

Linear Complementarity Problems over Symmetric Cones: Characterization of Q_b -transformations and Existence Results

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Abstract This paper is devoted to the study of the symmetric cone linear complementarity problem (SCLCP). Specifically, our aim is to characterize the class of linear transformations for which the SCLCP has always a nonempty and bounded solution set in terms of larger classes. For this, we introduce a couple of new classes of linear transformations in this SCLCP context. Then, we study them for concrete particular instances (such as second-order and semidefinite linear complementarity problems) and for specific examples (Lyapunov, Stein functions, among others). This naturally permits to establish coercive and noncoercive existence results for SCLCPs.

Keywords Euclidean Jordan algebra · Linear complementarity problem · Symmetric cone · Q_b -transformation · Q -transformation · García's transformation

1 Introduction

This paper is devoted to the study of the *symmetric cone linear complementarity problem* (SCLCP), which consists in solving a linear complementary problem over

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the set of squares elements in a Euclidean Jordan algebra. This problem is a particular case of a variational inequality problem (e.g., [1]) and provides a simple unified framework for various existing complementarity problems such as the linear complementarity problem over the nonnegative orthant (LCP) (e.g., [2]), the second-order cone linear complementarity problem (SOCLCP) (e.g., [3, 4]), and the semidefinite linear complementarity problem (SDLCP) (e.g., [5, 6]). Hence, it has extensive applications in engineering, economics, game theory, management science, and other fields; see [1, 7–9] and references therein.

In the last years, SCLCP has been studied by divers authors, with special emphasis in its particular cases, SOCLCP and SDLCP. For instance, Gowda et al. [10] extended P - and GUS -transformations from LCP to the SCLCP setting. These notions are further exploited in the papers [11–13].

A key issue in linear complementarity problems consists in finding necessary and sufficient conditions on the linear transformation involved in its definition that ensures the nonemptiness and boundedness of its solution set. A linear function satisfying this condition is called a Q_b -transformation. Indeed, the interest in studying the boundedness of the solution sets comes from its applications on the stability of LCPs with respect to perturbations to the data of the problem (see [14–16] and the recent reference [6] for SDLCPs).

The aim of this paper is to characterize the class Q_b in the context of SCLCP. More precisely, we are interested in finding large classes of linear transformations for which Q_b behaves similarly to larger classes, such as Q and R_0 . For this, on the one hand, we extend the class F from LCPs [17] and SDLCPs [18] to this SCLCP setting. Within this class, we prove that classes Q_b and Q coincide. Then, we consider subclasses of F (called F_1 and F_2) and study their connections as well as different examples of linear transformations belonging to them. We also extend a particular class, called T , which was originally defined in [19] in the LCP framework, and compare it with class F . Actually, we prove that T is contained in F_2 . We also specialize all these classes to particular SCLCPs such as LCP, SOCLCP, and SDLCP. On the other hand, we define the class of García's transformations for SCLCPs. The latter is an extension from LCPs to this setting (cf. [20]). See [6] for its extension to SDLCP. Within this class, we are able to prove that classes Q and R_0 coincide. This allows us to state some existence result for SCLCPs.

The existing literature on SCLCPs includes only some few works about the class Q_b . For instance, in one of these articles, Gowda and Tao [21] show that, within the class Z , classes Q and S behave similarly (see the definitions of S - and Z -transformations in Sect. 2.2).

This paper is organized as follows. Section 2 is devoted to the preliminaries. It is split into two subsections; the first one recalls basic results on Euclidean Jordan algebras, while the second one summarizes some classes of linear transformations with their respective connections. In Sects. 3 and 4, we establish our main results described above. Indeed, Sect. 3 is devoted to the study of linear transformations for which classes Q and Q_b coincide, while Sect. 4 is devoted to existence results for SCLCPs associated with García's transformations, for which we prove that R_0 and Q_b coincide.

2 Preliminaries

Consider a finite-dimensional vector space \mathbb{V} over the real field \mathbb{R} equipped with the inner product $\langle \cdot, \cdot \rangle$. Let $L : \mathbb{V} \rightarrow \mathbb{V}$ be a linear transformation (for short, $L \in \mathcal{L}(\mathbb{V})$), and $q \in \mathbb{V}$. In this paper we study the *symmetric cone linear complementarity problem*, which consists of finding an element \bar{x} such that

$$\bar{x} \in \mathcal{K}, \quad \bar{y} = L(\bar{x}) + q \in \mathcal{K}, \quad \text{and} \quad \langle \bar{y}, \bar{x} \rangle = 0. \tag{1}$$

Here, $\mathcal{K} := \{x \circ x : x \in \mathbb{V}\}$ denotes the set of square elements in a Euclidean Jordan algebra $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$, where \circ denotes the Jordan product. A review of Euclidean Jordan algebras is given below in Sect. 2.1.

In what follows, we denote problem (1) by $\text{LCP}(L, \mathcal{K}, q)$ and its solution by $\text{SOL}(L, \mathcal{K}, q)$. Also, its feasible set is defined to be $\text{FEAS}(L, \mathcal{K}, q) := \{x \in \mathcal{K} : L(x) + q \in \mathcal{K}\}$.

2.1 Euclidean Jordan Algebras Review

In this subsection, we briefly describe some concepts, properties, and results from Euclidean Jordan algebras that are needed in this paper and that have become important in the study of conic optimization; see, e.g., Schmieta and Alizadeh [22]. Most of this material can be found in Faraut and Korányi [23].

A *Euclidean Jordan algebra* is a triple $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$, where $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ is a finite-dimensional vector space over \mathbb{R} equipped with an inner product $\langle \cdot, \cdot \rangle$, and the Jordan product $(x, y) \mapsto x \circ y : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V}$ is a bilinear mapping satisfying the following three conditions:

- (i) $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$,
- (ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$, where $x^2 = x \circ x$, and
- (iii) $\langle x \circ y, z \rangle = \langle y, x \circ z \rangle$ for all $x, y, z \in \mathbb{V}$,

and there exists a (unique) unitary element $e \in \mathbb{V}$ such that $x \circ e = x$ for all $x \in \mathbb{V}$. Henceforth, we simply say that \mathbb{V} is a Euclidean Jordan algebra, and $x \circ y$ is called the *Jordan product* of x and y . A Euclidean Jordan algebra is said to be *simple* if it is not a direct sum of two Euclidean Jordan algebras.

In a Euclidean Jordan algebra \mathbb{V} , it is known that the set of squares $\mathcal{K} = \{x^2 : x \in \mathbb{V}\}$ is a *symmetric cone* (see [23, Theorem III.2.1]). This means that \mathcal{K} is a self-dual closed and convex cone with nonempty interior $\text{int}(\mathcal{K})$ and for any two elements $x, y \in \text{int}(\mathcal{K})$, there exists an invertible linear transformation $\Gamma : \mathbb{V} \rightarrow \mathbb{V}$ such that $\Gamma(\mathcal{K}) = \mathcal{K}$ and $\Gamma(x) = y$.

The *rank* of $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$ is defined as $r = \max\{\text{deg}(x) : x \in \mathbb{V}\}$, where $\text{deg}(x)$ is the degree of $x \in \mathbb{V}$ given by $\text{deg}(x) = \min\{k > 0 : \{e, x, x^2, \dots, x^k\}$ is linearly dependent.

Example 2.1 Typical examples of Euclidean Jordan algebras are the following.

- (i) *Euclidean Jordan algebra of n-dimensional vectors:*

$$\mathbb{V} = \mathbb{R}^n, \quad \mathcal{K} = \mathbb{R}_+^n, \quad r = n, \quad \langle x, y \rangle = \sum_{i=1}^n x_i y_i, \quad x \circ y = x * y,$$

where $x * y$ denotes the componentwise product of vectors x and y . Here, the unitary element is $e = (1, \dots, 1) \in \mathbb{R}^n$.

(ii) *Euclidean Jordan algebra of quadratic forms:*

$$\mathbb{V} = \mathbb{R}^n, \quad \mathcal{K} = \mathcal{L}_+^n := \{x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} : \|\bar{x}\| \leq x_1\}, \quad r = 2,$$

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i, \quad x \circ y = (x_1, \bar{x}) \circ (y_1, \bar{y}) = (\langle x, y \rangle, x_1 \bar{y} + y_1 \bar{x}),$$

where $\bar{x} = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$, and $\|\cdot\|$ denotes the Euclidean norm. In this algebra, the cone of squares is called the Lorentz cone (or the second-order cone). Moreover, the unitary element is $e = (1, 0, \dots, 0) \in \mathbb{R}^n$.

(iii) *Euclidean Jordan algebra of n -dimensional symmetric matrices:* Let \mathcal{S}^n be the set of all $n \times n$ real symmetric matrices, and \mathcal{S}_+^n be the cone of $n \times n$ symmetric positive semidefinite matrices.

$$\mathbb{V} = \mathcal{S}^n, \quad \mathcal{K} = \mathcal{S}_+^n, \quad r = n, \quad \langle X, Y \rangle = \text{tr}(XY), \quad X \circ Y = \frac{1}{2}(XY + YX).$$

Here tr denotes the trace of a matrix $X = (X_{ij}) \in \mathcal{S}^n$. In this setting, the identity matrix $I \in \mathbb{R}^{n \times n}$ is the unit element e .

Other examples are the set of $n \times n$ hermitian positive semidefinite matrices made of complex numbers, the set of $n \times n$ positive semidefinite matrices with quaternion entries, the set of 3×3 positive semidefinite matrices with octonion entries, and the exceptional 27-dimensional Albert octonion cone (see [23, 24]).

An element $c \in \mathbb{V}$ is an *idempotent* iff $c^2 = c$; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say that a finite set $\{e_1, \dots, e_r\}$ of primitive idempotents in \mathbb{V} is a *Jordan frame* if

$$e_i \circ e_j = 0 \quad \text{for all } i \neq j, \quad \text{and} \quad \sum_{i=1}^r e_i = e.$$

Note that $\langle e_i, e_j \rangle = \langle e_i \circ e_j, e \rangle = 0$ whenever $i \neq j$.

The following theorem gives us a spectral decomposition for the elements in a Euclidean Jordan algebra (see Theorem III.1.2 of [23]).

Theorem 2.1 (Spectral decomposition theorem) *Suppose that $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$ is a Euclidean Jordan algebra with rank r . Then, for every $x \in \mathbb{V}$, there exist a Jordan frame $\{e_1(x), \dots, e_r(x)\}$ and real numbers $\lambda_1(x), \dots, \lambda_r(x)$ such that $x = \lambda_1(x)e_1(x) + \dots + \lambda_r(x)e_r(x)$. The numbers $\lambda_i(x)$, called the eigenvalues of x , are uniquely determined.*

It is easy to show that $x \in \mathcal{K}$ (resp. $\text{int}(\mathcal{K})$) if and only if every eigenvalue $\lambda_i(x)$ of x is nonnegative (resp. positive). Due to the uniqueness of the eigenvalues $\lambda_i(x)$, we can define the trace of a element x as $\text{tr}(x) = \sum_{i=1}^r \lambda_i(x)$. Notice that the latter also implies that $\text{tr}(c) = 1$ for every primitive idempotent c in \mathbb{V} .

Remark 2.1 We recall that in any simple Euclidean Jordan algebra \mathbb{V} , there exists a $\theta > 0$ such that $\langle x, y \rangle = \theta \cdot \text{tr}(x \circ y)$ (see [23, Proposition III.4.1]). Hence, $\theta = \langle c, e \rangle = \|c\|^2$ for every primitive idempotent c in \mathbb{V} . In particular, we have $\|e_i\|^2 = \theta$ for every element e_i of a Jordan frame $\{e_1, \dots, e_r\}$.

For any $a \in \mathbb{V}$, the *Lyapunov transformation* $L_a : \mathbb{V} \rightarrow \mathbb{V}$ and the *quadratic representation* $P_a : \mathbb{V} \rightarrow \mathbb{V}$ are defined as

$$L_a(x) := a \circ x \quad \text{and} \quad P_a(x) := (2L_a^2 - L_{a^2})(x) = 2a \circ (a \circ x) - a^2 \circ x \quad \text{for all } x \in \mathbb{V}. \tag{2}$$

These transformations are linear and self-adjoint on \mathbb{V} (see [23]). In the following example, we describe these transformations in the Euclidean Jordan algebras defined in Example 2.1.

Example 2.2

- (i) For the Euclidean Jordan algebra of n -dimensional vectors, the above transformations are $L_a(x) = \text{Diag}(a)x$ and $P_a(x) = \text{Diag}(a^2)x$, where $\text{Diag}(q)$ denotes a diagonal matrix of size n whose diagonal entries are the entries of q .
- (ii) For the Euclidean Jordan algebra of quadratic forms, the above transformations are given by

$$L_a(x) = \begin{pmatrix} a_1 & \bar{a}^\top \\ \bar{a} & a_1 I \end{pmatrix} \begin{pmatrix} x_1 \\ \bar{x} \end{pmatrix},$$

$$P_a(x) = \begin{pmatrix} \|a\|^2 & 2a_1 \bar{a}^\top \\ 2a_1 \bar{a} & (a_1^2 - \|\bar{a}\|^2)I + 2\bar{a}\bar{a}^\top \end{pmatrix} \begin{pmatrix} x_1 \\ \bar{x} \end{pmatrix}.$$

- (iii) For the Euclidean Jordan algebra of n -dimensional symmetric matrices, the above transformations are $L_A(X) = A \circ X = \frac{1}{2}(AX + XA)$ and $P_A(X) = AXA$.

A useful tool in the theory of Euclidean Jordan algebras is the *Peirce decomposition* theorem, which is stated as follows (see Theorem IV.2.1 of [23]).

Theorem 2.2 *Let $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$ be a Euclidean Jordan algebra with rank r , and let $\{e_1, \dots, e_r\}$ be a Jordan frame in \mathbb{V} . For $i, j \in \{1, 2, \dots, r\}$, define the eigenspaces*

$$\mathbb{V}_{ii} := \{x \in \mathbb{V} : x \circ e_i = x\} = \mathbb{R}e_i, \quad \mathbb{V}_{ij} := \left\{ x \in \mathbb{V} : x \circ e_i = \frac{1}{2}x = x \circ e_j \right\}, \quad i \neq j.$$

Then, the space \mathbb{V} is the orthogonal direct sum of subspaces \mathbb{V}_{ij} ($i \leq j$). Furthermore,

- (a) $\mathbb{V}_{ij} \circ \mathbb{V}_{ij} \subseteq \mathbb{V}_{ii} + \mathbb{V}_{jj}$;
- (b) $\mathbb{V}_{ij} \circ \mathbb{V}_{jk} \subseteq \mathbb{V}_{ik}$ if $i \neq k$;
- (c) $\mathbb{V}_{ij} \circ \mathbb{V}_{kl} = \{0\}$ if $\{i, j\} \cap \{k, l\} = \emptyset$.

Thus, given any Jordan frame $\{e_1, \dots, e_r\}$, we can write any element $x \in \mathbb{V}$ as

$$x = \sum_{1 \leq i \leq j \leq r} x_{ij} = \sum_{i=1}^r x_i e_i + \sum_{1 \leq i < j \leq r} x_{ij}, \quad (3)$$

where $x_i \in \mathbb{R}$ and $x_{ij} \in \mathbb{V}_{ij}$. Equation (3) corresponds to the Peirce decomposition of x associated with $\{e_1, \dots, e_r\}$.

Example 2.3

- (i) Let $\mathbb{V} = \mathbb{R}^n$ and $\{e_1, \dots, e_n\}$ be the canonical basis of \mathbb{R}^n , that is, e_i is a vector with 1 in the i th entry and 0s elsewhere. It is easily seen that $\{e_1, \dots, e_n\}$ is a Jordan frame in \mathbb{R}^n , called *canonical Jordan frame*, and also that

$$\mathbb{V}_{ii} = \{\kappa e_i : \kappa \in \mathbb{R}\} \quad \text{for } i = 1, \dots, n, \quad \mathbb{V}_{ij} = \{0\}, \quad i \neq j.$$

Hence, any element $x \in \mathbb{R}^n$ can be written as $x = \sum_{i=1}^n \kappa_i e_i$, which denotes its Peirce decomposition associated with $\{e_1, \dots, e_n\}$.

- (ii) Let $\mathbb{V} = \mathbb{R}^n$, and $\{e_1, e_2\}$ be defined by $e_1 = (\frac{1}{2}, \frac{1}{2}, \mathbf{0}_{n-2})$, $e_2 = (\frac{1}{2}, -\frac{1}{2}, \mathbf{0}_{n-2})$, where $\mathbf{0}_{n-2}$ is a vector of zeros in \mathbb{R}^{n-2} . Clearly, this set is a Jordan frame in \mathbb{R}^n , called the *canonical Jordan frame*. It is easy to verify that

$$\mathbb{V}_{ii} = \{\kappa e_i : \kappa \in \mathbb{R}\} \quad \text{for } i = 1, 2, \quad \mathbb{V}_{12} = \{x \in \mathbb{R}^n : x_1 = x_2 = 0\}.$$

Thus, given an $x \in \mathbb{V}$, we can write it as

$$x = (x_1 + x_2)e_1 + (x_1 - x_2)e_2 + (0, 0, x_3, \dots, x_n),$$

which is its Peirce decomposition associated with $\{e_1, e_2\}$.

- (iii) Let $\mathbb{V} = \mathcal{S}^n$ and consider the set $\{E_1, \dots, E_n\}$, where E_i is the diagonal matrix with 1 in the (i, i) -entry and 0s elsewhere. It is easily seen that this set is a Jordan frame in \mathcal{S}^n , called the *canonical Jordan frame*. Also, associated with this Jordan frame, it is easy to verify that

$$\mathbb{V}_{ii} = \{\kappa E_i : \kappa \in \mathbb{R}\} \quad \text{for } i = 1, \dots, n, \quad \mathbb{V}_{ij} = \{\theta E_{ij} : \theta \in \mathbb{R}\}, \quad i \neq j,$$

where E_{ij} is a matrix with 1 in the (i, j) and (j, i) -entries and 0s elsewhere. Thus, any $X \in \mathcal{S}^n$ can be written as

$$X = \sum_{i=1}^n x_{ii} E_i + \sum_{1 \leq i < j \leq n} x_{ij} E_{ij}.$$

This expression denotes the Peirce decomposition of X associated with $\{E_1, \dots, E_n\}$.

Orthogonal Projection In \mathbb{V} , fix a Jordan frame $\{e_1, \dots, e_r\}$ and define

$$\mathbb{V}^{(\alpha)} := \{x \in \mathbb{V} : x \circ (e_1 + \dots + e_l) = x\}$$

for $\alpha = \{1, \dots, l\}$ with $1 \leq l \leq r$. This set is a subalgebra of \mathbb{V} with rank l (see [23, Proposition IV.1.1]). The symmetric cone in this subalgebra is defined by $\mathcal{K}^{(\alpha)} := \{y \circ y : y \in \mathbb{V}^{(\alpha)}\} = \mathbb{V}^{(\alpha)} \cap \mathcal{K}$ (see [12, Theorem 3.1]). Corresponding to $\mathbb{V}^{(\alpha)}$, we consider the (orthogonal) projection $P^{(\alpha)} : \mathbb{V} \rightarrow \mathbb{V}^{(\alpha)}$. Let $x \in \mathbb{V}$ be written as $x = u + v$, where $u \in \mathbb{V}^{(\alpha)}$ and $v \in (\mathbb{V}^{(\alpha)})^\perp$. Then, $P^{(\alpha)}(x) = u$. Now, we consider the Peirce decomposition of x given in (3) corresponding to $\{e_1, \dots, e_r\}$. Then (see [10, Lemma 20])

$$P^{(\alpha)}(x) = \sum_{i=1}^l x_i e_i + \sum_{1 \leq i < j \leq l} x_{ij}.$$

Note that, for a given Jordan frame $\{e_1, \dots, e_r\}$, we can permute the objects and select the first l objects (for any $1 \leq l \leq r$). Thus there are $2^r - 1$ projections $P^{(\alpha)}$ corresponding to a Jordan frame.

In a similar way one can define the subalgebra $\mathbb{V}^{(\bar{\alpha})}$ by using the set $\{e_{l+1}, \dots, e_r\}$ and also the projection $P^{(\bar{\alpha})}$ on $\mathbb{V}^{(\bar{\alpha})}$, where $\bar{\alpha} = \{1, \dots, r\} \setminus \alpha$.

Example 2.4 [10, Example 1.2] For $\mathbb{V} = \mathcal{S}^n$, consider the Jordan frame $\{E_1, \dots, E_n\}$ (defined in Example 2.3(iii)). Let $\alpha := \{1, \dots, l\}$ with $1 \leq l \leq n$. Then $X \in \mathbb{V}^{(\alpha)}$ has the form

$$X = \begin{pmatrix} X_{\alpha\alpha} & 0 \\ 0 & 0 \end{pmatrix},$$

where $X_{\alpha\alpha}$ is the principal submatrix of X corresponding to the index set α . Thus, we may view $\mathbb{V}^{(\alpha)}$ as $\mathcal{S}^{|\alpha|}$. Hence, the projection $P^{(\alpha)} : \mathcal{S}^n \rightarrow \mathbb{V}^{(\alpha)}$ is given by

$$P^{(\alpha)}(Y) = \begin{pmatrix} Y_{\alpha\alpha} & 0 \\ 0 & 0 \end{pmatrix}.$$

The following result characterizes all Euclidean Jordan algebras (See [23, Propositions III.4.4 and III.4.5, Theorem V.3.7]).

Theorem 2.3 *Any Euclidean Jordan algebra is, in a unique way, a direct sum of simple Euclidean Jordan algebras. Moreover, the symmetric cone in a given Euclidean Jordan algebra is, in a unique way, a direct sum of symmetric cones in the constituent simple Euclidean Jordan algebras.*

We note that the ‘‘direct sum’’ in the theorem refers to the orthogonal as well as the Jordan product direct sum. Thus, given a Euclidean Jordan algebra \mathbb{V} and the corresponding symmetric cone \mathcal{K} , we may write

$$\mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2 \times \dots \times \mathbb{V}_{\bar{j}} \quad \text{and} \quad \mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \dots \times \mathcal{K}_{\bar{j}},$$

where each \mathbb{V}_j is a simple Jordan Algebra with the corresponding symmetric cone \mathcal{K}_j . Moreover, for $x = (x^{(1)}, x^{(2)}, \dots, x^{(\bar{j})})$ and $y = (y^{(1)}, y^{(2)}, \dots, y^{(\bar{j})})$ in \mathbb{V} with $x^{(j)}, y^{(j)} \in \mathbb{V}_j$, we have

$$x \circ y = (x^{(1)} \circ y^{(1)}, \dots, x^{(\bar{j})} \circ y^{(\bar{j})}), \quad \langle x, y \rangle = \sum_{j=1}^{\bar{j}} \langle x^{(j)}, y^{(j)} \rangle, \quad \|x\|^2 = \sum_{j=1}^{\bar{j}} \|x^{(j)}\|^2.$$

Remark 2.2 When the Jordan Algebra \mathbb{V} is not simple (that is, when $\bar{j} > 1$ in the previous setting), it can be verified that every primitive idempotent element c of \mathbb{V} has necessarily the form $c = (0, 0, \dots, c^{(j)}, 0, \dots, 0)$ for some primitive idempotent element $c^{(j)}$ in \mathbb{V}_j .

In any Euclidean Jordan algebra \mathbb{V} , one can define automorphism groups in the following way (Faraut and Korányi [23]).

Definition 2.1 A linear transformation $\Lambda : \mathbb{V} \rightarrow \mathbb{V}$ is said to be an *automorphism* of \mathbb{V} iff Λ is invertible and

$$\Lambda(x \circ y) = \Lambda(x) \circ \Lambda(y) \quad \text{for all } x, y \in \mathbb{V}. \quad (4)$$

The set of all automorphisms of \mathbb{V} is denoted by $\text{Aut}(\mathbb{V})$.

Definition 2.2 A linear transformation $\Lambda : \mathbb{V} \rightarrow \mathbb{V}$ is said to be an *automorphism* of \mathcal{K} iff $\Lambda(\mathcal{K}) = \mathcal{K}$. Note that this transformation constrained to \mathcal{K} is necessarily invertible. We denote the set of all automorphisms of \mathcal{K} by $\text{Aut}(\mathcal{K})$ and each element of it by Γ .

It directly follows from (4) that $\text{Aut}(\mathbb{V}) \subseteq \text{Aut}(\mathcal{K})$. Moreover, if $\Gamma \in \text{Aut}(\mathcal{K})$, then Γ^{-1} and $\Gamma^\top \in \text{Aut}(\mathcal{K})$ [12, Proposition 4.1].

Example 2.5

- (i) For $\mathbb{V} = \mathbb{R}^n$, it is easily seen that $\text{Aut}(\mathbb{R}^n)$ consists of permutation matrices and any element in $\text{Aut}(\mathbb{R}_+^n)$ is a product of a permutation matrix and a diagonal matrix with positive diagonal entries.
- (ii) For $\mathbb{V} = \mathbb{R}^n$, it is known [10, Example 2.1] that any automorphism Λ in $\text{Aut}(\mathbb{R}^n)$ can be written as $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$, where D is an $(n-1) \times (n-1)$ orthogonal matrix. Also, an $n \times n$ matrix $\Gamma \in \text{Aut}(\mathbb{R}_+^n)$ iff there exists $\mu > 0$ such that $\Gamma^\top J \Gamma = \mu J$, where $J = \text{Diag}(1, -1, \dots, -1) \in \mathbb{R}^{n \times n}$.
- (iii) For $\mathbb{V} = \mathcal{S}^n$, it is known (see [25, Theorem 2]) that corresponding to any $\Lambda \in \text{Aut}(\mathcal{S}^n)$, there exists an orthogonal matrix U such that $\Lambda(X) = UXU^\top \forall X \in \mathcal{S}^n$. Also, for $\Gamma \in \text{Aut}(\mathcal{S}_+^n)$, there exists an invertible matrix $Q \in \mathbb{R}^{n \times n}$ such that $\Gamma(X) = QXQ^\top, \forall X \in \mathcal{S}^n$.

From now on, $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$ will be a Euclidean Jordan algebra of rank r , and $\{e_1, \dots, e_r\}$ will be a Jordan frame in \mathbb{V} .

We end this subsection by recalling properties that we shall employ throughout this paper. Their proofs and more details can be found in [10, 12, 22, 23, 26].

Proposition 2.1 *The following results hold:*

- (a) $x \in \mathcal{K}$ if and only if $\langle x, y \rangle \geq 0$ for all $y \in \mathcal{K}$. Moreover, $x \in \text{int}(\mathcal{K})$ if and only if $\langle x, y \rangle > 0$ for all $y \in \mathcal{K} \setminus \{0\}$.
- (b) For $x, y \in \mathcal{K}$, orthogonality condition $\langle x, y \rangle = 0$ is equivalent to $x \circ y = 0$. In this situation, the elements x and y operator commute, that is,

$$L_x L_y = L_y L_x.$$

- (c) The elements x and y operator commute if and only if x and y have their spectral decompositions with respect to a common Jordan frame.
- (d) If $x \in \mathcal{K}$ (resp. $\in \text{int}(\mathcal{K})$), then $P^{(\alpha)}(x) \in \mathcal{K}^{(\alpha)}$ (resp. $\in \text{int}(\mathcal{K}^{(\alpha)})$).
- (e) Suppose that $x \in \mathcal{K}$ and let $x = \sum_{i=1}^r x_i e_i + \sum_{1 \leq i < j \leq r} x_{ij}$ be its Peirce decomposition. If $x_k = 0$ for some index k , then

$$\sum_{1 \leq k < j \leq r} x_{kj} + \sum_{1 \leq i < k \leq r} x_{ik} = 0.$$

- (f) For any $x, y \in \mathbb{V}$, we have $\text{tr}(x \circ y) \leq \sum_{i=1}^r \lambda_i^\uparrow(x) \lambda_i^\uparrow(y)$, where $\lambda^\uparrow(x)$ denotes the vector of eigenvalues of x whose components are arranged in the nondecreasing order.
- (g) Let $x, y \in \mathcal{K}$. Then $\text{tr}(x \circ y) \geq 0$. Moreover, $\text{tr}(x \circ y) = 0$ if and only if $x \circ y = 0$.
- (h) The smallest and the largest eigenvalues of $x \in \mathbb{V}$ are given by

$$\lambda_{\min}(x) = \min_{u \neq 0} \frac{\text{tr}(x \circ u^2)}{\text{tr}(u^2)}, \quad \lambda_{\max}(x) = \max_{u \neq 0} \frac{\text{tr}(x \circ u^2)}{\text{tr}(u^2)}.$$

In particular, when \mathbb{V} is simple, these eigenvalues can be equivalently written as

$$\lambda_{\min}(x) = \min_{u \neq 0} \frac{\langle x, u^2 \rangle}{\|u\|^2}, \quad \lambda_{\max}(x) = \max_{u \neq 0} \frac{\langle x, u^2 \rangle}{\|u\|^2}.$$

2.2 Linear Transformations Review

The literature on symmetric cone LCP (see [10, 11, 13, 21]) has already been extended from the LCP theory. Most of the well-known classes of matrices used in that context have been extended to symmetric cone LCP. We list these classes here below to be employed in the sequel. Given a linear transformation $L \in \mathcal{L}(\mathbb{V})$, we say that:

- L has the Q -property iff $\text{SOL}(L, \mathcal{K}, q) \neq \emptyset$ for all $q \in \mathbb{V}$.
- L has the Q_b -property iff $\text{SOL}(L, \mathcal{K}, q) \neq \emptyset$ and bounded for all $q \in \mathbb{V}$.
- L is an R_0 -transformation iff $\text{SOL}(L, \mathcal{K}, 0) = \{0\}$.
- L is copositive (resp. strictly copositive) iff $\langle L(x), x \rangle \geq 0$ (resp. > 0) for all $x \in \mathcal{K}$ (resp. for all $x \in \mathcal{K}, x \neq 0$).
- L is monotone (resp. strongly monotone) iff $\langle L(x), x \rangle \geq 0$ (resp. > 0) for all $x \in \mathbb{V}$ (resp. for all $x \in \mathbb{V}, x \neq 0$).

- L has the P -property iff [x and $L(x)$ operator commute and $x \circ L(x) \in -\mathcal{K} \Rightarrow x = 0$].
- L has the Q_0 -property iff $[\text{FEAS}(L, \mathcal{K}, q) \neq \emptyset \Rightarrow \text{SOL}(L, \mathcal{K}, q) \neq \emptyset]$.
- L has the S -property iff there is an $x \in \text{int}(\mathcal{K})$ such that $L(x) \in \text{int}(\mathcal{K})$.
- L is normal iff L commutes with L^\top . Here, $L^\top : \mathbb{V} \rightarrow \mathbb{V}$ denotes the transpose of L , which is defined by $\langle L(x), y \rangle = \langle x, L^\top(y) \rangle$ for all $x, y \in \mathbb{V}$.
- L is a star-transformation iff $[v \in \text{SOL}(L, \mathcal{K}, 0) \Rightarrow L^\top(v) \in -\mathcal{K}]$.
- L has the Z -property iff $[x, y \in \mathcal{K}, \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leq 0]$.

It is easy to check that monotone (resp. strongly monotone) transformations are copositive (resp. strictly copositive) and that self-adjoint (that is, $L^\top = L$), skew-symmetric (that is, $L^\top = -L$), and orthogonal (that is, $L^{-1} = L^\top$) transformations are normal transformations.

Next proposition establishes some links between the classes mentioned above.

Proposition 2.2 *Let $L \in \mathcal{L}(\mathbb{V})$ and $q \in \mathbb{V}$ be given. The following relations hold:*

- (a) L is strongly monotone $\implies L \in P \implies L \in R_0$;
- (b) $L \in S \iff \text{FEAS}(L, \mathcal{K}, q) \neq \emptyset$ for all $q \in \mathbb{V}$;
- (c) $Q = Q_0 \cap S$;
- (d) If $L \in R_0$, or skew-symmetric, or monotone, or $-L$ is copositive, or $L \in Z$ and normal $\implies L$ is a star-transformation.

Proof Statement (a) is proven in [10]. The equality in (c) follows from (b).

(b): (\Leftarrow) Let $d \in \text{int}(\mathcal{K})$. By hypothesis $\text{FEAS}(L, \mathcal{K}, -d) \neq \emptyset$, that is, there exists $x \in \mathcal{K}$ such that $y = L(x) - d \in \mathcal{K}$. From this we obtain $L(x) = y + d \in \text{int}(\mathcal{K})$, since $\mathcal{K} + \text{int}(\mathcal{K}) = \text{int}(\mathcal{K})$. Hence $L \in S$.

(\Rightarrow) As \mathcal{K} is self-dual closed convex cone with $\text{int}(\mathcal{K}) \neq \emptyset$, by [27, Theorem 2.2.13] we conclude that \mathcal{K} has a closed bounded base, that is, $\mathcal{K} = \text{cone}(B)$, where B is a compact set such that $0 \notin B$. By hypothesis there is $x \in \mathcal{K}$ such that $L(x) \in \text{int}(\mathcal{K})$. Fix $q \in \mathbb{V}$. As B is compact, there exist e_1, e_2 such that $\min_{e \in B} \langle L(x), e \rangle = \langle L(x), e_1 \rangle > 0$ and $\min_{e \in B} \langle q, e \rangle = \langle q, e_2 \rangle$. Clearly, there is some $t > 0$ such that $t \langle L(x), e_1 \rangle + \langle q, e_2 \rangle > 0$. For each $y \in \mathcal{K}$, there exist $\gamma \geq 0$ and $e \in B$ such that $y = \gamma e$. Therefore,

$$\langle tL(x) + q, y \rangle = \gamma (t \langle L(x), e \rangle + \langle q, e \rangle) \geq \gamma (t \langle L(x), e_1 \rangle + \langle q, e_2 \rangle) > 0.$$

Then, as $y \in \mathcal{K}$ was arbitrary, by Proposition 2.1, Part (a) we obtain that $tL(x) + q = L(tx) + q \in \mathcal{K}$. Thus, $tx \in \text{FEAS}(L, \mathcal{K}, q)$. The desired equivalence follows.

(d): When $L \in R_0$ or L is skew-symmetric, the proof is trivial. For the other cases, let $v \in \text{SOL}(L, \mathcal{K}, 0)$, that is, $v, L(v) \in \mathcal{K}$ and $\langle v, L(v) \rangle = 0$. Firstly, suppose that L is monotone. Since $\langle L(tx - v), tx - v \rangle \geq 0$ for all $t \in \mathbb{R}$ and $x \in \mathcal{K}$, Proposition 2.1, Part (a), allows us to obtain that $(L + L^\top)(v) \in \mathcal{K} \cap (-\mathcal{K})$. Hence, since \mathcal{K} is pointed (because it is symmetric), i.e., $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$, it follows that $(L + L^\top)(v) = 0$. Consequently, $L^\top(v) = -L(v) \in -\mathcal{K}$. We thus conclude that L is a star-transformation. Secondly, if $-L$ is copositive, then $\langle L(tx + v), tx + v \rangle \leq 0$ for all $t > 0$ and $x \in \mathcal{K}$. Similarly to before, Proposition 2.1, Part (a), allows us to obtain that $L^\top(v) \in -\mathcal{K}$.

Thirdly, we assume that $L \in Z$ and it is normal. This amounts to $\langle L(v), L(v) \rangle \leq 0$, and consequently $L(v) = 0$. Moreover, the normality of L implies that

$$\|L^\top(v)\|^2 = \langle v, L(L^\top(v)) \rangle = \langle v, L^\top(L(v)) \rangle = \|L(v)\|^2 = 0,$$

which means that $L^\top(v) = 0$. It obviously belongs to $-\mathcal{K}$. □

3 Characterizations of Q - and Q_b -transformations

A direct consequence of [1, Proposition 2.5.6] is that, within the class R_0 , the classes Q and Q_b coincide. This result holds even when \mathcal{K} is a general closed convex solid cone in the SCLCP (cf. (1)). Let us recall this result.

Lemma 3.1 *Let $L \in R_0$. Then, $L \in Q_b \iff L \in Q$.*

Moreover, since $\text{SOL}(L, \mathcal{K}, 0)$ is always a cone, we also have the following:

Lemma 3.2 *It holds that $Q_b \subseteq R_0$. Consequently, $Q_b = Q \cap R_0$.*

The previous lemmas motivate us to study classes of linear transformations in $\mathcal{L}(\mathbb{V})$ larger than R_0 , for which the last results are fulfilled.

3.1 The Class of F -transformations and Its Subclasses

Inspired by [17] and [18, Definition 3.5], we introduce the next new class of linear transformations in $\mathcal{L}(\mathbb{V})$.

Definition 3.1 We say that $L \in \mathcal{L}(\mathbb{V})$ is an F -transformation or $L \in F$ iff for each $v \in \text{SOL}(L, \mathcal{K}, 0) \setminus \{0\}$, there exists $\chi_v \in \mathbb{V}$ such that

$$(i) \chi_v \in \mathcal{K}, \quad (ii) \langle \chi_v, v \rangle > 0, \quad (iii) L^\top(\chi_v) \in -\mathcal{K}. \tag{5}$$

Indeed, this class was defined in [17] in the LCP context as follows:

Definition 3.2 A matrix $M \in \mathbb{R}^{n \times n}$ is said to be an F_1 -matrix iff, for every $v \in \text{SOL}(M, 0) \setminus \{0\}$, there exists a nonnegative diagonal matrix Σ such that $\Sigma v \neq 0$ and $M^\top \Sigma v \in -\mathbb{R}_+^n$. Here $\text{SOL}(M, q)$ denotes, for given $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, the solution of problem $\text{LCP}(M, q)$.

Therein, the equivalence of Lemma 3.1 is proven within the class F_1 . This class turns to be larger than R_0 , which makes the result interesting to be analyzed for more general complementarity problems. So, in [18], this definition was extended to the SDLCP framework as well as the desired equivalence between classes Q and Q_b . As one can expect, both definitions, for the LCP and the SDLCP setting, are particular cases of Definition 3.1 given above. The rest of this section is dedicated to extend the desired equivalence within the class F and to study its different subclasses.

Given a linear transformation $L : \mathbb{V} \rightarrow \mathbb{V}$ and $\Lambda \in \text{Aut}(\mathbb{V})$, we define a linear transformation \tilde{L} on \mathbb{V} by $\tilde{L} := \Lambda^\top L \Lambda$.

Example 3.1 Consider in $\mathbb{V} = \mathcal{S}^n$ the automorphism $\Lambda \in \text{Aut}(\mathbb{V})$ defined in Example 2.5(iii). Then, $\tilde{L}(X) = U^\top L(UXU^\top)U$.

The next result shows that this class is invariant under automorphisms.

Lemma 3.3 *Let $L : \mathbb{V} \rightarrow \mathbb{V}$ be a linear transformation, and $\Lambda \in \text{Aut}(\mathbb{V})$ be orthogonal, that is, $\Lambda^{-1} = \Lambda^\top$. Then L has the F -property if and only if \tilde{L} has the F -property.*

Proof We first point out that $\Lambda^{-1}(\text{SOL}(L, \mathcal{K}, q)) = \text{SOL}(\tilde{L}, \mathcal{K}, \Lambda^\top q)$ (see [12, Theorem 5.1]).

(\Rightarrow): Let w be a nonzero solution of $\text{LCP}(\tilde{L}, \mathcal{K}, 0)$. Then, by the above equality, we get that for any $\Lambda \in \text{Aut}(\mathbb{V})$ orthogonal, $v = \Lambda(w)$ is nonzero solution of $\text{LCP}(L, \mathcal{K}, 0)$. So, by hypothesis there exists a $\chi_v \in \mathbb{V}$ such that (5) holds. Clearly, $\chi_w = \Lambda^\top(\chi_v) \in \mathcal{K}$. Also, $\langle \chi_w, w \rangle = \langle \chi_v, v \rangle > 0$. Finally, since $L^\top(\chi_v) \in -\mathcal{K}$, Λ is orthogonal and $\Lambda^\top(\mathcal{K}) = \mathcal{K}$, it follows that $\tilde{L}^\top(\chi_w) \in -\mathcal{K}$ and hence $\tilde{L} \in F$.

(\Leftarrow): Let v be a nonzero solution of $\text{LCP}(L, \mathcal{K}, 0)$. Then, by the above equality, we get that for any $\Lambda \in \text{Aut}(\mathbb{V})$ being orthogonal, $w = \Lambda^{-1}(v)$ is a nonzero solution of $\text{LCP}(\tilde{L}, \mathcal{K}, 0)$. Thus, by hypothesis there exists a $\chi_w \in \mathbb{V}$ such that (5) holds. Clearly, $\chi_v = \Lambda(\chi_w) \in \mathcal{K}$ and $\langle \chi_v, v \rangle = \langle \chi_w, \Lambda^\top(v) \rangle > 0$, since $\Lambda(\mathcal{K}) = \mathcal{K}$ and Λ is orthogonal. Finally, since $\tilde{L}^\top(\chi_w) \in -\mathcal{K}$, $\tilde{L}^\top(\chi_w) = \Lambda^\top L^\top(\chi_v)$, and $\Lambda^\top(\mathcal{K}) = \mathcal{K}$, it follows that $L^\top(\chi_v) \in -\mathcal{K}$ and hence $L \in F$. \square

We now establish the main properties of the class F . In particular, assertion (b) below extends Lemma 3.1 to this larger class.

Theorem 3.1 *Let $L \in \mathcal{L}(\mathbb{V})$ be given.*

- (a) *If $L \in F \cap S$, then $L \in R_0$;*
- (b) *Let $L \in F$. Then, $L \in Q_b \iff L \in Q$.*

Proof (a): Let $L \in F \cap S$. We argue by contradiction. Suppose that $L \notin R_0$, that is, there exists $v \in \text{SOL}(L, \mathcal{K}, 0) \setminus \{0\}$. Since $L \in F$, there exists a χ_v satisfying (i)–(iii) in Definition 3.1. This, together with Proposition 2.1, Part (a), implies that $\langle L(x) - v, \chi_v \rangle < 0$ for all $x \in \mathcal{K}$. Consequently, $L(x) - v \notin \mathcal{K}$ for all $x \in \mathcal{K}$. Therefore, by Proposition 2.2, Part (b), it follows that $L \notin S$, obtaining a contradiction.

(b): Obviously $L \in Q_b$ implies $L \in Q$. If $L \in Q$, then $L \in S$ (cf. Proposition 2.2, Part (c)). Thus, $L \in F \cap S$. By item (a) above we conclude that $L \in R_0$, and consequently $L \in Q \cap R_0$. We thus conclude that $L \in Q_b$ thanks to the equality $Q_b = Q \cap R_0$ established in Lemma 3.2. \square

Remark 3.1 Notice that neither $L \in F$ nor $L \in S$ is enough to ensure that $L \in R_0$. This was pointed out for LCPs in [17, p. 452].

To check whether a linear transformation L belongs to F can be a difficult task. This is mainly because there is no a clear guide about how to chose, for a given $v \in \text{SOL}(L, \mathcal{K}, 0) \setminus \{0\}$, a χ_v satisfying conditions (i)–(iii) in Definition 3.1. For this, we focus now on the subclass of F for which χ_v is chosen via the Schur product (see [28, 29] for more details) of two elements.

Definition 3.3 For any $A = (a_{ij}) \in \mathcal{S}^r$ and $x \in \mathbb{V}$, with Peirce decomposition $x = \sum_{i \leq j} x_{ij}$, we define the Schur product of A and x by

$$A \bullet x := \sum_{1 \leq i \leq j \leq r} a_{ij} x_{ij}. \tag{6}$$

Definition 3.4 Let $L : \mathbb{V} \rightarrow \mathbb{V}$ be a linear transformation. We say that L is an F_1 -transformation or $L \in F_1$ iff for each $v \in \text{SOL}(L, \mathcal{K}, 0) \setminus \{0\}$ with Peirce decomposition $v = \sum_{i \leq j} v_{ij}$, there exists a matrix $\mathcal{E} = (\xi_{ij}) \in \mathcal{S}_+^r$ such that

$$(i) \mathcal{E} \bullet v \in \mathcal{K}, \quad (ii) \langle \mathcal{E} \bullet v, v \rangle > 0, \quad (iii) L^\top(\mathcal{E} \bullet v) \in -\mathcal{K}.$$

Remark 3.2 Notice that, if \mathcal{E} is positive semidefnite and $x \in \mathcal{K}$, then $\mathcal{E} \bullet x \in \mathcal{K}$ (cf. [28, Proposition 2.2]). Hence, condition (i) above becomes superfluous.

We illustrate this concept in the Euclidean Jordan algebras defined in Example 2.1.

Example 3.2

- (i) For $\mathbb{V} = \mathbb{R}^n$ and $\mathcal{K} = \mathbb{R}_+^n$, we consider the Jordan frame $\{e_1, \dots, e_n\}$ (defined in Example 2.3(i)). Then, for $\mathcal{E} = (\xi_{ij}) \in \mathcal{S}_+^n$, the Schur product of \mathcal{E} and $v \in \text{SOL}(L, \mathbb{R}_+^n, 0) \setminus \{0\}$ reduces to

$$\mathcal{E} \bullet v = \sum_{i=1}^n \xi_{ii} v_i e_i = \text{Diag}(\xi_{11}, \dots, \xi_{nn})v,$$

where $v = \sum_{i=1}^n v_i e_i$ is its Peirce decomposition associated with $\{e_1, \dots, e_n\}$. Taking $\Sigma = \text{Diag}(\xi_{11}, \dots, \xi_{nn}) \in \mathcal{S}_+^n$, Definition 3.4 reduces to Definition 3.2 in the LCP context.

- (ii) For $\mathbb{V} = \mathbb{R}^n$, $\mathcal{K} = \mathcal{L}_+^n$, we consider the canonical Jordan frame $\{e_1, e_2\}$ (defined in Example 2.3(ii)). Then, for $\mathcal{E} = (\xi_{ij}) \in \mathcal{S}_+^2$, the Schur product of \mathcal{E} and $v \in \text{SOL}(L, \mathcal{L}_+^n, 0) \setminus \{0\}$ reduces to

$$\mathcal{E} \bullet v = \xi_{11}(v_1 + v_2)e_1 + \xi_{22}(v_1 - v_2)e_2 + \xi_{12}(0, 0, v_3, \dots, v_n),$$

where $v = (v_1 + v_2)e_1 + (v_1 - v_2)e_2 + (0, 0, v_3, \dots, v_n)$ is its Peirce decomposition associated with $\{e_1, e_2\}$. Taking into account this, condition (ii) of Definition 3.4 is reduced to

$$\langle \mathcal{E} \bullet v, v \rangle = \frac{1}{2}(\xi_{11}(v_1 + v_2)^2 + \xi_{22}(v_1 - v_2)^2) + \xi_{12} \|(v_3, \dots, v_n)\|^2 > 0.$$

(iii) For $\mathbb{V} = \mathcal{S}^n$ and $\mathcal{K} = \mathcal{S}^n_+$, we consider the canonical Jordan frame $\{E_1, \dots, E_n\}$ (defined in Example 2.3(iii)). Then, for $\mathcal{E} = (\xi_{ij}) \in \mathcal{S}^n_+$, the Schur product of \mathcal{E} and $V \in \text{SOL}(L, \mathcal{S}^n_+, 0) \setminus \{0\}$ coincide with the well-known Schur (or Hadamard) product of two symmetric matrices (see [30]). Thus, Definition 3.4 reduces to definition of F_1 -transformation given in [18] in the SDLCP context: $L \in \mathcal{L}(\mathcal{S}^n)$ is an F_1 -transformation iff for each $V \in \text{SOL}(L, \mathcal{S}^n_+, 0) \setminus \{0\}$, there exists a matrix $\Lambda \in \mathcal{S}^n_+$ such that

$$(i) \Lambda \bullet V \in \mathcal{S}^n_+, \quad (ii) \langle \Lambda \bullet V, V \rangle > 0, \quad (iii) L^\top(\Lambda \bullet V) \in -\mathcal{S}^n_+.$$

In the following proposition we list various classes of linear transformations that are contained in the class F_1 .

Proposition 3.1 *If L is a star-transformation, then $L \in F_1$.*

Proof Let $v \in \text{SOL}(L, \mathcal{K}, 0) \setminus \{0\}$ with Peirce decomposition $v = \sum_{i \leq j} v_{ij}$. Since L is a star-transformation, we have $L^\top(v) \in -\mathcal{K}$. Then, conditions (i)–(iii) of Definition 3.4 can be easily checked, provided that $\mathcal{E} = \mathbb{1}$, the $n \times n$ matrix whose entries are all equal to 1 (note that $\mathbb{1} \in \mathcal{S}^n_+$ and $\mathbb{1} \bullet v = v$). The result follows. \square

T-Transformation. In the following definition, we extend the notion of T -property for matrices given in [19] to our SCLCP context.

Definition 3.5 We say that $L \in \mathcal{L}(\mathbb{V})$ has the T -property iff for any $\Lambda \in \text{Aut}(\mathbb{V})$ being orthogonal and any index set $\alpha = \{1, \dots, l\}$ ($1 \leq l \leq r$), the existence of a solution $x \in \mathbb{V}$ to the system

$$\begin{aligned} P^{(\alpha)}(x) \in \text{int}(\mathcal{K}^{(\alpha)}), \quad (I - P^{(\alpha)})(x) = 0, \\ P^{(\alpha)}(\tilde{L}(x)) \in -\mathcal{K}^{(\alpha)}, \quad P^{(\bar{\alpha})}(\tilde{L}(x)) \in \mathcal{K}^{(\bar{\alpha})}, \quad (I - P^{(\alpha)} - P^{(\bar{\alpha})})(\tilde{L}(x)) = 0, \end{aligned} \tag{7}$$

$\bar{\alpha} = \{1, \dots, r\} \setminus \alpha$, implies that there is a nonzero $y \in \mathbb{V}$ satisfying

$$\begin{aligned} P^{(\alpha)}(y) \in \mathcal{K}^{(\alpha)}, \quad (I - P^{(\alpha)})(y) = 0, \\ P^{(\alpha)}(\tilde{L}^\top(y)) \in -\mathcal{K}^{(\alpha)}, \quad P^{(\bar{\alpha})}(\tilde{L}^\top(y)) \in -\mathcal{K}^{(\bar{\alpha})}, \quad (8) \\ (I - P^{(\alpha)} - P^{(\bar{\alpha})})(\tilde{L}^\top(y)) = 0, \quad \langle P^{(\alpha)}(y), P^{(\alpha)}(\tilde{L}(x)) \rangle = 0. \end{aligned}$$

We illustrate this concept in some known examples of Euclidean Jordan algebras.

Example 3.3

(i) For $\mathbb{V} = \mathbb{R}^n$ and $\mathcal{K} = \mathbb{R}^n_+$, we consider the Jordan frame $\{e_1, \dots, e_n\}$ (defined in Example 2.3(i)) and set $\alpha = \{1, \dots, l\}$ with $1 \leq l \leq n$. Then, it is easily seen that $\mathbb{V}^{(\alpha)} = \{(x_\alpha, 0) \in \mathbb{V} : x_\alpha \in \mathbb{R}^l\}$ and $\mathbb{V}^{(\bar{\alpha})} = \{(0, x_{\bar{\alpha}}) \in \mathbb{V} : x_{\bar{\alpha}} \in \mathbb{R}^{|\bar{\alpha}|}\}$. Hence, the projection $P^{(\alpha)}$ of $x \in \mathbb{V}$ on $\mathbb{V}^{(\alpha)}$ is given by $P^{(\alpha)}(x) = \sum_{i=1}^l x_i e_i$, where $x = \sum_{i=1}^n x_i e_i$ is its Peirce decomposition. On the other hand, taking

$I \in \text{Aut}(\mathbb{R}^n)$ (because $I(e) = e$), we get that $P^{(\bar{\alpha})}(\tilde{L}(x)) = \sum_{i=l+1}^n z_i e_i$, where $L(x) = \sum_{i=1}^n z_i e_i$ is its Peirce decomposition associated to $\{e_1, \dots, e_n\}$. Hence, taking into account the above and that $L(x) = Mx$ with $M \in \mathbb{R}^{n \times n}$, Definition 3.5 reduces to saying: A matrix $M \in \mathbb{R}^{n \times n}$ is said to have T -property if and only if for any nonempty set $\alpha = \{1, \dots, l\} \subseteq \{1, \dots, n\}$, the existence of a vector $x_\alpha \in \mathbb{R}^{|\alpha|}$ satisfying

$$x_\alpha > 0, \quad M_{\alpha\alpha}x_\alpha \leq 0, \quad \text{and} \quad M_{\bar{\alpha}\alpha}x_\alpha \geq 0 \tag{9}$$

implies that there exists a nonzero vector $y_\alpha \in \mathbb{R}_+^{|\alpha|}$ such that

$$y_\alpha^\top M_{\alpha\alpha} \leq 0, \quad y_\alpha^\top M_{\alpha\bar{\alpha}} \leq 0, \quad \text{and} \quad y_\alpha^\top (M_{\alpha\alpha}x_\alpha) = 0. \tag{10}$$

The last definition by using subclass coincides with that given by Aganagić and Cottle [19] in the LCP context.

- (ii) For $\mathbb{V} = \mathbb{R}^n$ and $\mathcal{K} = \mathcal{L}_+^n$, we consider the canonical Jordan frame $\{e_1, e_2\}$ (defined in Example 2.3(ii)). Then, corresponding to $\alpha = \{1, 2\}$, we have $\mathbb{V}^{(\alpha)} = \mathbb{V}$ (because $e_1 + e_2 = e$), and, associated with $\alpha = \{1\}$ and $\bar{\alpha} = \{2\}$, we have $\mathbb{V}^{(\alpha)} = \mathbb{V}_{11}$ and $\mathbb{V}^{(\bar{\alpha})} = \mathbb{V}_{22}$, respectively. Moreover,

$$\mathcal{K}^{(\alpha)} = \{\kappa e_1 : \kappa \in \mathbb{R}_+\}.$$

Hence, for $\alpha = \{1\}$, the projection $P^{(\alpha)}$ of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ on $\mathbb{V}^{(\alpha)}$ is given by $P^{(\alpha)}(x) = (x_1 + x_2)e_1$, where

$$x = (x_1 + x_2)e_1 + (x_1 - x_2)e_2 + (0, 0, x_3, \dots, x_n)$$

is its Peirce decomposition associated with $\{e_1, e_2\}$. On the other hand, since $\mathbb{V}^{(\alpha)}$ are one-dimensional spaces, their orthogonal projections are easily computed at the elements of $L(\mathbb{V}^{(1)})$, with $L \in \mathcal{L}(\mathbb{R}^n)$, as follows:

$$P^{(\alpha)}(L(\kappa e_1)) = \kappa \frac{\langle L(e_1), e_1 \rangle}{\|e_1\|^2} e_1, \quad P^{(\bar{\alpha})}(L(\kappa e_1)) = \kappa \frac{\langle L(e_1), e_2 \rangle}{\|e_2\|^2} e_2, \quad \kappa \in \mathbb{R}.$$

So, for $L(x) = Mx$ with $M \in \mathbb{R}^{n \times n}$ and for $x \in \mathbb{R}^n$ with Peirce decomposition $x = (x_1 + x_2)e_1 + (x_1 - x_2)e_2 + (0, 0, x_3, \dots, x_n)$, Definition 3.5, for $\alpha = \{1\}$, reduces to: A matrix $M : \mathcal{L}^n \rightarrow \mathcal{L}^n$ has T -property iff for any matrix $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$ with $D \in \mathbb{R}^{(n-1) \times (n-1)}$ being an orthogonal matrix, the existence of a solution $x \in \mathbb{R}^n$ to the system

$$\begin{aligned} x_1 = x_2 > 0, \quad x_3 = \dots = x_n = 0, \quad \langle \Lambda^\top M \Lambda(e_1), e_1 \rangle \leq 0, \quad p = 0, \\ \langle \Lambda^\top M \Lambda(e_1), e_2 \rangle \geq 0, \end{aligned}$$

where $\tilde{L}(x) = \kappa_1 e_1 + \kappa_2 e_2 + (0, 0, p)$, implies that there is a nonzero $y \in \mathbb{R}^n$ satisfying

$$\begin{aligned} y_1 = y_2 \geq 0, \quad y_3 = \dots = y_n = 0, \quad y_1 \langle \Lambda^\top M^\top \Lambda(e_1), e_1 \rangle \leq 0, \quad p' = 0, \\ y_1 \langle \Lambda^\top M^\top \Lambda(e_1), e_2 \rangle \leq 0, \\ y_1 \langle \Lambda^\top M \Lambda(e_1), e_1 \rangle = 0, \end{aligned}$$

where $\tilde{L}^\top(y) = \kappa'_1 e_1 + \kappa'_2 e_2 + (0, 0, p')$. Here we have used the Peirce decomposition of y with respect to $\{e_1, e_2\}$.

- (iii) For $\mathbb{V} = \mathcal{S}^n$ and $\mathcal{K} = \mathcal{S}^n_+$, we consider the Jordan frame $\{E_1, \dots, E_n\}$ (defined in Example 2.3(iii)) and set $\alpha = \{1, \dots, l\}$ with $1 \leq l \leq n$. Then, taking into account Examples 2.4, 2.5, and 3.1, Definition 3.5 reduces to saying:

A linear transformation $L : \mathcal{S}^n \rightarrow \mathcal{S}^n$ is said to have the T -property if for any orthogonal matrix $U \in \mathbb{R}^{n \times n}$ and for any index set $\alpha = \{1, \dots, k\}$ ($1 \leq k \leq n$), the existence of a solution $X \in \mathcal{S}^n$ to the system

$$X_{\alpha\alpha} \in \mathcal{S}^{|\alpha|}_+, \quad X_{ij} = 0, \quad \forall i, j \notin \alpha, \quad [\tilde{L}_U(X)]_{\alpha\alpha} \in -\mathcal{S}^{|\alpha|}_+, \quad [\tilde{L}_U(X)]_{\bar{\alpha}\bar{\alpha}} \in \mathcal{S}^{|\bar{\alpha}|}_+, \\ [\tilde{L}_U(X)]_{\alpha\bar{\alpha}} = 0,$$

implies that there is a nonzero matrix $Y \in \mathcal{S}^n$ satisfying

$$Y_{\alpha\alpha} \in \mathcal{S}^{|\alpha|}_+, \quad Y_{ij} = 0, \quad \forall i, j \notin \alpha, \quad [\tilde{L}_U^\top(Y)]_{\alpha\alpha} \in -\mathcal{S}^{|\alpha|}_+, \quad [\tilde{L}_U^\top(Y)]_{\bar{\alpha}\bar{\alpha}} \in -\mathcal{S}^{|\bar{\alpha}|}_+, \\ [\tilde{L}_U^\top(Y)]_{\alpha\bar{\alpha}} = 0, \quad \langle Y_{\alpha\alpha}, [\tilde{L}_U(X)]_{\alpha\alpha} \rangle = 0.$$

As far as we know, this is the first time that a T -transformation is extended to a nonpolyhedral cone \mathcal{K} .

The following result is an extension of [19, Proposition 2] to our SCLCP context.

Proposition 3.2 *If a linear transformation L is monotone, then it has the T -property.*

Proof Let Λ be an orthogonal automorphism of \mathbb{V} , $\alpha = \{1, \dots, l\}$ ($1 \leq l \leq n$) be a nonempty index set, and $x \in \mathbb{V}$ a solution of system (7). Clearly, \tilde{L} and \tilde{L}^\top are monotone. Then, from the inequality

$$0 \leq \langle x, \tilde{L}(x) \rangle = \langle P^{(\alpha)}(x), P^{(\alpha)}(\tilde{L}(x)) \rangle$$

and the fact that $P^{(\alpha)}(x) \in \text{int}(\mathcal{K}^{(\alpha)})$ and $P^{(\alpha)}(\tilde{L}(x)) \in -\mathcal{K}^{(\alpha)}$ we deduce that $P^{(\alpha)}(\tilde{L}(x)) = 0$. Hence, $\langle x, \tilde{L}(x) \rangle = \langle P^{(\alpha)}(x), P^{(\alpha)}(\tilde{L}(x)) \rangle = 0$.

We claim that

$$P^{(\alpha)}(\tilde{L}^\top(x)) \in -\mathcal{K}^{(\alpha)}, \quad P^{(\bar{\alpha})}(\tilde{L}^\top(x)) \in -\mathcal{K}^{(\bar{\alpha})}, \quad (I - P^{(\alpha)} - P^{(\bar{\alpha})})(\tilde{L}^\top(x)) = 0.$$

Indeed, since $\langle x, (\tilde{L} + \tilde{L}^\top)(x) \rangle = 0$ and $\tilde{L} + \tilde{L}^\top$ is a self-adjoint monotone linear transformation, it clearly follows that $(\tilde{L} + \tilde{L}^\top)(x) = 0$. But $\tilde{L}(x) \in \mathcal{K}$ (because $P^{(\alpha)}(\tilde{L}(x)) = 0$ and (7) holds), then $\tilde{L}^\top(x) \in -\mathcal{K}$. From this it follows that $P^{(\alpha)}(\tilde{L}^\top(x)) \in -\mathcal{K}^{(\alpha)}$ and $P^{(\bar{\alpha})}(\tilde{L}^\top(x)) \in -\mathcal{K}^{(\bar{\alpha})}$. On the other hand, since $P^{(\alpha)}(x) \in \text{int}(\mathcal{K}^{(\alpha)})$, $P^{(\alpha)}(\tilde{L}^\top(x)) \in -\mathcal{K}^{(\alpha)}$, and

$$\langle P^{(\alpha)}(\tilde{L}^\top(x)), P^{(\alpha)}(x) \rangle = \langle \tilde{L}^\top(x), x \rangle = 0,$$

it follows that $P^{(\alpha)}(\tilde{L}^\top(x)) = 0$. This, together with the condition $-\tilde{L}^\top(x) \in \mathcal{K}$ and Proposition 2.1, Part (e), implies that $(I - P^{(\bar{\alpha})})(\tilde{L}^\top(x)) = 0$. Thus, $y = x$ solves (8). □

F₂-Transformation

Definition 3.6 A linear transformation $L : \mathbb{V} \rightarrow \mathbb{V}$ is said to have the F_2 -property iff for any $\Lambda \in \text{Aut}(\mathbb{V})$ orthogonal and any index set $\alpha = \{1, \dots, l\}$ ($1 \leq l \leq r$), the existence of a solution $x \in \mathbb{V}$ to the system

$$\begin{aligned} P^{(\alpha)}(x) \in \text{int}(\mathcal{K}^{(\alpha)}), \quad (I - P^{(\alpha)})(x) = 0, \\ (I - P^{(\bar{\alpha})})(\tilde{L}(x)) = 0, \quad P^{(\bar{\alpha})}(\tilde{L}(x)) \in \mathcal{K}^{(\bar{\alpha})}, \end{aligned} \tag{11}$$

$\bar{\alpha} = \{1, \dots, r\} \setminus \alpha$, implies that there is a nonzero $y \in \mathbb{V}$ satisfying

$$\begin{aligned} P^{(\alpha)}(y) \in \mathcal{K}^{(\alpha)}, \quad (I - P^{(\alpha)})(y) = 0, \\ (I - P^{(\bar{\alpha})})(\tilde{L}^\top(y)) = 0, \quad P^{(\bar{\alpha})}(\tilde{L}^\top(y)) \in -\mathcal{K}^{(\bar{\alpha})}. \end{aligned} \tag{12}$$

We illustrate this definition in the following Euclidean Jordan algebras.

Example 3.4

- (i) For $\mathbb{V} = \mathbb{R}^n$ and $\mathcal{K} = \mathbb{R}^n_+$, we consider the Jordan frame $\{e_1, \dots, e_n\}$ (defined in Example 2.3(i)) and set $\alpha = \{1, \dots, l\}$ with $1 \leq l \leq n$. Taking into account Example 3.3, Part (i), and that $L(x) = Mx$ with $M \in \mathbb{R}^{n \times n}$, Definition 3.6 reduces to saying: A matrix $M \in \mathbb{R}^{n \times n}$ is an F_2 -matrix iff for any nonempty set $\alpha = \{1, \dots, l\} \subseteq \{1, \dots, n\}$, the existence of a vector $x_\alpha \in \mathbb{R}^{|\alpha|}$ satisfying

$$x_\alpha > 0, \quad M_{\alpha\alpha}x_\alpha = 0, \quad \text{and} \quad M_{\bar{\alpha}\alpha}x_\alpha \geq 0 \tag{13}$$

implies that there exists a nonzero vector $y_\alpha \in \mathbb{R}^{|\alpha|}$ such that

$$y_\alpha^\top M_{\alpha\alpha} = 0 \quad \text{and} \quad y_\alpha^\top M_{\alpha\bar{\alpha}} \leq 0. \tag{14}$$

Notice that, in the LCP context, both classes, F_1 and F_2 , coincide with the class introduced by Flores and López in [17].

- (ii) For $\mathbb{V} = \mathbb{R}^n$ and $\mathcal{K} = \mathcal{L}^n_+$, we consider the Jordan frame $\{e_1, e_2\}$ defined in Example 2.3(ii). Then, taking into account the ideas of Example 3.3(ii) and letting $L(x) = Mx$ with $M \in \mathbb{R}^{n \times n}$ for all $x \in \mathbb{R}^n$, Definition 3.6, for $\alpha = \{1\}$, reduces to saying: A matrix $M \in \mathbb{R}^{n \times n}$ is said to have the F_2 -property iff for any matrix $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$ with $D \in \mathbb{R}^{(n-1) \times (n-1)}$ an orthogonal matrix, the existence of a solution $x \in \mathbb{R}^n$ to the system

$$\begin{aligned} x_1 = x_2 > 0, \quad x_3 = \dots = x_n = 0, \quad \langle \Lambda^\top M \Lambda(e_1), e_1 \rangle = 0, \quad p = 0, \\ \langle \Lambda^\top M \Lambda(e_1), e_2 \rangle \geq 0, \end{aligned}$$

where $\tilde{L}(x) = \kappa_1 e_1 + \kappa_2 e_2 + (0, 0, p)$, implies that there is a nonzero $y \in \mathbb{R}^n$ satisfying

$$\begin{aligned} y_1 = y_2 \geq 0, \quad y_3 = \dots = y_n = 0, \quad y_1 \langle \Lambda^\top M^\top \Lambda(e_1), e_1 \rangle = 0, \quad p' = 0, \\ y_1 \langle \Lambda^\top M^\top \Lambda(e_1), e_2 \rangle \leq 0, \end{aligned}$$

where $\tilde{L}^\top(y) = \kappa'_1 e_1 + \kappa'_2 e_2 + (0, 0, p')$.

- (iii) For $\mathbb{V} = \mathcal{S}^n$ and $\mathcal{K} = \mathcal{S}^n_+$, we consider the Jordan frame $\{E_1, \dots, E_n\}$ (defined in Example 2.3(iii)) and set $\alpha = \{1, \dots, l\}$ with $1 \leq l \leq n$. Then, taking into account Examples 2.4, 2.5(iii), and 3.1, Definition 3.6 reduces to Definition of F_2 -transformation given in [18] in SDLCP context: A linear transformation $L : \mathcal{S}^n \rightarrow \mathcal{S}^n$ is said to have the F_2 -property iff for any orthogonal matrix $U \in \mathbb{R}^{n \times n}$ and for any index set $\alpha = \{1, \dots, k\}$ ($1 \leq k \leq n$), the existence of a solution $X \in \mathcal{S}^n$ to the system

$$X_{\alpha\alpha} \in \mathcal{S}^{|\alpha|}_{++}, \quad X_{ij} = 0, \quad \forall i, j \notin \alpha, \quad [\tilde{L}_U(X)]_{\alpha\alpha} = 0, \quad [\tilde{L}_U(X)]_{\alpha\bar{\alpha}} = 0, \\ [\tilde{L}_U(X)]_{\bar{\alpha}\bar{\alpha}} \in \mathcal{S}^{|\bar{\alpha}|}_+,$$

implies that there is a nonzero matrix $Y \in \mathcal{S}^n$ satisfying

$$Y_{\alpha\alpha} \in \mathcal{S}^{|\alpha|}_+, \quad Y_{ij} = 0, \quad \forall i, j \notin \alpha, \quad [\tilde{L}_U^\top(Y)]_{\alpha\alpha} = 0, \quad [\tilde{L}_U^\top(Y)]_{\alpha\bar{\alpha}} = 0, \\ [\tilde{L}_U^\top(Y)]_{\bar{\alpha}\bar{\alpha}} \in -\mathcal{S}^{|\bar{\alpha}|}_+.$$

Proposition 3.3 *If L has the F_2 -property, then L is an F -transformation.*

Proof Let v be a nonzero solution of $LCP(L, \mathcal{K}, 0)$. Consider an orthogonal automorphism $\Lambda \in \text{Aut}(\mathbb{V})$ such that

$$\Lambda^\top(v) = \Lambda^{-1}(v) = \sum_{i=1}^r \lambda_i(v) e_i = \lambda_1(v) e_1 + \dots + \lambda_l(v) e_l + 0e_{l+1} + \dots + 0e_r,$$

where $\{e_1, \dots, e_r\}$ is a Jordan frame of \mathbb{V} , and $\lambda_i(v) > 0$ for all $i = 1, \dots, l$, with $l \in \{1, \dots, r\}$. We proceed to show that $x = \Lambda^{-1}(v)$ is a solution of (11). It is immediate that $x = P^{(\alpha)}(x) \in \text{int}(\mathcal{K}^{(\alpha)})$, where $\alpha = \{1, \dots, l\}$, and hence x satisfies the first two conditions of (11). Also, $\tilde{L}(x) = \Lambda^\top(L(v))$. So, since $L(v) \in \mathcal{K}$, it follows that $P^{(\bar{\alpha})}(\tilde{L}(x)) \in \mathcal{K}^{(\bar{\alpha})}$ (cf. [12, Remark 4.1] and Proposition 2.1, Part (d)). Moreover, the condition $\langle L(v), v \rangle = 0$ implies that $\langle x, \tilde{L}(x) \rangle = 0$. Then, by using Proposition 2.1, Parts (b) and (c), we get that $\tilde{L}(x)$ has the following spectral decomposition:

$$\tilde{L}(x) = \sum_{i=1}^r \lambda_i(\tilde{L}(x)) e_i = 0e_1 + \dots + 0e_l + \lambda_{l+1}(\tilde{L}(x)) e_{l+1} + \dots + \lambda_r(\tilde{L}(x)) e_r,$$

where $\lambda_i(\tilde{L}(x)) e_i \geq 0$ for all $i = l + 1, \dots, r$. From this it follows that $\tilde{L}(x) \in \mathcal{K}^{(\bar{\alpha})}$ and hence $P^{(\bar{\alpha})}(\tilde{L}(x)) = \tilde{L}(x)$. Then, $\tilde{L}(x)$ satisfies the last two conditions of (11). Therefore, there exists a nonzero solution y of (12).

We claim that $\tau_v = \Lambda(y)$ satisfies conditions (i)–(iii) in (5). Indeed, it is obvious that $\tau_v \in \mathcal{K}$ because $y = P^{(\alpha)}(y) \in \mathcal{K}^{(\alpha)}$ and $\Lambda(\mathcal{K}) = \mathcal{K}$. So, due to that $x \in \text{int}(\mathcal{K}^{(\alpha)})$ and $y \in \mathcal{K}^{(\alpha)}$ with $y \neq 0$, from Proposition 2.1, Part (a), we get

$$\langle \tau_v, v \rangle = \langle y, x \rangle > 0.$$

Finally, since $L^\top(\tau_v) = (\Lambda^\top)^{-1}(\tilde{L}^\top(y))$ and $\tilde{L}^\top(y) \in -\mathcal{K}$ (a consequence of (12)), it follows that $L^\top(\tau_v) \in -\mathcal{K}$. We have thus deduced that L is an F -transformation. \square

In the following proposition we list various classes of linear transformations that are contained in the class F_2 .

Proposition 3.4 *If L is a star-transformation or L has T -property, then $L \in F_2$.*

Proof First, we assume that L is a star-transformation. Let Λ be an orthogonal automorphism of \mathbb{V} , $\alpha = \{1, \dots, l\}$ ($1 \leq l \leq n$) be a nonempty index set, and $x \in \mathbb{V}$ be a solution of system (11). Let us define $v = \Lambda(x)$. Clearly, $v, L(v) = (\Lambda^\top)^{-1}(\tilde{L}(x)) \in \mathcal{K}$ (because $x, \tilde{L}(x) \in \mathcal{K}$ and $\Lambda, (\Lambda^\top)^{-1}$ preserve \mathcal{K}) and $\langle v, L(v) \rangle = \langle x, \tilde{L}(x) \rangle = 0$. Hence, v is a nonzero solution of $\text{SOL}(L, \mathcal{K}, 0)$. Since L is a star-transformation, we have that $L^\top(v) \in -\mathcal{K}$. Then, $\tilde{L}^\top(x) = \Lambda^{-1}L^\top(v) \in -\mathcal{K}$ (because Λ^{-1} preserves \mathcal{K}). On the other hand, since $P^{(\alpha)}(x) \in \text{int}(\mathcal{K}^{(\alpha)})$, $P^{(\alpha)}(\tilde{L}^\top(x)) \in -\mathcal{K}^{(\alpha)}$ (cf. Proposition 2.1, Part (d)), and

$$\langle P^{(\alpha)}(\tilde{L}^\top(x)), P^{(\alpha)}(x) \rangle = \langle \tilde{L}^\top(x), x \rangle = \langle x, \tilde{L}(x) \rangle = 0,$$

it follows that $P^{(\alpha)}(\tilde{L}^\top(x)) = 0$. This, together with the condition $-\tilde{L}^\top(x) \in \mathcal{K}$ and Proposition 2.1, Part (e), implies that $(I - P^{(\alpha)})(\tilde{L}^\top(x)) = 0$. Therefore, $y = x$ solves (12). Thus, we conclude that L has the F_2 -property.

Second, if L has the T -property, then obviously $L \in F_2$, since (11) implies that $P^{(\alpha)}(\tilde{L}(x)) = 0$, and this implies $P^{(\alpha)}(\tilde{L}^\top(y)) = 0$. \square

3.2 Examples of Transformations

In this section, we present some transformations in $\mathcal{L}(\mathbb{V})$ that belong to subclasses F_1, F_2 , and T . These linear transformations are intensively studied in the LCP literature.

1. *Lyapunov transformation:* Let $a \in \mathbb{V}$ with \mathbb{V} any Euclidean Jordan algebra. The Lyapunov transformation L_a defined in (2) is a self-adjoint and Z -transformation [13]. By Proposition 2.2, Part (d), Proposition 3.1, and Proposition 3.4, we have that $L_a \in F_1 \cap F_2$. On the other hand, if $a \in \text{int}(\mathcal{K})$, then L_a is strongly monotone (cf. Proposition 2.1, Part(a)); thus, L_a has the T -property by Proposition 3.2.
2. *Quadratic representation:* Let \mathbb{V} be a Euclidean Jordan algebra and $a \in \mathbb{V}$. If, in addition, V is simple and $\pm a \in \text{int}(K)$, the transformation P_a is strongly monotone (see [13, Theorem 6.5]). By Proposition 2.2, Parts (a) and (d), Proposition 3.1, and Proposition 3.4, we have that $P_a \in F_1 \cap F_2$. On the other hand, under the same assumptions, P_a also has the T -property by Proposition 3.2.
3. *Stein transformation:* Let $a \in \mathbb{V}$ with \mathbb{V} any Euclidean Jordan algebra. Consider the Stein transformation S_a defined by $S_a = I - P_a$. If $\lambda_i(\pm a) \subseteq (-1, 1)$ for all i , then the transformation S_a is strongly monotone (see [31, Theorem 3.3]). Hence, by using Proposition 2.2, Parts (a) and (d), Proposition 3.1, and Proposition 3.4, we have that $S_a \in F_1 \cap F_2$. On the other hand, under the same assumptions, S_a also has the T -property by Proposition 3.2.

4. *The relaxation transformation:* Let $\{e_1, \dots, e_r\}$ be a Jordan frame in \mathbb{V} , and $A \in \mathbb{R}^{r \times r}$. We define $R_A : \mathbb{V} \rightarrow \mathbb{V}$ as follows. For any $x \in \mathbb{V}$, write the Peirce decomposition $x = \sum_{i=1}^r x_i e_i + \sum_{i < j} x_{ij}$. Then

$$R_A(x) = \sum_{i=1}^r y_i e_i + \sum_{i < j} x_{ij},$$

where $[y_1, y_2, \dots, y_r]^\top = A([x_1, x_2, \dots, x_r]^\top)$. This is a generalization of a concept introduced in [32] for $\mathbb{V} = \mathcal{S}^n$. Let A be a P -matrix (i.e., all its principal minors are positive). By [11, Proposition 5.1], the latter is equivalent to saying that R_A has the P -property, which in turn by Proposition 2.2, Parts (a) and (d), implies that R_A is a star-transformation. Hence, by using Proposition 3.1 and Proposition 3.4 we have that $R_A \in F_1 \cap F_2$. On the other hand, if A is a nonnegative diagonal matrix, then clearly R_A is a monotone transformation. Hence, by using Proposition 3.2 we have that R_A has the T -property.

4 Existence Results for Symmetric Cone LCP's

In this section, we present coercive and noncoercive existence results for symmetric cone SCLCPs. Our approach follows the same arguments of [16] for LCP's and of [6] for SDLCPs. For this, we recall that problem (1) is equivalent to the following variational inequality problem $\text{VIP}(L, \mathcal{K}, q)$: find an element \bar{x} such that

$$\bar{x} \in \mathcal{K} \quad \text{and} \quad \langle L(\bar{x}) + q, x - \bar{x} \rangle \geq 0 \quad \text{for all } x \in \mathcal{K}. \tag{15}$$

We approximate this problem by the following sequence of variational inequality problems $\text{VIP}(L, D_k, q)$: find an element x^k such that

$$x^k \in D_k \quad \text{and} \quad \langle L(x^k) + q, x - x^k \rangle \geq 0 \quad \text{for all } x \in D_k, \tag{16}$$

where $D_k := \{x \in \mathcal{K} : \langle d, x \rangle \leq \sigma_k\}$ with $d \in \text{int}(\mathcal{K})$ and $\sigma_k \rightarrow +\infty$. Since each set D_k is compact and convex, by the Hartman–Stampacchia theorem we have that (16) has a nonempty solution set $\text{SOL}(L, D_k, q)$. Moreover, it is clear that each solution x^k is a solution of (16) if and only if $x^k \in D_k$ is an optimal solution of the linear program

$$\inf_x [\langle L(x^k) + q, x \rangle : x \in \mathcal{K}, \langle d, x \rangle \leq \sigma_k].$$

Applying optimality conditions, we obtain that x^k is a solution of (16) if and only if there exists $\theta_k \in \mathbb{R}$ such that (x^k, θ_k) is a solution of the following problem, called the augmented symmetric cone LCP: find $x^k \in \mathcal{K}$ and $\theta_k \geq 0$ such that

$$\begin{aligned} y^k &:= L(x^k) + q + \theta_k d \in \mathcal{K}, & \langle d, x^k \rangle &\leq \sigma_k, \\ \langle y^k, x^k \rangle &= 0 & \text{and} & \theta_k (\sigma_k - \langle d, x^k \rangle) = 0. \end{aligned} \tag{ASCLCP}_k$$

From this we observe that

$$\langle d, x^k \rangle < \sigma_k \implies \theta_k = 0 \implies x^k \in \text{SOL}(L, \mathcal{K}, q). \tag{17}$$

Moreover, we have from (ASCLCP_k) that

$$\theta_k = -\left\langle L(x^k) + q, \frac{x^k}{\sigma_k} \right\rangle. \tag{18}$$

Implications (17) shows that only the case $\langle d, x^k \rangle = \sigma_k$ for all $k \in \mathbb{N}$ deserves further analysis. This analysis is carried out below by extending the arguments from [6, 16] to our symmetric cone framework via the spectral decomposition theorem. Since $\langle d, x^k \rangle = \sigma_k$ for all $k \in \mathbb{N}$, we are interested in obtaining asymptotic properties of the sequence $\{\frac{x^k}{\sigma_k}\}$.

Since $y^k, \frac{x^k}{\sigma_k} \in \mathcal{K}$ and $\langle y^k, \frac{x^k}{\sigma_k} \rangle = 0$ for all $k \in \mathbb{N}$, by Proposition 2.1, Parts (b) and (c), y^k and $\frac{x^k}{\sigma_k}$ share a Jordan frame, which for simplicity is denoted by $\{e_1^k, \dots, e_r^k\}$, for which it holds that

$$\frac{x^k}{\sigma_k} = \sum_{i=1}^r \lambda_i \left(\frac{x^k}{\sigma_k}\right) e_i^k, \quad y^k = \sum_{i=1}^r \lambda_i(y^k) e_i^k, \tag{19}$$

where $\lambda(\frac{x^k}{\sigma_k}) := (\lambda_1(\frac{x^k}{\sigma_k}), \dots, \lambda_r(\frac{x^k}{\sigma_k}))$ and $\lambda(y^k) := (\lambda_1(y^k), \dots, \lambda_r(y^k))$ denote the eigenvalues of $\frac{x^k}{\sigma_k}$ and y^k , respectively. Therefore,

$$\left\langle d, \frac{x^k}{\sigma_k} \right\rangle = \sum_{i=1}^r \lambda_i \left(\frac{x^k}{\sigma_k}\right) \langle d, e_i^k \rangle = 1,$$

and since $\lambda_i(\frac{x^k}{\sigma_k}) \langle d, e_i^k \rangle \geq 0$ for all $i \in \{1, \dots, r\}$, we conclude that, for all $k \in \mathbb{N}$,

$$\gamma^k := \left(\lambda_1 \left(\frac{x^k}{\sigma_k}\right) \langle d, e_1^k \rangle, \dots, \lambda_r \left(\frac{x^k}{\sigma_k}\right) \langle d, e_r^k \rangle \right) \in \Delta := \left\{ \gamma \in \mathbb{R}_+^r : \sum_{i=1}^r \gamma_i = 1 \right\}.$$

As stated in [33, Theorem 18.2], the simplex Δ can be decomposed as the disjoint union of the relative interior of its extreme faces

$$\Delta_{J_i} = \text{co}\left\{ \underbrace{(0, \dots, 0, 1, 0, \dots, 0)}_s : s \in J_i \right\},$$

with J_i being a nonempty subindex set of $\{1, \dots, r\}$ for each $i = 1, \dots, 2^r - 1$; that is to say,

$$\Delta = \bigsqcup_{i=1}^{2^r-1} \text{ri}(\Delta_{J_i}). \tag{20}$$

The next result describes the asymptotic behavior of the sequence $\{\frac{x^k}{\sigma_k}\}$.

Lemma 4.1 *Let $\{x^k\}$ be a sequence of solutions to (ASCLCP_k) such that $\langle d, x^k \rangle = \sigma_k$ for all $k \in \mathbb{N}$ and $\frac{x^k}{\sigma_k} \rightarrow v$ for some $v \in \mathcal{K}$. Then*

(a) $v \in \text{SOL}(L, \mathcal{K}, \tau_v d)$ with $\tau_v = -\langle L(v), v \rangle \geq 0$.

Moreover, there exist a nonempty subindex set $J_v \subseteq \{1, \dots, r\}$, a Jordan frame $\{e_1, \dots, e_r\}$, and a subsequence $\{k_m\}$ such that

- (b) $\{e_1^{k_m}, \dots, e_r^{k_m}\} \rightarrow \{e_1, \dots, e_r\}$ and $\lambda(\frac{x^{k_m}}{\sigma_{k_m}}) \rightarrow \lambda(v)$ as $m \rightarrow +\infty$; thus, $\gamma^{k_m} \rightarrow \gamma := (\lambda_1(v)\langle d, e_1 \rangle, \dots, \lambda_r(v)\langle d, e_r \rangle) \in \Delta$.
- (c) $\gamma^{k_m} \in \text{ri}(\Delta_{J_v})$; i.e., $\text{supp}\{\lambda(\frac{x^{k_m}}{\sigma_{k_m}})\} = J_v$ and $\lambda(y^{k_m})|_{J_v} = 0$ for all $m \in \mathbb{N}$. As a consequence, the vectors $\lambda(y^{k_m})$ have at least $|J_v|$ zeros, which implies that $\lambda_i^\uparrow(y^{k_m}) = 0$ for all $i = 1, \dots, |J_v|$, and $\text{supp}\{\lambda(v)\} \subseteq J_v$.

Finally, for every $z \in \mathcal{K} \setminus \{0\}$ with $\text{supp}\{\lambda(z)\} \subseteq J_v$, we have

- (d) $\langle y^{k_m}, z \rangle = 0$ for all $m \in \mathbb{N}$;
- (e) $\langle L(x^{k_m}) + q, \frac{z}{\langle d, z \rangle} \rangle = \langle L(x^{k_m}) + q, v \rangle$ for all $m \in \mathbb{N}$;
- (f) $\langle L(v), \frac{z}{\langle d, z \rangle} \rangle = \langle L(v), v \rangle$.

Proof (a): By dividing inequality (16) by σ_k^2 , setting $x = 0$ and $x = \frac{\sigma_k}{\langle d, z \rangle} z$ for $z \in \mathcal{K} \setminus \{0\}$, and taking the limit as $k \rightarrow +\infty$, we get $\langle L(v), v \rangle \leq 0$ and $\langle L(v), \frac{z}{\langle d, z \rangle} - v \rangle \geq 0$, respectively. The result follows from this since $\langle d, v \rangle = 1$.

(b): Let us consider the case where \mathbb{V} is not necessarily simple. Due to Theorem 2.3, it suffices to consider $\mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2 \times \dots \times \mathbb{V}_{\bar{j}}$, where each \mathbb{V}_j is a simple Jordan Algebra with the corresponding symmetric cone \mathcal{K}_j and rank r_j . As before, the superscript (j) is used to denote the j th block of a given vector in \mathbb{V} .

Remarks 2.1 and 2.2, applied to each \mathbb{V}_j , imply the existence of positive numbers $\theta_j, j = 1, \dots, \bar{j}$, such that either $\|(e_i^k)^{(j)}\|^2 = 0$ or $\|(e_i^k)^{(j)}\|^2 = \theta_j$ for all $i \in \{1, \dots, r\}, k \in \mathbb{N}$, where the latter holds for one and only one block j . Then $\|e_i^k\|^2 \leq \max\{\theta_j : j = 1, \dots, \bar{j}\}$ for all $i \in \{1, \dots, r\}, k \in \mathbb{N}$. Set $\bar{\theta} := \max\{\theta_j : j = 1, \dots, \bar{j}\}$. Therefore, there exists a Jordan frame $\{e_1, \dots, e_r\}$ and a subsequence $\{k_m\}$ such that $\{e_1^{k_m}, \dots, e_r^{k_m}\}$ converges to $\{e_1, \dots, e_r\}$. Moreover, Proposition 2.1, Part (h), yields¹

$$\lambda_{\min}(d) \leq \frac{\sum_{j=1}^{\bar{j}} \theta_j \langle d^{(j)}, (e_i^k)^{(j)} \rangle}{\sum_{j=1}^{\bar{j}} \theta_j \|(e_i^k)^{(j)}\|^2} \leq \frac{\bar{\theta} \sum_{j=1}^{\bar{j}} \langle d^{(j)}, (e_i^k)^{(j)} \rangle}{\sum_{j=1}^{\bar{j}} \theta_j^2} = \frac{\bar{\theta} \langle d, e_i^k \rangle}{\sum_{j=1}^{\bar{j}} \theta_j^2}$$

for all $i \in \{1, \dots, r\}$ and $k \in \mathbb{N}$. This, together with the equality $\sum_{i=1}^r \lambda_i(\frac{x^k}{\sigma_k}) \times \langle d, e_i^k \rangle = 1$, implies that the eigenvalues $\lambda_i(\frac{x^k}{\sigma_k}), i = 1, \dots, r$, are bounded. Hence,

¹This upper bound can be slightly improved thanks to Remark 2.2. Indeed, one can obtain $\lambda_{\min}(d) \leq \frac{\langle d, e_i^k \rangle}{\theta_{j_i, k}} \leq \frac{\langle d, e_i^k \rangle}{\min_j \theta_j}$, where j_i, k corresponds to the nonzero block of e_i^k .

passing to a subsequence if necessary, it follows that $\{\lambda(\frac{x^{k_m}}{\sigma_{k_m}})\}$ converges to $\lambda(v)$ as $m \rightarrow +\infty$. Consequently, $\gamma^{k_m} \rightarrow \gamma$ as $m \rightarrow +\infty$, and $\gamma \in \Delta$.

(c): Since $\gamma^k \in \Delta$ for all k , from decomposition (20) without loss of generality we may consider that there exists a nonempty subindex set $J_v \subseteq \{1, \dots, r\}$ such that $\gamma^{k_m} \in \text{ri}(\Delta_{J_v})$ for all $m \in \mathbb{N}$. Hence, we obtain that $\text{supp}\{\lambda(\frac{x^{k_m}}{\sigma_{k_m}})\} = J_v$ for such m (see [35, Exercise 2.28(e)]). From this we prove that $\text{supp}\{\lambda(v)\} \subseteq J_v$. From the spectral decompositions (19) we get

$$\begin{aligned} 0 &= \left\langle \frac{x^{k_m}}{\sigma_{k_m}}, y^{k_m} \right\rangle = \sum_{i,j=1}^r \lambda_i \left(\frac{x^{k_m}}{\sigma_{k_m}} \right) \lambda_j(y^{k_m}) (e_i^{k_m}, e_j^{k_m}) \\ &= \sum_{i=1}^r \lambda_i \left(\frac{x^{k_m}}{\sigma_{k_m}} \right) \lambda_i(y^{k_m}) \|e_i^{k_m}\|^2 \\ &= \sum_{i \in J_v} \lambda_i \left(\frac{x^{k_m}}{\sigma_{k_m}} \right) \lambda_i(y^{k_m}) \|e_i^{k_m}\|^2, \end{aligned}$$

and thus $\lambda(y^{k_m})|_{J_v} = 0$ since $\lambda_i(\frac{x^{k_m}}{\sigma_{k_m}}) > 0$ for $i \in J_v$.

(d): Let $z \in \mathcal{K}$ be such that $\text{supp}\{\lambda(z)\} \subseteq J_v$. This yields $\lambda_i^\uparrow(z) = 0$ for all $i = |J_v| + 1, \dots, r$. By applying Parts (f) and (g) of Proposition 2.1 and item (c) above, we obtain:

$$0 \leq \text{tr}(y^{k_m} \circ z) \leq \sum_{i=1}^r \lambda_i^\uparrow(y^{k_m}) \lambda_i^\uparrow(z) = \sum_{i=1}^{|J_v|} \lambda_i^\uparrow(y^{k_m}) \lambda_i^\uparrow(z) = 0.$$

Therefore, $\text{tr}(y^{k_m} \circ z) = 0$ for all $m \in \mathbb{N}$. The desired result follows from Parts (g) and (b) of Proposition 2.1.

(e): If $z \in \mathcal{K} \setminus \{0\}$ is such that $\text{supp}\{\lambda(z)\} \subseteq J_v$, then Eq. (18) and item (d) yield

$$\left\langle L(x^{k_m}) + q, \frac{x^{k_m}}{\sigma_{k_m}} \right\rangle = \left\langle L(x^{k_m}) + q, \frac{z}{\langle d, z \rangle} \right\rangle.$$

Replacing z by v , we obtain item (e).

(f): After dividing the equality in item (e) by σ_{k_m} and taking the limit as $m \rightarrow +\infty$, we obtain the desired result. □

The proof of Lemma 4.1, Part (a), shows us that the sets $\text{SOL}(L, \mathcal{K}, \tau d)$ for $\tau \geq 0$ play an important role in our analysis. Conditions imposed to these sets allow us to extend to SCLCPs the following classes of linear transformations that were introduced for LCPs by García (see [16] and the references therein) and that were extended to SDLCPs in [6].

Definition 4.1 Let $L : \mathbb{V} \rightarrow \mathbb{V}$ be a linear transformation.

- L is a García's transformation iff there exists a $d \in \text{int}(\mathcal{K})$ such that $\text{SOL}(L, \mathcal{K}, \tau d) = \{0\}$ for all $\tau > 0$. In this case we say that L is a G -transformation with respect to d , or simply $L \in G(d)$.
- L is a $\#$ -transformation iff $[v \in \text{SOL}(L, \mathcal{K}, 0) \implies (L + L^\top)(v) \in \mathcal{K}]$.
- L is a $G^\#$ -transformation iff $L \in G$ and it is a $\#$ -transformation. Similarly, for a given $d \in \text{int}(\mathcal{K})$, L is a $G(d)^\#$ -transformation if $L \in G(d)$ and it is a $\#$ -transformation.

Example 4.1

- (i) Monotone and copositive transformations are G -transformations.
- (ii) Proceeding exactly as in [6, Proposition 4.8], one can prove that $L \in \#$ if any of the following conditions is satisfied: L is self-adjoint; L is skew-symmetric; $L \in R_0$; L is copositive; $-L$ is a star-transformation; and L is a star-transformation and $-L^\top \in Z$.

The next result shows that the class G is invariant under automorphisms.

Lemma 4.2 *Let $L : \mathbb{V} \rightarrow \mathbb{V}$ be a linear transformation. For $\Gamma \in \text{Aut}(\mathcal{K})$, define $\widehat{L} = \Gamma^\top L \Gamma$. Then, L is a G -transformation with respect to d if and only if \widehat{L} is a G -transformation with respect to $\Gamma^\top(d)$.*

Proof The result follows directly from $\Gamma^{-1}(\text{SOL}(L, \mathcal{K}, d)) = \text{SOL}(\widehat{L}, \mathcal{K}, \Gamma^\top(d))$ (see [12, Theorem 5.1]) and $\Gamma^\top \in \text{Aut}(\mathcal{K})$. \square

The next proposition provides two characterizations of the class of García's linear transformations. This is a symmetric cone version of [16, Proposition 3.1] and [6, Proposition 4.6] proved for LCPs and SDLCPs, respectively.

Proposition 4.1 *Let $d \in \text{int}(\mathcal{K})$, and $L : \mathbb{V} \rightarrow \mathbb{V}$ be a linear transformation. Then, the following are equivalent:*

- (a) $L \in G(d)$;
- (b) $[v \in \mathcal{K}, L(v) - \langle L(v), v \rangle d \in \mathcal{K}, \langle d, v \rangle = 1] \implies \langle L(v), v \rangle \geq 0$;
- (c) $[v \in \mathcal{K}, \langle d, v \rangle = 1, \langle L(v), v \rangle < 0] \implies L(v) - \langle L(v), v \rangle d \notin \mathcal{K}$.

Proof (a) \implies (b): We argue by contradiction. Suppose that the left-hand side of item (b) holds and $\langle L(v), v \rangle < 0$. It follows that $\langle L(v) - \langle L(v), v \rangle d, v \rangle = 0$ and hence $v \in \text{SOL}(L, \mathcal{K}, \tau d)$ with $\tau = -\langle L(v), v \rangle > 0$. By linearity we have $v/\tau \in \text{SOL}(L, \mathcal{K}, d)$, which implies that $v = 0$ by item (a), obtaining a contradiction with the fact that $\langle d, v \rangle = 1$.

(b) \implies (c): We argue by contradiction. Suppose that the left-hand side of item (c) holds and that $L(v) - \langle L(v), v \rangle d \in \mathcal{K}$. Then, by using item (b) we conclude that $\langle L(v), v \rangle \geq 0$, obtaining a contradiction.

(c) \implies (a): We argue by contradiction. Suppose that for some $\tau > 0$, there exists a $v \neq 0$ such that $v/\tau \in \text{SOL}(L, \mathcal{K}, d)$. By changing τ if necessary we may assume that $\langle d, v \rangle = 1$. By assumption we have that $v, L(v) + \tau d \in \mathcal{K}$ and $\langle v, L(v) + \tau d \rangle = 0$.

From this we deduce that $\langle L(v), v \rangle = -\tau < 0$. Then, by using item (c) we conclude that $L(v) - \langle L(v), v \rangle d \notin \mathcal{K}$, obtaining a contradiction with the fact that $L(v) + \tau d \in \mathcal{K}$. □

Recall that the positive polar cone and the asymptotic cone of a given set $A \subseteq \mathbb{R}^n$ are respectively defined by $A^+ := \{y \in \mathbb{R}^n : \langle y, z \rangle \geq 0 \forall z \in A\}$ and

$$A^\infty := \left\{ d \in \mathbb{R}^n : \exists t_k \rightarrow +\infty, \exists x_k \in A \text{ with } \lim_{k \rightarrow +\infty} \frac{x_k}{t_k} = d \right\}.$$

We now obtain a bound for the asymptotic cone of the solution set to symmetric cone LCP’s for $G^\#$ -transformations.

Proposition 4.2 *If $L \in G^\#$, then $\text{SOL}(L, \mathcal{K}, q)^\infty \subseteq \text{SOL}(L, \mathcal{K}, 0) \cap \{-q\}^+$.*

Proof Let $d \in \text{int}(\mathcal{K})$ be such that $L \in G^\#(d)$ and $v \in \text{SOL}(L, \mathcal{K}, q)^\infty$. If $v = 0$, then the inclusion is trivial. So, we consider that $v \neq 0$. Without loss of generality we assume that $\langle d, v \rangle = 1$. By definition there exist $\{x^k\}$ and $\{t_k\}$ such that $x^k \in \text{SOL}(L, \mathcal{K}, q)$ for all $k \in \mathbb{N}$, $t_k \rightarrow +\infty$, and $\frac{x^k}{t_k} \rightarrow v$ as $k \rightarrow +\infty$. By defining $\sigma_k := \langle d, x^k \rangle$ for all $k \in \mathbb{N}$ it is easy to check that $\sigma_k \rightarrow +\infty$ and $\frac{x^k}{\sigma_k} \rightarrow v$. By Lemma 4.1, Part (a), we have $v \in \text{SOL}(L, \mathcal{K}, \tau_v d) \setminus \{0\}$ with $\tau_v = -\langle L(v), v \rangle \geq 0$. If $\tau_v > 0$, then we get a contradiction to L being a G -transformation. Therefore, $\tau_v = 0$, and we have $v \in \text{SOL}(L, \mathcal{K}, 0)$. From this and from Lemma 4.1, Part (f), for $z = \frac{x^{k_m}}{\sigma_{k_m}}$, we get $\langle L(v), \frac{x^{k_m}}{\sigma_{k_m}} \rangle = 0$. Then,

$$0 = \left\langle L(x^{k_m}) + q, \frac{x^{k_m}}{\sigma_{k_m}} \right\rangle = \langle L(x^{k_m}) + q, v \rangle = \langle x^{k_m}, (L + L^\top)(v) \rangle + \langle q, v \rangle,$$

where we have used Lemma 4.1, Part (d), and the fact that each x^{k_m} is a solution to problem (1). As $L \in \#$, we have $(L + L^\top)(v) \in \mathcal{K}$, which in turn by Proposition 2.1, Part (a), implies that $\langle x^{k_m}, (L + L^\top)(v) \rangle \geq 0$. Hence, from the above equality we get $\langle q, v \rangle \leq 0$. □

We now obtain existence results that extend [34, Theorems 9 and 11] given for LCPs and [6, Theorem 5.1] given for SDLCPs.

Theorem 4.1 *Let $q \in \mathbb{V}$ and $L \in G^\#$.*

- (a) *If $q \in \text{SOL}(L, \mathcal{K}, 0)^+$, then $\text{SOL}(L, \mathcal{K}, q)$ is nonempty (possibly unbounded);*
- (b) *If $q \in \text{int}[\text{SOL}(L, \mathcal{K}, 0)^+]$, then $\text{SOL}(L, \mathcal{K}, q)$ is nonempty and compact.*

Proof Let $d \in \text{int}(\mathcal{K})$ be such that $L \in G^\#(d)$. (a): Let $\{(x^k, \theta_k)\}$ be a sequence of solutions to problems (ASCLCP_k). If there exists $k \in \mathbb{N}$ such that $\langle d, x^k \rangle < \sigma_k$, then by implication (17) we have that $x^k \in \text{SOL}(L, \mathcal{K}, q)$, and we are done. On the contrary, if $\langle d, x^k \rangle = \sigma_k$ for all $k \in \mathbb{N}$, then up to subsequences, $\frac{x^k}{\sigma_k} \rightarrow v$ for some v .

By Lemma 4.1, Part (a), we have $v \in \text{SOL}(L, \mathcal{K}, \tau_v d)$. Proceeding as in Proposition 4.2, we prove that $\tau_v = 0$; thus, $v \in \text{SOL}(L, \mathcal{K}, 0)$. From this, by Lemma 4.1 there exist a nonempty subindex set $J_v \subseteq \{1, \dots, r\}$ and a subsequence $\{\frac{x^{k_m}}{\sigma_{k_m}}\}$ such that $\text{supp}\{\lambda(\frac{x^{k_m}}{\sigma_{k_m}})\} = J_v$ and $\langle L(v), x^{k_m} \rangle = 0$ for all $m \in \mathbb{N}$. By using this, Eq. (18), and Lemma 4.1, Part (e), we obtain

$$\begin{aligned} 0 \leq \theta_{k_m} &= -\left\langle L(x^{k_m}) + q, \frac{x^{k_m}}{\sigma_{k_m}} \right\rangle = -\langle L(x^{k_m}) + q, v \rangle \\ &= -\langle x^{k_m}, (L + L^\top)(v) \rangle - \langle q, v \rangle. \end{aligned}$$

As $v \in \text{SOL}(L, \mathcal{K}, 0)$, by hypothesis we get $\langle q, v \rangle \geq 0$ and $(L + L^\top)(v) \in \mathcal{K}$. Thus, $\langle x^{k_m}, (L + L^\top)(v) \rangle \geq 0$ by Proposition 2.1, Part (a). Consequently, $\theta_{k_m} = 0$, and by implication (17) we conclude that $x^{k_m} \in \text{SOL}(L, \mathcal{K}, q)$, and we are done.

(b): From item (a) we conclude that $\text{SOL}(L, \mathcal{K}, q) \neq \emptyset$. To prove that this set is bounded, it is sufficient to show that $\text{SOL}(L, \mathcal{K}, q)^\infty = \{0\}$. This follows from Proposition 4.2 since by hypothesis $\text{SOL}(L, \mathcal{K}, 0) \cap \{-q\}^+ = \{0\}$. Indeed, if on the contrary we suppose that there exists $u \neq 0$ such that $u \in \text{SOL}(L, \mathcal{K}, 0)$ and $\langle q, u \rangle \leq 0$, then as

$$q \in \text{int}[\text{SOL}(L, \mathcal{K}, 0)^+] = \text{SOL}(L, \mathcal{K}, 0)^{s+}$$

(see [35, Exercise 6.22]), we obtain $\langle q, u \rangle > 0$, a contradiction. □

The last theorem directly implies the following result.

Corollary 4.1 *If $L \in G$, then $L \in R_0$ if and only if $L \in Q_b$.*

Proof Clearly $L \in Q_b$ implies that $L \in R_0$. In the opposite direction, it suffices to note that if $L \in R_0$, then $L \in \#$ (see Example 4.1). We conclude thus from Part (b) of Theorem 4.1. □

Remark 4.1 The hypothesis on q of Theorem 4.1, Part (a), implies the following necessary condition:

$$q \in \text{SOL}(L, \mathcal{K}, 0)^+ \implies \lambda_{\max}(q) \geq 0.$$

Indeed, if $q \in \text{SOL}(L, \mathcal{K}, 0)^+$, then $\langle q, x \rangle \geq 0$ for all $x \in \text{SOL}(L, \mathcal{K}, 0)$. In the case where \mathbb{V} is not simple, we denote by $\mathbb{V}_j, j = 1, \dots, \bar{j}$, the simple Jordan Algebra located in the j th position, by \mathcal{K}_j its corresponding symmetric cone, by r_j its rank, and by $x^{(j)}$ and $q^{(j)}$ the j th block of x and q , respectively. Thus, Proposition 2.1, Part (f), applied to \mathbb{V}_j , implies that $\sum_{j=1}^{\bar{j}} \sum_{i=1}^{r_j} \lambda_i^\uparrow(q^{(j)}) \lambda_i^\uparrow(x^{(j)}) \geq \langle q, x \rangle \geq 0$. But, since $\lambda_i^\uparrow(x^{(j)}) \geq 0$ for all $i = 1, \dots, r_j$ and $j = 1, \dots, \bar{j}$ (because $x^{(j)} \in \mathcal{K}_j$), it necessarily follows that $\lambda_{\max}(q) \geq 0$.

By taking into account Example 4.1 we now list some conditions under which the linear transformations defined in Sect. 3.2 are G -transformations.

Example 4.2 Let $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$ be a Euclidean Jordan algebra, and $a \in \mathbb{V}$.

1. If $a \in \text{int}(\mathcal{K})$, then L_a is strongly monotone (cf. Proposition 2.1, Part(a)). Thus, $L_a \in G(d) \cap R_0$ for any $d \in \text{int}(\mathcal{K})$.
2. If \mathbb{V} is simple and $\pm a \in \text{int}(\mathcal{K})$, the quadratic transformation P_a is strongly monotone by [13, Theorem 6.5]. Thus, $P_a \in G(d) \cap R_0$ for any $d \in \text{int}(\mathcal{K})$.
3. If $\lambda_i(\pm a) \subseteq (-1, 1)$, for all i , then the Stein transformation S_a is strongly monotone (see [31, Theorem 3.3]). Thus, $S_a \in G(d) \cap R_0$ for any $d \in \text{int}(\mathcal{K})$.
4. Let $A \in \mathbb{R}^{r \times r}$ be a nonnegative matrix. Then the relaxation transformation R_A is copositive. Thus, $R_A \in G(d)$ for any $d \in \text{int}(\mathcal{K})$.

5 Concluding Remarks

In this paper, we introduce a new class of linear transformations called F , and, within this new class, we characterize the class of Q_b -transformations in terms of larger classes, such as Q and R_0 . Next, we provide conditions to ensure that Lyapunov, Quadratic, Stein, and Relaxation linear transformations belong to this class. Finally, we extend the notion of García's transformations to the SCLCP setting, which is used to establish coercive and noncoercive existence results for SCLCPs.

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