}$ and $U_{ji} = \beta_I U_{jn} \le U_{jn} \le U_{jj}$. Hence $\alpha_I \beta_I = 1$, a contradiction. Therefore no two columns in U are the same. $\square$

Now we may present the proof of Theorem 2.2.

Proof of Theorem 2.2. We use induction with respect to the size of the matrix U. For $n = 2$, it is easy to see that $\det(U) = (1 - \alpha_I \beta_I)\alpha_1 \alpha_2 > 0$ and

$$U^{-1} = \begin{pmatrix} \alpha_1 & \alpha_1 \alpha_I \\ \beta_I \alpha_2 & \alpha_2 \end{pmatrix}^{-1} = \frac{1}{\det(U)} \begin{pmatrix} \alpha_2 & -\alpha_1 \alpha_I \\ -\beta_I \alpha_2 & \alpha_1 \end{pmatrix}.$$

Hence U^{-1} is a column diagonally dominant M-matrix. Assume that the assertion holds for less than n. For n, by Theorem 3.2, we may assume that

$$U = \begin{pmatrix} U_{JJ} & \alpha_I b_J e_K^T \\ \beta_I b_K e_J^T & U_{KK} \end{pmatrix}.$$

By Lemma 4.1, U_{JJ} and U_{KK} do not contain a row of zeros and no two columns in U_{JJ} and U_{KK} are the same. By the induction hypothesis, U_{JJ} and U_{KK} are nonsingular. Further, U_{JJ}^{-1} and U_{KK}^{-1} are column diagonally dominant M-matrices. So $\mu_J^t = e^T U_{JJ}^{-1} \ge 0$ and $\mu_K^t = e^T U_{KK}^{-1} \ge 0$. By $\alpha_I \beta_I < 1$ and the Sherman–Morrison formula (see [11]), we have

$$U^{-1} = \begin{pmatrix} U_{JJ}^{-1} + \frac{\alpha_I \beta_I}{1 - \alpha_I \beta_I} \varepsilon_J \mu_J^T & -\frac{\alpha_I}{1 - \alpha_I \beta_I} \varepsilon_J \mu_K^T \\ -\frac{\beta_I}{1 - \alpha_I \beta_I} \varepsilon_K \mu_J^T & U_{KK}^{-1} + \frac{\alpha_I \beta_I}{1 - \alpha_I \beta_I} \varepsilon_K \mu_K^T \end{pmatrix} := \begin{pmatrix} C & D \\ E & F \end{pmatrix},$$

where $\varepsilon_J = (0, \ldots, 0, 1)^T$ and $\varepsilon_K = (0, \ldots, 0, 1)^T$. It is easy to see that $D \le 0$ and $E \le 0$. Since $\alpha_I \beta_I \le \beta_J$ and $\alpha_I \beta_I < 1$, by Lemma 3.3, $U_{JJ} - \alpha_I \beta_I b_J e_J^T$ is still a $\mathcal{W}$ matrix. In addition,

$$C = U_{JJ}^{-1} + \frac{\alpha_I \beta_I}{1 - \alpha_I \beta_I} \varepsilon_J \mu_J^T = (U_{JJ} - \alpha_I \beta_I b_J e^T)^{-1}$$

is nonsingular. By the induction hypothesis, C is a column diagonally dominant M-matrix. By a similar argument, we may prove that F is a column diagonally

dominant M-matrix. Therefore U^{-1} is an M-matrix. Moreover,

$$e_J^T C + e_K^T E = e_J^T U_{JJ}^{-1} + \frac{\alpha_I \beta_I}{1 - \alpha_I \beta_I} e_J^T \varepsilon_J \mu_J^T + \frac{-\beta_I}{1 - \alpha_I \beta_I} e_K^T \varepsilon_K \mu_J^T = \frac{1 - \beta_I}{1 - \alpha_I \beta_I} \mu_J^T \geq 0,$$

$$e_J^T D + e_K^T F = \frac{-\alpha_I}{1 - \alpha_I \beta_I} e_J^T \varepsilon_J \mu_K^T + e_K^T U_{KK}^{-1} + \frac{\alpha_I \beta_I}{1 - \alpha_I \beta_I} e_K^T \varepsilon_K \mu_K^T = \frac{1 - \alpha_I}{1 - \alpha_I \beta_I} \mu_K^T \geq 0.$$

Hence U^{-1} is a column diagonally dominant M-matrix.   □

Remark 4.2. Neumann in [15] conjectured that the Hadamard product $A \circ A$ is an inverse M-matrix if A is an inverse M-matrix. Clearly, this conjecture is true for $A \in \mathcal{W}$ since $A \circ A \in \mathcal{W}$ (moreover for any $n \geq 1$, $A^{\circ n} \in \mathcal{W}$).

Example 4.3. Let T be a dyadic tree with $\overrightarrow{\alpha}, \overrightarrow{\beta}$ defined by Figure 1.
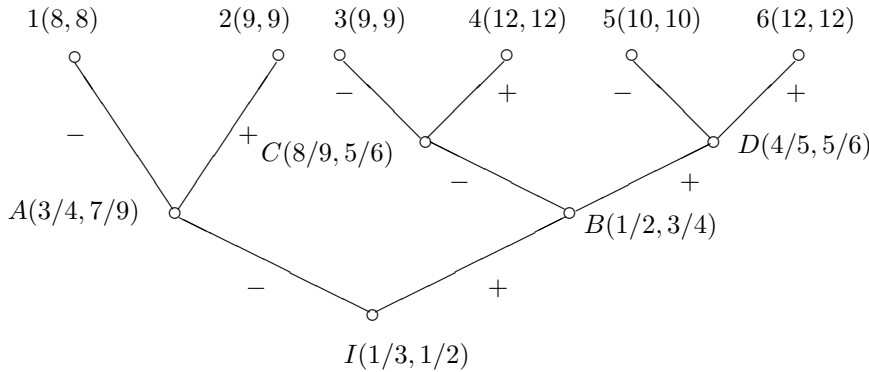

Fig. 1.

Then the matrix U, associated with tree T, and the inverse of U are

$$U = \begin{pmatrix} 8 & 6 & 2 & 2 & 2 & 2 \\ 7 & 9 & 3 & 3 & 3 & 3 \\ 2 & 2 & 9 & 8 & 4 & 4 \\ 3 & 3 & 10 & 12 & 6 & 6 \\ 4 & 4 & 6 & 6 & 10 & 8 \\ 6 & 6 & 9 & 9 & 10 & 12 \end{pmatrix}$$

and

$$U^{-1} = \begin{pmatrix} 0.3000 & -0.2000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 \\ -0.2200 & 0.2800 & -0.0114 & -0.0057 & -0.0160 & -0.0160 \\ -0.0000 & -0.0000 & 0.4286 & -0.2857 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & -0.3143 & 0.3429 & -0.0400 & -0.0400 \\ -0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.3000 & -0.2000 \\ -0.0400 & -0.0400 & -0.0800 & -0.0400 & -0.2120 & 0.2880 \end{pmatrix},$$

which is a column diagonally dominant M-matrix.

Remark 4.4. Nabben in [12] described a class of inverse M-matrices whose nested block form is similar to GUMs (generally ultrametric matrices) with the major change being that in the $(2,1)$-block the ee^T was replaced by ce^T, where b corresponds to the

last column of the $(2, 2)$-block. From Theorems 3.2 and 2.2, the quasi-nested block form in $\mathcal{W}$ is also similar to GUMs with the major changes being that the $(1, 2)$-block was replaced by be^T and the $(2, 1)$-block was replaced by ce^T, where b and c are the last columns of the $(1, 1)$-block and $(2, 2)$-block, respectively. Hence it is natural that the following two questions were proposed.

QUESTION 4.5. *Is it possible to use be^T in the off diagonal blocks, where b is any column of the corresponding diagonal block? Are there any other vectors that will work?*

QUESTION 4.6. *Is it possible to use be^T and ee^T alternately in the nested block form, or must one use one or the other only?*

The following two examples illustrate that the above questions are answered in a negative way.

Example 4.7. Let A be

$$
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A_{11} = \begin{pmatrix} 8 & 8 \times \frac{1}{8} \\ 10 \times 1 & 10 \end{pmatrix}, \quad A_{12} = \frac{1}{2} \begin{pmatrix} 8 \\ 10 \times 1 \end{pmatrix} (1\ 1),
$$

$$
A_{21} = \frac{1}{2} \begin{pmatrix} 10 \times \frac{1}{2} \\ 9 \end{pmatrix} (1\ 1), \quad A_{22} = \begin{pmatrix} 10 & 10 \times \frac{1}{2} \\ 9 \times \frac{2}{3} & 9 \end{pmatrix}.
$$

But

$$
A^{-1} = \begin{pmatrix} 0.1429 & 0.0190 & -0.0333 & -0.0556 \\ -0.1429 & 0.1143 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.1500 & -0.0833 \\ 0.0000 & 0.0667 & -0.0833 & 0.1944 \end{pmatrix}
$$

is not an M-matrix. Hence in general, we cannot use be^T in the off diagonal blocks for b not being the last column of the corresponding block.

Example 4.8. Let B be

$$
B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad B_{11} = \begin{pmatrix} 10 & 5 & \frac{2}{5} \times 5 \\ 6 & 10 & \frac{2}{5} \times 10 \\ \frac{1}{2} \times 10 & \frac{1}{2} \times 10 & 10 \end{pmatrix},
$$

$$
B_{12} = ee^T, \quad B_{21} = 2e^Te, \quad B_{22} = 5,
$$

a 4×4 matrix. But

$$
B^{-1} = \begin{pmatrix} 0.1439 & -0.0701 & 0.0023 & -0.0152 \\ -0.0708 & 0.1615 & -0.0487 & -0.0084 \\ -0.0350 & -0.0438 & 0.1264 & -0.0095 \\ -0.0152 & -0.0192 & -0.0320 & 0.2133 \end{pmatrix}
$$

is not an M-matrix. Hence in general, we cannot use be^T and ee^T alternately in the nested block form.

5. Combinatorial aspects of a $\mathcal{W}$ matrix in quasi-nested block form. In section 3, we have proved that each $\mathcal{W}$ matrix can be put into quasi-nested block form after a suitable permutation. In this section, we try to describe the set of permutations preserving a $\mathcal{W}$ matrix in quasi-nested block form, which is related to the behavior of a sub-Markov chain. The reader is referred to [2] and [3].

We assume that U is a $\mathcal{W}$ matrix in quasi-nested block form with supporting tree T and vectors $\overrightarrow{\alpha}, \overrightarrow{\beta}$, where $I = \{1, 2, \ldots, n\}$. The root of tree T is I and its successors are $I^- = J$ and $I^+ = K$. We also denote $|J| = m$ and write $U[i_1, \ldots, i_t]$ for the principal submatrix of U whose rows and columns are indexed by $1 \leq i_1 < i_2 < \cdots < i_t \leq n$.

Let $U[i_1, i_2, i_3, i_4]$ be the principal submatrix of U. It is easy to see that $U[i_1, i_2, i_3, i_4]$ is not a $\mathcal{W}$ matrix, in general. But we can obtain a $\mathcal{W}$ in quasi-nested form from $U[i_1, i_2, i_3, i_4]$ by changing the diagonal entries of $U[i_1, i_2, i_3, i_4]$. In fact, without loss of generality, we may assume that $i_1 \wedge i_2 \wedge i_3 \wedge i_4 = P$, $i_1 \wedge i_2 \wedge i_3 = M$, $i_1 \wedge i_2 = N$; $i_1, i_2, i_3 \in P^-$, $i_4 \in P^+$; $i_1, i_2 \in M^-$, $i_3 \in M^+$; $i_1 \in N^-$, $i_2 \in N^+$ (for the other cases, we may show the same result by a similar argument). Let $\gamma_1 = \alpha_{i_1} \prod_{(l,l^-) \in geod(N^-,i_1)} \alpha_l$, $\gamma_2 = \alpha_{i_2} \prod_{(l,l^-) \in geod(N,i_2)} \alpha_l$, $\gamma_3 = \alpha_{i_3} \prod_{(l,l^-) \in geod(M,i_3)} \alpha_l$, and $\gamma_4 = \alpha_{i_4} \prod_{(l,l^-) \in geod(P,i_4)} \alpha_l$; $\gamma_P = \prod_{(l,l^-) \in geod(M,P)} \alpha_l$, $\gamma_M = \prod_{(l,l^-) \in geod(M,N)} \alpha_l$, $\gamma_N = \alpha_N$; $\delta_P = \beta_P$, $\delta_M = \beta_M$, $\delta_N = \beta_N$. Then

$$V_1 = \begin{pmatrix} \gamma_1 & \gamma_1 \gamma_N & \gamma_1 \gamma_N \gamma_M & \gamma_1 \gamma_N \gamma_M \gamma_P \\ \delta_N \gamma_2 & \gamma_2 & \gamma_2 \gamma_M & \gamma_2 \gamma_M \gamma_P \\ \delta_M \gamma_3 & \delta_M \gamma_3 & \gamma_3 & \gamma_3 \gamma_P \\ \delta_P \gamma_4 & \delta_P \gamma_4 & \delta_P \gamma_4 & \gamma_4 \end{pmatrix}$$

is a $\mathcal{W}$ matrix in quasi-nested block form. Hence we may choose a support tree T_1 for V_1 such that the partial order relationship in T_1 is consistent with the partial order relationship in T. Moreover, if $\gamma_t = 1$ or $\delta_t = 1$ for $t \in T_1$, then for the corresponding t in T, we have $\alpha_t = 1$ or $\beta_t = 1$. Hence V_1 is called the *induced $\mathcal{W}$ matrix in quasi-nested block form* from $U[i_1, i_2, i_3, i_4]$. For the principal submatrix $U[i_1, i_2, i_3]$ of U, there is a similar result.

In the rest of this section, we assume U is nonsingular. Hence by Lemma 4.1, $\alpha_t \beta_t < 1$ for any $t \in T \setminus I$. Moreover, we shall also assume that $\varphi : I \mapsto I$ is a permutation such that $U^\varphi := (U_{\varphi(i), \varphi(j)})$ is a $\mathcal{W}$ matrix in quasi-nested block form with support tree T^φ and vectors $\overrightarrow{\alpha^\varphi}, \overrightarrow{\beta^\varphi}$. Let $U^\varphi[i_1, i_2, i_3, i_4]$ be the principal submatrix of U^φ with $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$. Then there exists a 4×4 permutation matrix P_1 corresponding to rearranging $\varphi^{-1}(i_1), \varphi^{-1}(i_2), \varphi^{-1}(i_3), \varphi^{-1}(i_4)$ in their natural order such that $P_1 U^\varphi[i_1, i_2, i_3, i_4] P_1^T$ is the principal submatrix of U whose rows and columns are indexed by $j_1 < j_2 < j_3 < j_4$, where j_1, j_2, j_3, j_4 are obtained by rearranging $\varphi^{-1}(i_1), \varphi^{-1}(i_2), \varphi^{-1}(i_3), \varphi^{-1}(i_4)$ into their natural order. Hence we have the induced $\mathcal{W}$ matrix V_1 in quasi-nested block form from $U[j_1, j_2, j_3, j_4]$ associated with tree T_1 and $\overrightarrow{\gamma}, \overrightarrow{\delta}$. Moreover, the partial order relationship of $\{\varphi^{-1}(i_1), \varphi^{-1}(i_2), \varphi^{-1}(i_3), \varphi^{-1}(i_4)\}$ in the support tree T_1 is consistent with the partial order relationship of $\{\varphi^{-1}(i_1), \varphi^{-1}(i_2), \varphi^{-1}(i_3), \varphi^{-1}(i_4)\}$ in the support tree T. Therefore, for any $t \in V(T_1)$, $\gamma_t = 1$ ($\delta_t = 1$) implies $\alpha_t = 1$ ($\beta_t = 1$). Moreover, $P_1^T V_1 P_1 := V$ is the induced $\mathcal{W}$ matrix in quasi-nested block form from $U^\varphi[i_1, i_2, i_3, i_4]$.

LEMMA 5.1. *Let $|J| = m$ and $|K| \geq 2$. If there exist $1 \leq f < g \leq n$ such that $\varphi(f) = n$ and $\varphi(g) = m + 1$, then $\varphi(J) = J$ and $\varphi(K) = K$.*

Proof. We first prove the following claim: There does not exist $f < i < g$ such

that $\varphi(i) := p \le m$.

Assume there exists $f < i < g$ such that $\varphi(i) = p \le m$. Clearly, $p \in I^-$ and $(m+1) \wedge n = K$. Then the induced $\mathcal{W}$ matrix of order 3 in quasi-nested block form from $U^\varphi[f, i, g]$ is

$$V = \begin{pmatrix} \gamma_n & \gamma_n \delta_I & \gamma_n \delta_K \\ \gamma_p \gamma_I & \gamma_p & \gamma_p \gamma_I \\ \gamma_{m+1} \gamma_K & \gamma_{m+1} \gamma_K \delta_I & \gamma_{m+1} \end{pmatrix}.$$

If $f, i \in (f \wedge^\varphi i \wedge^\varphi g)^-$, then $\gamma_I = \delta_I = 1$. Hence $\alpha_I = \beta_I = 1$, a contradiction. If $i, g \in (f \wedge^\varphi i \wedge^\varphi g)^+$, then $\gamma_K = \delta_K = 1$. Hence $\alpha_K = \beta_K = 1$, a contradiction.

By a similar argument, we may prove that there does not exist $i > g$ such that $\varphi(i) = p \le m$. Now let $\varphi(h) > m + 1$ and $\varphi(i) \le m$ for $i = 1, \ldots, h - 1$, where $h \le f$. By a similar argument as used in the proof of the claim, there does not exist $i > h$ such that $\varphi(i) \le m$. Therefore $\varphi(J) = J$ and $\varphi(K) = K$. $\quad\square$

LEMMA 5.2. *Let $|J| = m$ and $|K| \ge 2$. If there exists $1 \le f < g \le n$ such that $\varphi(f) = n$ and $\varphi(g) = m + 1$, then $\varphi(i) = i$ for $i \in J$.*

Proof. By Lemma 5.1, $\varphi(J) = J$ and $\varphi(K) = K$. If there exists $1 \le i < j \le m$ such that $\varphi(i) := p > \varphi(j) := q$, then the induced $\mathcal{W}$ matrix of order 4 in quasi-nested block form from $U^\varphi[i, j, f, g]$ is

$$V = \begin{pmatrix} \gamma_p & \gamma_p \delta_L & \gamma_p \gamma_I & \gamma_p \gamma_I \\ \gamma_q \gamma_L & \gamma_q & \gamma_q \gamma_L \gamma_I & \gamma_q \gamma_L \gamma_I \\ \gamma_n \delta_I & \gamma_n \delta_I & \gamma_n & \gamma_n \delta_K \\ \gamma_{m+1} \gamma_K \delta_I & \gamma_{m+1} \gamma_K \delta_I & \gamma_{m+1} \gamma_K & \gamma_{m+1} \end{pmatrix},$$

where $p \wedge q = L$, since $p, q \in I^-$ and $m + 1, n \in I^+$. If $j, f, g \in (i \wedge^\varphi j \wedge^\varphi f \wedge^\varphi g)^+$, then $\gamma_K \delta_I \delta_K = \delta_I$, which implies $\gamma_K = \delta_K = 1$. Thus $\alpha_K = \beta_K = 1$, a contradiction. If $i, j \in (i \wedge^\varphi j \wedge^\varphi f \wedge^\varphi g)^-$ and $f, g \in (i \wedge^\varphi j \wedge^\varphi f \wedge^\varphi g)^+$, or $i, j, f \in (i \wedge^\varphi j \wedge^\varphi f \wedge^\varphi g)^-$, then by a similar argument it is easy to see that $\gamma_K = \delta_K = 1$ or $\gamma_I = \delta_I = 1$. Both are contradictions. Hence $\varphi(i) = i$ for $i \in J$. $\quad\square$

COROLLARY 5.3. *If $\alpha_t < 1$, $\beta_t < 1$ for all $t \in V \setminus I$ and $|K| \ge 2$, then there does not exist $f < g$ such that $\varphi(f) = n$ and $\varphi(g) = m + 1$.*

Proof. Suppose that there exists $f < g$ such that $\varphi(f) = n$ and $\varphi(g) = m+1$. By Lemma 5.2, $\varphi(i) = i$ for any $i \in J$. Moreover, $f > m$. Hence the induced $\mathcal{W}$ matrix of order 3 in quasi-nested block form from $U^\varphi[1, f, g]$ is

$$V = \begin{pmatrix} \gamma_1 & \gamma_1 \gamma_I & \gamma_1 \gamma_I \\ \gamma_n \delta_I & \gamma_n & \gamma_n \delta_K \\ \gamma_{m+1} \gamma_K \delta_I & \gamma_{m+1} \gamma_K & \gamma_{m+1} \end{pmatrix}.$$

If $1, f \in (1 \wedge^\varphi f \wedge^\varphi g)^-$, then $\delta_K = 1$. If $f, g \in (1 \wedge^\varphi f \wedge^\varphi g)^+$, then $\delta_I = 1$, a contradiction. Hence the assertion holds. $\quad\square$

LEMMA 5.4. *Let $\alpha_t < 1$, $\beta_t < 1$ for all $t \in V \setminus I$ and $|K| \ge 2$. If there exists $1 \le f < g \le n$ such that $\varphi(f) = m + 1$ and $\varphi(g) = n$, then there does not exist $f < i < g$ such that $\varphi(i) = p \le m$.*

Proof. Suppose that there exists $f < i < g$ such that $\varphi(i) = p \le m$. Then the induced $\mathcal{W}$ matrix of order 3 in quasi-nested block form from $U^\varphi[f, i, g]$ is

$$V = \begin{pmatrix} \gamma_{m+1} & \gamma_{m+1} \gamma_K \delta_I & \gamma_{m+1} \gamma_K \\ \gamma_p \gamma_I & \gamma_p & \gamma_p \gamma_I \\ \gamma_n \delta_K & \gamma_n \delta_I & \gamma_n \end{pmatrix}.$$

By the definition of $\mathcal{W}$ in quasi-nested block form, it is easy to see that $\delta_I = 1$, a contradiction. Hence the assertion holds.   □

LEMMA 5.5. *Let $\alpha_t < 1$, $\beta_t < 1$ for all $t \in V \setminus I$ and $|K| \geq 2$. If there exists $1 \leq f < g \leq n$ such that $\varphi(f) = m+1$ and $\varphi(g) = n$, then $\varphi(i) \leq m$ for all $i < f$ and $i > g$.*

Proof. We consider the following two cases.

Case 1. Suppose that there exists $i < f$ such that $\varphi(i) = p > m+1$.

If $p, n \in ((m+1) \wedge p \wedge n)^+$, then the induced $\mathcal{W}$ matrix of order 3 in quasi-nested block form from $U^\varphi[i, f, g]$ is

$$
V = \begin{pmatrix} \gamma_p & \gamma_p \gamma_L \delta_K & \gamma_p \gamma_L \\ \gamma_{m+1}\gamma_K & \gamma_{m+1} & \gamma_{m+1}\gamma_K \\ \gamma_n \delta_L & \gamma_n \delta_K & \gamma_n \end{pmatrix},
$$

where $p \wedge n := L$. By the definition of $\mathcal{W}$ in quasi-nested block form, it is easy to see that $\delta_K = 1$. Hence $\beta_K = 1$ and it is a contradiction.

If $m+1, p \in ((m+1) \wedge p \wedge n)^-$, then denote it by $(m+1) \wedge p := M$, and by a similar argument we have $\delta_M = 1$. Hence $\beta_M = 1$ and it is a contradiction.

Case 2. Suppose that there exists $i > g$ such that $\varphi(i) = p > m+1$. By a similar argument as used in the proof of Case 1, it is a contradiction.   □

LEMMA 5.6. *Let $\alpha_t < 1$, $\beta_t < 1$ for all $t \in V \setminus I$ and $|K| \geq 2$. If there exists $1 \leq f < g \leq n$ such that $\varphi(f) = m+1$ and $\varphi(g) = n$, then there does not exist a pair (i, j) such that $i < f$, $j > g$ and $\varphi(i) \leq m$, $\varphi(j) \leq m$.*

Proof. Suppose that there exist $i < f$ and $j > g$ such that $\varphi(i) := p \leq m$ and $\varphi(j) := q \leq m$. If $p < q$, then the induced $\mathcal{W}$ matrix of order 3 in quasi-nested block form from $U^\varphi[i, j, g]$ is

$$
V = \begin{pmatrix} \gamma_p & \gamma_p \gamma_L \gamma_I & \gamma_p \gamma_L \\ \gamma_n \delta_I & \gamma_n & \gamma_n \delta_I \\ \gamma_q \delta_L & \gamma_q \gamma_I & \gamma_q \end{pmatrix},
$$

where $p \wedge q = L$. By the definition of $\mathcal{W}$ in quasi-nested block form, it is easy to see that $\gamma_I = 1$. Hence $\alpha_I = 1$ and it is a contradiction.

If $p > q$, it is a contradiction by a similar argument. Hence the assertion holds.   □

LEMMA 5.7. *Let $\alpha_t < 1, \beta_t < 1$ for all $t \in V \setminus I$ and $|K| \geq 2$. If there exists $1 \leq f < g \leq n$ such that $\varphi(f) = m+1$ and $\varphi(g) = n$, then either $\varphi(i) = i$ for all $i \in I$ or $\varphi(i) = m+i \pmod{n}$ for all $i \in I$ and $\alpha_I \leq \min\{\beta_J, \beta_K\}$.*

Proof. By Lemmas 5.4 and 5.5, we have $\varphi(i) \leq m$ for all $i < f$ and $i > g$ and $\varphi(i) > m+1$ for $f < i < g$. Hence we need only consider the following two cases.

Case 1. There exists $1 \leq h < f$ such that $\varphi(h) \leq m$. Then by Lemma 5.6, there does not exist $i > f$ such that $\varphi(i) \leq m$. Further, for $1 \leq i < j < f$, $\varphi(i) := p < \varphi(j) := q$. In fact, if $p > q$, then the induced $\mathcal{W}$ matrix of order 3 in quasi-nested block form from $U^\varphi[i, j, g]$ is

$$
V = \begin{pmatrix} \gamma_p & \gamma_p \delta_L & \gamma_p \gamma_I \\ \gamma_q \gamma_L & \gamma_q & \gamma_q \gamma_L \gamma_I \\ \gamma_n \delta_I & \gamma_n \delta_I & \gamma_n \end{pmatrix},
$$

where $p \wedge q = L$. By the definition of $\mathcal{W}$ in quasi-nested block form, it is easy to see that $\gamma_I = 1$ or $\gamma_L = 1$. Hence $\alpha_L = 1$ or $\alpha_I = 1$. Both are contradictions.

Hence $\varphi(i) = i$ for $i = 1, \ldots, m$. Moreover, it is easy to show that $\varphi(i) < \varphi(j)$ for all $m < i < j \leq n$. Therefore $\varphi(i) = i$ for $i = 1, \ldots, n$.

Case 2. There exists $h > g$ such that $\varphi(h) \leq m$. Then $\varphi(i) \geq m+1$ for all $i < g$ and $\varphi(i) \leq m$ for any $i > g$ by Lemma 5.6. Furthermore, it is easy to show that $\varphi(i) < \varphi(j)$ for all $g < i < j$, and $\varphi(i) < \varphi(j)$ for all $1 \leq i < j \leq g$. Hence $\varphi(i) = m + i \pmod{n}$ for all $i \in I$. Moreover, since U^φ is a $\mathcal{W}$ matrix in quasi-nested block form, then $\alpha_I \leq \min\{\beta_J, \beta_K\}$, and the proof is completed. □

LEMMA 5.8. *Let $\alpha_t < 1$, $\beta_t < 1$ for all $t \in V \setminus I$. If $|K| = 1$, then φ is the identity permutation, or $\varphi(1) = m+1$ and $\varphi(i) = i-1$ for all $i = 2, \ldots, m+1$ with $\alpha_I \leq \beta_J$.*

Proof. Since $|K| = 1$, $n = m+1$. Let $f \in I$ such that $\varphi(f) = m+1$. We consider the following three cases.

Case 1. $f = 1$. Then for any $1 < i < j$, $\varphi(i) < \varphi(j)$. In fact, if $\varphi(i) := p > \varphi(j) := q$, then the induced $\mathcal{W}$ matrix of order 3 in quasi-nested block form from $U^\varphi[1, i, j]$ is

$$V = \begin{pmatrix} \gamma_{m+1} & \gamma_{m+1}\delta_I & \gamma_{m+1}\delta_I \\ \gamma_p\gamma_I & \gamma_p & \gamma_p\delta_L \\ \gamma_q\gamma_L\gamma_I & \gamma_q\gamma_L & \gamma_q \end{pmatrix},$$

where $i \wedge j = L$. It is easy to see that $\gamma_L = 1$, which yields $\alpha_L = 1$, a contradiction. Hence $\varphi(1) = m+1$ and $\varphi(i) = i-1$ for $i = 2, \ldots, m+1$. Moreover, $\alpha_I \leq \beta_J$, since U^φ is a $\mathcal{W}$ matrix in quasi-nested block form.

Case 2. $1 < f < m+1$. Then there exists $i < f < j$ such that $\varphi(i) := p$, $\varphi(j) := q \leq m$. Without loss of generality, we may assume that $p > q$. Then the induced $\mathcal{W}$ matrix of order 3 in quasi-nested block form from $U^\varphi[i, f, j]$ is

$$V = \begin{pmatrix} \gamma_p & \gamma_p\gamma_I & \gamma_p\delta_L \\ \gamma_{m+1}\delta_I & \gamma_{m+1} & \gamma_{m+1}\delta_I \\ \gamma_q\gamma_L & \gamma_q\gamma_L\gamma_I & \gamma_q \end{pmatrix},$$

where $i \wedge j = L$. It is easy to see that $\gamma_I = 1$, which implies that $\alpha_I = 1$, a contradiction.

Case 3. $f = m+1$. By an argument similar to the proof of Case 1, it is easy to see that φ is the identity permutation. □

Now we present the main result of this section.

THEOREM 5.9. *Let U be a $\mathcal{W}$ matrix of order n in quasi-nested block form with support tree T and defined by $\overrightarrow{\alpha}, \overrightarrow{\beta}$ on T. The root of the support tree is $I = \{1, 2, \ldots, n\}$, and $I^- = J$, $I^+ = K$. Denote $|J| = m$. If $\alpha_t < 1$, $\beta_t < 1$ for all $t \in V \setminus I$ and φ is a permutation on I, then $U^\varphi := (U_{\varphi(i),\varphi(j)})$ is a $\mathcal{W}$ matrix in quasi-nested block form if and only if φ is the identity permutation on I or $\alpha_I \leq \min\{\beta_J, \beta_K\}$ with $\varphi(i) = m + i \pmod{n}$ for $i = 1, \ldots, n$.*

Proof. If $U^\varphi := (U_{\varphi(i),\varphi(j)})$ is a $\mathcal{W}$ matrix in quasi-nested block form, it follows from Corollary 5.3 and Lemmas 5.7 and 5.8 that the assertion holds. Conversely, it is easy to show that the assertion holds by the definition of a $\mathcal{W}$ matrix in quasi-nested block form. □

Remark 5.10. Theorem 5.9 does not hold in general, as we will see in the following example, if we cancel the conditions $\alpha_t < 1$, $\beta_t < 1$.

Example 5.11. Let U be a $\mathcal{W}$ matrix of order 6 as follows:

$$U = \begin{pmatrix} \alpha_1 & \alpha_1\alpha_J & \alpha_1\alpha_J & \alpha_1\alpha_J & \alpha_1\alpha_J & \alpha_1\alpha_J\alpha_I \\ \alpha_2\alpha_M\beta_J & \alpha_2 & \alpha_2\alpha_M & \alpha_2\alpha_M & \alpha_2\alpha_M & \alpha_2\alpha_M\alpha_I \\ \alpha_3\alpha_L\alpha_N\beta_J & \alpha_3\alpha_L\alpha_N\beta_M & \alpha_3 & \alpha_3\alpha_L & \alpha_3\alpha_L\alpha_N & \alpha_3\alpha_L\alpha_N\alpha_I \\ \alpha_4\alpha_N\beta_J & \alpha_4\alpha_N\beta_M & \alpha_4\beta_L & \alpha_4 & \alpha_4\alpha_N & \alpha_4\alpha_L\alpha_I \\ \alpha_5\beta_J & \alpha_5\beta_M & \alpha_5\beta_N & \alpha_5\beta_N & \alpha_5 & \alpha_5\alpha_I \\ \alpha_6\beta_I & \alpha_6\beta_I & \alpha_6\beta_I & \alpha_6\beta_I & \alpha_6\beta_I & \alpha_6 \end{pmatrix}.$$

If $\alpha_I = 1$ and $\beta_J = \beta_M = \beta_N = \beta_L = 1$, then U^φ is a $\mathcal{W}$ matrix in quasi-nested block form for $\varphi(1) = 6$, $\varphi(2) = 2$, $\varphi(3) = 1$, $\varphi(4) = 5$, $\varphi(5) = 3$, $\varphi(6) = 4$.

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