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A modified Lagrange–Galerkin method for a fluid-rigid system with discontinuous density

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Abstract In this paper, we propose a new characteristics method for the discretization of the two dimensional fluid-rigid body problem in the case where the densities of the fluid and the solid are different. The method is based on a global weak formulation involving only terms defined on the whole fluid-rigid domain. To take into account the material derivative, we construct a special characteristic function which maps the approximate rigid body at the $(k+1)$-th discrete time level into the approximate rigid body at k-th time. Convergence results are proved for both semi-discrete and fully-discrete schemes.

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1 Introduction

The aim of this paper is to present a modified characteristics method for the discretization of the equations modelling the motion of a rigid solid immersed into a viscous incompressible fluid. Our method is a generalisation of the numerical scheme

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presented in San Martín et al. [19] to the case where the fluid and the solid have different densities. The fluid-rigid system occupies a bounded, convex and regular domain $\mathcal{O} \subset \mathbb{R}^2$. The solid is assumed to be a ball of radius 1 whose center, at time t, is denoted by $\boldsymbol{\zeta}(t)$. The fluid fills the part $\Omega(t) = \mathcal{O} \backslash B(\boldsymbol{\zeta}(t))$ at time t. The velocity field $\mathbf{u}(\mathbf{x}, t)$ and the pressure $p(\mathbf{x}, t)$ of the fluid, the center of mass $\boldsymbol{\zeta}(t)$ and the angular velocity $\omega(t)$ of the ball satisfy the following Navier–Stokes system coupled with Newton's laws:

$$\rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) - \mu \Delta \mathbf{u} + \nabla p = \rho_f \mathbf{f}, \mathbf{x} \in \Omega(t), t \in [0, T], \tag{1.1}$$

$$\operatorname{div} \mathbf{u} = 0, \mathbf{x} \in \Omega(t), t \in [0, T], \tag{1.2}$$

$$\mathbf{u} = 0, \mathbf{x} \in \partial \mathcal{O}, t \in [0, T], \tag{1.3}$$

$$\mathbf{u} = \boldsymbol{\zeta}'(t) + \omega(t)(\mathbf{x} - \boldsymbol{\zeta}(t))^{\perp}, \mathbf{x} \in \partial B(\boldsymbol{\zeta}(t)), t \in [0, T], \tag{1.4}$$

$$m\boldsymbol{\zeta}''(t) = - \int\limits_{\partial B(\boldsymbol{\zeta}(t))} \boldsymbol{\sigma} \mathbf{n} \, d\Gamma + \rho_s \int\limits_{B(\boldsymbol{\zeta}(t))} \mathbf{f}(\mathbf{x}, t) d\mathbf{x}, t \in [0, T], \tag{1.5}$$

$$\begin{aligned} J\omega'(t) = &- \int\limits_{\partial B(\boldsymbol{\zeta}(t))} (\mathbf{x} - \boldsymbol{\zeta}(t))^{\perp} \cdot \boldsymbol{\sigma} \mathbf{n} \, d\Gamma \\ &+ \rho_s \int\limits_{B(\boldsymbol{\zeta}(t))} (\mathbf{x} - \boldsymbol{\zeta}(t))^{\perp} \cdot \mathbf{f}(\mathbf{x}, t) d\mathbf{x}, t \in [0, T]. \end{aligned} \tag{1.6}$$

In the above system, $\boldsymbol{\sigma} = -p\,\mathbf{Id} + 2\mu \mathrm{D}(\mathbf{u})$ denotes the Cauchy stress tensor with $\mathrm{D}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$ and $\nabla \mathbf{u}^T$ means the transpose of $\nabla \mathbf{u}$. The positive constant μ is the dynamic viscosity of the fluid and the constants m and J are the mass and the moment of inertia of the rigid body. Throughout this article, we will use the notation $\mathbf{x}^{\perp} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$ for all $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$. System (1.1)–(1.6) is completed with initial conditions:

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega(0), \tag{1.7}$$

$$\boldsymbol{\zeta}(0) = \boldsymbol{\zeta}_0 \in \mathbb{R}^2, \quad \boldsymbol{\zeta}'(0) = \boldsymbol{\zeta}_1 \in \mathbb{R}^2, \quad \omega(0) = \omega_0 \in \mathbb{R}. \tag{1.8}$$

In this paper, we suppose that the density ρ_f of the fluid and the density ρ_s of the solid are constant, but not equal, that is

$$\rho_f \neq \rho_s.$$

The fluid-structure interaction problem (1.1)–(1.8) is characterized by the strong coupling between the nonlinear equations of the fluid and those of the structure, as well as the fact that the equations of the fluid are written in a variable domain in time, which depends on the displacement of the structure. From the numerical point of view, in this kind of problems it is necessary to solve equations on moving domains.

For this reason, in recent years various authors have proposed a number of different techniques, some of which are the level set method (see Osher and Sethian [13]), the fictitious domain method (see Glowinski, Pan, Hesla, Joseph and Périaux [7,8]), the immersed boundary method (see Peskin [14]) and the Arbitrary Lagrangian Eulerian (ALE) method (see Formaggia and Nobile [4], Gastaldi [5], Maury [11], Maury and Glowinski [12]).

In the sequel, we briefly recall some reference about the numerical convergence for Navier–Stokes equations, when the domain is independent of time. The Lagrange–Galerkin method has been proposed for the numerical treatment of convection-dominated equations and it is based on combining a Galerkin finite element procedure with a special discretisation of the material derivative along trajectories. Pironneau in [15] has given a detailed analysis of the method for the Navier–Stokes equations and Süli [22] has proved optimal error estimates for the Lagrange–Galerkin mixed finite element approximation of Navier–Stokes equations in a velocity/pressure formulation. We also mention the work of Achdou and Guermond [1], where convergence analysis of a finite element projection/Lagrange–Galerkin method for the incompressible Navier–Stokes equations is done.

The numerical analysis of some time decoupling algorithms for the simulation of the interaction between a fluid and a structure in the case where the deformation of the structure induces an evolution in the fluid domain has been developed by Grandmont et al. [9] (one dimensional problem). For the ALE method, the numerical analysis of the unsteady Stokes equations in a time dependent domain when the motion of the domain is given has been studied in San Martín et al. [21]. Moreover, Legendre and Takahashi [10] have combined the method of characteristics with a finite element approximation to derive error estimates in the ALE formulation of a two-dimensional problem describing the motion of a rigid body in a viscous fluid. In San Martín et al. [18,19], the authors have proved the convergence of a numerical method based on finite elements with a fixed mesh for a two dimensional fluid-rigid body problem with the densities of the fluid and the solid equal, i.e. $\rho_f = \rho_s$. Their numerical scheme is based on a standard characteristic function resulting from the classical formulation of the material derivative in the Navier–Stokes equations. The method introduced in [19] cannot be easily extended to our case $\rho_f \neq \rho_s$, where the global density is discontinuous, by using the same characteristic function. In this paper, we introduce crucial modifications on the characteristic function, and we propose a new numerical scheme in order to prove a similar convergence result as in [19]. We think that this modification on the characteristic function should be useful to obtain convergent algorithms for the simulation of aquatic organisms in two and three dimensional cases (see San Martín et al. [20]). We refer the reader to Remark 5.4 below for a comparison between our modified characteristic function and other possible choices. Let us also cite our preliminary version [17], where we have introduced the semi-discrete formulation of the problem and we state the convergence result.

The paper is organized as follows. In the next section we introduce some notation and the functional spaces we work on. In Sect. 3 we discretize the fluid-structure interaction problem (1.1)–(1.8) in time variable and we state our first main result given in Theorem 3.2 which consists in the convergence of the semi-discretization scheme. Section 4 is dedicated to the full discretization in time and space variables and then we state our second main result given in Theorem 4.4 which concerns an error estimate for the fully-discrete formulation. Section 5 is devoted to some crucial properties on the characteristic functions associated with our schemes. The last two sections are focused on the proofs of the convergence results for both semi-discrete and fully-discrete formulations.

2 Notation and functional spaces

Throughout this paper, we shall use the classical Sobolev spaces $H^s(\mathcal{O})$, $H_0^s(\mathcal{O})$, $H^{-s}(\mathcal{O})$, $s \geqslant 0$ and the space of Lipschitz continuous functions $C^{0,1}(\overline{\mathcal{O}})$ on the closure of $\mathcal{O}$. We also define

$$L_0^2(\mathcal{O}) = \left\{ f \in L^2(\mathcal{O}) \mid \int_{\mathcal{O}} f \, \mathrm{d}\mathbf{x} = 0 \right\}.$$

The usual inner product in $L^2(\mathcal{O})^2$ will be denoted by

$$(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{O}} \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} \quad \forall \mathbf{u}, \mathbf{v} \in L^2(\mathcal{O})^2. \tag{2.1}$$

If $\mathbf{A}$ is a matrix, we denote by $\mathbf{A}^T$ its transpose. For any 2×2 matrices $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{2\times 2}$, we denote by $\mathbf{A} : \mathbf{B}$ their inner product $\mathbf{A} : \mathbf{B} = \operatorname{Trace}(\mathbf{A}^T\mathbf{B})$, and by $|\mathbf{A}|$ the corresponding norm. For convenience, we use the same notation as in (2.1) for the inner product in $L^2(\mathcal{O}, \mathcal{M}_{2\times 2})$, that is

$$(\mathbf{A}, \mathbf{B}) = \int_{\mathcal{O}} \mathbf{A} : \mathbf{B} \, \mathrm{d}\mathbf{x} \quad \forall \mathbf{A}, \mathbf{B} \in L^2(\mathcal{O}, \mathcal{M}_{2\times 2}).$$

For $\boldsymbol{\zeta} \in \mathcal{O}$, we introduce the space of rigid functions in $B(\boldsymbol{\zeta}) = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x} - \boldsymbol{\zeta}| \leq 1\}$,

$$\mathcal{K}(\boldsymbol{\zeta}) = \left\{ \mathbf{u} \in H_0^1(\mathcal{O})^2 \mid \mathrm{D}(\mathbf{u}) = 0 \text{ in } B(\boldsymbol{\zeta}) \right\}, \tag{2.2}$$

the space of rigid functions in $B(\boldsymbol{\zeta})$ with free divergence in the whole domain $\mathcal{O}$,

$$\widehat{\mathcal{K}}(\boldsymbol{\zeta}) = \left\{ \mathbf{u} \in \mathcal{K}(\boldsymbol{\zeta}) \mid \operatorname{div} \mathbf{u} = 0 \text{ in } \mathcal{O} \right\}, \tag{2.3}$$

and the space of the pressure

$$M(\boldsymbol{\zeta}) = \left\{ p \in L_0^2(\mathcal{O}) \mid p = 0 \text{ in } B(\boldsymbol{\zeta}) \right\}. \tag{2.4}$$

Remark 2.1 For convenience, in the remainder of the paper, any velocity field in $\mathcal{K}(\boldsymbol{\zeta})$ will be extended by zero outside of $\mathcal{O}$.

According to Lemma 1.1 of [23, p. 18], for any $\mathbf{u} \in \mathcal{K}(\boldsymbol{\zeta})$, there exist $\mathbf{l}_u \in \mathbb{R}^2$ and $\omega_{\mathbf{u}} \in \mathbb{R}$ such that

$$\mathbf{u}(\mathbf{y}) = \mathbf{l}_{\mathbf{u}} + \omega_{\mathbf{u}} (\mathbf{y} - \boldsymbol{\zeta})^{\perp} \quad \forall \mathbf{y} \in B(\boldsymbol{\zeta}). \tag{2.5}$$

In addition, we define the density ρ by the following piecewise constant function

$$\rho(\mathbf{x}) = \begin{cases} \rho_s & \text{if } \mathbf{x} \in B(\boldsymbol{\zeta}), \\ \rho_f & \text{if } \mathbf{x} \in \mathcal{O} \backslash B(\boldsymbol{\zeta}). \end{cases}$$

We notice that, by using the above definitions, for any $\mathbf{u}, \mathbf{v} \in \mathcal{K}(\boldsymbol{\zeta})$ we have

$$(\rho \mathbf{u}, \mathbf{v}) = \int_{\mathcal{O} \backslash B(\boldsymbol{\zeta})} \rho_f \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x} + M \mathbf{l}_{\mathbf{u}} \cdot \mathbf{l}_{\mathbf{v}} + J \omega_{\mathbf{u}} \, \omega_{\mathbf{v}}. \tag{2.6}$$

The spaces (2.2)–(2.3) are specific to our problem. In fact, if the solution $\mathbf{u}$ of (1.1)–(1.8) is extended by

$$\mathbf{u}(\mathbf{x}, t) = \boldsymbol{\zeta}'(t) + \omega(t)(\mathbf{x} - \boldsymbol{\zeta}(t))^{\perp} \quad \forall \mathbf{x} \in B(\boldsymbol{\zeta}(t)),$$

then, we easily see that $\mathbf{u}(t) \in \widehat{\mathcal{K}}(\boldsymbol{\zeta}(t))$. In the remainder of this paper, the solution $\mathbf{u}$ of (1.1)–(1.8) will be extended as above.

An important ingredient of the numerical method we use is given by the characteristic functions corresponding to the integral curves of the velocity field. More precisely (see, for instance, [15,22]) the characteristic function $\widetilde{\boldsymbol{\psi}} : [0, T]^2 \times \mathcal{O} \to \mathcal{O}$ is defined as the solution of the initial value problem

$$\begin{cases} \dfrac{d}{dt} \widetilde{\boldsymbol{\psi}}(t; s, \mathbf{x}) = \mathbf{u}(\widetilde{\boldsymbol{\psi}}(t; s, \mathbf{x}), t) & \forall t \in [0, T], \\ \widetilde{\boldsymbol{\psi}}(s; s, \mathbf{x}) = \mathbf{x}. \end{cases} \tag{2.7}$$

It is well-known that the material derivative $D_t \mathbf{u} = \partial \mathbf{u}/\partial t + (\mathbf{u} \cdot \nabla)\mathbf{u}$ of $\mathbf{u}$ at instant t_0 satisfies:

$$D_t \mathbf{u}(\mathbf{x}, t_0) = \frac{d}{dt} \left[\mathbf{u}(\widetilde{\boldsymbol{\psi}}(t; t_0, \mathbf{x}), t) \right]_{|t=t_0}. \tag{2.8}$$

Remark 2.2 By using a classical result of Liouville (see, for instance, [2, p. 251]), if

$$\boldsymbol{\zeta} \in H^2(0, T)^2, \quad \omega \in H^1(0, T), \quad \mathbf{u} \in C([0, T]; \widehat{\mathcal{K}}(\boldsymbol{\zeta}(t))),$$

then we have that

$$\det \mathbf{J}_{\widetilde{\boldsymbol{\psi}}} = 1, \tag{2.9}$$

where we have denoted by

$$\mathbf{J}_{\widetilde{\boldsymbol{\psi}}} = \left(\frac{\partial \widetilde{\psi}_i}{\partial x_j}\right)_{i,j}$$

the jacobian matrix of the transformation $\mathbf{x} \mapsto \widetilde{\boldsymbol{\psi}}(t; s, \mathbf{x})$.

In the following lemma we give a weak formulation of the system (1.1)–(1.8) which will be then used to discretize the problem with respect to time.

Lemma 2.3 *Assume that*

$$\mathbf{u} \in L^2\big(0, T; H^2(\Omega(t))^2\big) \cap H^1\big(0, T; L^2(\Omega(t))^2\big) \cap C\big([0, T]; H^1(\Omega(t))^2\big),$$
$$p \in L^2\big(0, T; H^1(\Omega(t))\big), \quad \boldsymbol{\zeta} \in H^2(0, T)^2, \quad \omega \in H^1(0, T)$$

and that $\mathbf{u}$ *is extended by*

$$\mathbf{u}(\mathbf{x}, t) = \boldsymbol{\zeta}'(t) + \omega(t)(\mathbf{x} - \boldsymbol{\zeta}(t))^{\perp} \quad \forall \mathbf{x} \in B(\boldsymbol{\zeta}(t)).$$

Then $(\mathbf{u}, p, \boldsymbol{\zeta}, \omega)$ *is the solution of* (1.1)–(1.8) *if and only if for all* $t \in [0, T]$, $\mathbf{u}(\cdot, t) \in \mathcal{K}(\boldsymbol{\zeta}(t))$, $p(\cdot, t) \in M(\boldsymbol{\zeta}(t))$ *and* $(\mathbf{u}, p)$ *satisfies*

$$\left(\rho \frac{d}{dt}\left[\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}\right](t), \boldsymbol{\varphi}\right) + a(\mathbf{u}, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, p) = (\rho\, \mathbf{f}(t), \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in \mathcal{K}(\boldsymbol{\zeta}(t)), \tag{2.10}$$

$$b(\mathbf{u}, q) = 0 \quad \forall q \in M(\boldsymbol{\zeta}(t)), \tag{2.11}$$

where

$$a(\mathbf{u}, \mathbf{v}) = 2\mu \int_{\mathcal{O}} \mathrm{D}(\mathbf{u}) : \mathrm{D}(\mathbf{v})\, \mathrm{d}\mathbf{x} \quad \forall \mathbf{u}, \mathbf{v} \in H^1(\mathcal{O})^2 \tag{2.12}$$

and

$$b(\mathbf{u}, p) = -\int_{\mathcal{O}} \mathrm{div}\,(\mathbf{u}) p\, \mathrm{d}\mathbf{x} \quad \forall \mathbf{u} \in H^1(\mathcal{O})^2, \quad \forall p \in L^2_0(\mathcal{O}). \tag{2.13}$$

We skip the proof of Lemma 2.3 since it is similar to the proof of the corresponding result for the classical Navier–Stokes system (see, for instance, [16, Ch.12]).

In the remainder of the paper, we suppose that $\mathbf{f}$ and $\mathbf{u}_0$ satisfy

$$\begin{aligned} &\mathbf{f} \in C([0, T]; H^1(\mathcal{O})^2), \quad \mathbf{u}_0 \in H^2(\Omega)^2, \quad \mathrm{div}\,(\mathbf{u}_0) = 0 \ \text{ in } \Omega, \\ &\mathbf{u}_0 = 0 \ \text{ on } \partial\mathcal{O}, \quad \mathbf{u}_0(\mathbf{x}) = \boldsymbol{\zeta}_1 + \omega_0(\mathbf{x} - \boldsymbol{\zeta}_0)^{\perp} \ \text{ on } \partial B(\boldsymbol{\zeta}_0), \end{aligned} \tag{2.14}$$

where $\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1 \in \mathbb{R}^2$, $\omega_0 \in \mathbb{R}$ and $\Omega = \mathcal{O} \backslash B(\boldsymbol{\zeta}_0)$.

Let us also assume that the corresponding solution $(\mathbf{u}, p, \boldsymbol{\zeta}, \omega)$ of problem (1.1)–(1.8) satisfies

$$\begin{cases} \mathbf{u} \in C([0,T]; H^2(\Omega(t))^2) \cap H^1(0,T; L^2(\Omega(t))^2), \\ D_t^2\mathbf{u} \in L^2(0,T; L^2(\Omega(t))^2), \quad \mathbf{u} \in C([0,T]; C^{0,1}(\overline{\mathcal{O}})^2) \\ p \in C\big([0,T]; H^1(\Omega(t))\big), \quad \boldsymbol{\zeta} \in H^3(0,T)^2, \quad \omega \in H^2(0,T) \end{cases} \tag{2.15}$$

and

$$\text{dist}\,(B(\boldsymbol{\zeta}(t)), \partial\mathcal{O}) > 0 \quad \forall t \in [0,T]. \tag{2.16}$$

Remark 2.4 The hypotheses (2.15) and (2.16) imply the existence of $\eta > 0$ such that

$$\text{dist}\,(B(\boldsymbol{\zeta}(t)), \partial\mathcal{O}) > 3\eta \quad \forall t \in [0,T]. \tag{2.17}$$

3 Semi-discretization scheme and statement of the first main result

By using the weak formulation (2.10)–(2.11) we can derive a semi-discrete version of our system. For $N \in \mathbb{N}^*$ we denote $\Delta t = T/N$ and $t_k = k\Delta t$ for $k = 0, \ldots, N$. Denote by $(\mathbf{u}^k, \boldsymbol{\zeta}^k) \in \big(\widehat{\mathcal{K}}(\boldsymbol{\zeta}^k) \cap C^0(\overline{\mathcal{O}})^2\big) \times \mathcal{O}$ the approximation of the solution of (1.1)–(1.8) at the time $t = t_k$. For $k = 0$, we define

$$\mathbf{u}^0(\cdot) = \mathbf{u}(\cdot, 0) \quad \text{and} \quad \boldsymbol{\zeta}^0 = \boldsymbol{\zeta}(0), \tag{3.1}$$

then using the initial conditions (1.7)–(1.8), we have

$$\mathbf{u}^0(\mathbf{x}) = \begin{cases} \mathbf{u}_0(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega(0) \\ \boldsymbol{\zeta}_1 + \omega_0(\mathbf{x} - \boldsymbol{\zeta}_0)^\perp & \text{if } \mathbf{x} \in B(\boldsymbol{\zeta}_0) \end{cases} \quad \text{and} \quad \boldsymbol{\zeta}^0 = \boldsymbol{\zeta}_0.$$

Let us note that due to hypothesis (2.14), $\mathbf{u}^0(\cdot)$ is a continuous function.

In the sequel, we shall use the notation

$$\widetilde{\mathbf{X}}(\mathbf{x}) = \widetilde{\boldsymbol{\psi}}(t_k; t_{k+1}, \mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{O}. \tag{3.2}$$

We approximate the position of the rigid ball at instant t_{k+1} by $\boldsymbol{\zeta}^{k+1}$ which is defined by the relation

$$\boldsymbol{\zeta}^{k+1} = \boldsymbol{\zeta}^k + \mathbf{u}^k(\boldsymbol{\zeta}^k)\Delta t. \tag{3.3}$$

We then define the characteristic function $\overline{\boldsymbol{\psi}}$ associated with the semi-discretized velocity field as the solution of

$$\begin{cases} \dfrac{d}{dt}\overline{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{x}) = \mathbf{u}^k(\overline{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{x})) - \mathbf{u}^k(\boldsymbol{\zeta}^k) \quad & \forall t \in [t_k, t_{k+1}], \\ \overline{\boldsymbol{\psi}}(t_{k+1}; t_{k+1}, \mathbf{x}) = \mathbf{x} - \mathbf{u}^k(\boldsymbol{\zeta}^k)\Delta t, \end{cases} \tag{3.4}$$

and we denote

$$\overline{\mathbf{X}}^k(\mathbf{x}) = \overline{\boldsymbol{\psi}}(t_k; t_{k+1}, \mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{O}. \tag{3.5}$$

In Eq. (3.4), the velocity $\mathbf{u}^k$ is extended by zero outside of the domain $\mathcal{O}$ as it was noted in Remark 2.1. This extension is necessary because we have $\overline{\mathbf{X}}^k(\mathcal{O}) \nsubseteq \mathcal{O}$. Indeed, we observe that due to the initial condition in (3.4), if we consider $\mathbf{x} \in \mathcal{O}$, then $\overline{\boldsymbol{\psi}}(t_{k+1}; t_{k+1}, \mathbf{x})$ does not necessarily belong to $\mathcal{O}$. Nevertheless, one can easily check that $\overline{\mathbf{X}}^k(\mathcal{O}) \subseteq \mathcal{O} + B(0, |\mathbf{u}^k(\boldsymbol{\zeta}^k)|\Delta t)$. We emphasize that the Cauchy problem (3.4) is well-posed and then the characteristic function $\overline{\mathbf{X}}^k$ is also well defined. Indeed, since $\mathbf{u}^k \in H_0^1(\mathcal{O})^2$ with $\operatorname{div} \mathbf{u}^k = 0$ in $\mathcal{O}$ and $\mathbf{u}^k = 0$ in $\mathbb{R}^2 \backslash \mathcal{O}$, the problem (3.4) admits a unique solution $\overline{\boldsymbol{\psi}}(\cdot; t_{k+1}, \mathbf{x}) \in C^1([t_k, t_{k+1}])$ for almost everywhere $\mathbf{x} \in \mathbb{R}^2$, which satisfies the following measure preserving property (see [3, Sect. III]),

$$\int_{\mathcal{A}} f\left(\overline{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{x})\right) \mathrm{d}\mathbf{x} = \int_{\overline{\boldsymbol{\psi}}(t; t_{k+1}, \mathcal{A})} f(\mathbf{y})\, \mathrm{d}\mathbf{y}, \tag{3.6}$$

for all function $f \in L^1(\mathbb{R}^2)$ and for all $t \in [t_k, t_{k+1}]$. Moreover, since $\mathbf{u}^k \in C^0(\overline{\mathcal{O}})^2$, the characteristic function $\overline{\boldsymbol{\psi}}(\cdot; t_{k+1}, \mathbf{x})$ is actually well defined in $[t_k, t_{k+1}]$, for all $\mathbf{x} \in \mathbb{R}^2$.

We next define $\mathbf{u}^{k+1} \in \widehat{\mathcal{K}}(\boldsymbol{\zeta}^{k+1})$ as the solution of the following Stokes type system

$$\left(\rho^{k+1} \frac{\mathbf{u}^{k+1} - \mathbf{u}^k \circ \overline{\mathbf{X}}^k}{\Delta t}, \boldsymbol{\varphi}\right) + a\left(\mathbf{u}^{k+1}, \boldsymbol{\varphi}\right) = (\rho^{k+1}\mathbf{f}^{k+1}, \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in \widehat{\mathcal{K}}(\boldsymbol{\zeta}^{k+1}), \tag{3.7}$$

where $\mathbf{f}^{k+1} = \mathbf{f}(t_{k+1})$ and ρ^{k+1} is defined by

$$\rho^{k+1}(\mathbf{x}) = \begin{cases} \rho_s & \text{if } \mathbf{x} \in B(\boldsymbol{\zeta}^{k+1}), \\ \rho_f & \text{if } \mathbf{x} \in \mathcal{O} \backslash B(\boldsymbol{\zeta}^{k+1}). \end{cases}$$

The above equation can be rewritten by using a mixed formulation. It is clear that (3.7) is equivalent to the following system

$$\left(\rho^{k+1} \frac{\mathbf{u}^{k+1} - \mathbf{u}^k \circ \overline{\mathbf{X}}^k}{\Delta t}, \boldsymbol{\varphi}\right) + a(\mathbf{u}^{k+1}, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, p^{k+1}) = (\rho^{k+1}\mathbf{f}^{k+1}, \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in \mathcal{K}(\boldsymbol{\zeta}^{k+1}), \tag{3.8}$$

$$b(\mathbf{u}^{k+1}, q) = 0 \quad \forall q \in M(\boldsymbol{\zeta}^{k+1}), \tag{3.9}$$

of unknowns $(\mathbf{u}^{k+1}, p^{k+1}) \in \mathcal{K}(\boldsymbol{\zeta}^{k+1}) \times M(\boldsymbol{\zeta}^{k+1})$.

It is well-known (see, for example, [6, Corollary I.4.1, p. 61]) that the mixed formulation (3.8)–(3.9) is a well-posed problem, provided that the spaces $\mathcal{K}(\boldsymbol{\zeta})$, $M(\boldsymbol{\zeta})$

and the bilinear form b satisfy an *inf-sup* condition. The fact that this *inf-sup* condition is satisfied in our case follows from the result below.

Lemma 3.1 *Suppose that* $\boldsymbol{\zeta} \in \mathcal{O}$ *is such that* $d(\boldsymbol{\zeta}, \partial\mathcal{O}) = 1 + \eta$, *with* $\eta > 0$. *Then there exists a constant* $\beta > 0$, *depending only on* η *and on* $\mathcal{O}$, *such that for all* $q \in M(\boldsymbol{\zeta})$ *there exists* $\mathbf{u} \in \mathcal{K}(\boldsymbol{\zeta})$ *with*

$$\int_{\mathcal{O}} \operatorname{div}(\mathbf{u})\, q \, d\mathbf{x} \geq \beta \|\mathbf{u}\|_{H^1(\mathcal{O})^2} \|q\|_{L^2(\mathcal{O})}. \tag{3.10}$$

The proof of the result above can be obtained by slightly modifying the approach used for the mixed formulation of the standard Stokes system (see, for instance [6, p. 81]), therefore it is left to the reader. In addition, it can be easily proved that $\mathbf{u}^{k+1}$ is continuous in $\overline{\mathcal{O}}$. To see this, we remark that $(\mathbf{u}^{k+1} p^{k+1})$ satisfies a Stokes problem in the fluid part $\mathcal{O}\backslash B(\boldsymbol{\zeta}^{k+1})$ with a rigid velocity boundary condition on $\partial B(\boldsymbol{\zeta}^{k+1})$. Then assuming $\mathbf{f}^{k+1} \in L^2(\mathcal{O})^2$, we get $\mathbf{u}^{k+1} \in H^2(\mathcal{O}\backslash B(\boldsymbol{\zeta}^{k+1}))^2$ and we deduce that

$$\mathbf{u}^{k+1} \in C^0(\overline{\mathcal{O}})^2. \tag{3.11}$$

Let us now state our first main result concerning the convergence of the semi-discrete scheme (3.8)–(3.9):

Theorem 3.2 *Suppose that* $\mathcal{O}$ *is an open and convex smooth bounded domain in* $\mathbb{R}^2$, $\mathbf{f}$ *and* $\mathbf{u}_0$ *satisfy* (2.14) *and* $(\mathbf{u}, p, \boldsymbol{\zeta}, \omega)$ *is a solution of* (1.1)–(1.8) *satisfying* (2.15)–(2.16). *Then there exist two positive constants* K *and* τ^* *not depending on* Δt *such that for all* $0 < \Delta t \leq \tau^*$ *the solution* $(\mathbf{u}^k, p^k, \boldsymbol{\zeta}^k)$ *of the semi-discretization problem* (3.8)–(3.9) *satisfies*

$$\sup_{1\leq k\leq N} \left(|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}^k| + \|\mathbf{u}(t_k) - \mathbf{u}^k\|_{L^2(\mathcal{O})^2}\right) \leq K\Delta t. \tag{3.12}$$

Remark 3.3 The constants K and τ^* from Theorem 3.2 are dependent on T as follows: there exists two positive constants C and C_1, independent on T and Δt, such that

$$K = C \exp(C_1 T)\left(1 + \left\|\frac{\partial \mathbf{u}}{\partial t}\right\|_{L^2(\mathcal{O}\times(0,T))^2} + \left\|\frac{d^2}{dt^2}[\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}]\right\|_{L^2(\mathcal{O}\times(0,T))^2}\right),$$

$$\tau^* = \min\left\{\frac{\eta}{4C_2}, \sqrt{\frac{\eta \min\{\sqrt{\rho_f}, \sqrt{\rho_s}\}}{4C_2}}, \sqrt{\frac{\eta}{4C_1}}\right\},$$

where $C_2 = K/C = \exp(C_1 T)\left(1 + \left\|\frac{\partial \mathbf{u}}{\partial t}\right\|_{L^2(\mathcal{O}\times(0,T))^2} + \left\|\frac{d^2}{dt^2}[\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}]\right\|_{L^2(\mathcal{O}\times(0,T))^2}\right)$ and η is the positive constant given in (2.17).

4 Fully discrete formulation and statement of the second main result

In order to discretize the problem (3.8)–(3.9) with respect to the space variable, we introduce two families of finite element spaces which approximate the spaces $\mathcal{K}(\boldsymbol{\zeta})$ and $M(\boldsymbol{\zeta})$ defined in (2.2) and (2.4). To this end, we consider the discretization parameter $0 < h < 1$.

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of the domain $\mathcal{O}$. We denote by $\mathcal{W}_h$ the $\mathbb{P}_1$+bubble finite elements space associated with $\mathcal{T}_h$ for the velocity field in the Stokes problem and by E_h the $\mathbb{P}_1$-finite elements space for the pressure, that is

$$\mathcal{W}_h = \left\{ \boldsymbol{\varphi} \in \mathscr{C}(\overline{\mathcal{O}})^2 : \forall T \in \mathcal{T}_h,\ \boldsymbol{\varphi}_{|T} \in \mathbb{P}_1 + \text{bubble}(T) \right\},$$
$$E_h = \left\{ q \in \mathscr{C}(\overline{\mathcal{O}}) : \forall T \in \mathcal{T}_h,\ q_{|T} \in \mathbb{P}_1 \right\}.$$

Then, we define the following finite element spaces for a conform approximation of the fluid-rigid system:

$$\mathcal{K}_h(\boldsymbol{\zeta}) = \mathcal{W}_h \cap \mathcal{K}(\boldsymbol{\zeta}) \qquad \forall \boldsymbol{\zeta} \in \mathcal{O},$$
$$M_h(\boldsymbol{\zeta}) = E_h \cap M(\boldsymbol{\zeta}) \qquad \forall \boldsymbol{\zeta} \in \mathcal{O}.$$

Let us recall an approximation property of the projection on $K_h(\boldsymbol{\zeta}) \times M_h(\boldsymbol{\zeta})$ (see [19]).

Lemma 4.1 *Suppose that* $\mathbf{V} \in \mathcal{K}(\boldsymbol{\zeta})$ *and that* $P \in M(\boldsymbol{\zeta})$. *Then there exists a unique couple* $(\overline{\mathbf{V}}_h, \overline{P}_h)$ *in* $\mathcal{K}_h(\boldsymbol{\zeta}) \times M_h(\boldsymbol{\zeta})$ *such that:*

$$\begin{cases} a\left(\mathbf{V} - \overline{\mathbf{V}}_h, \boldsymbol{\varphi}\right) + b\left(\boldsymbol{\varphi}, P - \overline{P}_h\right) = 0 & \forall \boldsymbol{\varphi} \in \mathcal{K}_h(\boldsymbol{\zeta}), \\ b\left(\mathbf{V} - \overline{\mathbf{V}}_h, q\right) = 0 & \forall q \in M_h(\boldsymbol{\zeta}). \end{cases} \tag{4.1}$$

Moreover, if we suppose in addition that $\mathbf{V}_{|\mathcal{O}\setminus B(\boldsymbol{\zeta})} \in H^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}))^2$ *and that* $P_{|\mathcal{O}\setminus B(\boldsymbol{\zeta})} \in H^1(\mathcal{O}\setminus B(\boldsymbol{\zeta}))$, *then there exists a positive constant* C, *independent of* h, *such that*

$$\|\mathbf{V} - \overline{\mathbf{V}}_h\|_{L^2(\mathcal{O})^2} \leq Ch.$$

In order to define the approximate characteristics, let us denote by F_h the $\mathbb{P}_2$-finite element space associated with the triangulation $\mathcal{T}_h$ and we introduce the space:

$$\mathcal{R}_h(\boldsymbol{\zeta}) = \{\nabla^{\perp}\varphi_h : \varphi_h \in F_h,\ \varphi_h = 0 \text{ on } \partial\mathcal{O}\} \cap \mathcal{K}(\boldsymbol{\zeta}) \qquad \forall \boldsymbol{\zeta} \in \mathcal{O},$$

where $\nabla^{\perp}\varphi_h = \begin{pmatrix} -\dfrac{\partial \varphi_h}{\partial x_2} \\ \dfrac{\partial \varphi_h}{\partial x_1} \end{pmatrix}$.

We denote by $\mathbf{P}(\boldsymbol{\zeta})$ the orthogonal projection from $L^2(\mathcal{O})^2$ onto $\mathcal{R}_h(\boldsymbol{\zeta})$, i.e. if $\mathbf{u} \in L^2(\mathcal{O})^2$ then $\mathbf{P}(\boldsymbol{\zeta})\mathbf{u} \in \mathcal{R}_h(\boldsymbol{\zeta})$ is such that $(\mathbf{u} - \mathbf{P}(\boldsymbol{\zeta})\mathbf{u}, \mathbf{r}_h) = 0$ for all $\mathbf{r}_h \in \mathcal{R}_h(\boldsymbol{\zeta})$.

Let N be a positive integer. We denote $\Delta t = T/N$ and $t_k = k\Delta t$ for all $k \in \{0, \ldots, N\}$.

For $k = 0$, we define

$$\mathbf{u}_h^0(\cdot) = \overline{\mathbf{u}(\cdot, 0)} \quad \text{and} \quad \boldsymbol{\zeta}_h^0 = \boldsymbol{\zeta}_0, \tag{4.2}$$

where $\left(\overline{\mathbf{u}(\cdot, 0)}, \overline{p(\cdot, 0)}\right) \in \mathcal{K}_h(\boldsymbol{\zeta}_0) \times M_h(\boldsymbol{\zeta}_0)$ is the projection of the initial condition $\left(\mathbf{u}(\cdot, 0), p(\cdot, 0)\right)$ on $\mathcal{K}_h(\boldsymbol{\zeta}_0) \times M_h(\boldsymbol{\zeta}_0)$ defined in (4.1).

Assume that the approximate solution $(\mathbf{u}_h^k, p_h^k, \boldsymbol{\zeta}_h^k)$ of (1.1)–(1.8) at $t = t_k$ is known. We describe below the numerical scheme allowing to determinate the approximate solution $(\mathbf{u}_h^{k+1}, p_h^{k+1}, \boldsymbol{\zeta}_h^{k+1})$ at $t = t_{k+1}$. First, we compute $\boldsymbol{\zeta}_h^{k+1} \in \mathbb{R}^2$ by

$$\boldsymbol{\zeta}_h^{k+1} = \boldsymbol{\zeta}_h^k + \mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)\Delta t. \tag{4.3}$$

We consider the approximated characteristic function $\overline{\boldsymbol{\psi}}_h^k$ defined as the solution of

$$\begin{cases} \dfrac{d}{dt}\overline{\boldsymbol{\psi}}_h^k(t; t_{k+1}, \mathbf{x}) = \mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k(\overline{\boldsymbol{\psi}}_h^k(t; t_{k+1}, \mathbf{x})) - \mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k(\boldsymbol{\zeta}_h^k) \quad \forall t \in [t_k, t_{k+1}], \\ \overline{\boldsymbol{\psi}}_h^k(t_{k+1}; t_{k+1}, \mathbf{x}) = \mathbf{x} - \mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)\Delta t. \end{cases} \tag{4.4}$$

Then, we define

$$\overline{\mathbf{X}}_h^k(\mathbf{x}) = \overline{\boldsymbol{\psi}}_h^k(t_k; t_{k+1}, \mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{O}. \tag{4.5}$$

We remark that, since $\operatorname{div}\left(\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k(\overline{\boldsymbol{\psi}}_h^k(t; t_{k+1}, \cdot)) - \mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)\right) = 0$ and $\nabla(\mathbf{x} - \mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)\Delta t) = \mathbf{Id}$, we have

$$\det \mathbf{J}_{\overline{\boldsymbol{\psi}}_h^k} = 1. \tag{4.6}$$

In the sequel, we shall split the mesh into the union of 4 different types of triangle's subsets. We first introduce A_h as the union of all triangles intersecting the ball $B(\boldsymbol{\zeta}_h^k)$, i.e.

$$A_h = \bigcup_{\substack{T \in \mathcal{T}_h \\ \overset{\circ}{T} \cap \overset{\circ}{B}(\boldsymbol{\zeta}_h^k) \neq \emptyset}} T.$$

We also denote by Q_h the union of all triangles such that all their vertices are contained in $\overline{A_h}$. The triangles of $\mathcal{T}_h$ are then split into the following four categories (see Fig. 1):

- $\mathcal{F}_1$ is the subset of $\mathcal{T}_h$ formed by all triangles $T \in \mathcal{T}_h$ such that $\overline{T} \subset B(\boldsymbol{\zeta}_h^k)$.
- $\mathcal{F}_2$ is the subset formed by all triangles $T \in \mathcal{T}_h \backslash \mathcal{F}_1$ such that $\overline{T} \subset \overline{Q_h}$.
- $\mathcal{F}_3$ is the subset formed by all triangles $T \in \mathcal{T}_h$ such that $\overline{T} \cap \overline{Q_h} \neq \emptyset$ and $T \not\subset \overline{Q_h}$.

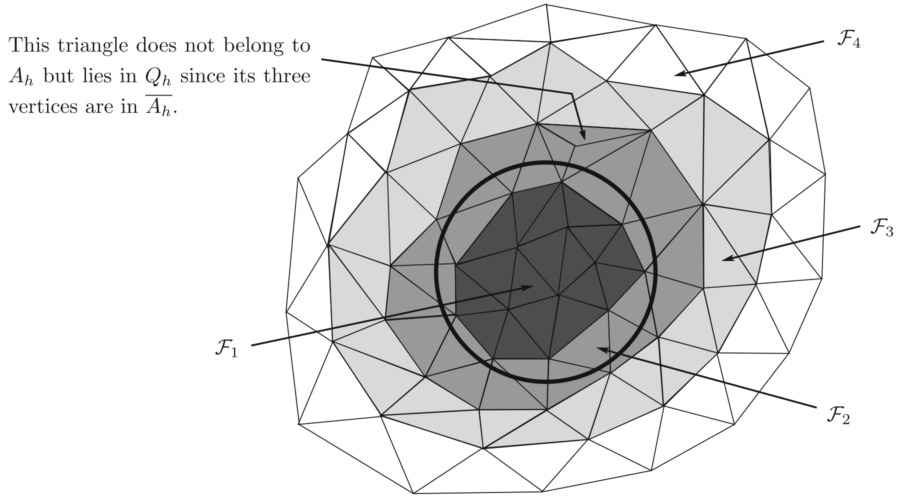

Fig. 1 The rigid ball and the related splitting of the triangulation

- $\mathcal{F}_4 = \mathcal{T}_h \backslash (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3)$.

We introduce two approximated density functions ρ_h^k and $\overline{\rho}_h^k$ as follows:

$$\rho_h^k(\mathbf{x}) = \begin{cases} \rho_s & \text{if } \mathbf{x} \in B(\boldsymbol{\zeta}_h^k), \\ \rho_f & \text{if } \mathbf{x} \in \mathcal{O} \backslash B(\boldsymbol{\zeta}_h^k) \end{cases} \tag{4.7}$$

and $\overline{\rho}_h^k$ is the continuous function in $\overline{\mathcal{O}}$ which is piecewise linear on triangles of $\mathcal{T}_h$ and satisfies

$$\overline{\rho}_h^k(\mathbf{x}) = \begin{cases} \rho_s & \text{if } \mathbf{x} \in \overline{Q_h}, \\ \rho_f & \text{if } \mathbf{x} \in \mathcal{F}_4. \end{cases} \tag{4.8}$$

With these notations, we consider the following mixed variational fully discrete formulation: Find $(\mathbf{u}_h^{k+1}, p_h^{k+1}) \in \mathcal{K}_h(\boldsymbol{\zeta}_h^{k+1}) \times M_h(\boldsymbol{\zeta}_h^{k+1})$ such that

$$\left(\rho_h^{k+1} \frac{\mathbf{u}_h^{k+1} - \mathbf{u}_h^k \circ \overline{\mathbf{X}}_h^k}{\Delta t}, \boldsymbol{\varphi}\right) + a(\mathbf{u}_h^{k+1}, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, p_h^{k+1})$$
$$= (\overline{\rho}_h^{k+1} \mathbf{f}_h^{k+1}, \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in \mathcal{K}_h(\boldsymbol{\zeta}_h^{k+1}), \tag{4.9}$$

$$b(\mathbf{u}_h^{k+1}, q) = 0 \quad \forall q \in M_h(\boldsymbol{\zeta}_h^{k+1}), \tag{4.10}$$

where $\mathbf{f}_h^{k+1}$ is the $L^2(\mathcal{O})^2$-projection of $\mathbf{f}^{k+1} = \mathbf{f}(t_{k+1})$ on $(E_h)^2$.

Remark 4.2 The density ρ_h^{k+1} is not a finite element function. Nevertheless, making use of this density function is justified through the following two points of view: the first integral term in (4.9) involving the density ρ_h^{k+1} can be exactly computed (see Remark 5.7 below). On the other hand, ρ_h^{k+1} has important properties that guarantee the stability and the convergence of our method (see identity (5.8) below). It is important to remark that the density $\overline{\rho}_h^{k+1}$ does not satisfy a similar property to (5.8). Therefore, if we use $\overline{\rho}_h^{k+1}$ instead of ρ_h^{k+1} in the first term of our scheme (4.9)–(4.10), the analysis becomes harder to handle and we didn't succeed to overcome the induced difficulties.

Remark 4.3 On the right side of our numerical scheme (4.9)–(4.10), we use the function $\overline{\rho}_h^{k+1}$ because it is the natural $\mathbb{P}_1$ finite element density corresponding to ρ_h^{k+1}. This choice is not mandatory. In fact, we can use any other density function associated with finite element mesh provided that estimate (7.28) below holds. For instance, it is possible to use the discontinuous $\mathbb{P}_0$ (piecewise constant) density approximation of ρ_h^{k+1}.

Let us now state the second main result of this paper which asserts the convergence of the fully-discrete scheme (4.9)–(4.10):

Theorem 4.4 *Let $\mathcal{O}$ be a bounded convex domain with a polygonal boundary. Suppose that $\mathbf{f}$ and $\mathbf{u}_0$ satisfy the conditions (2.14) and that $(\mathbf{u}, p, \boldsymbol{\zeta}, \omega)$ is a solution of (1.1)–(1.8) satisfying the regularity properties (2.15) and such that (2.16) holds. Let $C_0 > 0$ and $0 < \alpha \leq 1$ be two fixed constants. Then there exist two positive constants K and τ^* independent of h and Δt such that for all $0 < \Delta t \leq \tau^*$ and for all $h \leq C_0 \Delta t^{1+\alpha}$ we have*

$$\sup_{1\leq k\leq N} \left(|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + \|\mathbf{u}(t_k) - \mathbf{u}_h^k\|_{L^2(\mathcal{O})^2} \right) \leq K \Delta t^{\alpha}.$$

Remark 4.5 The constants K and τ^* from Theorem 4.4 are dependent on T as follows: there exists two positive constants C and C_3, independent on T, Δt and h, such that

$$K = C(C_4 + C_0),$$
$$\tau^* = \min \left\{ \frac{\eta}{4C_3}, \sqrt{\frac{\eta}{4C_3}}, 1, \sqrt{\rho_f}, \sqrt{\rho_s}, \left(\frac{\eta}{4C_4}\right)^{1/\alpha} \right\},$$

where $C_4 = \exp(C_3 T)\left(1+\sqrt{C_0}+C_0+\left\|\frac{\partial \mathbf{u}}{\partial t}\right\|_{L^2(\mathcal{O}\times(0,T))^2}+\left\|\frac{d^2}{dt^2}[\mathbf{u}\circ\widetilde{\boldsymbol{\psi}}]\right\|_{L^2(\mathcal{O}\times(0,T))^2}\right)$, C_0 is the constant given in the statement of Theorem 4.4 and η is the positive constant given in (2.17).

Remark 4.6 In order to get an approximation of first order in time (i.e. $O(\Delta t)$), we have to choose $\alpha = 1$. In this case, the corresponding condition on h becomes $h \leq C_0 \Delta t^2$ which is similar to the one obtained in [19, Th.3.2] in the case of equal densities $\rho_f = \rho_s$.

Remark 4.7 Let us give some comments on the condition on h and Δt required for the convergence result in Theorem 4.4. First, we emphasize that the same type of condition appears in several works for approximation in a Lagrangian framework of the Navier–Stokes equations without any rigid body. We may cite [15] where the convergence is obtained under the condition $h \le C_0\Delta t$ and [22] where h and Δt are chosen such that $h^2 \le C\Delta t \le C_1 h^\sigma$ and $\sigma > 1/2$ (with h and Δt small enough). We also mention [21] for an ALE scheme applied to Stokes equations in a time-dependent domain, where the authors obtain an error estimate of order $\mathcal{O}(\Delta t)$ under the condition $h \le C\Delta t^{3/4}$.

Remark 4.8 It can be easily shown that the fully-discrete scheme (4.9)–(4.10) is unconditionally stable. This is namely due to the fact that (4.6) holds and also due to the stability property (5.8) below fulfilled by the density function.

5 Properties on the characteristic function

In this section, we prove some properties on the new characteristic function which are essential for the proof of our main results.

Lemma 5.1 *For any free divergence velocity field* $\mathbf{v} \in H_0^1(\mathcal{O})^2 \cap C^0(\overline{\mathcal{O}})^2$ *extended by zero outside of* $\mathcal{O}$*, and for any differentiable function* $\mathbf{R} : \mathcal{O} \to \mathbb{R}^2$ *such that* $\det(\nabla\mathbf{R}) = 1$ *and* $\mathbf{R}(\overline{S}_{k+1}) = \overline{S}_k$*, where* S_k *and* S_{k+1} *are two open smooth subsets of* $\mathcal{O}$*, we consider the characteristic function as the solution of problem*

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{\psi}(t;t_{k+1},\mathbf{x}) = \mathbf{v}(\boldsymbol{\psi}(t;t_{k+1},\mathbf{x})) & \forall t \in [t_k,t_{k+1}], \\ \boldsymbol{\psi}(t_{k+1};t_{k+1},\mathbf{x}) = \mathbf{R}(\mathbf{x}) \end{cases} \tag{5.1}$$

and we denote

$$\mathbf{X}(\mathbf{x}) = \boldsymbol{\psi}(t_k;t_{k+1},\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{O}. \tag{5.2}$$

If $\mathbf{v}(\mathbf{z})\cdot\mathbf{n} = 0$ *for any* $\mathbf{z} \in \partial S_k$*, then the characteristic function satisfies the following properties:*

(1) $\mathbf{X}\big(\overline{S}_{k+1}\big) = \overline{S}_k$*;*

(2) *For any* $f \in L^2(\mathbb{R}^2)$ *such that* $f = 0$ *in* $\mathbb{R}^2\backslash\mathcal{O}$*, we have*

$$\big\| f \circ \boldsymbol{\psi}(t;t_{k+1},\cdot)\big\|_{L^2(\mathcal{O})} \le \big\| f \big\|_{L^2(\mathcal{O})} \qquad \forall t \in [t_k,t_{k+1}]. \tag{5.3}$$

Proof Let us first remark that the Cauchy problem (5.1) is well-posed. To see this, we transform problem (5.1) by making use of the following change of unknown:

$$\boldsymbol{\psi}(t;t_{k+1},\mathbf{x}) = \boldsymbol{\varphi}(t;t_{k+1},\mathbf{R}(\mathbf{x})), \tag{5.4}$$

where $\boldsymbol{\varphi}$ satisfies

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{\varphi}(t;t_{k+1},\mathbf{y}) = \mathbf{v}(\boldsymbol{\varphi}(t;t_{k+1},\mathbf{y})) & \forall t \in [t_k,t_{k+1}], \\ \boldsymbol{\varphi}(t_{k+1};t_{k+1},\mathbf{y}) = \mathbf{y} \quad \forall \mathbf{y} \in \mathbb{R}^2. \end{cases} \tag{5.5}$$

According to [3, Sect. III], the Cauchy problem (5.5) admits a unique solution $\boldsymbol{\varphi}(\cdot\,;t_{k+1},\mathbf{y}) \in C^1(\mathbb{R})^2$ for almost everywhere $\mathbf{y} \in \mathbb{R}^2$ and satisfies the following measure preserving property

$$\int_{\mathcal{A}} F\left(\boldsymbol{\varphi}(t;t_{k+1},\mathbf{y})\right) \mathrm{d}\mathbf{y} = \int_{\boldsymbol{\varphi}(t;t_{k+1},\mathcal{A})} F(\mathbf{x})\mathrm{d}\mathbf{x}, \tag{5.6}$$

for any subset $\mathcal{A} \subset \mathbb{R}^2$, for all function $F \in L^1(\mathbb{R}^2)$ and for all $t \in [t_k, t_{k+1}]$. Since the velocity field $\mathbf{v}$ is continuous in $\mathbb{R}^2$, then $\boldsymbol{\varphi}(\cdot\,; t_{k+1}, \mathbf{y})$ is actually defined for all $\mathbf{y} \in \mathbb{R}^2$. Moreover, due to the hypothesis $\mathbf{v} \cdot \mathbf{n} = 0$ on ∂S_k, we have that for all $t \in [t_k, t_{k+1}]$, $\boldsymbol{\varphi}(t; t_{k+1}, \cdot)$ maps $\overline{S}_k$ onto itself (see [3, Sect. IV]). In particular, we get that $\boldsymbol{\varphi}(t_k; t_{k+1}, \cdot)$ maps $\overline{S}_k$ onto itself.

We can now prove the equality (i). In fact, we have that

$$\mathbf{X}(\overline{S}_{k+1}) = \boldsymbol{\varphi}(t_k; t_{k+1}, \mathbf{R}(\overline{S}_{k+1})) = \boldsymbol{\varphi}(t_k; t_{k+1}, \overline{S}_k) = \overline{S}_k.$$

Let us turn to the proof of (ii). Under the assumption $\det(\nabla\mathbf{R}) = 1$ and using the property (5.6), we obtain

$$\begin{aligned}
\|f \circ \boldsymbol{\psi}(t; t_{k+1}, \cdot)\|^2_{L^2(\mathcal{O})} &= \int_{\mathcal{O}} |f(\boldsymbol{\psi}(t; t_{k+1}, \mathbf{x}))|^2 \mathrm{d}\mathbf{x} = \int_{\mathcal{O}} |f(\boldsymbol{\varphi}(t; t_{k+1}, \mathbf{R}(\mathbf{x})))|^2 \mathrm{d}\mathbf{x} \\
&= \int_{\mathbf{R}(\mathcal{O})} |f(\boldsymbol{\varphi}(t; t_{k+1}, \mathbf{y}))|^2 \mathrm{d}\mathbf{y} = \int_{\boldsymbol{\psi}(t;t_{k+1},\mathcal{O})} |f(\mathbf{z})|^2 \mathrm{d}\mathbf{z}.
\end{aligned}$$

On the other hand, since $f = 0$ in $\mathbb{R}^2 \backslash \mathcal{O}$ we have

$$\int_{\boldsymbol{\psi}(t;t_{k+1},\mathcal{O})} |f(\mathbf{z})|^2 \mathrm{d}\mathbf{z} = \int_{\boldsymbol{\psi}(t;t_{k+1},\mathcal{O})\cap\mathcal{O}} |f(\mathbf{z})|^2 \mathrm{d}\mathbf{z} \leq \int_{\mathcal{O}} |f(\mathbf{z})|^2 \mathrm{d}\mathbf{z}.$$

Therefore, we conclude the result (ii). □

In the sequel, we state two corollaries of the above lemma which state the properties on the characteristic functions associated with the semi-discretized and full-discretized velocity fields:

Corollary 5.2 *For any $k \in \{0, \ldots, N\}$, the characteristic function $\overline{\boldsymbol{\psi}}$ defined in (3.4)–(3.5) satisfies the following properties:*

(i) $\overline{\mathbf{X}}^k\left(B(\boldsymbol{\zeta}^{k+1})\right) = B(\boldsymbol{\zeta}^k)$;

(ii) *If we extend by ρ_f the density field ρ^k outside of $\mathcal{O}$, we have*

$$\rho^{k+1} = \rho^k \circ \overline{\mathbf{X}}^k;$$

(iii) *For any $f \in L^2(\mathbb{R}^2)$ such that $f = 0$ in $\mathbb{R}^2\backslash\mathcal{O}$, we have*

$$\left\| f \circ \overline{\boldsymbol{\psi}}(t; t_{k+1}, \cdot)\right\|_{L^2(\mathcal{O})} \leq \left\| f \right\|_{L^2(\mathcal{O})} \qquad \forall t \in [t_k, t_{k+1}]. \tag{5.7}$$

Proof The properties (i) and (iii) are direct consequences of Lemma 5.1. In fact, we have that the function $\mathbf{R}(\mathbf{x}) = \mathbf{x} - \mathbf{u}^k(\boldsymbol{\zeta}^k)\Delta t$ maps $B(\boldsymbol{\zeta}^{k+1})$ onto $B(\boldsymbol{\zeta}^k)$ and $\nabla\mathbf{R} = \mathbf{Id}$. Moreover, the velocity field

$$\mathbf{v}(\mathbf{z}) = \mathbf{u}^k(\mathbf{z}) - \mathbf{u}^k(\boldsymbol{\zeta}^k) \qquad \forall \mathbf{z} \in \mathbb{R}^2$$

is a free divergence field and for any $\mathbf{z} \in B(\boldsymbol{\zeta}^k)$ the decomposition (2.5) allows us to get that

$$\mathbf{v}(\mathbf{z}) = \omega_{\mathbf{u}^k}(\mathbf{z} - \boldsymbol{\zeta}^k)^\perp,$$

which implies that the hypothesis $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial B(\boldsymbol{\zeta}^k)$ holds.

The equality (ii) is a direct consequence of (i) and the extension of ρ^k by ρ_f outside of $\mathcal{O}$. In fact, we have $(\rho^k \circ \overline{\mathbf{X}}^k)(\mathbf{x}) = \rho_s$ if and only if $\overline{\mathbf{X}}^k(\mathbf{x}) \in B(\boldsymbol{\zeta}^k)$ which is equivalent to $\mathbf{x} \in B(\boldsymbol{\zeta}^{k+1})$ due to identity (i). □

Corollary 5.3 *For any $k \in \{0, \dots, N\}$ and $h \in (0, 1)$, the characteristic function $\overline{\boldsymbol{\psi}}_h^k$ defined in* (4.4)–(4.5) *satisfies the following properties:*

(i) $\overline{\mathbf{X}}_h^k\big(B(\boldsymbol{\zeta}_h^{k+1})\big) = B(\boldsymbol{\zeta}_h^k)$;

(ii) *If we extend by ρ_f the density field ρ_h^k outside of $\mathcal{O}$, we have*

$$\rho_h^{k+1} = \rho_h^k \circ \overline{\mathbf{X}}_h^k; \tag{5.8}$$

(iii) *For any $f \in L^2(\mathbb{R}^2)$ such that $f = 0$ in $\mathbb{R}^2 \backslash \mathcal{O}$, we have*

$$\big\| f \circ \overline{\boldsymbol{\psi}}_h^k(t; t_{k+1}, \cdot)\big\|_{L^2(\mathcal{O})^2} \le \|f\|_{L^2(\mathcal{O})^2} \qquad \forall t \in [t_k, t_{k+1}]. \tag{5.9}$$

Remark 5.4 The property (5.8) is one of the key ingredients that guarantee the stability and the convergence of our numerical method for the case $\rho_s \ne \rho_f$. In fact, the numerical scheme studied in [19] for the case $\rho_s = \rho_f$ cannot be easily extended to the case with different densities. If we naively add different densities in the scheme of [19], as for instance the $\mathbb{P}_1$-density function $\overline{\rho}_h^{k+1}$, and preserve their characteristic function, the corresponding property (5.8) could be false (for some velocity fields). In this case, the error estimate involves a term of type $\big\|\rho_h^{k+1} - \rho_h^k \circ \overline{\mathbf{X}}_h^k\big\|_{L^2(\mathcal{O})}$ which cannot be conveniently estimated in order to prove either the convergence or the stability of the method. For this reason, we suspect that the resulting numerical scheme does not converge and is unstable. Even incorporating the discontinuous density ρ_h^{k+1} and still preserving their characteristic function don't allow to overcome the difficulties. In our case, in order to get the property (5.8), we propose a suitable combination of a discontinuous density function with a modified characteristic function.

Proof The proof is similar to the proof of Corollary 5.2. It is enough to observe that the initial condition from equation (4.4), $\mathbf{R}(\mathbf{x}) = \mathbf{x} - \mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)\Delta t$ maps $B(\boldsymbol{\zeta}_h^{k+1})$ onto $B(\boldsymbol{\zeta}_h^k)$ and $\nabla\mathbf{R} = \mathbf{Id}$. The velocity field

$$\mathbf{v}(\mathbf{z}) = \mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k(\mathbf{z}) - \mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k(\boldsymbol{\zeta}_h^k) \qquad \forall \mathbf{z} \in \mathbb{R}^2$$

is free divergence and for any $\mathbf{z} \in B(\boldsymbol{\zeta}_h^k)$ the decomposition (2.5) gives us

$$\mathbf{v}(\mathbf{z}) = \omega_{\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k}(\mathbf{z} - \boldsymbol{\zeta}_h^k)^{\perp}, \tag{5.10}$$

where $\omega_{\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k}$ is the angular velocity associated with the rigid velocity field $\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k$ in $B(\boldsymbol{\zeta}_h^k)$. This implies that

$$\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \partial B(\boldsymbol{\zeta}_h^k). \tag{5.11}$$

With these remarks, the hypotheses of Lemma 5.1 are satisfied and thus the proof is concluded. □

Remark 5.5 In the case of a general rigid body, not necessarily ball, the definition of the characteristic function $\overline{\boldsymbol{\psi}}$ has to be modified in order to take into account the rotation effects. To this end, we denote by $B(\boldsymbol{\zeta}^k, \theta^k)$ the rigid body with the center of mass $\boldsymbol{\zeta}^k$ and the orientation angle θ^k. We also denote ω_k the approximate angular velocity and $\mathcal{R}_\theta$ will stand for the rotation matrix of angle θ. The characteristic function $\overline{\boldsymbol{\psi}}$ is now defined as the solution of

$$\begin{cases} \dfrac{d}{dt}\overline{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{x}) = \mathbf{u}^k\left(\overline{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{x})\right) - \mathbf{u}_R^k\left(\overline{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{x})\right), \\ \overline{\boldsymbol{\psi}}(t_{k+1}; t_{k+1}, \mathbf{x}) = \mathcal{R}_{-\omega_k(t_{k+1}-t_k)}(\mathbf{x} - \boldsymbol{\zeta}^{k+1}) + \boldsymbol{\zeta}^k, \end{cases} \tag{5.12}$$

where $\mathbf{u}^k$ is extended by zero outside of $\mathcal{O}$ as in Remark 2.1 and $\mathbf{u}_R^k$ is the rigid velocity field defined as follows

$$\mathbf{u}_R^k(\mathbf{x}) = \mathbf{u}^k(\boldsymbol{\zeta}^k) + \omega_k(\mathbf{x} - \boldsymbol{\zeta}^k)^{\perp} \qquad \forall \mathbf{x} \in \mathbb{R}^2. \tag{5.13}$$

We also define the function $\overline{\mathbf{X}}^k$ by

$$\overline{\mathbf{X}}^k(\mathbf{x}) = \overline{\boldsymbol{\psi}}(t_k; t_{k+1}, \mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{O}. \tag{5.14}$$

With these definitions, the hypotheses of Lemma 5.1 are still fulfilled and then Corollary 5.2 holds for the general case of a rigid body.

Let us finish this section by giving an important property on the characteristic function $\overline{X}_h^k$. This result is very useful for computational and practical considerations (see Remark 5.7 below).

Corollary 5.6 *For any* $k \in \{0, \ldots, N\}$ *and* $h \in (0, 1)$*, the characteristic function* $\overline{\boldsymbol{\psi}}_h^k$ *defined in* (4.4)–(4.5) *satisfies the following property:*

$$\overline{\mathbf{X}}_h^k(\mathbf{x}) - \boldsymbol{\zeta}_h^k = \mathcal{R}_{-\omega_{\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k}\Delta t}(\mathbf{x} - \boldsymbol{\zeta}_h^{k+1}) \qquad \forall \mathbf{x} \in B(\boldsymbol{\zeta}_h^{k+1}). \tag{5.15}$$

Proof Let us fix $\mathbf{x} \in B(\boldsymbol{\zeta}_h^{k+1})$. Since the identity (5.11) holds, we can apply results from [3, Sect. IV] and we get that the solution $\overline{\boldsymbol{\psi}}_h^k$ of problem (4.4) satisfies

$$\overline{\boldsymbol{\psi}}_h^k(t; t_{k+1}, \mathbf{x}) \in B(\boldsymbol{\zeta}_h^k) \qquad \text{for all } t \in [t_k, t_{k+1}].$$

Due to identity (5.10), the differential equation (4.4) reduces to

$$\begin{cases} \dfrac{d}{dt}\overline{\boldsymbol{\psi}}_h^k(t; t_{k+1}, \mathbf{x}) = \omega_{\mathbf{P}(\zeta_h^k)\mathbf{u}_h^k}(\overline{\boldsymbol{\psi}}_h^k(t; t_{k+1}, \mathbf{x}) - \boldsymbol{\zeta}_h^k)^{\perp} \quad \forall t \in [t_k, t_{k+1}], \\ \overline{\boldsymbol{\psi}}_h^k(t_{k+1}; t_{k+1}, \mathbf{x}) = \mathbf{x} - \mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)\Delta t. \end{cases} \tag{5.16}$$

The solution of this equation is given by

$$\begin{aligned} \overline{\boldsymbol{\psi}}_h^k(t; t_{k+1}, \mathbf{x}) &= \boldsymbol{\zeta}_h^k + \mathcal{R}_{\omega_{\mathbf{P}(\zeta_h^k)\mathbf{u}_h^k}(t-t_{k+1})}(\overline{\boldsymbol{\psi}}_h^k(t_{k+1}; t_{k+1}, \mathbf{x}) - \boldsymbol{\zeta}_h^k) \\ &= \boldsymbol{\zeta}_h^k + \mathcal{R}_{\omega_{\mathbf{P}(\zeta_h^k)\mathbf{u}_h^k}(t-t_{k+1})}(\mathbf{x} - \boldsymbol{\zeta}_h^{k+1}), \end{aligned}$$

for all $t \in [t_k, t_{k+1}]$. Taking $t = t_k$ leads to the desired relation (5.15). □

Remark 5.7 An important feature in our fully discrete scheme (4.9)–(4.10) is the use of ρ_h^{k+1} inside the first integral term in (4.9). Despite of the fact that the density ρ_h^{k+1} is not a finite element function associated to the mesh $\mathcal{T}_h$, this integral term can be exactly computed using the following decomposition:

$$\begin{aligned} \left(\rho_h^{k+1}\frac{\mathbf{u}_h^{k+1} - \mathbf{u}_h^k \circ \overline{\mathbf{X}}_h^k}{\Delta t}, \boldsymbol{\varphi}\right) &= \frac{\rho_f}{\Delta t}\int_{\mathcal{O}} (\mathbf{u}_h^{k+1} - \mathbf{u}_h^k \circ \overline{\mathbf{X}}_h^k) \cdot \boldsymbol{\varphi}\, d\mathbf{x} \\ &\quad + \frac{\rho_s - \rho_f}{\Delta t}\int_{B(\zeta_h^{k+1})} (\mathbf{u}_h^{k+1} - \mathbf{u}_h^k \circ \overline{\mathbf{X}}_h^k) \cdot \boldsymbol{\varphi}\, d\mathbf{x} \quad \forall \boldsymbol{\varphi} \in \mathcal{K}_h(\boldsymbol{\zeta}_h^{k+1}). \end{aligned}$$

The first integral term is computed as usual by using the finite element spaces. The second integral term is computed using the following properties: for all $\mathbf{x} \in B(\boldsymbol{\zeta}_h^{k+1})$, we have

$$\begin{aligned} \mathbf{u}_h^{k+1}(\mathbf{x}) &= \ell_1 + \omega_1(\mathbf{x} - \boldsymbol{\zeta}_h^{k+1})^{\perp}, \\ (\mathbf{u}_h^k \circ \overline{\mathbf{X}}_h^k)(\mathbf{x}) &= \ell_2 + \omega_2(\overline{\mathbf{X}}_h^k(\mathbf{x}) - \boldsymbol{\zeta}_h^k)^{\perp}, \\ \boldsymbol{\varphi}(\mathbf{x}) &= \ell_\varphi + \omega_\varphi(\mathbf{x} - \boldsymbol{\zeta}_h^{k+1})^{\perp}, \end{aligned}$$

then we get

$$\int_{B(\boldsymbol{\zeta}_h^{k+1})} \mathbf{u}_h^{k+1} \cdot \boldsymbol{\varphi} d\mathbf{x} = \pi \ell_1 \cdot \ell_\varphi + \frac{\pi}{2}\omega_1\omega_\varphi,$$

$$\int_{B(\boldsymbol{\zeta}_h^{k+1})} \mathbf{u}_h^k \circ \overline{\mathbf{X}}_h^k \cdot \boldsymbol{\varphi} d\mathbf{x} = \pi \ell_2 \cdot \ell_\varphi + \omega_2\omega_\varphi \int_{B(\boldsymbol{\zeta}_h^{k+1})} (\overline{\mathbf{X}}_h^k(\mathbf{x}) - \boldsymbol{\zeta}_h^k) \cdot (\mathbf{x} - \boldsymbol{\zeta}_h^{k+1}) d\mathbf{x}.$$

In order to evaluate the last integral, we take into account the property (5.15) of the characteristic function proved in Corollary 5.6 and we get that

$$\int_{B(\boldsymbol{\zeta}_h^{k+1})} \mathbf{u}_h^k \circ \overline{\mathbf{X}}_h^k \cdot \boldsymbol{\varphi} d\mathbf{x} = \pi \ell_2 \cdot \ell_\varphi + \frac{\pi}{2}\omega_2\omega_\varphi \cos(\omega_{\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k} \Delta t).$$

6 Proof of the first main result

Let us now prove our first main result stated in Theorem 3.2 concerning the convergence of the semi-discretization scheme (3.8)–(3.9). For this purpose, we first introduce the transformed system (6.6)–(6.7) below, and then in Sect. 6.2 we give the proof of the convergence result. Since our scheme is a generalization of the method previously introduced in [19], some properties are very similar to those of [19] and no proofs are given since only minor technical changes appear. For these properties, the reader is referred to the corresponding results in [19]. However, in this present study there are some completely new steps for which we give entire proofs (see Lemma 6.2). In the remainder of the section, C will denote any positive constant independent of k, Δt and T.

6.1 Transformed system

Let $k \in \{0, \dots, N-1\}$ be a fixed integer. We need to compare the exact solution $\mathbf{u}(t_k) \in \mathcal{K}(\boldsymbol{\zeta}(t_k))$, which is a rigid velocity field in $B(\boldsymbol{\zeta}(t_k))$ with $\mathbf{u}^k \in \mathcal{K}(\boldsymbol{\zeta}^k)$ which is a rigid velocity field in $B(\boldsymbol{\zeta}^k)$. To this end, we use the change of variable $X_{\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2}$ defined in [19, Sect. 5], which maps the ball $B(\boldsymbol{\zeta}_1)$ into the ball $B(\boldsymbol{\zeta}_2)$. In order to define this change of variables, we need to make the following assumption. Let $\eta > 0$ be the positive constant given in (2.17). We suppose that

$$\left|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}^k\right| < \eta \quad \text{and} \quad \left|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}^{k+1}\right| < \eta. \tag{6.1}$$

This hypothesis and (2.17) imply that

$$d(B(\boldsymbol{\zeta}^k), \partial\mathcal{O}) > 2\eta \quad \text{and} \quad d(B(\boldsymbol{\zeta}^{k+1}), \partial\mathcal{O}) > 2\eta. \tag{6.2}$$

Under the assumption (6.1) which implies (6.2), we are in position to define

$$\mathbf{X}^k = \mathbf{X}_{\boldsymbol{\zeta}^k,\boldsymbol{\zeta}(t_k)}, \quad \mathbf{Y}^k = \mathbf{Y}_{\boldsymbol{\zeta}^k,\boldsymbol{\zeta}(t_k)},$$

where Y_{ζ_1,ζ_2} is the inverse mapping of X_{ζ_1,ζ_2}. We also define

$$\mathbf{U}^k(\mathbf{y}) = \mathbf{J}_{\mathbf{Y}^k}(\mathbf{X}^k(\mathbf{y}))\mathbf{u}(\mathbf{X}^k(\mathbf{y}), t_k), \quad P^k(\mathbf{y}) = p(\mathbf{X}^k(\mathbf{y}), t_k), \tag{6.3}$$

where $\mathbf{J}_{\mathbf{Y}^k}$ is the determinant of the jacobian matrix of $\mathbf{Y}^k$. We recall that $\mathbf{U}^k \in \widehat{\mathcal{K}}(\boldsymbol{\zeta}^k)$ and $P^k \in M(\boldsymbol{\zeta}^k)$.

Let us also introduce the following notations that will be useful in the sequel:

$$\widehat{\mathbf{X}} = \mathbf{Y}^k \circ \widetilde{\mathbf{X}} \circ \mathbf{X}^{k+1} \tag{6.4}$$

and

$$\widehat{\mathbf{J}} = \left(\mathbf{J}_{\mathbf{Y}^{k+1}} \circ \mathbf{X}^{k+1}\right)\left(\mathbf{J}_{\mathbf{X}^k} \circ \widehat{\mathbf{X}}\right), \tag{6.5}$$

where $\widetilde{\mathbf{X}}$ is defined in (3.2). The characteristics functions satisfy the properties depicted on the following diagram:

$$\begin{array}{ccc} B(\boldsymbol{\zeta}^{k+1}) & \xrightarrow{\mathbf{X}^{k+1}} & B(\boldsymbol{\zeta}(t_{k+1})) \\ \widehat{\mathbf{X}}\big\downarrow & & \big\downarrow\widetilde{\mathbf{X}} \\ B(\boldsymbol{\zeta}^k) & \xleftarrow[\mathbf{Y}^k]{} & B(\boldsymbol{\zeta}(t_k)) \end{array}$$

We point out that the following relation holds

$$\rho^{k+1} = \rho^k \circ \widehat{\mathbf{X}}.$$

The transformed functions $\mathbf{U}^{k+1}$ and P^{k+1} satisfy a mixed weak formulation with test functions in $\mathcal{K}(\boldsymbol{\zeta}^{k+1})$ and $M(\boldsymbol{\zeta}^{k+1})$. Precisely, we have the following result which can be obtained as in [19, Proposition 6.2] with a very slight modification of the proof.

Proposition 6.1 *The functions* $(\mathbf{U}^{k+1}, P^{k+1})$ *defined by* (6.3) *satisfy*

$$\frac{1}{\Delta t}\left(\rho^{k+1}\left[\mathbf{U}^{k+1} - \widehat{\mathbf{J}}\left(\mathbf{U}^k \circ \widehat{\mathbf{X}}\right)\right], \boldsymbol{\varphi}\right) + a(\mathbf{U}^{k+1}, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, P^{k+1}) \\ = (\rho^{k+1}\mathbf{f}^{k+1}, \boldsymbol{\varphi}) + (\mathbf{A}_k, \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in \mathcal{K}(\boldsymbol{\zeta}^{k+1}), \tag{6.6}$$

$$b(\mathbf{U}^{k+1}, q) = 0 \quad \forall q \in M(\boldsymbol{\zeta}^{k+1}), \tag{6.7}$$

with

$$\|\mathbf{A}_k\|_{L^2(\mathcal{O})^2} \leq C\Big(|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}^{k+1}| + \Delta t + \sqrt{\Delta t}\Big\|\frac{d^2}{dt^2}[\mathbf{u}\circ\widetilde{\boldsymbol{\psi}}]\Big\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2}\Big). \tag{6.8}$$

Let us prove an approximation property of the function $\overline{\mathbf{X}}^k$ and we also recall an useful property on the change of variables which is given in [19].

Lemma 6.2 *The functions $\overline{\mathbf{X}}^k$, $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{J}}$ defined in (3.5), (6.4) and (6.5) respectively, satisfy the following estimates:*

$$\|\widehat{\mathbf{X}} - \overline{\mathbf{X}}^k\|_{L^2(\mathcal{O})^2} \leq C\left(\Delta t^2 + \Delta t\|\mathbf{U}^k - \mathbf{u}^k\|_{L^2(\mathcal{O})^2} + \sqrt{\Delta t}\,\|\boldsymbol{\delta}_k\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2}\right), \tag{6.9}$$

$$\Big\|\widehat{\mathbf{J}} - \mathbf{Id}\Big\|_{L^2(\mathcal{O})^2} \leq C\Big(\Delta t^2 + \Delta t\Big\|\mathbf{U}^k - \mathbf{u}^k\Big\|_{L^2(\mathcal{O})^2} + \sqrt{\Delta t}\,\|\boldsymbol{\delta}_k\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2} + \Delta t|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}^k|\Big), \tag{6.10}$$

with $\|\boldsymbol{\delta}_k\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2} \leq C\Delta t\left\|\frac{\partial\mathbf{u}}{\partial t}\right\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2}$ *and C a positive constant independent of k.*

Proof Let us define a new characteristic function $\overline{\overline{\boldsymbol{\psi}}}$ associated with the semi-discretized velocity field as the solution of

$$\begin{cases} \dfrac{d}{dt}\overline{\overline{\boldsymbol{\psi}}}(t;t_{k+1},\mathbf{x}) = \mathbf{U}^k(\overline{\overline{\boldsymbol{\psi}}}(t;t_{k+1},\mathbf{x})), \\ \overline{\overline{\boldsymbol{\psi}}}(t_{k+1};t_{k+1},\mathbf{x}) = \mathbf{x} \end{cases} \tag{6.11}$$

and let us denote

$$\overline{\overline{\mathbf{X}}}^k(\mathbf{x}) = \overline{\overline{\boldsymbol{\psi}}}(t_k;t_{k+1},\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{O}. \tag{6.12}$$

With a very slight modification of the proof of Lemma 6.5 from [19], we get

$$\|\widehat{\mathbf{X}} - \overline{\overline{\mathbf{X}}}^k\|_{L^2(\mathcal{O})^2} \leq C\left(\Delta t^2 + \sqrt{\Delta t}\,\|\boldsymbol{\delta}_k\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2}\right). \tag{6.13}$$

The characteristic Eqs. (3.4) and (6.11) can be written as follows:

$$\overline{\boldsymbol{\psi}}(t;t_{k+1},\mathbf{x}) = \mathbf{x} - \mathbf{u}^k(\boldsymbol{\zeta}^k)(t - t_k) + \int_{t_{k+1}}^{t} \mathbf{u}^k(\overline{\boldsymbol{\psi}}(s;t_{k+1},\mathbf{x}))\,\mathrm{d}s,$$

$$\overline{\overline{\boldsymbol{\psi}}}(t;t_{k+1},\mathbf{x}) = \mathbf{x} + \int_{t_{k+1}}^{t} \mathbf{U}^k(\overline{\overline{\boldsymbol{\psi}}}(s;t_{k+1},\mathbf{x}))\,\mathrm{d}s,$$

for all $t \in [t_k, t_{k+1}]$. Subtracting the previous identities, we get

$$\begin{aligned}&\overline{\boldsymbol{\psi}}(t;t_{k+1},\mathbf{x}) - \overline{\overline{\boldsymbol{\psi}}}(t;t_{k+1},\mathbf{x})\\ &\quad= -\mathbf{u}^k(\boldsymbol{\zeta}^k)(t-t_k) + \int_{t_{k+1}}^{t} \Big(\mathbf{u}^k(\overline{\boldsymbol{\psi}}(s;t_{k+1},\mathbf{x})) - \mathbf{U}^k(\overline{\overline{\boldsymbol{\psi}}}(s;t_{k+1},\mathbf{x}))\Big)\,\mathrm{d}s.\end{aligned}$$

Taking the $L^2(\mathcal{O})^2$-norm, we obtain that

$$\begin{aligned}\big\|\overline{\boldsymbol{\psi}}(t;t_{k+1},\cdot) - \overline{\overline{\boldsymbol{\psi}}}(t;t_{k+1},\cdot)\big\|_{L^2(\mathcal{O})^2} &\le C|\mathbf{u}^k(\boldsymbol{\zeta}^k)|(t-t_k)\\ &\quad+ \int_t^{t_{k+1}} \big\|\mathbf{u}^k(\overline{\boldsymbol{\psi}}(s;t_{k+1},\cdot)) - \mathbf{U}^k(\overline{\boldsymbol{\psi}}(s;t_{k+1},\cdot))\big\|_{L^2(\mathcal{O})^2}\,\mathrm{d}s\\ &\quad+ \int_t^{t_{k+1}} \big\|\mathbf{U}^k(\overline{\boldsymbol{\psi}}(s;t_{k+1},\cdot)) - \mathbf{U}^k(\overline{\overline{\boldsymbol{\psi}}}(s;t_{k+1},\cdot))\big\|_{L^2(\mathcal{O})^2}\,\mathrm{d}s.\end{aligned}$$

By using the property (5.7) and the regularity hypothesis (2.15), we get

$$\begin{aligned}&\big\|\overline{\boldsymbol{\psi}}(t;t_{k+1},\cdot) - \overline{\overline{\boldsymbol{\psi}}}(t;t_{k+1},\cdot)\big\|_{L^2(\mathcal{O})^2} \le C|\mathbf{u}^k(\boldsymbol{\zeta}^k)|(t-t_k)\\ &\quad+\big\|\mathbf{u}^k - \mathbf{U}^k\big\|_{L^2(\mathcal{O})^2}(t_{k+1}-t) + \int_t^{t_{k+1}} C\big\|\overline{\boldsymbol{\psi}}(s;t_{k+1},\cdot) - \overline{\overline{\boldsymbol{\psi}}}(s;t_{k+1},\cdot)\big\|_{L^2(\mathcal{O})^2}\,\mathrm{d}s.\end{aligned}$$

Then, due to Gronwall inequality, the above estimate yields

$$\begin{aligned}\big\|\overline{\boldsymbol{\psi}}(t;t_{k+1},\cdot) - \overline{\overline{\boldsymbol{\psi}}}(t;t_{k+1},\cdot)\big\|_{L^2(\mathcal{O})^2} &\le C|\mathbf{u}^k(\boldsymbol{\zeta}^k)|(t-t_k) + \big\|\mathbf{u}^k - \mathbf{U}^k\big\|_{L^2(\mathcal{O})^2}(t_{k+1}-t)\\ &\quad+ C\Delta t^2\Big(|\mathbf{u}^k(\boldsymbol{\zeta}^k)| + \big\|\mathbf{u}^k - \mathbf{U}^k\big\|_{L^2(\mathcal{O})^2}\Big),\end{aligned}$$

for all $t \in [t_k, t_{k+1}]$. In particular, for $t = t_k$ we deduce that

$$\begin{aligned}\big\|\overline{\mathbf{X}}^k - \overline{\overline{\mathbf{X}}}^k\big\|_{L^2(\mathcal{O})^2} &\le C\Delta t^2|\mathbf{u}^k(\boldsymbol{\zeta}^k)| + C\Delta t\big\|\mathbf{u}^k - \mathbf{U}^k\big\|_{L^2(\mathcal{O})^2}\\ &\le C\Delta t^2|\mathbf{U}^k(\boldsymbol{\zeta}^k)| + C\Delta t^2|\mathbf{u}^k(\boldsymbol{\zeta}^k) - \mathbf{U}^k(\boldsymbol{\zeta}^k)| + C\Delta t\big\|\mathbf{u}^k - \mathbf{U}^k\big\|_{L^2(\mathcal{O})^2}.\end{aligned}$$

Combining the above inequality with (6.13) and using again the regularity hypothesis (2.15), we deduce the result (6.9).

The proof of (6.10) is done in [19, Eq. (7.6)]. □

6.2 Error estimate

In this subsection, we give the proof of our first main result stated in Theorem 3.2. To this end, for any $k \in \{0, \dots, N\}$, we denote the numerical error of our scheme at step k as follows:

$$E^k = \left\| \sqrt{\rho^k} \left(\mathbf{U}^k - \mathbf{u}^k \right) \right\|_{L^2(\mathcal{O})^2} + \left| \boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}^k \right|.$$

6.2.1 First step

In this subsection, we prove that if $k \in \{0, \dots, N-1\}$ is such that the hypothesis (6.1) holds, then there exists a positive constant C_1, independent of k, Δt and T, such that

$$E^{k+1} \leq (1 + C_1 \Delta t) E^k + C_1 \Big(\Delta t^2 + (\Delta t)^{3/2} \left\| \frac{\partial \mathbf{u}}{\partial t} \right\|_{L^2(\mathcal{O} \times (t_k, t_{k+1}))^2} + (\Delta t)^{3/2} \left\| \frac{d^2}{dt^2} [\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}] \right\|_{L^2(\mathcal{O} \times (t_k, t_{k+1}))^2} \Big). \tag{6.14}$$

To this end, let us subtract (3.8)–(3.9) from (6.6)–(6.7) and we obtain

$$\begin{aligned} &\frac{1}{\Delta t} \left(\rho^{k+1} (\mathbf{U}^{k+1} - \mathbf{u}^{k+1}), \boldsymbol{\varphi} \right) + a(\mathbf{U}^{k+1} - \mathbf{u}^{k+1}, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, P^{k+1} - p^{k+1}) \\ &\quad = \frac{1}{\Delta t} \left(\rho^{k+1} \left(\widehat{\mathbf{J}} (\mathbf{U}^k \circ \widehat{\mathbf{X}}) - \mathbf{u}^k \circ \overline{\mathbf{X}}^k \right), \boldsymbol{\varphi} \right) + (\mathbf{A}_k, \boldsymbol{\varphi}) \qquad \forall \boldsymbol{\varphi} \in \mathcal{K}(\boldsymbol{\zeta}^{k+1}), \\ &\quad b(\mathbf{U}^{k+1} - \mathbf{u}^{k+1}, q) = 0 \quad \forall q \in M(\boldsymbol{\zeta}^{k+1}). \end{aligned}$$

We choose the test functions $\boldsymbol{\varphi} = \mathbf{U}^{k+1} - \mathbf{u}^{k+1} \in \mathcal{K}(\boldsymbol{\zeta}^{k+1})$ and $q = P^{k+1} - p^{k+1} \in M(\boldsymbol{\zeta}^{k+1})$ and we get that

$$\begin{aligned} &\left(\rho^{k+1} \left(\mathbf{U}^{k+1} - \mathbf{u}^{k+1} \right), \mathbf{U}^{k+1} - \mathbf{u}^{k+1} \right) + \Delta t \, a(\mathbf{U}^{k+1} - \mathbf{u}^{k+1}, \mathbf{U}^{k+1} - \mathbf{u}^{k+1}) \\ &\quad = \left(\rho^{k+1} \left(\widehat{\mathbf{J}} \left(\mathbf{U}^k \circ \widehat{\mathbf{X}} \right) - \mathbf{u}^k \circ \overline{\mathbf{X}}^k \right), \mathbf{U}^{k+1} - \mathbf{u}^{k+1} \right) + \Delta t (\mathbf{A}_k, \mathbf{U}^{k+1} - \mathbf{u}^{k+1}), \end{aligned}$$

then due to Cauchy–Schwarz inequality, there exists a positive constant C independent of k and T such that

$$\left\| \sqrt{\rho^{k+1}} (\mathbf{U}^{k+1} - \mathbf{u}^{k+1}) \right\|_{L^2(\mathcal{O})^2} \leq \left\| \sqrt{\rho^{k+1}} \left(\widehat{\mathbf{J}} \left(\mathbf{U}^k \circ \widehat{\mathbf{X}} \right) - \mathbf{u}^k \circ \overline{\mathbf{X}}^k \right) \right\|_{L^2(\mathcal{O})^2} + C \Delta t \left\| \mathbf{A}_k \right\|_{L^2(\mathcal{O})^2}. \tag{6.15}$$

In order to estimate the first term in the right hand side of (6.15), we observe that

$$\begin{aligned}\sqrt{\rho^{k+1}}\Big(\widehat{\mathbf{J}}\left(\mathbf{U}^k\circ\widehat{\mathbf{X}}\right)-\mathbf{u}^k\circ\overline{\mathbf{X}}^k\Big) &= \sqrt{\rho^{k+1}}(\widehat{\mathbf{J}}-\mathbf{Id})\big(\mathbf{U}^k\circ\widehat{\mathbf{X}}\big)\\ &+\sqrt{\rho^{k+1}}\Big(\mathbf{U}^k\circ\widehat{\mathbf{X}}-\mathbf{U}^k\circ\overline{\mathbf{X}}^k\Big)\\ &+\sqrt{\rho^{k+1}}\big(\mathbf{U}^k-\mathbf{u}^k\big)\circ\overline{\mathbf{X}}^k,\end{aligned}$$

then using the regularity hypothesis (2.15) and the definition of $\mathbf{U}^k$ given in (6.3), we easily deduce

$$\begin{aligned}\Big\|\sqrt{\rho^{k+1}}\Big(\widehat{\mathbf{J}}\left(\mathbf{U}^k\circ\widehat{\mathbf{X}}\right)-\mathbf{u}^k\circ\overline{\mathbf{X}}^k\Big)\Big\|_{L^2(\mathcal{O})^2} &\le C\big\|\widehat{\mathbf{J}}-\mathbf{Id}\big\|_{L^2(\mathcal{O})^2}\\ &+C\big\|\widehat{\mathbf{X}}-\overline{\mathbf{X}}^k\big\|_{L^2(\mathcal{O})^2}+\Big\|\sqrt{\rho^{k+1}}\big(\mathbf{U}^k-\mathbf{u}^k\big)\circ\overline{\mathbf{X}}^k\Big\|_{L^2(\mathcal{O})^2}.\end{aligned}$$

Then, by using the inequality (5.7) from Proposition 5.2, we observe that

$$\begin{aligned}\Big\|\sqrt{\rho^{k+1}}\Big(\widehat{\mathbf{J}}\left(\mathbf{U}^k\circ\widehat{\mathbf{X}}\right)-\mathbf{u}^k\circ\overline{\mathbf{X}}^k\Big)\Big\|_{L^2(\mathcal{O})^2} &\le C\big\|\widehat{\mathbf{J}}-\mathbf{Id}\big\|_{L^2(\mathcal{O})^2}\\ &+C\big\|\widehat{\mathbf{X}}-\overline{\mathbf{X}}^k\big\|_{L^2(\mathcal{O})^2}+\Big\|\sqrt{\rho^{k}}\big(\mathbf{U}^k-\mathbf{u}^k\big)\Big\|_{L^2(\mathcal{O})^2}.\end{aligned}$$

Due to the above estimate, the inequality (6.15) becomes

$$\begin{aligned}\Big\|\sqrt{\rho^{k+1}}(\mathbf{U}^{k+1}-\mathbf{u}^{k+1})\Big\|_{L^2(\mathcal{O})^2} &\le \Big\|\sqrt{\rho^{k}}\big(\mathbf{U}^k-\mathbf{u}^k\big)\Big\|_{L^2(\mathcal{O})^2}+C\big\|\widehat{\mathbf{J}}-\mathbf{Id}\big\|_{L^2(\mathcal{O})^2}\\ &+C\big\|\widehat{\mathbf{X}}-\overline{\mathbf{X}}^k\big\|_{L^2(\mathcal{O})^2}+C\Delta t\big\|\mathbf{A}_k\big\|_{L^2(\mathcal{O})^2},\end{aligned}\tag{6.16}$$

then, taking into account the estimates (6.8) from Proposition 6.1, (6.9)–(6.10) from Lemma 6.2, we obtain

$$\begin{aligned}&\Big\|\sqrt{\rho^{k+1}}(\mathbf{U}^{k+1}-\mathbf{u}^{k+1})\Big\|_{L^2(\mathcal{O})^2} \le \Big\|\sqrt{\rho^{k}}\big(\mathbf{U}^k-\mathbf{u}^k\big)\Big\|_{L^2(\mathcal{O})^2}\\ &+C\Big(\Delta t\big\|\mathbf{U}^k-\mathbf{u}^k\big\|_{L^2(\mathcal{O})^2}+\Delta t\big|\boldsymbol{\zeta}(t_k)-\boldsymbol{\zeta}^k\big|+\Delta t\big|\boldsymbol{\zeta}(t_{k+1})-\boldsymbol{\zeta}^{k+1}\big|\\ &+\Delta t^2+\sqrt{\Delta t}\,\|\boldsymbol{\delta}_k\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2}+(\Delta t)^{3/2}\Big\|\frac{d^2}{dt^2}[\mathbf{u}\circ\widetilde{\boldsymbol{\psi}}]\Big\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2}\Big).\end{aligned}\tag{6.17}$$

Since

$$\|\boldsymbol{\delta}_k\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2}\le C\Delta t\Big\|\frac{\partial\mathbf{u}}{\partial t}\Big\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2},$$

we get that

$$\left\|\sqrt{\rho^{k+1}}(\mathbf{U}^{k+1}-\mathbf{u}^{k+1})\right\|_{L^2(\mathcal{O})^2} \leq \left\|\sqrt{\rho^k}\left(\mathbf{U}^k-\mathbf{u}^k\right)\right\|_{L^2(\mathcal{O})^2}$$
$$+C\Big(\Delta t\left\|\mathbf{U}^k-\mathbf{u}^k\right\|_{L^2(\mathcal{O})^2}+\Delta t\left|\boldsymbol{\zeta}(t_k)-\boldsymbol{\zeta}^k\right|+\Delta t\left|\boldsymbol{\zeta}(t_{k+1})-\boldsymbol{\zeta}^{k+1}\right|+\Delta t^2$$
$$+(\Delta t)^{3/2}\left\|\frac{\partial \mathbf{u}}{\partial t}\right\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2}+(\Delta t)^{3/2}\left\|\frac{d^2}{dt^2}[\mathbf{u}\circ\widetilde{\boldsymbol{\psi}}]\right\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2}\Big). \tag{6.18}$$

Let us now use identity (3.3) and we get

$$\boldsymbol{\zeta}(t_{k+1})-\boldsymbol{\zeta}^{k+1}=\boldsymbol{\zeta}(t_k)-\boldsymbol{\zeta}^k-\mathbf{u}^k(\boldsymbol{\zeta}^k)\Delta t+\int_{t_k}^{t_{k+1}}\mathbf{u}(\boldsymbol{\zeta}(s),s)ds,$$

then

$$|\boldsymbol{\zeta}(t_{k+1})-\boldsymbol{\zeta}^{k+1}|\leq|\boldsymbol{\zeta}(t_k)-\boldsymbol{\zeta}^k|+\Delta t|\mathbf{u}(\boldsymbol{\zeta}(t_k),t_k)-\mathbf{u}^k(\boldsymbol{\zeta}^k)|+\int_{t_k}^{t_{k+1}}|\boldsymbol{\zeta}'(s)-\boldsymbol{\zeta}'(t_k)|ds.$$

Since the equality $\mathbf{u}(\boldsymbol{\zeta}(t_k),t_k)=\mathbf{U}^k(\boldsymbol{\zeta}^k)$ holds and since the function $\mathbf{U}^k-\mathbf{u}^k$ is a rigid velocity in $B(\boldsymbol{\zeta}^k)$, we deduce that

$$|\boldsymbol{\zeta}(t_{k+1})-\boldsymbol{\zeta}^{k+1}|\leq|\boldsymbol{\zeta}(t_k)-\boldsymbol{\zeta}^k|+\frac{\Delta t}{\sqrt{\pi}}\left\|\mathbf{U}^k-\mathbf{u}^k\right\|_{L^2(\mathcal{O})^2}+\frac{\Delta t^2}{2}\left\|\boldsymbol{\zeta}''\right\|_{L^\infty(0,T)^2}. \tag{6.19}$$

Combining the estimates (6.18) and (6.19), it follows that there exists $C_1>0$ independent of k, Δt and T such that the estimate (6.14) holds. Thus, we conclude the first step of the proof.

Additionally, using the same constant C_1, estimate (6.19) can be rewritten as follows:

$$|\boldsymbol{\zeta}(t_{k+1})-\boldsymbol{\zeta}^{k+1}|\leq|\boldsymbol{\zeta}(t_k)-\boldsymbol{\zeta}^k|+\Delta t\left\|\mathbf{U}^k-\mathbf{u}^k\right\|_{L^2(\mathcal{O})^2}+C_1\Delta t^2. \tag{6.20}$$

6.2.2 Second step

In this subsection, we are going to prove that there exists $\tau^*>0$ such that if $\Delta t\in(0,\tau^*]$, then the following estimate holds

$$\left|\boldsymbol{\zeta}(t_k)-\boldsymbol{\zeta}^k\right|<\eta \qquad \forall k\in\{0,\ldots,N\}. \tag{6.21}$$

To this end, let us define the constant

$$C_2 = \exp(C_1 T)\Big(1 + \Big\|\frac{\partial \mathbf{u}}{\partial t}\Big\|_{L^2(\mathcal{O}\times(0,T))^2} + \Big\|\frac{d^2}{dt^2}[\mathbf{u}\circ\widetilde{\boldsymbol{\psi}}]\Big\|_{L^2(\mathcal{O}\times(0,T))^2}\Big), \tag{6.22}$$

where C_1 is the positive constant independent of k, Δt and T, which appears in (6.14). We define

$$\tau^* \overset{(\text{def})}{=} \min\left\{\frac{\eta}{4C_2}, \sqrt{\frac{\eta\min\{\sqrt{\rho_f}, \sqrt{\rho_s}\}}{4C_2}}, \sqrt{\frac{\eta}{4C_1}}\right\} \tag{6.23}$$

and we choose $0 < \Delta t \leq \tau^*$.

In order to prove (6.21), we proceed by induction. Let us consider the statement

$$\mathscr{P}(j) : \left|\boldsymbol{\zeta}(t_j) - \boldsymbol{\zeta}^j\right| < \eta. \tag{6.24}$$

- First, we remark that $\mathscr{P}(0)$ is true due to the initial conditions (1.8) and (3.1).
- Secondly, taking $k = 0$ in (6.20), we get that

$$|\boldsymbol{\zeta}(t_1) - \boldsymbol{\zeta}^1| \leq C_1 \Delta t^2.$$

  Since $\Delta t \leq \tau^* < \sqrt{\frac{\eta}{C_1}}$, we deduce that

$$|\boldsymbol{\zeta}(t_1) - \boldsymbol{\zeta}^1| < \eta, \tag{6.25}$$

  that is, $\mathscr{P}(1)$ is also true.
- Finally, we prove that if $j \geq 1$ is such that the statements $\mathscr{P}(0), \mathscr{P}(1), \ldots, \mathscr{P}(j)$ hold true, then $\mathscr{P}(j+1)$ is also true. To this end, we first remark that the induction hypothesis implies that condition (6.1) holds for any $k \in \{0, \ldots, j-1\}$. Then, using the first step of the proof we deduce that estimate (6.14) holds for any $k \in \{0, \ldots, j-1\}$. Hence, by applying the discrete Gronwall Lemma, and using that error E^0 is equal to zero (see initial conditions (3.1)), we deduce that for any $k \in \{0, \ldots, j\}$,

$$E^k \leq \exp(C_1 T)\Big(\Delta t + \Delta t\Big\|\frac{\partial \mathbf{u}}{\partial t}\Big\|_{L^2(\mathcal{O}\times(0,T))^2} + \Delta t\Big\|\frac{d^2}{dt^2}[\mathbf{u}\circ\widetilde{\boldsymbol{\psi}}]\Big\|_{L^2(\mathcal{O}\times(0,T))^2}\Big).$$

  Then using the hypothesis (2.15), we get that for any $k \in \{0, \ldots, j\}$, we have

$$\left\|\sqrt{\rho^k}(\mathbf{U}^k - \mathbf{u}^k)\right\|_{L^2(\mathcal{O})^2} + \left|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}^k\right| \leq C_2 \Delta t, \tag{6.26}$$

  where C_2 is defined in (6.22).

Combining (6.26) with (6.20), we deduce that for any $k \in \{0, \ldots, j\}$,

$$|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}^{k+1}| \leq C_2 \Delta t + \frac{C_2 \Delta t^2}{\min\{\sqrt{\rho_f}, \sqrt{\rho_s}\}} + C_1 \Delta t^2.$$

Since $0 < \Delta t \leq \tau^* < \min\left\{\frac{\eta}{3C_2}, \sqrt{\frac{\eta \min\{\sqrt{\rho_f}, \sqrt{\rho_s}\}}{3C_2}}, \sqrt{\frac{\eta}{3C_1}}\right\}$, we deduce

$$|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}^{k+1}| < \eta \qquad \forall k \in \{0, \ldots, j\},$$

which in particular implies that $\mathscr{P}(j+1)$ is true. Thus, the induction process is finished and the proof of (6.21) is completed.

6.2.3 Third step

From the previous steps, we deduce that if $0 < \Delta t \leq \tau^*$, the following estimate holds:

$$\left\|\mathbf{U}^k - \mathbf{u}^k\right\|_{L^2(\mathcal{O})^2} + \left|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}^k\right| \leq \frac{C_2 \Delta t}{\min\{\sqrt{\rho_f}, \sqrt{\rho_s}\}} \qquad \forall k \in \{0, \ldots, N\}. \tag{6.27}$$

We conclude the proof of the theorem as follows. For all $k \in \{0, \ldots, N\}$, we have

$$\begin{aligned} &\left\|\mathbf{u}(t_k) - \mathbf{u}^k\right\|_{L^2(\mathcal{O})^2} + \left|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}^k\right| \\ &\quad \leq \left\|\mathbf{u}(t_k) - \mathbf{U}^k\right\|_{L^2(\mathcal{O})^2} + \left\|\mathbf{U}^k - \mathbf{u}^k\right\|_{L^2(\mathcal{O})^2} + \left|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}^k\right| \end{aligned} \tag{6.28}$$

Using the definition of $\mathbf{U}^k$ from (6.3) and the properties on the change of variables $\mathbf{X}^k$ given in [19, Lemmas 5.5–5.6], which are true due to (6.21), we obtain that there exists a positive constant C independent on T and Δt such that

$$\left\|\mathbf{u}(t_k) - \mathbf{U}^k\right\|_{L^2(\mathcal{O})^2} \leq C\left|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}^k\right|. \tag{6.29}$$

Combining (6.28) together with (6.27) and (6.29), we conclude that

$$\left\|\mathbf{u}(t_k) - \mathbf{u}^k\right\|_{L^2(\mathcal{O})^2} + \left|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}^k\right| \leq (C+1)\frac{C_2 \Delta t}{\min\{\sqrt{\rho_f}, \sqrt{\rho_s}\}}.$$

Using of the expression (6.22) for the constant C_2, we introduce the positive constant K defined by

$$K = \frac{C+1}{\min\{\sqrt{\rho_f}, \sqrt{\rho_s}\}} \exp(C_1 T)\left(1 + \left\|\frac{\partial \mathbf{u}}{\partial t}\right\|_{L^2(\mathcal{O}\times(0,T))^2} + \left\|\frac{d^2}{dt^2}[\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}]\right\|_{L^2(\mathcal{O}\times(0,T))^2}\right)$$

and the proof of Theorem 3.2 is completed. □

7 Proof of the second main result

We turn to the proof of the second main result stated in Theorem 4.4 which concerns the convergence of the full discretization scheme (4.9)–(4.10). To this end, we first introduce the transformed system (7.6)–(7.7) below and we prove some important estimates on the transform velocity field. Then we prove the second main result of this paper. Since our fully discrete scheme is a generalization of the method previously introduced in [19], some properties are similar to the ones proved in [19] (with just minor technical changes). For this reason, some properties below refer the reader to the corresponding results in [19]. However, there are several steps that are completely new for our algorithm and therefore we give their proofs (see Lemmas 7.2, 7.4, 7.5 and Proposition 7.3). In the remainder of the section, for the sake of simplicity, C will denote any positive constant independent of h, k, Δt and T.

7.1 Preliminaries

Let us fix $k \in \{0, \dots, N-1\}$. Similarly to the proof of the first result, we need to compare the exact solution $\mathbf{u}(t_k) \in \mathcal{K}(\boldsymbol{\zeta}(t_k))$, which is a rigid velocity field in $B(\boldsymbol{\zeta}(t_k))$ with $\mathbf{u}_h^k \in \mathcal{K}(\boldsymbol{\zeta}_h^k)$ which is a rigid velocity field in $B(\boldsymbol{\zeta}_h^k)$. To this end, we will make use the change of variable X_{ζ_1,ζ_2} defined in [19, Sect. 5], which maps the ball $B(\boldsymbol{\zeta}_1)$ into the ball $B(\boldsymbol{\zeta}_2)$.

Let $\eta > 0$ be the positive constant given in (2.17). We suppose that

$$\left|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k\right| < \eta \quad \text{and} \quad \left|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}\right| < \eta. \tag{7.1}$$

This hypothesis and (2.17) imply that

$$d(B(\boldsymbol{\zeta}_h^k), \partial\mathcal{O}) > 2\eta \quad \text{and} \quad d(B(\boldsymbol{\zeta}_h^{k+1}), \partial\mathcal{O}) > 2\eta. \tag{7.2}$$

Then, under the assumption (7.1) which implies (7.2), we are in position to define

$$\mathbf{X}_h^k = \mathbf{X}_{\boldsymbol{\zeta}_h^k, \boldsymbol{\zeta}(t_k)}, \quad \mathbf{Y}_h^k = \mathbf{Y}_{\boldsymbol{\zeta}_h^k, \boldsymbol{\zeta}(t_k)},$$

where Y_{ζ_1,ζ_2} is the inverse mapping of X_{ζ_1,ζ_2}. We point out that $\mathbf{X}_h^k$ maps the ball $B(\boldsymbol{\zeta}_h^k)$ into the ball $B(\boldsymbol{\zeta}(t_k))$. We also define

$$\mathbf{U}_h^k(\mathbf{y}) = \mathbf{J}_{\mathbf{Y}_h^k}(\mathbf{X}_h^k(\mathbf{y}))\mathbf{u}(\mathbf{X}_h^k(\mathbf{y}), t_k), \quad P_h^k(\mathbf{y}) = p(\mathbf{X}_h^k(\mathbf{y}), t_k), \tag{7.3}$$

where $\mathbf{J}_{\mathbf{Y}_h^k}$ is the determinant of the jacobian matrix of $\mathbf{Y}_h^k$. We recall that $\mathbf{U}_h^k \in \widehat{\mathcal{K}}(\boldsymbol{\zeta}_h^k)$ and $P_h^k \in M(\boldsymbol{\zeta}_h^k)$.

Let us introduce the following notations that will be useful in the sequel:

$$\widehat{\mathbf{X}}_h = \mathbf{Y}_h^k \circ \widetilde{\mathbf{X}} \circ \mathbf{X}_h^{k+1} \tag{7.4}$$

and

$$\widehat{\mathbf{J}}_h = \left(\mathbf{J}_{\mathbf{Y}_h^{k+1}} \circ \mathbf{X}_h^{k+1}\right) \left(\mathbf{J}_{\mathbf{X}_h^k} \circ \widehat{\mathbf{X}}_h\right). \tag{7.5}$$

We observe that the characteristics functions satisfy the properties depicted on the following diagram:

$$\begin{array}{ccc} B(\boldsymbol{\zeta}_h^{k+1}) & \xrightarrow{\mathbf{X}_h^{k+1}} & B(\boldsymbol{\zeta}(t_{k+1})) \\ \widehat{\mathbf{X}}_h \big\downarrow & & \big\downarrow \widetilde{\mathbf{X}} \\ B(\boldsymbol{\zeta}_h^k) & \xleftarrow[\mathbf{Y}_h^k]{} & B(\boldsymbol{\zeta}(t_k)) \end{array}$$

The transformed functions $\mathbf{U}_h^{k+1}$ and P_h^{k+1} satisfy a mixed weak formulation with test functions in $\mathcal{K}(\boldsymbol{\zeta}_h^{k+1})$ and $M(\boldsymbol{\zeta}_h^{k+1})$.

Proposition 7.1 *The functions* $(\mathbf{U}_h^{k+1}, P_h^{k+1})$ *defined by* (7.3) *satisfy*

$$\frac{1}{\Delta t}\left(\rho_h^{k+1}\left[\mathbf{U}_h^{k+1} - \widehat{\mathbf{J}}_h\left(\mathbf{U}_h^k \circ \widehat{\mathbf{X}}_h\right)\right], \boldsymbol{\varphi}\right) + a(\mathbf{U}_h^{k+1}, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, P_h^{k+1}) = (\rho_h^{k+1}\mathbf{f}_h^{k+1}, \boldsymbol{\varphi}) + (\mathbf{A}_h^k, \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in \mathcal{K}(\boldsymbol{\zeta}_h^{k+1}), \tag{7.6}$$

$$b(\mathbf{U}_h^{k+1}, q) = 0 \quad \forall q \in M(\boldsymbol{\zeta}_h^{k+1}), \tag{7.7}$$

with

$$\|\mathbf{A}_h^k\|_{L^2(\mathcal{O})^2} \leq C\left(|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}| + h + \Delta t + \sqrt{\Delta t}\left\|\frac{d^2}{dt^2}[\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}]\right\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2}\right). \tag{7.8}$$

The above result can be obtained as in [19, Proposition 6.2] with a very slight modification of the proof.

In the following lemma, we state an important result on the transformed velocity field. Precisely, we prove the existence of a velocity field $\mathbf{U}_{h,ext}^k$ near $\mathbf{U}_h^k$ which is rigid in a h-neighbourhood of the ball $B(\boldsymbol{\zeta}_h^k)$. Moreover, this function is a rigid velocity field in Q_h.

Lemma 7.2 *For any* $k \in \{0, \ldots, N\}$ *and* $h \in (0, 1)$*, there exists a velocity field* $\mathbf{U}_{h,ext}^k \in H_0^1(\mathcal{O})^2$ *such that*

$$\mathbf{U}_{h,ext}^k(\mathbf{x}) = \mathbf{U}_h^k(\mathbf{x}) \qquad \forall \mathbf{x} \in B(\boldsymbol{\zeta}_h^k), \tag{7.9}$$

$$\mathrm{D}(\mathbf{U}_{h,ext}^k) = 0 \text{ in } \left\{\mathbf{x} \in \mathcal{O} : |\mathbf{x} - \boldsymbol{\zeta}_h^k| < 1 + h\right\}, \tag{7.10}$$

$$\|\mathbf{U}_h^k - \mathbf{U}_{h,ext}^k\|_{L^2(\mathcal{O})^2} \leq Ch^{3/2}, \tag{7.11}$$

$$\|\mathbf{U}^k_{h,ext}\|_{H^1(\mathcal{O})^2} \leq C, \tag{7.12}$$

where C is a positive constant independent of h and k.

Proof Since $\mathbf{U}^k_h \in H^1_0(\mathcal{O})^2$ and $\operatorname{div} \mathbf{U}^k_h = 0$, there exists a stream function $\Phi \in H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})$ such that $\mathbf{U}^k_h = \nabla^\perp \Phi$.

It clearly suffices to prove that there exists a stream function $\Phi_{ext} \in H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})$, such that $\nabla^\perp \Phi_{ext}$ satisfies the conditions (7.9)–(7.12).

To this end, let us observe that since $\mathrm{D}(\mathbf{U}^k_h) = 0$ in $B(\boldsymbol{\zeta}^k_h)$, there exist some constants $a, c \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^2$ such that

$$\Phi(\mathbf{x}) = a + \mathbf{b} \cdot \mathbf{x} + c|\mathbf{x}|^2 \qquad \forall \mathbf{x} \in B(\boldsymbol{\zeta}^k_h).$$

We denote

$$w(\mathbf{x}) = \Phi(\mathbf{x}) - \left(a + \mathbf{b} \cdot \mathbf{x} + c|\mathbf{x}|^2\right) \qquad \forall \mathbf{x} \in \mathcal{O}, \tag{7.13}$$

then it is clear that

$$w(\mathbf{x}) = 0 \qquad \forall \mathbf{x} \in B(\boldsymbol{\zeta}^k_h). \tag{7.14}$$

Let us define the stream function Φ_{ext} as follows:

$$\Phi_{ext}(\mathbf{x}) = \Phi(\mathbf{x}) - w(\mathbf{x})\rho(|\mathbf{x} - \boldsymbol{\zeta}^k_h|) \qquad \forall \mathbf{x} \in \mathcal{O}, \tag{7.15}$$

where the real function $\rho \in H^2(\mathbb{R})$ is given by the following formula

$$\rho(s) = \begin{cases} 1 & \text{if } s < 1 + h, \\ \frac{1}{2}\left(\cos\left(\frac{s-(1+h)}{h}\pi\right) + 1\right) & \text{if } 1 + h \leq s \leq 1 + 2h, \\ 0 & \text{if } 1 + 2h < s. \end{cases}$$

Using this definition, one can easily check that $\Phi_{ext}(\mathbf{x}) = \Phi(\mathbf{x})$ for all $\mathbf{x}$ such that $|\mathbf{x} - \boldsymbol{\zeta}^k_h| \leq 1$ or $|\mathbf{x} - \boldsymbol{\zeta}^k_h| \geq 1 + 2h$. Then $\nabla^\perp \Phi_{ext} \in H^1_0(\mathcal{O})^2$ and satisfies the identity (7.9). Additionally, if $|\mathbf{x} - \boldsymbol{\zeta}^k_h| < 1 + h$, we have $\Phi_{ext}(\mathbf{x}) = a + \mathbf{b} \cdot \mathbf{x} + c|\mathbf{x}|^2$ and this identity implies that $\nabla^\perp \Phi_{ext}$ satisfies (7.10).

In order to prove that $\nabla^\perp \Phi_{ext}$ satisfies the estimates (7.11) and (7.12), we first remark that

$$(\Phi - \Phi_{ext})(\mathbf{x}) = 0 \qquad \forall \mathbf{x} \in \mathcal{O} \backslash \mathcal{A}_{1,1+2h}, \tag{7.16}$$

where we have denoted by $\mathcal{A}_{1,1+2h}$ the annulus enclosed between the circles of radius 1, respectively $1 + 2h$ and with center at $\boldsymbol{\zeta}^k_h$.

Let us now differentiate the identity (7.15) and for any $i, j \in \{1, 2\}$, we get that

$$\frac{\partial(\Phi - \Phi_{ext})}{\partial x_i} = \frac{\partial w}{\partial x_i}\rho + w\rho' \frac{x_i - \zeta^k_{h,i}}{|\mathbf{x} - \boldsymbol{\zeta}^k_h|} \tag{7.17}$$

and

$$\frac{\partial^2(\Phi - \Phi_{ext})}{\partial x_i \partial x_j} = \frac{\partial^2 w}{\partial x_i \partial x_j}\rho + \frac{\partial w}{\partial x_i}\rho' \frac{x_j - \zeta^k_{h,j}}{|\mathbf{x} - \boldsymbol{\zeta}^k_h|} + \frac{\partial w}{\partial x_j}\rho' \frac{x_i - \zeta^k_{h,i}}{|\mathbf{x} - \boldsymbol{\zeta}^k_h|}$$
$$+ w\rho'' \frac{(x_i - \zeta^k_{h,i})(x_j - \zeta^k_{h,j})}{|\mathbf{x} - \boldsymbol{\zeta}^k_h|^2} + w\rho' \left(\frac{\delta_{ij}}{|\mathbf{x} - \boldsymbol{\zeta}^k_h|} - \frac{(x_i - \zeta^k_{h,i})(x_j - \zeta^k_{h,j})}{|\mathbf{x} - \boldsymbol{\zeta}^k_h|^3} \right). \tag{7.18}$$

Taking the $L^2(\mathcal{O})$-norm in the estimates (7.17)–(7.18), using the identity (7.16) and the properties $|\rho| \leq 1$, $|\rho'| \leq 2/h$ and $|\rho''| \leq 5/h^2$, we obtain that for all $i, j \in \{1, 2\}$,

$$\left\| \frac{\partial(\Phi - \Phi_{ext})}{\partial x_i} \right\|_{L^2(\mathcal{O})} \leq \left\| \frac{\partial w}{\partial x_i} \right\|_{L^2(\mathcal{A}_{1,1+2h})} + \frac{2}{h} \|w\|_{L^2(\mathcal{A}_{1,1+2h})} \tag{7.19}$$

and

$$\left\| \frac{\partial^2(\Phi - \Phi_{ext})}{\partial x_i \partial x_j} \right\|_{L^2(\mathcal{O})} \leq \left\| \frac{\partial^2 w}{\partial x_i \partial x_j} \right\|_{L^2(\mathcal{A}_{1,1+2h})} + \frac{2}{h} \left\| \frac{\partial w}{\partial x_i} \right\|_{L^2(\mathcal{A}_{1,1+2h})}$$
$$+ \frac{2}{h} \left\| \frac{\partial w}{\partial x_j} \right\|_{L^2(\mathcal{A}_{1,1+2h})} + \frac{5}{h^2} \|w\|_{L^2(\mathcal{A}_{1,1+2h})} + \frac{4}{h} \|w\|_{L^2(\mathcal{A}_{1,1+2h})}. \tag{7.20}$$

In order to finish the proof, we need to estimate the different norms of w on the annulus $\mathcal{A}_{1,1+2h}$. To this end, let us take an arbitrary $\mathbf{x} \in \mathcal{A}_{1,1+2h}$. It is easy to see that there exists $\mathbf{y} \in B(\boldsymbol{\zeta}^k_h)$ such that $|\mathbf{x} - \mathbf{y}| \leq 2h$. Since $\nabla^{\perp}\Phi$ is a Lipschitz function (see the hypothesis (2.15) and the definition (7.3)), we get that ∇w is also a Lipschitz function, with the Lipschitz constant L independent of h. Using this property and (7.14), we have

$$|\nabla w(\mathbf{x})| = |\nabla w(\mathbf{x}) - \nabla w(\mathbf{y})| \leq L|\mathbf{x} - \mathbf{y}| \leq 2Lh \tag{7.21}$$

and

$$|w(\mathbf{x})| = |w(\mathbf{x}) - w(\mathbf{y})| \leq |\nabla w(\mathbf{y} + \lambda(\mathbf{x} - \mathbf{y}))| \cdot |\mathbf{x} - \mathbf{y}|$$
$$= \left| \nabla w(\mathbf{y} + \lambda(\mathbf{x} - \mathbf{y})) - \nabla w(\mathbf{y}) \right| \cdot |\mathbf{x} - \mathbf{y}| \leq L\lambda|\mathbf{x} - \mathbf{y}|^2 \leq 4\lambda L h^2, \tag{7.22}$$

for some $\lambda \in (0, 1)$. Taking the $L^2(\mathcal{A}_{1,1+2h})$-norm in the estimates (7.21) and (7.22), we deduce

$$\|w\|_{L^2(\mathcal{A}_{1,1+2h})} \leq Ch^{5/2}, \tag{7.23}$$

$$\|\nabla w\|_{L^2(\mathcal{A}_{1,1+2h})^2} \leq Ch^{3/2}. \tag{7.24}$$

Combining (7.19) with (7.23)–(7.24), we obtain the inequality (7.11). Moreover, the estimate (7.12) is a direct consequence of (7.20), (7.23)–(7.24) and the fact that $\|w\|_{H^2(\mathcal{O})}$ is independent of h (see the definition of w given in (7.13)). Thus, we conclude the proof of Lemma 7.2. □

Using the above lemma, let us prove the following crucial estimate:

Proposition 7.3 *For any $k \in \{0, 1, \ldots, N\}$ and $h \in (0, 1)$, the following estimate holds:*

$$\left\| \mathbf{U}_h^k - \mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{U}_h^k \right\|_{L^2(\mathcal{O})^2} \leq Ch, \tag{7.25}$$

where C is a positive constant independent of h and k.

Proof Using Lemma 7.2, there exists $\mathbf{U}_{h,ext}^k \in H_0^1(\mathcal{O})^2$ satisfying (7.9)–(7.12). We can write

$$\begin{aligned}\left\| \mathbf{U}_h^k - \mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{U}_h^k \right\|_{L^2(\mathcal{O})^2} \leq \left\| \mathbf{U}_h^k - \mathbf{U}_{h,ext}^k \right\|_{L^2(\mathcal{O})^2} + \left\| \mathbf{U}_{h,ext}^k - \mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{U}_{h,ext}^k \right\|_{L^2(\mathcal{O})^2} \\ + \left\| \mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{U}_{h,ext}^k - \mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{U}_h^k \right\|_{L^2(\mathcal{O})^2},\end{aligned}$$

then, since $\mathbf{P}(\boldsymbol{\zeta}_h^k)$ is an orthogonal projection from $L^2(\mathcal{O})^2$ onto $\mathcal{R}_h(\boldsymbol{\zeta}_h^k)$, we get

$$\left\| \mathbf{U}_h^k - \mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{U}_h^k \right\|_{L^2(\mathcal{O})^2} \leq 2\left\| \mathbf{U}_h^k - \mathbf{U}_{h,ext}^k \right\|_{L^2(\mathcal{O})^2} + \left\| \mathbf{U}_{h,ext}^k - \mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{U}_{h,ext}^k \right\|_{L^2(\mathcal{O})^2}. \tag{7.26}$$

Let Φ_{ext} be the stream function corresponding to $\mathbf{U}_{h,ext}^k$ and Φ_I be the $\mathbb{P}_2$-Lagrange interpolated function of Φ_{ext} on the triangulation $\mathcal{T}_h$. Since $\mathbf{U}_{h,ext}^k$ is a rigid velocity field on Q_h, the function Φ_{ext} is quadratic on Q_h and thus

$$\Phi_I(\mathbf{x}) = \Phi_{ext}(\mathbf{x}) \qquad \forall \mathbf{x} \in Q_h$$

and $\nabla^\perp \Phi_I$ is a rigid velocity field in Q_h. This implies that $\nabla^\perp \Phi_I \in \mathcal{R}_h(\boldsymbol{\zeta}_h^k)$.

By using the definition of the orthogonal projection and due to the classical estimates of the interpolated functions (see, for instance, [6, Lemma A.2, p. 99]), we deduce that

$$\begin{aligned}\left\|\mathbf{U}_{h,ext}^{k}-\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{U}_{h,ext}^{k}\right\|_{L^2(\mathcal{O})^2} &\leq \left\|\mathbf{U}_{h,ext}^{k}-\nabla^{\perp}\Phi_I\right\|_{L^2(\mathcal{O})^2}\\ &= \left\|\nabla^{\perp}\Phi_{ext}-\nabla^{\perp}\Phi_I\right\|_{L^2(\mathcal{O})^2}\\ &\leq Ch\left\|\Phi_{ext}\right\|_{H^2(\mathcal{O})^2} \leq Ch\left\|\mathbf{U}_{h,ext}^{k}\right\|_{H^1(\mathcal{O})^2}. \qquad (7.27)\end{aligned}$$

Let us now conclude the proof of our result by noting that the estimate (7.25) is a direct consequence of the inequality (7.26) combined with (7.27) and the estimates (7.11)–(7.12) from Lemma 7.2. □

Let us now state an important estimate on the $L^2(\mathcal{O})$-norm of the difference between the density functions ρ_h^k and $\overline{\rho}_h^k$ defined in (4.7) and (4.8), respectively.

Lemma 7.4 *There exists a positive constant C, independent of h and k, such that*

$$\|\rho_h^k-\overline{\rho}_h^k\|_{L^2(\mathcal{O})} \leq C\sqrt{h}. \qquad (7.28)$$

Proof Using the definitions of ρ_h^k and $\overline{\rho}_h^k$ given in (4.7) and (4.8) respectively, we have that

$$\rho_h^k-\overline{\rho}_h^k=0 \quad \text{in } \mathcal{F}_4\cup B(\boldsymbol{\zeta}_h^k).$$

Taking the $L^2(\mathcal{O})$-norm, we deduce that

$$\|\rho_h^k-\overline{\rho}_h^k\|_{L^2(\mathcal{O})}^2 = \int_{\mathcal{F}_2\cup\mathcal{F}_3}\left|\rho_h^k(\mathbf{x})-\overline{\rho}_h^k(\mathbf{x})\right|^2 d\mathbf{x} \leq \left|\rho_f-\rho_s\right|^2 \operatorname{mes}(\mathcal{F}_2\cup\mathcal{F}_3). \qquad (7.29)$$

The region $\mathcal{F}_2\cup\mathcal{F}_3$ is contained into the annulus of center $\boldsymbol{\zeta}_h^k$ with radius $r_1=1-h$ and $r_2=1+2h$ (see Fig. 1). Then, the area of the region $\mathcal{F}_2\cup\mathcal{F}_3$ can be estimated as follows:

$$\operatorname{mes}(\mathcal{F}_2\cup\mathcal{F}_3) \leq \pi\left((1+2h)^2-(1-h)^2\right) = 3\pi h(h+2).$$

Combining this estimate with (7.29), we conclude that the estimate (7.28) holds. □

Let us now prove approximation properties of the characteristic function defined in (4.4) and also properties on the change of variables.

Lemma 7.5 *The functions $\overline{\mathbf{X}}_h^k$, $\widehat{\mathbf{X}}_h$ and $\widehat{\mathbf{J}}_h$ defined by (4.5), (7.4) and (7.5) respectively, satisfy the following estimates:*

$$\begin{aligned}\|\widehat{\mathbf{X}}_h-\overline{\mathbf{X}}_h^k\|_{L^2(\mathcal{O})^2} \leq C\Big(&\Delta t^2+h\Delta t+\Delta t\left\|\mathbf{U}_h^k-\mathbf{u}_h^k\right\|_{L^2(\mathcal{O})^2}\\ &+\sqrt{\Delta t}\,\|\boldsymbol{\delta}_k\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2}\Big), \qquad (7.30)\\ \left\|\widehat{\mathbf{J}}_h-\mathbf{Id}\right\|_{L^2(\mathcal{O})^2} \leq C\Big(&\Delta t^2+\Delta t\left\|\mathbf{U}_h^k-\mathbf{u}_h^k\right\|_{L^2(\mathcal{O})^2}+\sqrt{\Delta t}\,\|\boldsymbol{\delta}_k\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2}\\ &+\Delta t|\boldsymbol{\zeta}(t_k)-\boldsymbol{\zeta}_h^k|\Big), \qquad (7.31)\end{aligned}$$

with

$$\|\boldsymbol{\delta}_k\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2} \leq C\Delta t \left\| \frac{\partial \mathbf{u}}{\partial t} \right\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2}. \tag{7.32}$$

Proof Analogous to the proof of Lemma 6.2, we define the characteristic function $\overline{\overline{\boldsymbol{\psi}}}{}_h^k$ associated with the fully-discrete velocity field as the solution of

$$\begin{cases} \dfrac{d}{dt}\overline{\overline{\boldsymbol{\psi}}}{}_h^k(t;t_{k+1},\mathbf{x}) = \mathbf{U}_h^k(\overline{\overline{\boldsymbol{\psi}}}{}_h^k(t;t_{k+1},\mathbf{x})), \\ \overline{\overline{\boldsymbol{\psi}}}{}_h^k(t_{k+1};t_{k+1},\mathbf{x}) = \mathbf{x} \end{cases} \tag{7.33}$$

and we denote

$$\overline{\overline{\mathbf{X}}}{}_h^k(\mathbf{x}) = \overline{\overline{\boldsymbol{\psi}}}{}_h^k(t_k;t_{k+1},\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{O}. \tag{7.34}$$

With a very slight modification of the proof of Lemma 6.5 in [19], we get

$$\|\widehat{\mathbf{X}}_h - \overline{\overline{\mathbf{X}}}{}_h^k\|_{L^2(\mathcal{O})^2} \leq C\left(\Delta t^2 + \sqrt{\Delta t}\,\|\boldsymbol{\delta}_k\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2}\right). \tag{7.35}$$

Let us observe that the characteristic Eqs. (4.4) and (7.33) can be written as follows: for any $t \in [t_k, t_{k+1}]$, we have

$$\begin{aligned} \overline{\boldsymbol{\psi}}{}_h^k(t;t_{k+1},\mathbf{x}) &= \mathbf{x} - \mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)\Delta t + \int_{t_{k+1}}^{t} \mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k(\overline{\boldsymbol{\psi}}{}_h^k(s;t_{k+1},\mathbf{x}))\,\mathrm{d}s \\ &\quad -\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)(t-t_{k+1}), \\ \overline{\overline{\boldsymbol{\psi}}}{}_h^k(t;t_{k+1},\mathbf{x}) &= \mathbf{x} + \int_{t_{k+1}}^{t} \mathbf{U}_h^k(\overline{\overline{\boldsymbol{\psi}}}{}_h^k(s;t_{k+1},\mathbf{x}))\,\mathrm{d}s. \end{aligned}$$

Subtracting the previous identities, we obtain

$$\begin{aligned} \overline{\boldsymbol{\psi}}{}_h^k(t;t_{k+1},\mathbf{x}) - \overline{\overline{\boldsymbol{\psi}}}{}_h^k(t;t_{k+1},\mathbf{x}) &= -\mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)\Delta t - \mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)(t-t_{k+1}) \\ &\quad + \int_{t_{k+1}}^{t} \left(\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k(\overline{\boldsymbol{\psi}}{}_h^k(s;t_{k+1},\mathbf{x})) - \mathbf{U}_h^k(\overline{\overline{\boldsymbol{\psi}}}{}_h^k(s;t_{k+1},\mathbf{x}))\right)\mathrm{d}s, \end{aligned}$$

then taking the $L^2(\mathcal{O})^2$-norm, we deduce that

$$\begin{aligned}
&\left\|\overline{\boldsymbol{\psi}}_h^k(t;t_{k+1},\cdot)-\overline{\overline{\boldsymbol{\psi}}}_h^k(t;t_{k+1},\cdot)\right\|_{L^2(\mathcal{O})^2}\\
&\leq C|\mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)|(t-t_k)+C\left|\left(\mathbf{u}_h^k-\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k\right)(\boldsymbol{\zeta}_h^k)\right|(t_{k+1}-t)\\
&\quad+\int_t^{t_{k+1}}\left\|\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k(\overline{\boldsymbol{\psi}}_h^k(s;t_{k+1},\cdot))-\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{U}_h^k(\overline{\boldsymbol{\psi}}_h^k(s;t_{k+1},\cdot))\right\|_{L^2(\mathcal{O})^2}\mathrm{d}s\\
&\quad+\int_t^{t_{k+1}}\left\|\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{U}_h^k(\overline{\boldsymbol{\psi}}_h^k(s;t_{k+1},\cdot))-\mathbf{U}_h^k(\overline{\boldsymbol{\psi}}_h^k(s;t_{k+1},\cdot))\right\|_{L^2(\mathcal{O})^2}\mathrm{d}s\\
&\quad+\int_t^{t_{k+1}}\left\|\mathbf{U}_h^k(\overline{\boldsymbol{\psi}}_h^k(s;t_{k+1},\cdot))-\mathbf{U}_h^k(\overline{\overline{\boldsymbol{\psi}}}_h^k(s;t_{k+1},\cdot))\right\|_{L^2(\mathcal{O})^2}\mathrm{d}s.
\end{aligned}$$

By using (5.9) and the hypothesis (2.15), the above estimate yields

$$\begin{aligned}
&\left\|\overline{\boldsymbol{\psi}}_h^k(t;t_{k+1},\cdot)-\overline{\overline{\boldsymbol{\psi}}}_h^k(t;t_{k+1},\cdot)\right\|_{L^2(\mathcal{O})^2}\\
&\leq C|\mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)|(t-t_k)+C\left|\left(\mathbf{u}_h^k-\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k\right)(\boldsymbol{\zeta}_h^k)\right|(t_{k+1}-t)\\
&\quad+\left\|\mathbf{u}_h^k-\mathbf{U}_h^k\right\|_{L^2(\mathcal{O})^2}(t_{k+1}-t)+\left\|\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{U}_h^k-\mathbf{U}_h^k\right\|_{L^2(\mathcal{O})^2}(t_{k+1}-t)\\
&\quad+C\int_t^{t_{k+1}}\left\|\overline{\boldsymbol{\psi}}_h^k(s;t_{k+1},\cdot)-\overline{\overline{\boldsymbol{\psi}}}_h^k(s;t_{k+1},\cdot)\right\|_{L^2(\mathcal{O})^2}\mathrm{d}s.
\end{aligned}$$

Then, applying the Gronwall inequality to the above estimate, for all $t\in[t_k,t_{k+1}]$ we deduce that

$$\begin{aligned}
&\left\|\overline{\boldsymbol{\psi}}_h^k(t;t_{k+1},\cdot)-\overline{\overline{\boldsymbol{\psi}}}_h^k(t;t_{k+1},\cdot)\right\|_{L^2(\mathcal{O})^2}\\
&\leq C|\mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)|(t-t_k)+C\left|\left(\mathbf{u}_h^k-\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k\right)(\boldsymbol{\zeta}_h^k)\right|(t_{k+1}-t)\\
&\quad+\left\|\mathbf{u}_h^k-\mathbf{U}_h^k\right\|_{L^2(\mathcal{O})^2}(t_{k+1}-t)+\left\|\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{U}_h^k-\mathbf{U}_h^k\right\|_{L^2(\mathcal{O})^2}(t_{k+1}-t)\\
&\quad+C\Delta t^2\Big(|\mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)|+\left|\left(\mathbf{u}_h^k-\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k\right)(\boldsymbol{\zeta}_h^k)\right|\\
&\quad+\left\|\mathbf{u}_h^k-\mathbf{U}_h^k\right\|_{L^2(\mathcal{O})^2}+\left\|\mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{U}_h^k-\mathbf{U}_h^k\right\|_{L^2(\mathcal{O})^2}\Big),
\end{aligned}$$

and in particular, taking $t=t_k$, we get

$$
\begin{aligned}
\|\overline{\mathbf{X}}_h^k - \overline{\overline{\mathbf{X}}}_h^k\|_{L^2(\mathcal{O})^2} &\leq C\Delta t^2 |\mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)| + C\Delta t \Big(\Big| \Big(\mathbf{u}_h^k - \mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{u}_h^k \Big)(\boldsymbol{\zeta}_h^k) \Big| \\
&\quad + \|\mathbf{u}_h^k - \mathbf{U}_h^k\|_{L^2(\mathcal{O})^2} + \Big\| \mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{U}_h^k - \mathbf{U}_h^k \Big\|_{L^2(\mathcal{O})^2} \Big) \\
&\leq C\Delta t^2 \|\mathbf{U}_h^k\|_{L^2(\mathcal{O})^2} + C\Delta t \|\mathbf{u}_h^k - \mathbf{U}_h^k\|_{L^2(\mathcal{O})^2} \\
&\quad + C\Delta t \Big\| \mathbf{P}(\boldsymbol{\zeta}_h^k)\mathbf{U}_h^k - \mathbf{U}_h^k \Big\|_{L^2(\mathcal{O})^2}.
\end{aligned}
$$

Combining the above inequality with the estimates (7.35) and (7.25) from Proposition 7.3, and using the hypothesis (2.15), we conclude (7.30).

The proof of (7.31) is completely similar to [19, Eq. (7.6)]. □

7.2 Error estimate

In this section, we give the proof of our second main result stated in Theorem 4.4. First of all, let us observe that according to Lemma 4.1, there exists a unique couple $(\overline{\mathbf{U}}_h^{k+1}, \overline{P}_h^{k+1}) \in \mathcal{K}_h(\boldsymbol{\zeta}_h^{k+1}) \times M_h(\boldsymbol{\zeta}_h^{k+1})$ such that

$$
\begin{cases}
a\big(\mathbf{U}_h^{k+1} - \overline{\mathbf{U}}_h^{k+1}, \boldsymbol{\varphi}\big) + b\big(\boldsymbol{\varphi}, P_h^{k+1} - \overline{P}_h^{k+1}\big) = 0 & \forall \boldsymbol{\varphi} \in \mathcal{K}_h(\boldsymbol{\zeta}_h^{k+1}), \\
b\big(\mathbf{U}_h^{k+1} - \overline{\mathbf{U}}_h^{k+1}, q\big) = 0 & \forall q \in M_h(\boldsymbol{\zeta}_h^{k+1})
\end{cases}
\tag{7.36}
$$

and moreover, there exists a positive constant C, independent of h and k, such that the following estimate holds:

$$
\|\mathbf{U}_h^{k+1} - \overline{\mathbf{U}}_h^{k+1}\|_{L^2(\mathcal{O})^2} \leq Ch \qquad \forall k \in \{0, \dots, N-1\}. \tag{7.37}
$$

For any $k \in \{0, \dots, N\}$, let us denote

$$
E_h^k = \|\sqrt{\rho_h^k}(\overline{\mathbf{U}}_h^k - \mathbf{u}_h^k)\|_{L^2(\mathcal{O})^2} + |\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k|.
$$

7.2.1 First step

In this subsection we prove that if $k \in \{0, \dots, N-1\}$ is such that the hypothesis (7.1) holds then there exists a positive constant C_3, independent of k, h, Δt and T, such that

$$
\begin{aligned}
E_h^{k+1} &\leq (1 + C_3\Delta t)E_h^k + C_3 \Big(\Delta t^2 + \Delta t\sqrt{h} + h + (\Delta t)^{3/2} \Big\| \frac{\partial \mathbf{u}}{\partial t} \Big\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2} \\
&\quad + (\Delta t)^{3/2} \Big\| \frac{d^2}{dt^2}[\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}] \Big\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2} \Big).
\end{aligned}
\tag{7.38}
$$

To this end, let us subtract Eqs. (4.9)–(4.10) from (7.6)–(7.7). Using the definition of the projection $(\overline{\mathbf{U}}_h^{k+1}, \overline{P}_h^{k+1})$ on the finite element spaces $\mathcal{K}_h(\boldsymbol{\zeta}_h^{k+1}) \times M_h(\boldsymbol{\zeta}_h^{k+1})$ given in (7.36), we get

$$
\begin{aligned}
&\frac{1}{\Delta t}\left(\rho_h^{k+1}(\mathbf{U}_h^{k+1}-\mathbf{u}_h^{k+1}),\boldsymbol{\varphi}\right)+a(\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1},\boldsymbol{\varphi})+b(\boldsymbol{\varphi},\overline{P}_h^{k+1}-p_h^{k+1})\\
&\qquad=\frac{1}{\Delta t}\left(\rho_h^{k+1}\widehat{\mathbf{J}}_h\big(\mathbf{U}_h^k\circ\widehat{\mathbf{X}}_h\big)-\rho_h^{k+1}\mathbf{u}_h^k\circ\overline{\mathbf{X}}_h^k,\boldsymbol{\varphi}\right)\\
&\qquad\quad+\left((\rho_h^{k+1}-\overline{\rho}_h^{k+1})\mathbf{f}_h^{k+1},\boldsymbol{\varphi}\right)+(\mathbf{A}_h^k,\boldsymbol{\varphi})\qquad\forall\boldsymbol{\varphi}\in\mathcal{K}_h(\boldsymbol{\zeta}_h^{k+1}),\\
&\qquad b(\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1},q)=0\qquad\forall q\in M_h(\boldsymbol{\zeta}_h^{k+1}).
\end{aligned}
$$

We choose the test functions

$$
\boldsymbol{\varphi}=\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1}\in\mathcal{K}_h(\boldsymbol{\zeta}_h^{k+1})\quad\text{and}\quad q=\overline{P}_h^{k+1}-p_h^{k+1}\in M_h(\boldsymbol{\zeta}_h^{k+1}),
$$

then we obtain the following identity

$$
\begin{aligned}
&\left(\rho_h^{k+1}(\mathbf{U}_h^{k+1}-\mathbf{u}_h^{k+1}),\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1}\right)+\Delta t\,a(\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1},\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1})\\
&\quad=\left(\rho_h^{k+1}\widehat{\mathbf{J}}_h\big(\mathbf{U}_h^k\circ\widehat{\mathbf{X}}_h\big)-\rho_h^{k+1}\mathbf{u}_h^k\circ\overline{\mathbf{X}}_h^k,\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1}\right)\\
&\qquad+\Delta t\left((\rho_h^{k+1}-\overline{\rho}_h^{k+1})\mathbf{f}_h^{k+1},\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1}\right)+\Delta t(\mathbf{A}_h^k,\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1}),
\end{aligned}
$$

which can be written as follows:

$$
\begin{aligned}
&\left(\rho_h^{k+1}(\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1}),\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1}\right)+\Delta t\,a(\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1},\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1})\\
&\quad=\left(\rho_h^{k+1}\widehat{\mathbf{J}}_h\big(\mathbf{U}_h^k\circ\widehat{\mathbf{X}}_h\big)-\rho_h^{k+1}\mathbf{u}_h^k\circ\overline{\mathbf{X}}_h^k,\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1}\right)\\
&\qquad+\left(\rho_h^{k+1}(\overline{\mathbf{U}}_h^{k+1}-\mathbf{U}_h^{k+1}),\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1}\right)\\
&\qquad+\Delta t\left((\rho_h^{k+1}-\overline{\rho}_h^{k+1})\mathbf{f}_h^{k+1},\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1}\right)+\Delta t(\mathbf{A}_h^k,\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1}).
\end{aligned}
$$

By using the Cauchy–Schwarz inequality, there exists a positive constant C, independent of h and k, such that

$$
\begin{aligned}
&\left\|\sqrt{\rho_h^{k+1}}(\overline{\mathbf{U}}_h^{k+1}-\mathbf{u}_h^{k+1})\right\|_{L^2(\mathcal{O})^2}\le\left\|\sqrt{\rho_h^{k+1}}\Big(\widehat{\mathbf{J}}_h\big(\mathbf{U}_h^k\circ\widehat{\mathbf{X}}_h\big)-\mathbf{u}_h^k\circ\overline{\mathbf{X}}_h^k\Big)\right\|_{L^2(\mathcal{O})^2}\\
&\quad+C\Big(\big\|\overline{\mathbf{U}}_h^{k+1}-\mathbf{U}_h^{k+1}\big\|_{L^2(\mathcal{O})^2}+\Delta t\big\|\overline{\rho}_h^{k+1}-\rho_h^{k+1}\big\|_{L^2(\mathcal{O})}+\Delta t\big\|\mathbf{A}_h^k\big\|_{L^2(\mathcal{O})^2}\Big).
\end{aligned}
\tag{7.39}
$$

Let us now estimate the first term in the right hand side of the inequality (7.39). To this end, we remark that since $\sqrt{\rho_h^{k+1}}=\sqrt{\rho_h^k}\circ\overline{\mathbf{X}}_h^k$ (see (5.8)), one can write

$$\sqrt{\rho_h^{k+1}}\Big(\widehat{\mathbf{J}}_h\big(\mathbf{U}_h^k \circ \widehat{\mathbf{X}}_h\big) - \mathbf{u}_h^k \circ \overline{\mathbf{X}}_h^k\Big)$$
$$= \sqrt{\rho_h^{k+1}}(\widehat{\mathbf{J}}_h - \mathbf{Id})\mathbf{U}_h^k \circ \widehat{\mathbf{X}}_h + \sqrt{\rho_h^{k+1}}\Big(\mathbf{U}_h^k \circ \widehat{\mathbf{X}}_h - \mathbf{U}_h^k \circ \overline{\mathbf{X}}_h^k\Big)$$
$$+\sqrt{\rho_h^{k+1}}(\mathbf{U}_h^k - \overline{\mathbf{U}}_h^k) \circ \overline{\mathbf{X}}_h^k + \Big(\sqrt{\rho_h^k}(\overline{\mathbf{U}}_h^k - \mathbf{u}_h^k)\Big) \circ \overline{\mathbf{X}}_h^k.$$

Then, by using the hypothesis (2.15) and the fact that $\det \mathbf{J}_{\overline{\mathbf{X}}_h^k} = 1$, we deduce the following estimate:

$$\Big\|\sqrt{\rho_h^{k+1}}\Big(\widehat{\mathbf{J}}_h\big(\mathbf{U}_h^k \circ \widehat{\mathbf{X}}_h\big) - \mathbf{u}_h^k \circ \overline{\mathbf{X}}_h^k\Big)\Big\|_{L^2(\mathcal{O})^2} \le C\|\widehat{\mathbf{J}}_h - \mathbf{Id}\|_{L^2(\mathcal{O})^4}$$
$$+C\|\widehat{\mathbf{X}}_h - \overline{\mathbf{X}}_h^k\|_{L^2(\mathcal{O})^2} + C\|\overline{\mathbf{U}}_h^k - \mathbf{U}_h^k\|_{L^2(\mathcal{O})^2} + \|\sqrt{\rho_h^k}(\overline{\mathbf{U}}_h^k - \mathbf{u}_h^k)\|_{L^2(\mathcal{O})^2}. \tag{7.40}$$

Combining the estimates (7.39) together with (7.40), we deduce that

$$\Big\|\sqrt{\rho_h^{k+1}}(\overline{\mathbf{U}}_h^{k+1} - \mathbf{u}_h^{k+1})\Big\|_{L^2(\mathcal{O})^2} \le \|\sqrt{\rho_h^k}(\overline{\mathbf{U}}_h^k - \mathbf{u}_h^k)\|_{L^2(\mathcal{O})^2}$$
$$+C\Big(\|\widehat{\mathbf{J}}_h - \mathbf{Id}\|_{L^2(\mathcal{O})^4} + \|\widehat{\mathbf{X}}_h - \overline{\mathbf{X}}_h^k\|_{L^2(\mathcal{O})^2} + \|\overline{\mathbf{U}}_h^k - \mathbf{U}_h^k\|_{L^2(\mathcal{O})^2}$$
$$+\|\overline{\mathbf{U}}_h^{k+1} - \mathbf{U}_h^{k+1}\|_{L^2(\mathcal{O})^2} + \Delta t\|\overline{\rho}_h^{k+1} - \rho_h^{k+1}\|_{L^2(\mathcal{O})} + \Delta t\|\mathbf{A}_h^k\|_{L^2(\mathcal{O})^2}\Big).$$

Due to Lemma 7.5 (see (7.30)–(7.31)) and Lemma 7.4, the above estimate yields

$$\|\sqrt{\rho_h^{k+1}}(\overline{\mathbf{U}}_h^{k+1} - \mathbf{u}_h^{k+1})\|_{L^2(\mathcal{O})^2} \le \|\sqrt{\rho_h^k}(\overline{\mathbf{U}}_h^k - \mathbf{u}_h^k)\|_{L^2(\mathcal{O})^2}$$
$$+C\Big(\Delta t|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + \Delta t\|\mathbf{U}_h^k - \mathbf{u}_h^k\|_{L^2(\mathcal{O})^2} + \sqrt{\Delta t}\|\boldsymbol{\delta}_k\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2}$$
$$+\Delta t^2 + \Delta t\sqrt{h} + \|\overline{\mathbf{U}}_h^k - \mathbf{U}_h^k\|_{L^2(\mathcal{O})^2} + \|\overline{\mathbf{U}}_h^{k+1} - \mathbf{U}_h^{k+1}\|_{L^2(\mathcal{O})^2} + \Delta t\|\mathbf{A}_h^k\|_{L^2(\mathcal{O})^2}\Big).$$

Let us now use the estimate (7.37) of the projection, the estimates (7.8) and (7.32) of $\mathbf{A}_h^k$, respectively $\boldsymbol{\delta}_k$, then the above inequality becomes

$$\|\sqrt{\rho_h^{k+1}}(\overline{\mathbf{U}}_h^{k+1} - \mathbf{u}_h^{k+1})\|_{L^2(\mathcal{O})^2} \le (1 + C\Delta t)\|\sqrt{\rho_h^k}(\overline{\mathbf{U}}_h^k - \mathbf{u}_h^k)\|_{L^2(\mathcal{O})^2}$$
$$+C\Big(\Delta t^2 + \Delta t\sqrt{h} + h + \Delta t\big|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k\big| + \Delta t\big|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}\big|$$
$$+(\Delta t)^{3/2}\Big\|\frac{\partial \mathbf{u}}{\partial t}\Big\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2} + (\Delta t)^{3/2}\Big\|\frac{d^2}{dt^2}[\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}]\Big\|_{L^2(\mathcal{O}\times(t_k,t_{k+1}))^2}\Big). \tag{7.41}$$

Let us now use identity (4.3) and we get

$$\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1} = \boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k - \mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)\Delta t + \int_{t_k}^{t_{k+1}} \mathbf{u}(\boldsymbol{\zeta}(s), s)ds,$$

then

$$|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}| \\ \leq |\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + \Delta t \Big|\mathbf{u}(\boldsymbol{\zeta}(t_k), t_k) - \mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)\Big| + \int_{t_k}^{t_{k+1}} \big|\boldsymbol{\zeta}'(s) - \boldsymbol{\zeta}'(t_k)\big| ds.$$

Since the equality $\mathbf{u}(\boldsymbol{\zeta}(t_k), t_k) = \mathbf{U}_h^k(\boldsymbol{\zeta}_h^k)$ holds and the function $\mathbf{u}_h^k - \mathbf{U}_h^k$ is a rigid velocity in $B(\boldsymbol{\zeta}_h^k)$, then we deduce that

$$|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}| \leq |\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + \frac{\Delta t}{\sqrt{\pi}} \|\mathbf{U}_h^k - \mathbf{u}_h^k\|_{L^2(\mathcal{O})^2} + \frac{\Delta t^2}{2} \|\boldsymbol{\zeta}''\|_{L^\infty(0,T)^2}.$$

Thus, due to the estimate of the projection (7.37), we get

$$|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}| \\ \leq |\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + \Delta t \Big\|\overline{\mathbf{U}}_h^k - \mathbf{u}_h^k\Big\|_{L^2(\mathcal{O})^2} + C h \Delta t + \frac{\Delta t^2}{2} \|\boldsymbol{\zeta}''\|_{L^\infty(0,T)^2}. \tag{7.42}$$

Combining estimates (7.41) and (7.42) one can easily deduce the existence of a positive constant C_3 independent of T such that (7.38) holds. Thus, we conclude the first step of the proof.

Additionally, using the same constant C_3, estimate (7.42) can be rewritten as follows:

$$|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}| \leq |\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + \frac{\Delta t}{\min\{\sqrt{\rho_f}, \sqrt{\rho_s}\}} \Big\|\sqrt{\rho_h^k}(\overline{\mathbf{U}}_h^k - \mathbf{u}_h^k)\Big\|_{L^2(\mathcal{O})^2} \\ + C_3 h \Delta t + C_3 \Delta t^2. \tag{7.43}$$

7.2.2 Second step

Let $C_0 > 0$ be a fixed constant. In this subsection, we are going to prove that there exists $\tau^* > 0$ such that if $\Delta t \in (0, \tau^*]$ and $h \leq C_0 \Delta t^{1+\alpha}$, then the following estimate holds

$$\Big|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k\Big| < \eta \qquad \forall k \in \{0, \ldots, N\}. \tag{7.44}$$

To this end, using the hypothesis (2.15) we can define the constant

$$C_4 = \exp(C_3 T)\Big(1 + \sqrt{C_0} + C_0 + \Big\|\frac{\partial \mathbf{u}}{\partial t}\Big\|_{L^2(\mathcal{O}\times(0,T))^2} + \Big\|\frac{d^2}{dt^2}[\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}]\Big\|_{L^2(\mathcal{O}\times(0,T))^2}\Big), \tag{7.45}$$

where C_3 is the positive constant independent of k, h, Δt and T, which appears in (7.38) and (7.43). We define

$$\tau^* \overset{\text{(def)}}{=} \min\left\{\frac{\eta}{4C_3}, \sqrt{\frac{\eta}{4C_3}}, 1, \sqrt{\rho_f}, \sqrt{\rho_s}, \left(\frac{\eta}{4C_4}\right)^{1/\alpha}\right\} \tag{7.46}$$

and we take $0 < \Delta t \leq \tau^*$.

In order to prove (7.44), we proceed by induction. Let us consider the statement

$$\mathscr{P}(j): \quad \left|\boldsymbol{\zeta}(t_j) - \boldsymbol{\zeta}_h^j\right| < \eta. \tag{7.47}$$

- First, we remark that $\mathscr{P}(0)$ is true due to the initial conditions (1.8) and (4.2).
- Secondly, taking $k = 0$ in (7.43), we get that

  $$|\boldsymbol{\zeta}(t_1) - \boldsymbol{\zeta}_h^1| \leq C_3 h \Delta t + C_3 \Delta t^2.$$

  Since $h < 1$ and $\Delta t \leq \tau^* < \min\left\{\frac{\eta}{2C_3}, \sqrt{\frac{\eta}{2C_3}}\right\}$, we deduce

  $$|\boldsymbol{\zeta}(t_1) - \boldsymbol{\zeta}_h^1| < \eta, \tag{7.48}$$

  that is, $\mathscr{P}(1)$ is also true.
- Finally, we prove that if $j \geq 1$ is such that the statements $\mathscr{P}(0), \mathscr{P}(1), \ldots, \mathscr{P}(j)$ hold true, then $\mathscr{P}(j+1)$ is also true. Firstly, we infer from the induction hypothesis that condition (7.1) holds for any $k \in \{0, \ldots, j-1\}$. Then, using the first step of the proof we deduce that estimate (7.38) holds for any $k \in \{0, \ldots, j-1\}$. Hence, by applying the discrete Gronwall Lemma, and using that the error E_h^0 is equal to zero (see initial conditions (4.2)), we deduce that for any $k \in \{0, \ldots, j\}$,

  $$E_h^k \leq \exp(C_3 T)\Big(\Delta t + \sqrt{h} + \frac{h}{\Delta t} + \Delta t \left\|\frac{\partial \mathbf{u}}{\partial t}\right\|_{L^2(\mathcal{O}\times(0,T))^2} + \Delta t \left\|\frac{d^2}{dt^2}[\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}]\right\|_{L^2(\mathcal{O}\times(0,T))^2}\Big).$$

  Then using the fact that $h \leq C_0 \Delta t^{1+\alpha}$ and $\Delta t \leq \tau^* \leq 1$, we have

  $$E_h^k \leq C_4 \Delta t^\alpha, \tag{7.49}$$

  for any $k \in \{0, \ldots, j\}$, where C_4 is defined in (7.45).
  Let us observe that from the estimate (7.43), for $\Delta t \leq \tau^* \leq \min\{\sqrt{\rho_f}, \sqrt{\rho_s}\}$, it follows

  $$|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}| \leq E^k + C_3 h \Delta t + C_3 \Delta t^2. \tag{7.50}$$

Combining (7.49) with (7.50), and taking into account that $h < 1$,we deduce for any $k \in \{0, \ldots, j\}$,

$$|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}^{k+1}| \leq C_4 \Delta t^{\alpha} + C_3 \Delta t + C_3 \Delta t^2.$$

Since $\Delta t \leq \tau^* < \min\left\{(\frac{\eta}{3C_4})^{1/\alpha}, \frac{\eta}{3C_3}, \sqrt{\frac{\eta}{3C_3}}\right\}$, we deduce that

$$|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}| < \eta \qquad k \in \{0, \ldots, j\},$$

that is $\mathscr{P}(j+1)$ is true. Thus, the induction process is finished and the proof of (7.44) is completed.

7.2.3 Third step

From the previous steps and the fact that $\sqrt{\rho_h^k} \geq \min\{\sqrt{\rho_f}, \sqrt{\rho_s}\} > 0$, we conclude that if $0 < \Delta t \leq \tau^*$ and $h \leq C_0 \Delta t^{1+\alpha}$, the following estimate holds:

$$\|\overline{\mathbf{U}}_h^k - \mathbf{u}_h^k\|_{L^2(\mathcal{O})^2} + |\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| \leq \frac{C_4 \Delta t^{\alpha}}{\min\{\sqrt{\rho_f}, \sqrt{\rho_s}\}} \qquad \forall k \in \{0, \ldots, N\}. \tag{7.51}$$

Using the estimate of the projection (7.37), we get

$$\begin{aligned}\|\mathbf{U}_h^k - \mathbf{u}_h^k\|_{L^2(\mathcal{O})^2} + |\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| &\leq \frac{C_4 \Delta t^{\alpha}}{\min\{\sqrt{\rho_f}, \sqrt{\rho_s}\}} + Ch \\ &\leq \left(\frac{C_4}{\min\{\sqrt{\rho_f}, \sqrt{\rho_s}\}} + CC_0\right)\Delta t^{\alpha} \qquad \forall k \in \{0, \ldots, N\}.\end{aligned} \tag{7.52}$$

The above estimate together with the definition of $\mathbf{U}_h^k$ from (7.3) and properties on the change of variables $\mathbf{X}_h^k$ given in [19, Lemmas 5.5–5.6], yield to the conclusion of Theorem 4.4. □

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