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Euler scheme for solutions of a countable system of stochastic differential equations

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Abstract

We consider a countable system of stochastic differential equation. Euler scheme for approximating these solutions is used, and the global error is estimated. Solutions are approximated by means of a process which takes values in a finite dimensional space. Finally, we expand the global error for a class of smooth functions in powers of the discretization step size. © 2001 Elsevier Science B.V. All rights reserved

1. Introduction

The purpose of this note is to study the Euler scheme in order to approximate solutions of a countable system of stochastic differential equations (in short CSSDE). The global error of approximating these solutions is estimated, by showing that the standard technique used to study the Euler scheme on the finite dimensional case can be carried out. The only problem here is that under suitable assumptions the constants appearing in the usual bounds can be independent of the dimension. We also present a way of approximating the infinite dimensional case by a sequence of finite dimensional processes. Finally, expansion of the global error for a class of smooth functions in powers of the discretization step size is computed in the same way as in Talay (1986) and Talay and Tubaro (1990). Let us consider the following CSSDE:

$$X(t)=X(0)+\int_0^t a(s,X(s))\,\mathrm{d}s+\int_0^t b(s,X(s))\,\mathrm{d}W(s), \tag{1}$$

or equivalently

$$x_i(t)=x_i(0)+\int_0^t a_i(s,X(s))\,\mathrm{d}s+\sum_{j=1}^m\int_0^t b_i^j(s,X(s))\,\mathrm{d}W_i^j(s),$$

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where S is a countable set and $X(t)=(x_i(t))_{i\in S}$ is a stochastic process in $(\mathbb{R}^d)^S$. $W(t)$ is a Wiener process in $(\mathbb{R}^m)^S$; $a=(a_i)_{i\in S}$ where $i\in S$, $a_i:[0,T]\times(\mathbb{R}^d)^S\to\mathbb{R}^d$ and for $i\in S$, $1\leqslant j\leqslant m$, $b_i^j:[0,T]\times(\mathbb{R}^d)^S\to\mathbb{R}^d\times\mathbb{R}^m$. $|x_i|$ denotes the d-dimensional Euclidean norm of x_i and $|b_i|^2=\sum_{j=1,\dots,m}\sum_{k=1,\dots,d}|b_i^{j,k}|^2$, where $b_i^{j,k}$ is the (j,k) component of b_i.

Applications of CSSDE can be found for example in Genetics, where $X(t)$ represents the proportion at time t of one of two possible alleles of a certain gene, S the set of colonies, and the change of gene frequencies is caused by random sampling, mutation and migration. Such application can be found in Kloeden and Platen (1995) or Shiga and Shimizu (1980).

In what follows, we assume the functions a and b satisfy certain regularity conditions, so that the existence and uniqueness of this solution are ensured. Such conditions can be found in Shiga and Shimizu (1980).

We will denote by L the infinitesimal generator associated to Eq. (1) as

$$L=\frac{1}{2}\sum_{i,j\in S}\sigma_j^i(t,x)\partial_{ij}+\sum_{i\in S}a^i(t,x)\partial_i, \tag{2}$$

where $\sigma(t,x)=b(t,x)b^t(t,x)$.

Let $\gamma=\{\gamma_i\}_{i\in S}$ be a sequence of real numbers and consider

$$l_\gamma^2=\left\{X\in(\mathbb{R}^d)^S;\ \|X\|_\gamma^2:=\sum_{i\in S}\gamma_i|x_i|^2<\infty\right\}.$$

Let $0=t_0\leqslant t_1,\dots,\leqslant t_n=T$ be a discretization of $[0,T]$ and δ the time step such that $\delta=\delta_n=T/n$. The process $Y^\delta=\{Y^\delta(t),0\leqslant t\leqslant T\}$ defined below will be considered to approximate the solution X. First we define Y^δ at t_k recursively as follows:

$$Y^\delta(0)=y^\delta(0),$$

$$Y^\delta(t_{k+1})=Y^\delta(t_k)+a(t_k,Y^\delta(t_k))\delta+b(t_k,Y^\delta(t_k))(W(t_{k+1})-W(t_k))\quad\text{for } k=0,\dots,n-1.$$

Next, $Y^\delta(t)$ can be defined for each $t\in[t_k,t_{k+1}[$, $k=0,1,\dots,n-1$ as the following linear interpolation

$$Y^\delta(t)=Y^\delta(t_k)+\int_{t_k}^t a(t_k,Y^\delta(t_k))\,\mathrm{d}s+\int_{t_k}^t b(t_k,Y^\delta(t_k))\,\mathrm{d}W(s). \tag{3}$$

2. Main results

In this section, main results are stated and their proofs are postponed to the next section. The first theorem is concerned with a bound for the global error when the Euler scheme is used. We will make use of the following standing assumptions throughout the paper.

(A1) $\|X(0)\|_\gamma^2<\infty$,
(A2) $\|X(0)-Y^\delta(0)\|_\gamma^2\leqslant K_1\delta$,
(A3) $\|a(t,x)-a(t,y)\|_\gamma^2+\|b(t,x)-b(t,y)\|_\gamma^2\leqslant K_2\|x-y\|_\gamma^2$,
(A4) $\|a(t,x)\|_\gamma^2+\|b(t,x)\|_\gamma^2\leqslant K_3(1+\|x\|_\gamma^2)$,
(A5) $\|a(s,x)-a(t,x)\|_\gamma^2+\|b(s,x)-b(t,x)\|_\gamma^2\leqslant K_4(1+\|x\|_\gamma^2)|s-t|$,

for all $x,y\in l^2(\gamma)$, $s,t\in[0,T]$ where the constants $K_1,\dots,K_4$ do not depend on δ.

Theorem 1. *Assume* (A1)–(A5) *hold. Then there exists two positive constants A and B not depending on* δ *such that*

$$\mathbb{E}\left(\sup_{0\leqslant t\leqslant T}\|X(t)-Y^\delta(t)\|_\gamma^2\right)\leqslant\delta A\mathrm{e}^{BT}.$$

In our next result, we give a finite dimensional approximation, and for that reason we need a truncation of a and b.

Let $1 \leqslant i \leqslant N$, and $a_i^N : [0,T] \times (\mathbb{R}^d)^N \to \mathbb{R}^d$ defined as $a_i(t,x_1,\ldots,x_N) = a_i(t,x_1,\ldots,x_N,0,\ldots,0,\ldots)$, $b_i^{N,j}$: $[0,T] \times (\mathbb{R}^d)^N \to \mathbb{R}^d \times \mathbb{R}^m$ defined as $b_i^j(t,x_1,\ldots,x_N) = b_i^j(x_1,\ldots,x_N,0,\ldots,0,\ldots)$, and $Y^{\delta,N}$ be the process defined on t_k recursively as

$$y_i^{\delta,N}(0) = y_i^{\delta}(0) \quad \forall 1 \leqslant i \leqslant N,$$

$$y_i^{\delta,N}(t_{k+1}) = y_i^{\delta,N}(t_k) + a_i^N(t_k, Y^{\delta,N}(t_k))\delta + \sum_{j=1}^m b_i^{N,j}(t_k, Y^{\delta,N}(t_k))(W_i^j(t_{k+1}) - W_i^j(t_k)),$$

for $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = T$. Again for $t \in [t_k, t_{k+1}[,\ k = 0,1,\ldots,n-1,\ 1 \leqslant i \leqslant N$ we define

$$y_i^N(t) = y_i^{\delta,N}(t_k) + \int_{t_k}^t a_i^N(t_k, Y^{\delta,N}(t_k))\,\mathrm{d}s + \sum_{j=1}^m \int_{t_k}^t b_i^{N,j}(t_k, Y^{\delta,N}(t_k))\,\mathrm{d}W_i^j(s). \tag{4}$$

Theorem 2. *Assume* (A1)–(A5), *and the additional assumptions hold*
(H1) $\sup_{i\in S} |y_i^{\delta}(0)|^2 < K_5$,
(H2) $\sum_{i\in S} \gamma_i < \infty$,
(H3) $\sup_{i\in S} (|a_i(s,x)|^2 + |b_i(s,x)|^2) < K_6(1 + \|x\|_\gamma^2)$.
Then

$$\mathbb{E}\left(\sup_{0\leqslant t\leqslant T} \|Y^{\delta}(t) - Y^{\delta,N}(t)\|_\gamma^2\right) \leqslant K_7 \sum_{i=N}^{\infty} \gamma_i.$$

In the following theorem, we assume for simplicity $d = 1$, and for technical difficulties $\gamma_i = 1$ for all $i \in S$. Let us now consider the class $\mathscr{PG}(\mathbb{R}^S)$ of C^∞ functions $\phi : \mathbb{R}^S \to \mathbb{R}$ with polynomial growth in all the derivatives (with respect to $\|x\|_\gamma$).

Let us define for $f \in \mathscr{PG}$ the global error for CSSDE as

$$\mathrm{Err}(T,\delta) = \mathbb{E}(f(X(T))) - \mathbb{E}(f(Y^{\delta}(T))). \tag{5}$$

Theorem 3. *Let us assume that the functions a and b are C^∞, whose derivatives of any order are bounded, and $Y^{\delta}(0) = X(0)$. Then for the Euler method, the Global error is given by*

$$\mathrm{Err}(T,\delta) = -\,\delta \int_0^T \mathbb{E}\psi(s,X(s))\,\mathrm{d}s + \mathcal{O}(\delta^2),$$

where the function ψ is defined as

$$\begin{aligned}\psi(t,x) = {} & \frac{1}{2}\sum_{i,j\in S} a_i(t,x)a_j(t,x)\partial_{ij}u(t,x) + \frac{1}{2}\sum_{i,j,k\in S} a_i(t,x)\sigma_{jk}(t,x)\partial_{ijk}u(t,x) \\ & + \frac{1}{8}\sum_{i,j,k,l\in S} \sigma_{i,j}(t,x)\sigma_{k,l}(t,x)\partial_{ijkl}u(t,x) + \frac{1}{2}\frac{\partial^2}{\partial t^2}u(t,x) \\ & + \sum_{i\in S} a_i(t,x)\frac{\partial}{\partial t}\partial_i u(t,x) + \frac{1}{2}\sum_{i,j\in S} \sigma_{i,j}(t,x)\frac{\partial}{\partial t}\partial_{ij}u(t,x),\end{aligned} \tag{6}$$

and $u(t,x)=\mathbb{E}(f(X^{t,x}(T)))=\mathbb{E}_{t,x}(f(X(T)))$ *verifies the following equation*:

$$\begin{cases} \dfrac{\partial u}{\partial t}+Lu=0, \\ u(T,x)=f(x), \quad x\in\mathbb{R}^S. \end{cases} \tag{7}$$

3. Proofs

The proof of Theorem 1 is based upon the following Lemmas 3.1–3.3, whose proofs are standard using Cauchy–Schwarz's inequality, Doob's inequality and Gronwall's Lemma.

Lemma 3.1. *Let* $X(t)$ *be the process satisfying Eq.* (1). *Then under* (A1)–(A5) *there exists two positive constants* C_1 *and* C_2 *such that*

$$\mathbb{E}\left(\sup_{0\leqslant t\leqslant T}\|X(t)\|_\gamma^2\right)\leqslant C_1(1+\|X(0)\|_\gamma^2)\mathrm{e}^{C_2T}.$$

Proof. Since $X(t)$ satisfies the Eq. (1) we have

$$|x_i(t)|^2\leqslant 3|x_i(0)|^2+3\left|\int_0^t a_i(s,X(s))\,\mathrm{d}s\right|^2+3m\sum_{j=1}^m\left|\int_0^t b_i^j(s,X(s))\,\mathrm{d}W_i^j(s)\right|^2.$$

From hypothesis (A1) and growth bound (A4) the following inequality holds for $0\leqslant R\leqslant T$

$$\sum_{i\in S}\gamma_i\mathbb{E}\left(\sup_{0\leqslant t\leqslant R}|x_i(t)|^2\right)\leqslant 3\|X(0)\|_\gamma^2+3TK_3(T+4m)+3K_3(T+4m)\int_0^R\mathbb{E}\left(\sup_{0\leqslant u\leqslant s}\sum_{i\in S}\gamma_i|x_i(u)|^2\right)\,\mathrm{d}s.$$

By applying Gronwall's inequality we obtain

$$\sum_{i\in S}\gamma_i\mathbb{E}\left(\sup_{0\leqslant u\leqslant T}|x_i(u)|^2\right)\leqslant C_1(1+\|X(0)\|_\gamma^2)\mathrm{e}^{C_2T},$$

where $C_1=\max\{3,3K_3T(T+4m)\}$ and $C_2=3K_3(T+4m)$. Finally, since

$$\mathbb{E}\left(\sup_{0\leqslant t\leqslant T}\|X(t)\|_\gamma^2\right)\leqslant\sum_{i\in S}\gamma_i\mathbb{E}\left(\sup_{0\leqslant t\leqslant T}|x_i(t)|^2\right),$$

we obtain the result. □

The following two lemmas are proved in the same way, whose proofs are left to the reader.

Lemma 3.2. *Under* (A1)–(A5), *there exists two positive constants* C_3 *and* C_4 *such that*

$$\mathbb{E}\left(\sup_{0\leqslant t\leqslant T}\|Y^\delta(t)\|_\gamma^2\right)\leqslant C_3(1+\|Y^\delta(0)\|_\gamma^2)\mathrm{e}^{C_4T}.$$

Lemma 3.3. *Under* (A1)–(A5), *there exists a positive constant* C_5 *such that the solution* $X(t)$ *of Eq.* (1) *satisfies*

$$\mathbb{E}\|X(t)-X(s)\|_\gamma^2\leqslant C_5(t-s)(1+\|X(0)\|_\gamma^2).$$

Proof of Theorem 1. Let $Z(T)=\mathbb{E}\{\sup_{0\leqslant t\leqslant T}(\|X(t)-Y^{\delta}(t)\|_{\gamma}^{2})\}$ and $c(s)=[sn]/n,\ s\in[0,T]$. We have

$$Z(T)\leqslant \mathbb{E}\left(\sum_{i\in S}\gamma_i\sup_{0\leqslant t\leqslant T}|x_i(t)-y_i^{\delta}(t)|^2\right)$$

$$\leqslant 4\mathbb{E}\left(\sum_{i\in S}\gamma_i\sup_{0\leqslant t\leqslant T}(|x_i(0)-y_i^{\delta}(0)|^2+\mathscr{I}_i(t)+\mathscr{J}_i(t)+\mathscr{K}_i(t))\right),$$

where

$$\mathscr{I}_i(t)=\Bigg|\int_0^t a_i(c(s),X(c(s)))-a_i(c(s),Y^{\delta}(c(s)))\,\mathrm{d}s$$
$$+\sum_{j=1}^m\int_0^t(b_i^j(c(s),X(c(s))-b_i^j(c(s),Y^{\delta}(c(s)))\,\mathrm{d}W_i^j(s)\Bigg|^2,$$

$$\mathscr{J}_i(t)=\Bigg|\int_0^t(a_i(c(s),X(s))-a_i(c(s),X(c(s)))\,\mathrm{d}s$$
$$+\sum_{j=1}^m\left\{\int_0^t(b_i^j(c(s),X(s)-b_i^j(c(s),X(c(s)))\,\mathrm{d}W_i^j(s)\right\}\Bigg|^2,$$

$$\mathscr{K}_i(t)=\Bigg|\int_0^t(a_i(s,X(s))-a_i(c(s),X(s)))\,\mathrm{d}s$$
$$+\sum_{j=1}^m\left\{\int_0^t(b_i^j(s,X(s))-b_i^j(c(s),X(s)))\,\mathrm{d}W_i^j(s)\right\}\Bigg|^2.$$

We have the following estimates

$$\mathbb{E}\left(\sum_{i\in S}\gamma_i\sup_{0\leqslant t\leqslant T}\mathscr{I}_i(t)\right)\leqslant(2T+8m)K_2\int_0^T Z(s)\,\mathrm{d}s,$$

$$\mathbb{E}\left(\sum_{i\in S}\gamma_i\sup_{0\leqslant t\leqslant T}\mathscr{J}_i(t)\right)\leqslant\delta(2T+8m)TK_2C_5(1+\|X(0)\|_{\gamma}^2),$$

$$\mathbb{E}\left(\sum_{i\in S}\gamma_i\sup_{0\leqslant t\leqslant T}\mathscr{K}_i(t)\right)\leqslant(2T+8m)K_4\delta T(1+C_1(1+\|X(0)\|_{\gamma}^2)\mathrm{e}^{C_2T}).$$

By combining these estimates and using hypothesis (A2) we obtain

$$Z(T)\leqslant A\delta+B\int_0^T Z(s)\,\mathrm{d}s, \tag{8}$$

where $A=4K_1+(8T+32m)K_2C_5T(1+\|X(0)\|_{\gamma}^2)+(8T+32m)K_4T(1+C_1(1+\|X(0)\|_{\gamma}^2)\mathrm{e}^{C_2T})$ and $B=(8T+32m)K_2$. By applying Gronwall inequality to Eq. (8) we have $Z(T)\leqslant\delta A\mathrm{e}^{BT}$. □

Proof of Theorem 2. From relations (3) and (4) we get as before

$$\sum_{i\in S}\gamma_i\,\mathbb{E}\left(\sup_{0\leqslant t\leqslant T}|y_i^\delta(t)-y_i^{\delta,N}(t)|^2\right)\leqslant 3\mathbb{E}\left(\sum_{n=N+1}^{\infty}\gamma_i|y_i^\delta(0)|^2\right)$$

$$+3T\mathbb{E}\left(\int_0^T\left(\sum_{i=1}^{N}\gamma_i|a_i(c(s),Y^\delta(c(s)))-a_i^N(c(s),Y^{\delta,N}(c(s)))|^2+\sum_{i=N+1}^{\infty}\gamma_i|a_i(c(s),Y^\delta(c(s)))|^2\right)\mathrm{d}s\right)$$

$$+12m\mathbb{E}\left(\int_0^T\left(\sum_{j=1}^{m}\sum_{i=1}^{N}\gamma_i|b_i^j(c(s),Y^\delta(c(s)))-b_i^{N,j}(c(s),Y^{\delta,N}(c(s)))|^2\right.\right.$$

$$\left.\left.+\sum_{i=N+1}^{\infty}\gamma_i|b_i^j(c(s),Y^\delta(c(s)))|^2\right)\mathrm{d}s\right).$$

From the Lipschitz condition (A3), hypothesis (H1)–(H3), Lemma 3.2, and since $\sum_{i=1}^N\gamma_i|x_i|^2\leqslant\|X\|_\gamma^2$ we obtain

$$\sum_{i\in S}\mathbb{E}\left(\gamma_i\sup_{0\leqslant t\leqslant T}|y_i^\delta(t)-y_i^{\delta,N}(t)|^2\right)\leqslant D\sum_{i=N+1}^{\infty}\gamma_i+E\int_0^T\mathbb{E}\left(\sup_{0\leqslant u\leqslant s}\|Y^\delta(u)-Y^{\delta,N}(u)\|_\gamma^2\right)\mathrm{d}s,$$

where $D=(3K_5+K_6T(3T+12m))(1+C_3(1+\|Y^\delta(0)\|_\gamma^2)\mathrm{e}^{C_4T})$ and $E=(3T+12m)K_2$. Finally, by applying Gronwall's inequality to $\sum_{i\in S}\gamma_i\mathbb{E}\left(\sup_{0\leqslant t\leqslant T}|y_i^\delta(t)-y_i^{\delta,N}(t)|^2\right)$ we obtain

$$\mathbb{E}\left(\sup_{0\leqslant t\leqslant T}\|Y^\delta(t)-Y^{\delta,N}(t)\|_\gamma^2\right)\leqslant K_7\sum_{i=N+1}^{\infty}\gamma_i,$$

where $K_7=D\mathrm{e}^{ET}$. □

The proof of Theorem 3 is based upon the following Lemmas 3.4–3.8, following the same ideas of Talay and Tubaro (1990). Let $X^{t,x}(s)$, $t\leqslant s\leqslant T$ be the strong solution of the following stochastic differential equation:

$$X(s)=x+\int_t^s a(r,X(r))\,\mathrm{d}r+\int_t^s b(r,X(r))\,\mathrm{d}W(r).\tag{9}$$

Lemma 3.4. *Let us suppose that the functions a and b are C^∞, whose derivatives of any order are bounded. Let f in $\mathscr{PG}(\mathbb{R}^S)$, then $u(t,x)=\mathbb{E}(f(X^{t,x}(T)))$ is in $\mathscr{PG}([0,T]\times\mathbb{R}^S)$.*

Proof. We analyze the case $\alpha=\alpha_1$, i.e., we consider the first derivative, the general case follows by induction. It is well known that under the assumed regularity on a and b there exists a smooth version of the stochastic flow defined in Eq. (9). We assume $X^{t,x}(s)$ is this smooth version. Moreover for all integer $k>0$, the family of the processes equal to the partial derivatives of the flow up to the order k, solves a system of stochastic differential equations with Lipschitz conditions (see Protter (1990) and Karatzas and Shreve (1988)). Then

$$\partial_i u(t,x)=\partial_i(\mathbb{E}(f(X^{t,x}(T))))=\mathbb{E}\left(\sum_{j\in S}\partial_j f(X^{t,x}(T))\partial_i(X_j^{t,x}(T))\right).$$

A small modification of Lemma 3.1 allow us to conclude that

$$\sup_{x\in\mathbb{R}^S} \mathbb{E}\left(\sup_{0\leqslant t\leqslant T}\sum_{i\in S}\sum_{j\in S}|\partial_i X_j^{t,x}(T)|^2\right) < \infty.$$

Therefore, we have $\sup_{0\leqslant t\leqslant T}\sum_{i\in S}|\partial_i u(t,x)|^2$ has the same polynomial growth in x as $\sum_{j\in S}|\partial_j f(x)|^2$, and the result follows. □

Lemma 3.5. *Let f in $\mathscr{PG}$, then u is a smooth solution of Eq.* (7).

Proof. From Lemma 3.4 we only have to prove the function $u(t,x)=\mathbb{E}(f(X^{t,x}(T)))$ satisfies the Eq. (7). First, for $t=T$ we have

$$u(T,x)=\mathbb{E}(f(X^{T,x}(T)))=\mathbb{E}(f(x))=f(x).$$

By applying Itô's formula to $f(X^{t,x}(T))$ we obtain

$$f(X^{t,x}(T))=f(x)+\int_t^T Lf(X^{t,x}(s))\,\mathrm{d}s+M(t),$$

where M is a martingale. Taking expected value we have

$$u(t,x)=f(x)+\int_t^T \mathbb{E}(Lf(X^{t,x}(s)))\,\mathrm{d}s.$$

Differentiating with respect to t and evaluating in $t=T$ we obtain

$$\left.\frac{\partial u(t,x)}{\partial t}\right|_{t=T} = -\,\mathbb{E}(Lf(X^{t,x}(T)))|_{t=T} = -\,Lu(T,x). \tag{10}$$

Now, from the Markov property, we have $u(t,x)=\mathbb{E}(u(s,X^{t,x}(s)))$, for $s\geqslant t$ and from Eq. (10) we obtain

$$\left.\frac{\partial u(t,x)}{\partial t}\right|_{t=s} = -\,Lu(s,x).$$

This proves that $u(t,x)=\mathbb{E}(f(X^{t,x}(T)))$ verifies the Eq. (7). □

Lemma 3.6. *There exists a positive constant $C(T)$, which does not depend on δ, such that*

$$\mathbb{E}(u(T,Y^\delta(T)))=\mathbb{E}(u(0,X(0)))+\delta^2\sum_{j=0}^{n-1}\mathbb{E}(\psi(j\delta,Y^\delta(j\delta)))+\delta^2\mathscr{R}(\delta),$$

and $\delta\sum_{j=0}^{n-1}\mathbb{E}|\psi(j\delta,Y^\delta(j\delta))|\leqslant C(T)$, $\mathscr{R}(\delta)\leqslant C(T)$, *where ψ is defined on Eq.* (6).

Proof. Since $f\in\mathscr{PG}$ and $u(t,x)=\mathbb{E}(f(X^{t,x}(T)))$ solves the Eq. (7) with final condition $u(T,x)=f(x)$, we have $\mathrm{Err}(T,\delta)=\mathbb{E}(u(T,Y^\delta(T)))-\mathbb{E}(u(0,X(0)))$. By computing an expansion in Taylor series for u at the point $((n-1)\delta, Y^\delta((n-1)\delta))$ we obtain

$$\mathbb{E}(u(T,Y^\delta(T)))=\mathbb{E}(u((n-1)\delta,Y^\delta((n-1)\delta)))+\delta^2\mathbb{E}(\psi((n-1)\delta,Y^\delta((n-1)\delta)))+\delta^3R_n(\delta).$$

It is not hard to prove using the fact u solves Eq. (7) that there exists a constant $C(T)$ which does not depend on δ such that $|R_n(\delta)|\leqslant C(T)$. Continuing in this way n times, we arrive at

$$\mathbb{E}(u(T,Y^\delta(T)))=\mathbb{E}(u(0,X(0)))+\delta^2\sum_{j=0}^{n-1}\mathbb{E}(\psi(j\delta,Y^\delta(j\delta)))+\delta^2\mathscr{R}(\delta),$$

where $\mathscr{R}(\delta)=\delta\sum_{j=0}^{n-1}R_j^\delta\leqslant C(T)$. We decompose $\delta^2\sum_{j=0}^{n-1}\mathbb{E}(\psi((j\delta,Y^\delta(j\delta)))$ according to Eq. (6). We analyze the first term, being the others completely analogous. Since

$$\delta\sum_{k=0}^{n-1}\mathbb{E}\left|\sum_{i,j\in S}a_i(t,Y^\delta(k\delta))a_j(t,Y^\delta(k\delta))\partial_{i,j}u(t,Y^\delta(k\delta))\right|\leqslant\delta\sum_{k=0}^{n-1}\|a\|^2\mathbb{E}\left(\sum_{i,j\in S}|\partial_{i,j}u(t,Y^\delta(k\delta))|^2\right).$$

From Lemma 3.4 and a generalization of Lemma 3.2, we deduce

$$\delta\sum_{k=0}^{n-1}\mathbb{E}\left|\sum_{i,j\in S}a_i(t,Y^\delta(k\delta))a_j(t,Y^\delta(k\delta))\partial_{i,j}u(t,Y^\delta(k\delta))\right|\leqslant\delta nK(T)=TK(T).$$

Using the same technique, we obtain a bound for $\psi(j\delta,Y^\delta(j\delta))$, and therefore exists a constant $C(T)$ independent of δ such that

$$\delta\sum_{j=0}^{n-1}\mathbb{E}|\psi(j\delta,Y^\delta(j\delta))|\leqslant C(T).\qquad\square$$

Lemma 3.7. *Under the assumptions of Theorem* 3

$$\left|\delta\sum_{j=0}^{n-1}\mathbb{E}(\psi(j\delta,Y^\delta(j\delta)))-\int_0^T\mathbb{E}(\psi(s,X(s)))\,\mathrm{d}s\right|=\mathcal{O}(\delta).$$

Proof. First note that

$$\left|\delta\sum_{j=0}^{n-1}\mathbb{E}(\psi(j\delta,Y^\delta(j\delta)))-\int_0^T\mathbb{E}(\psi(s,X(s)))\,\mathrm{d}s\right|$$
$$\leqslant\delta\left|\sum_{j=0}^{n-1}\mathbb{E}(\psi(j\delta,Y^\delta(j\delta)))-\mathbb{E}(\psi(j\delta,X(j\delta)))\right|+\left|\delta\sum_{j=0}^{n-1}\mathbb{E}(\psi(j\delta,X(j\delta)))-\int_0^T\mathbb{E}(\psi(s,X(s))\,\mathrm{d}s)\right|.$$

Since the function ψ belongs to $\mathscr{PG}([0,T]\times\mathbb{R}^S)$, Lemma 3.6 implies that

$$|\mathbb{E}(\psi(j\delta,Y^\delta(j\delta)))-\mathbb{E}(\psi(j\delta,X(j\delta)))|\leqslant C(T)\delta,$$

and therefore

$$\left|\sum_{j=0}^{n-1}\mathbb{E}(\psi(j\delta,Y^\delta(j\delta)))-\mathbb{E}(\psi(j\delta,X(j\delta)))\right|=\mathcal{O}(1),\tag{11}$$

On the other hand, since the function $s\to\mathbb{E}(\psi(s,X_s))$ has a continuous first derivative, one concludes

$$\left|\delta\sum_{j=0}^{n-1}\mathbb{E}(\psi(j\delta,X(j\delta)))-\int_0^T\mathbb{E}(\psi(s,X(s))\,\mathrm{d}s)\right|=\mathcal{O}(\delta).\qquad\square$$

Proof of Theorem 3. Follows immediately from Lemmas 3.7–3.9. $\square$

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References

Karatzas, I., Shreve, S.E., 1988. Brownian Motion and Stochastic Calculus. Springer, Berlin.

Kloeden, P.E., Platen, E., 1995. Numerical Solution of Stochastic Differential Equations. Springer, Berlin.

Protter, P., 1990. Stochastic Integration and Differential Equations. Springer, Berlin.

Shiga, T., Shimizu, A., 1980. Infinite dimensional stochastic differential equations and their applications. J. Math. Kyoto Univ. (JMKYAZ) 20 (3), 395–416.

Talay, D., 1986. Discrétisation d'une équation différentielle stochastique et calcul approché d'espérances de fonctionelles de la solution. Math. Model. Numer. Anal. 20, 141–179.

Talay, D., Tubaro, L., 1990. Expansion of the global error for numerical schemes solving Stochastic differential equations. Stochastic Anal. Appl. 4, 483–509.