Prewhitening of Climatological Time Series

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(Manuscript received 7 June 1988, in final form 12 September 1989)

ABSTRACT

The most common procedures to prewhite a climatological time series are considered and compared with those based on Fourier analysis. In particular, advantages and shortcomings of the anomaly method and the seasonal differences are noted. Some modifications for the Fourier prewhitened synthesis are introduced to preserve the smoothness of the spectra. A simple recursion filter with prewhitening ability is presented and compared with the other filters using synthetic and a real time series. Special attention is given to the case of short time series. The anomaly method and the recursion filter appear as suitable procedures to replace a Fourier technique giving both similar outputs but differing in the details. The seasonal differences, although very simple to apply, introduce substantial modifications in the output and should be avoided. When the data series contains a few times the cycle to be eliminated the recursion filter seems to be safer than the anomaly method.

1. Introduction

In the analysis of climatological time series it is often desirable to extract strong periodicities; outstanding examples are the annual and daily cycles. The process by which an important cyclic fluctuation is removed from a time series is known as prewhitening. For this purpose several digital techniques are available, well-known examples being the anomaly method and seasonal differences. When prediction is envisaged the latter method is recommended by Box and Jenkins (1979) because it only uses past input. Its frequency response function is found in textbooks [for instance in Koopmans (1974)] and is included in Table 1. Recently Fuenzalida and Rosenblüth (1986) have examined the anomaly method as a digital filter obtaining an expression for its frequency response which is also listed in the table. They concluded that the method is well behaved when the input series encompasses several times the cycle to be rejected, say over ten times the cycle. The method has some drawbacks, however, that become more important as the time series gets shorter. In this article the shortcomings of the anomaly filter are explained in terms of the higher harmonics rejection, a recursion filter that avoids such a problem is presented and a more comprehensive survey of prewhitening filters is attempted.

In time domain a digital filter is a set of numerical weights that when applied over elements of an input series produces a term of the output time series. Filter design consists of determining proper values for the weights in order to attain some desirable feature in the output, our present interest being the suppression of a strong periodic component. In general this component will be not a pure sinusoid and it will possess its own Fourier spectrum, although in most cases there will be a dominant harmonic, say diurnal or annual. When, through Fourier Transforms, the time relation is carried over to the frequency space, the output can be expressed as the simple multiplication of two complex functions: the input transform and the filter frequency response. In general terms such frequency response can be expressed as an infinite Fourier series but in practice only a finite number of terms can be used. If such truncation introduces first order discontinuities, spurious oscillations appear in the outcome of the filtering process (Gibbs phenomenon).

According to what is to be eliminated prewhitening digital filters can be classified into three categories: 1) those that suppress all components with frequency higher than a certain value, sometimes called low-pass or smoothing filters; 2) those that reject a given frequency and its higher harmonics; and 3) those that eliminate a single frequency. In terms of a Fourier decomposition and subsequent synthesis they are equivalent, respectively, to a truncated sum, a Fourier surgery where a finite set of frequencies have been deleted in the synthesis, and a Fourier sum where a single component has been omitted.

In the first group are the most commonly used moving average filters, weighted or unweighted, and other more complex smoothing filters. They belong to the simplest of the three groups since they sweep out all high frequencies. This would be the crudest way to get rid of a periodic component and because they are well

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TABLE 1. Prewhitening filters.

Filters	Seasonal differences	Anomalies	Recursion
Mathematical expression	$y_t = x_t - x_{t-s}$	$y_t = x_t - 1/k \sum_{j=0}^{k-1} x_{\mathrm{mod}(t,s)+js}$	$v_t = c(x_t - 2\cos\phi x_{t-1} + x_{t-2}) + 2c\cos\phi y_{t-1} - c^2 y_{t-2}$ $y_t = c(v_t - 2\cos\phi v_{t+1} + v_{t+2}) + 2c\cos\phi y_{t+1} - c^2 y_{t+2}$ $\phi = 2\pi fd$
Frequency response	$e^{j\pi(fs+1/2)}\,\mathrm{sen}\,\pi fs$	$1 - e^{j\pi fs(k+1-2p)}(\mathrm{sen}\,k\pi fs)/(k\,\mathrm{sen}\,\pi fs)$	$c(1 - 2\cos\phi Z^{-1} + Z^{-2})/(1 - 2c\cos\phi Z^{-1} + c^2 Z^{-2})$ $Z = \exp(i2\pi fd)$
Rejected bandwidth	$1/4s$	$1.92/\pi ks$	$1/ks$ adjustable
Phase	$\pi(fs + 1/2)$	Zero for $k \rightarrow \infty$	Zero
Short series behavior	Deficient	Deficient	Good

documented in textbooks they will not be considered hereafter.

In the second category, suppressing a frequency and its harmonics along with the mean value, are the anomaly filter and the seasonal differences, the latter also eliminates trends. The anomaly filter preserves all Fourier components other than the main frequency to be eliminated and its higher harmonics. It is equivalent to a synthesis where a frequency, its harmonics and the zero frequency have been deleted. When using short time series, the frequency interval represented by one harmonic component is wide, this fact can generate important losses in series with significant contributions from periods that are not an exact submultiple of the total number of points in the data. The worst situation is one in which an important component frequency coincides with one of the suppressed harmonics. Since the analyst is not aware offhand of such a condition, filters that eliminates a single frequency are to be preferred.

Among the filters that reject a single frequency or several frequencies are those based on a Fourier synthesis with some components deleted. Nevertheless, if a single component is made equal to zero, a discontinuity is introduced in the spectrum and some care must be exercised to avoid its consequences. This point will be taken up later. Another kind of filter that eliminates a single frequency is recursion filters. They differ from the so-called convolution filters, in a frequency response that can be expressed as rational algebraic functions. These filters reject a frequency band as narrow as desired and are easier to apply than Fourier analysis followed by an incomplete synthesis. In the following sections the properties and design procedure for such filters are considered and a very simple prewhitening example is found.

Section 2 deals with some general problems in filtering that need to be kept in mind. Section 3 contains a brief comment on how to avoid Gibbs phenomenon when performing Fourier synthesis with some frequency deleted. In section 4 the recursion filter is introduced and in section 5 it is applied to synthetic as well as to real input and its performance is compared against those of the anomaly filter, the seasonal difference filter and Fourier incomplete synthesis.

2. Some general problems in filtering

A well-known problem in the filtering of finite length records is the widening of spectral lines present in the data, spreading their variance contribution contaminating neighboring frequencies. The shorter the input series the wider the corresponding frequency band appearing in the spectrum of the time series. For instance, a single sinusoidal component of frequency f present in a series of length T appears in the amplitude spectrum as $\sin(\pi f T)/\pi f$, a feature that has a typical width of $1/T$. In this respect the anomaly filter takes care of the widening by itself because the rejection bandwidth decreases with T, or more precisely with the number of times k the rejected period is present in the record (see Table 1). This is not the case with the seasonal differences filter, whose rejection bandwith remains constant as T increases. From this point of view it is convenient to use a filter with a rejection band that closely matches the contaminated band, a parameter to be defined more precisely later.

Another fact that must be considered is the phase response of the filter, also listed in Table 1. Ideally the filter should have a zero phase response for all frequencies but an acceptable alternative is a linear phase response that shifts the time coordinate in a constant value but does not destroy the sequence of events. The seasonal difference filter has a linear phase response and the anomaly filter, when used with a long enough input, approaches a zero phase response (Fuenzalida and Rosenblüth 1986).

As already mentioned, an additional desirable feature in a filter is a smoothed frequency response to avoid spurious Gibbs oscillations.

3. Incomplete Fourier synthesis

The most obvious prewhitening procedure, although somewhat lengthy, is to sum a Fourier series, obtained with a number of data points that makes the frequency to be eliminated coincide with one harmonic, leaving aside one component. This method will be used as a reference case to compare results for other kinds of filters; however, this procedure is equivalent to using a filter with a discontinuous response equal to 1 for all frequencies but 0 for the rejected one. To avoid Gibbs oscillations it is better to replace the amplitude and phase at the rejection frequency by an average of the nearest components both at high and low frequency sides. This procedure will be designated as smoothed spectral line filter, hereafter SSL filter if applied to a set of harmonic frequencies or SSSL filter if applied to a single frequency.

4. A recursion filter for prewhitening

Recursion digital filters have been used for quite a long time in geophysics (Shanks 1967) and are described in several textbooks, for instance in Kanasewich (1973) and Koopmans (1974). The latter points out that recursion filters must perform better than those designed by convolution because their impulse response fall off exponentially and after a brief initial period the actual outcome from the filter will approach closely to that obtained from an input extending very far in the past. They have been applied to time series arising in seismic exploration. In these cases the number of terms in the series are counted by thousands and tapering windows are normally applied at both ends so that widening and end effects do not require special attention. This is not the case with climatic series; often the number of terms in the series is not large and the final part must be preserved because it represents the threshold to future events. What follows is a rather conventional design of a simple recursion filter except that it takes proper advantage of any freedom at disposal to match the rejection bandwidth to the widened spectral lines. Because this kind of filter is not frequently used in meteorology, some general introduction may be in order.

Recursion filters are based on autoregressive-moving average (ARMA) processes that are defined by a linear relation between a predictand y_t and a predictor variable x_t of the kind:

$$y_r + a_1 y_{r-1} + \cdots + a_n y_{r-n} = b_0 x_r + b_1 x_{r-1} + \cdots + b_m x_{r-m}. \quad (1)$$

Both variables, x and y, are discrete and observed with a constant time interval d. Such an expression defines a digital recursion filter with input x_t and output y_t. The frequency response H of such a filter can be found applying the Finite Fourier Transform to (1). In terms of a complex variable $Z = \exp(i2\pi fd)$ the transformed relation is the product of the frequency response and the Fourier Transform of the input X,

$$Y(Z) = H(Z)X(Z),$$

where

$$H(Z) = (b_0 + b_1 Z^{-1} + \cdots + b_m Z^{-m}) / (1 + a_1 Z^{-1} + \cdots + a_n Z^{-n}).$$

Since $H(Z)$ is a rational function, it can be expressed in terms of the zeros of polynomials appearing in the numerator and denominator; in other words, in terms of its m zeros Z_{0k} and n poles Z_{pk}:

$$H(Z) = b_0(Z - Z_{01})(Z - Z_{02}) \cdots (Z - Z_{0m}) / [(Z - Z_{p1})(Z - Z_{p2}) \cdots (Z - Z_{pn})]\, Z^{n-m}. \quad (2)$$

As frequency f varies from 0 to the Nyquist limit $(1/2d)$ the complex variable Z maps the upper semicircle of radius one, so that to every value of f corresponds a value of the polar angle φ between 0 and 180°. For instance, if the time series is composed of monthly values of some climatic variable ($d = 1$ month) the annual cycle is characterized by frequency $f = 1/12$ and a polar angle of 30°. Negative frequencies, introduced for mathematical convenience, are mapped on the lower part of the unit circle. According to (2), the magnitude of the frequency response, or amplitude response, varies in direct relation to the distances of zeros to the point on the unit circle corresponding to the frequency under consideration and in inverse proportion to distances to poles.

The design of a rejection filter for some frequency f_0 reduces to the proper choice of zeros and poles in the complex plane because when these are known the coefficients, a_i and b_i, are also determined. The simplest recursion filter will have only one zero and one pole on the upper half of the complex plane. To eliminate contributions from a single frequency, it is necessary to locate a zero on the unit circle at the point corresponding to frequency f_0; to narrow the rejection bandwidth, a pole must be located on the corresponding radius near the zero, a small distance inside or outside the unit circle. Thus the magnitude of the response function will be flat and close to 1 at frequencies differing in small amount from that to be rejected but 0 at f_0 itself. To make the filter coefficients real, additional zeros and poles must be located at the complex conjugate positions of those already defined. Therefore the response function should be:

$$H(Z) = b_0(Z - Z_{01})(Z - Z_{01}^*) / [(Z - cZ_{01})(Z - cZ_{01}^*)], \quad (3)$$

where asterisks denote complex conjugate quantities, c is a real number close to 1 and the value of b_0 can be used to make the magnitude of $H(Z)$ equal to 1 at frequencies far from the value to be rejected.

Because $H(Z)$ is complex, the resulting filter has a nonzero phase response. To attain a zero phase filter, the output from a first application should be fed to a recursion inverse filter with a frequency response $H(Z^{-1})$ so that the total response be real and equal to the product $H(Z)H(Z^{-1}) = H(Z)H^*(Z) = |H(Z)|^2$.

Defining $\varphi = 180° f_0/f_N$, where f_N is the Nyquist limit, Z_{01} can be written as $Z_{01} = \exp(i\varphi) = \cos\varphi + i \sin\varphi$, and (3) becomes

$$H(Z) = b_0(1 - 2\cos\varphi Z^{-1} + Z^{-2})/(1 - 2c\cos\varphi Z^{-1} + c^2 Z^{-2}). \quad (4)$$

In the time domain the corresponding filter is

$$y_r = b_0 x_r - 2b_0 \cos\varphi x_{r-1} + b_0 x_{r-2} + 2c\cos\varphi y_{r-1} - c^2 y_{r-2}. \quad (5)$$

To attain a zero phase filter, the y_r time series is now passed through the inverse filter obtaining the final output v_r from

$$v_r = b_0 y_r - 2b_0 \cos\varphi y_{r+1} + b_0 y_{r+2} + 2c\cos\varphi v_{r+1} - c^2 v_{r+2}. \quad (6)$$

Let us now see how a proper choice of the arbitrary constant c can take care of the widening of spectral lines and how the starting of the filtering procedure can be implemented.

a. Contamination and pole location

Let the Fourier Transform of a continuous infinitely long signal $s(t)$ be $S(f)$ while $S_T(f)$ represents the transform of $s_T(t)$, the same signal sampled over a finite interval of length T. Then as shown in Jenkins and Watts (1968),

$$S_T(f) = \int S(f-g)\,\mathrm{sinc}(g)dg, \quad \text{where}$$

$$\mathrm{sinc}(g) = \sin(\pi g T)/(\pi g).$$

The function sinc(g) has a central lobe of height T and width $2/T$, which is the distance separating the first two zeros around the central frequency f_0, along with strongly damped lateral lobes. Therefore $S_T(f_0)$ receives contributions mainly from frequencies within the interval $(f_0 - 1/T, f_0 + 1/T)$, and $2/T$ can be taken as a measure of the contamination bandwidth.

On the other hand, if f_0 is the frequency to be eliminated by the filter, the rejection bandwidth can be defined in terms of $|1 - c|$. This is the distance separating a pole from the corresponding zero in the complex plane. Only within a frequency interval (or arc length along the unit circle in the complex plane) less than or comparable to $|1 - c|$ the different location of the pole and the zero will be relevant for the value of $|H(Z)|$. For frequencies out of such interval both points, zero and pole, look very close to one another. Straightforward geometry can be used to show that at a distance $|1 - c|$ from the f_0 point, $|H|^2$ is approximately equal to 1/2 (see Appendix). Therefore the rejection bandwidth will be defined as $\Delta\varphi = 2|1 - c|$, or in terms of frequency ($\Delta\varphi = 2\pi d\Delta f$): $\Delta f = |1 - c|/\pi d$. When the rejection bandwidth is made equal to the contamination bandwidth for a time series of length $T = Nd$ the value of c, less than 1, is determined by the simple rule

$$c = 1 - 2\pi/N. \quad (7)$$

A c value closer to 1 than (7) will pass part of the contamination and a lower value will reject additional side frequencies which are not supposed to be filtered out.

It is important to note the time domain implication in the choice of c. Because the regressive character of the filter, its output at a given time depends on the previous values of the whole input series. Because such dependence is proportional to powers of c, however, the farther from 1 its value is, the lower the dependence of the output from distant values in the input.

Relation (7) indicates that the distance separating zero and pole depends on the length of the input series. Thus when a long series is available the filter can be very selective, eliminating a narrow interval of frequencies around f_0, but if a short set of data is used the filter will destroy a rather wide frequency interval.

In the case of annual cycle suppression from a 5 year sequence of monthly mean values $f_0 = 0.083$, $N = 60$ and (7) gives $c = 0.895$. Then the rejection half-width is $1/N = 0.017$ with a suppression interval from frequency 0.066 to 0.100 affecting periods between 10 and 15 months, but if 10 years of data are used, $N = 120$, $c = 0.948$, and only the periods in the interval 11 to 13 months will be modified.

b. Initialization and normalization of the filtering process

A disadvantage of recursion filters is the need for a starting procedure for the filtering process since (5) indicates that y_r depends on previous values of input and output series. Such values do not exist for $r = 1$ and 2 and the filter cannot be properly applied. The simple alternative of a truncated start, with zeros in place of the unavailable values, introduces wide oscillations in the first part of the output because the cycle to be suppressed is allowed to pass in the starting procedure. In general, a suitable way to evaluate y_1 and y_2 should be a zero-phase filter able to eliminate the cycle. This is possible using the anomaly filter, although there may remain some deficiencies when the input

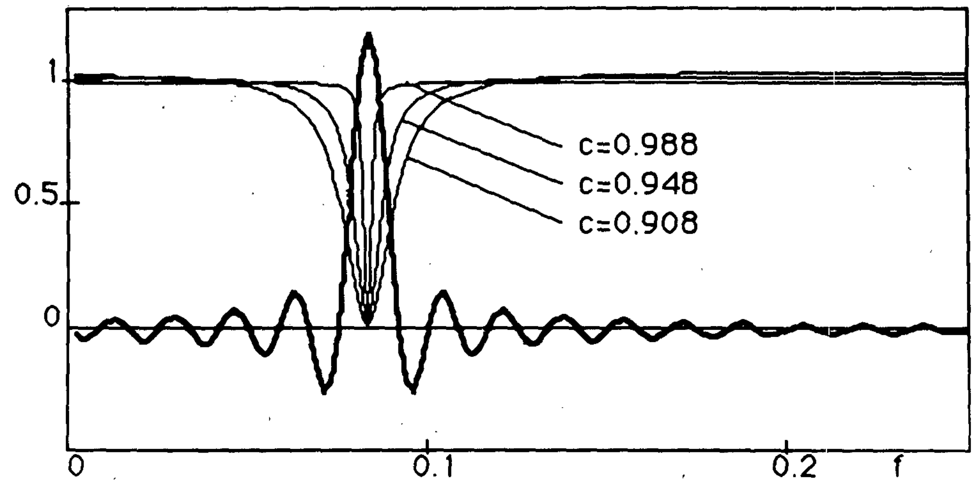


FIG. 1. Fourier transform of the annual sinusoid and amplitude response for the recursion filter for three values of c: one tuned to suppress the annual cycle in a series made of $N = 120$ monthly mean values, $c = 0.948$, and two other values, one smaller, $c = 0.908$ and another larger, $c = 0.988$.

series is short, besides the higher harmonics of the rejected frequency and slow trends will always be affected in the beginning of the output. In any event the recursion filter will perform with advantage over other filters under these conditions.

An additional source of concern is the evaluation of (6) near the end points, $r = (N - 1), N$. The following procedure works quite satisfactorily: assume that $v_N = y_N$ and $v_{N-1} = y_{N-1}$ and evaluation using (6) starts at v_{N-2} to end with v_2 and v_1. The truncation in this case is not so important because the input series has already been filtered once.

Finally, as to the b_0 value is concerned, some geometry developed in the Appendix shows that a suitable normalization factor is $b_0 = c$.

5. Some numerical experiments

The first experiments with the filter were done using a synthetic time series composed of two sinusoids:

$$s_r = A_1 \sin(2\pi r/P_1 + \beta_1) + A_0 \sin(2\pi r/P_0 + \beta_0),$$
$$r = 0, 1, 2 \cdots (N - 1). \qquad (8)$$

The first term representing the variations to be retained after filtering and the second term being the cycle to be extracted from a sample of N points. If the last one corresponds to an annual component, $P_0 = 12$, $\varphi = 30°$ and $f_0 = 1/12$. For simplicity additional parameters will be fixed as follows: $A_0 = A_1 = 1$, $\beta_0 = 1.5\pi$, $\beta_1 = 0.5\pi$, $N = 100$ and $P_1 = 144$, unless otherwise specified. Applying (7) for $N = 120$ a value $c = 0.948$ is obtained. Figure 1 shows the amplitude response (4) for this value of c and $\varphi = 30°$ together with the Fourier Transform of the annual sinusoid. It can be seen that the response has low magnitude for the frequency range where this component makes most of its contribution, thus the rejection bandwidth has been tuned to the spectral widening according to the length of the input series. In the same figure, two other amplitude responses, corresponding to c values of 0.908 and 0.988, are compared with the annual cycle transform. It can be appreciated that low values of c produce too wide a rejection band and that high values of c let part of the annual component pass through the filter.

An application of the filter, using $N = 100$ and $c = 0.937$ is presented in Fig. 2. Due to the already mentioned starting problem, a slight discrepancy is present

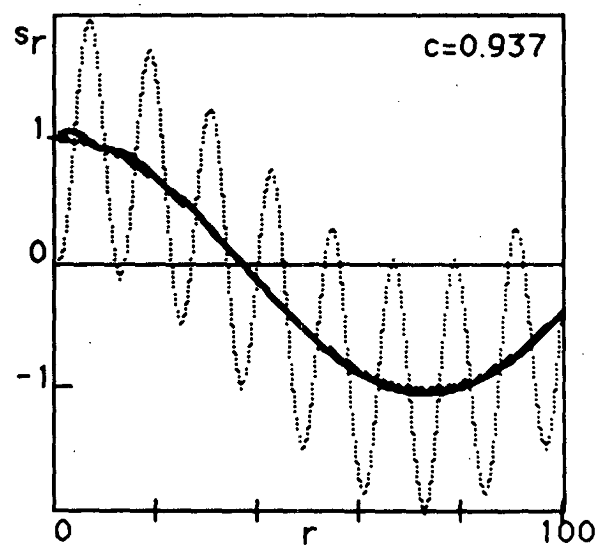


FIG 2. Output of the recursion filter for $c = 0.937$ applied to time series (8) with $A_0 = A_1 = 1$, $\beta_0 = 1.5\pi$, $\beta_1 = 0.5\pi$, $N = 100$, $P_0 = 12$ and $P_1 = 144$ (black line) compared with the sinusoid to be isolated (gray line).

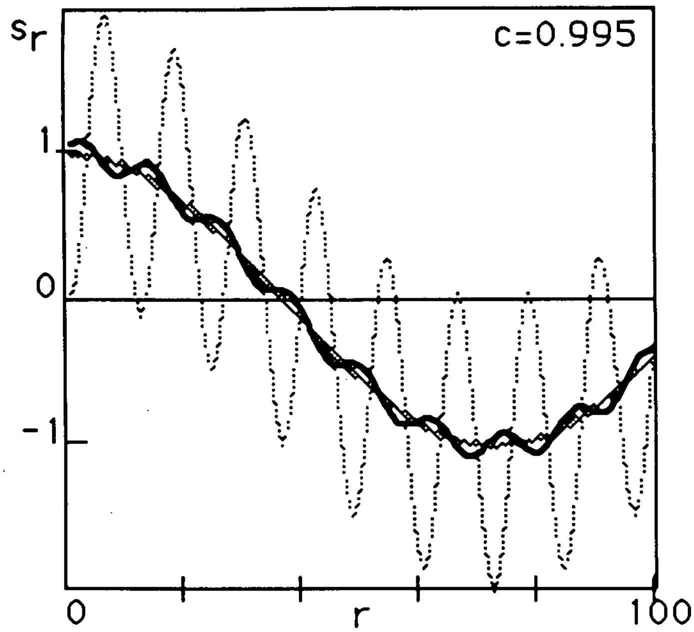


FIG. 3. Output of the recursion filter for $c = 0.995$ applied to time series (8) with $A_0 = A_1 = 1$, $\beta_0 = 1.5\pi$, $\beta_1 = 0.5\pi$, $N = 100$, $P_0 = 12$ and $P_1 = 144$ (black line) compared with the sinusoid to be isolated (gray line).

in the beginning of the output series and some small error in the normalization is also apparent but on the whole the sinusoid passes the filter almost intact while the annual cycle has been completely suppressed.

To illustrate the consequences of using a relatively high value for c, Fig. 3 presents the outcome of filtering for the same sequence (7) but with $c = 0.995$. Now the filter rejection band is too narrow and part of the annual cycle contamination is retained in the output.

Higher harmonics of the rejection frequency f are slightly affected by the recursion filter just described. To illustrate this, a second case for $f = 1/6$ is shown

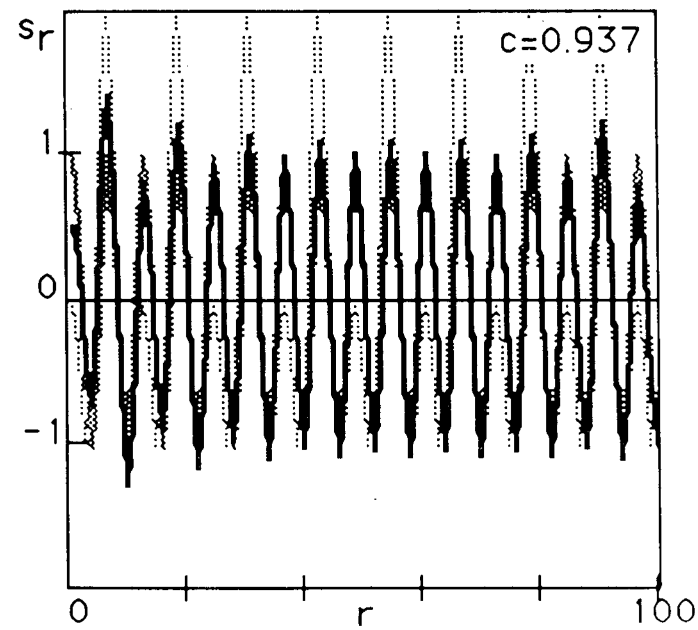


FIG. 4. Output of the recursion filter for $c = 0.937$ applied to time series (8) with $A_0 = A_1 = 1$, $\beta_0 = 1.5\pi$, $\beta_1 = 0.5\pi$, $N = 100$, $P_0 = 12$ and $P_1 = 6$ (black line) compared with the sinusoid to be isolated (gray line).

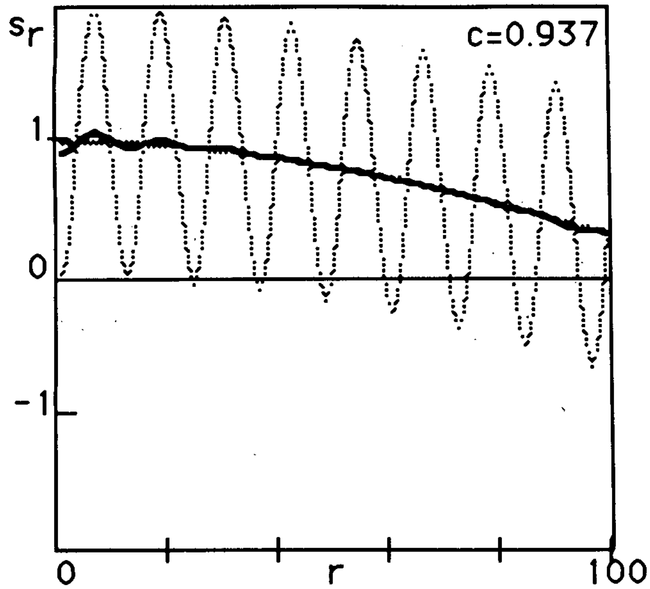
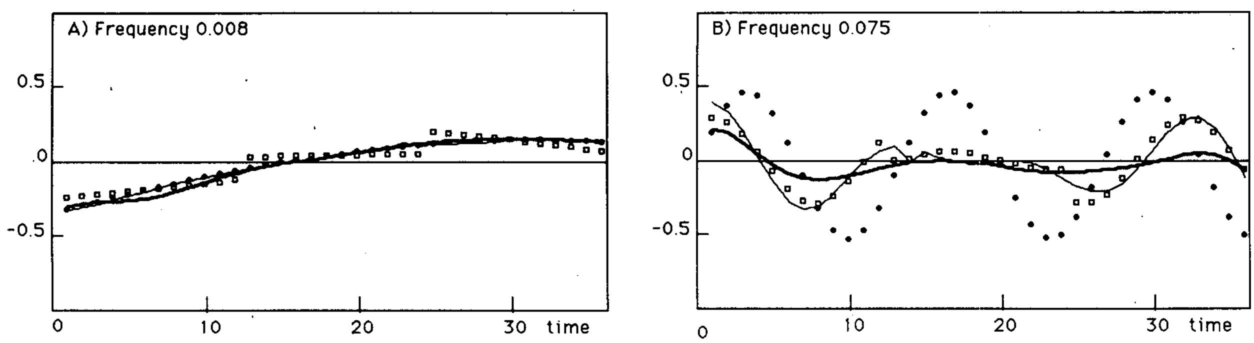


FIG. 5. Output of the recursion filter for $c = 0.937$ applied to time series (8) with $A_0 = A_1 = 1$, $\beta_0 = 1.5\pi$, $\beta_1 = 0.5\pi$, $N = 100$, $P_0 = 12$ and $P_1 = 500$ (black line) compared with the sinusoid to be isolated (gray line).

in Fig. 4. In this respect the recursion filter presents an important improvement in relation with the seasonal differences and the anomaly filters since it passes the fast oscillation with only minor discrepancies in the amplitude of the first few and last cycles. Notice that the recursion filter does not affect the phase of the series in any appreciable manner.

Finally, the case of a slow variation is shown in Fig. 5. Here again the initialization produces some ripples in the first and last part of the output but the slow variation is followed quite closely. In this case it is important to take the mean value from the series before the filter application in order to keep the oscillations small; however, other simpler smoothing techniques



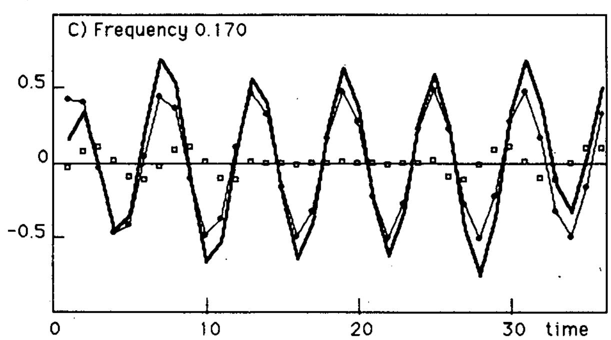


FIG. 6. Output of the recursion filter (thick line), anomaly filter (squares), SSSL filter (thin line) against the cycle to be retained (dots) with frequency 0.008 (case a), 0.075 (case b) and 0.170 (case c) applied to short time series (8) with $A_0 = A_1 = 1$, $\beta_0 = \beta_1 = 0$, $N = 36$, $P_0 = 12$.

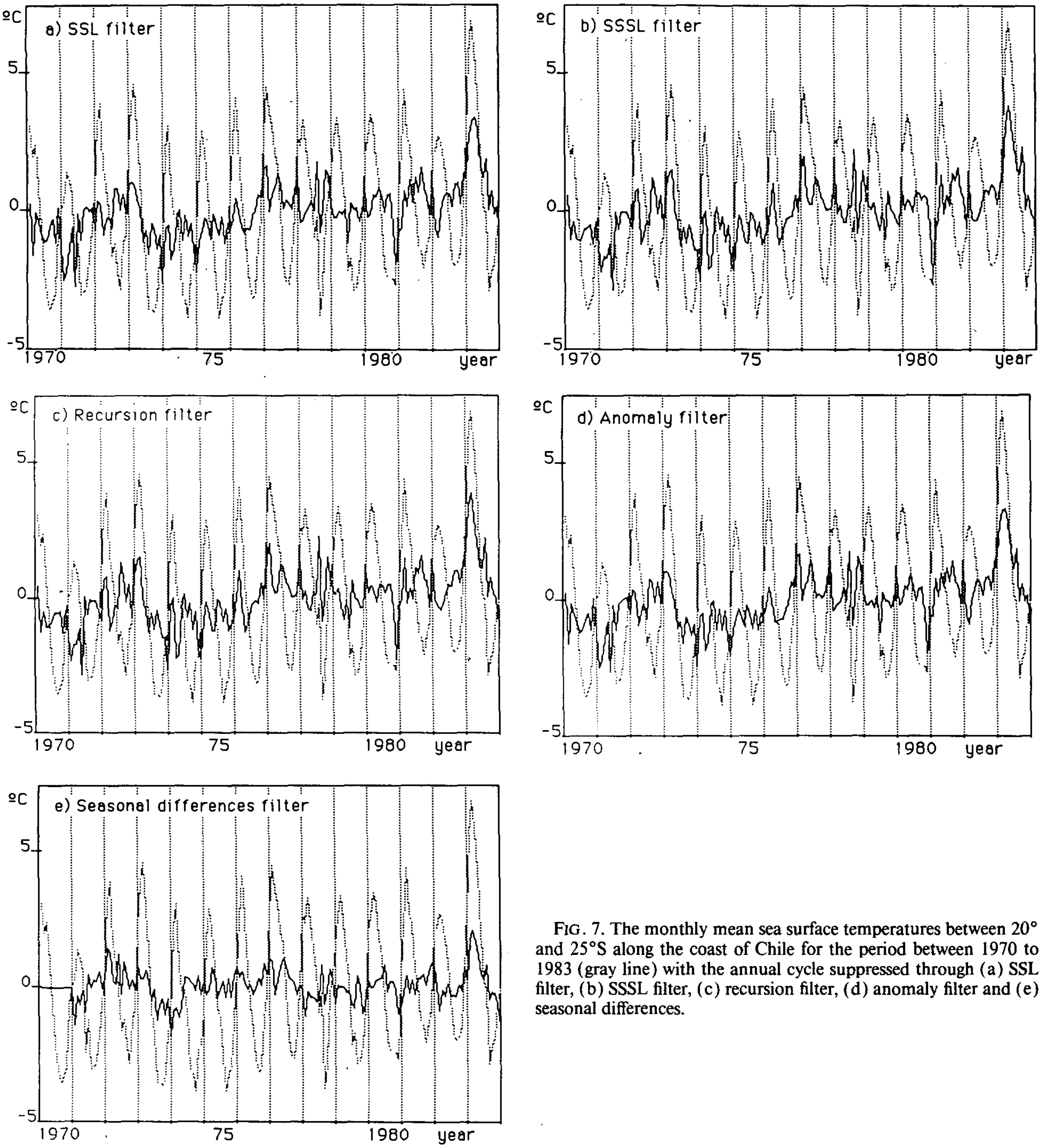


FIG. 7. The monthly mean sea surface temperatures between 20° and 25°S along the coast of Chile for the period between 1970 to 1983 (gray line) with the annual cycle suppressed through (a) SSL filter, (b) SSSL filter, (c) recursion filter, (d) anomaly filter and (e) seasonal differences.

adequate for the detection of very slow variations, such as the moving average filters, exist.

To compare the behavior of different prewhitening filters in short series which contain periods that are not submultiples of the total number of points in the sample, three series with 36 points each were used. A set of 36 values were evaluated through (8) with β_0 and β_1 zero and $A_0 = 1$ and $A_1 = 0.5$. In these series the first term represents the variation to be retained and the second term is the cycle to be extracted ($P_0 = 12$, $f_0 = 0.083$). The values used for $f_1 = 1/P_1$ were 0.008, 0.075 and 0.170. They correspond to frequencies that are lower than, similar to and higher than the frequency to be extracted, respectively. In the last case the frequency to be retained is a multiple of that to be suppressed.

Figure 6 contains the outcome of passing the three synthetic series through three different filters: anomaly, SSSL and recursive filters. The upper portion of the figure corresponds to the low frequency case ($f_1 = 0.008$). Here the anomaly filter produces a useless result but the outcomes of the recursion filter and the SSSL filter agree quite well with the variation to be isolated. In particular the SSSL filter shows a remarkably good agreement while the recursion filter exhibits important discrepancies in the first part of the series. The central part of the figure is the case of a frequency value close to the annual cycle ($f_1 = 0.075$). Now none of the three filters is able to produce a good result because the cycle to be retained is lost due to the proximity between both frequencies. The lower part is related to the high frequency case ($f_1 = 0.170$). Here again the anomaly filter destroys the signal but the recursion filter and the SSSL filter work well, particularly the latter one.

6. An application case

To illustrate the performance of the recursion filter and to compare it with other prewhitening methods, a 14 year series of monthly mean values of sea surface temperature from the U.S. National Climatic Center at Asheville, North Carolina, has been chosen. They are space averages for a quadrangle bounded by latitudes 20° and 25°S, longitude 75°W and the South American coast from 1970–1983. Deviations from the 14 year mean are considered. The series, reproduced with a dotted line in Fig. 7, has a strong annual cycle and includes three large warm events associated with three El Niño occurrences in the austral summer of 1972, 1976 (aborted) and 1982. They emerge as the warmest summers but the annual cycle does not permit to see any detail behind.

Deviations from the mean value were first subjected to a Fourier analysis and then synthesized smoothing a single line, the annual cycle, (SSSL filter) and smoothing the annual cycle and its higher harmonics (SSL filter). The results are shown in Figs. 7a and 7b, together with the input series as a dotted line.

The outcome of a filtering process using the recursion filter together with the original deviations series is shown in Fig. 7c. For comparison Figs. 7d and 7e present the output of filtering processes by means of the anomaly method and the seasonal differences method. The coincidence of the anomaly method with the SSL filter is apparent and the same is true for the recursion filter and the SSSL filter. In all cases the annual cycle has been effectively suppressed revealing a finer structure in the series. The residues show very clearly the strong warming starting at the end of 1982 with a peak in early 1983. The other warm events are less conspicuous and show a different structure. Furthermore there are other maxima that are comparable in height but do not peak in the right season (austral summer). The outcome of the seasonal differences filter differs from the others. Apart from the one-year shortening of the time series, Fig. 7e shows variations of smaller amplitude than other filter outputs and also affects their relative magnitude. For instance, it suggests that in the 1972 austral summer temperature would have been higher than 12 months later, when a warm event was well underway. This is not shown in the output of the

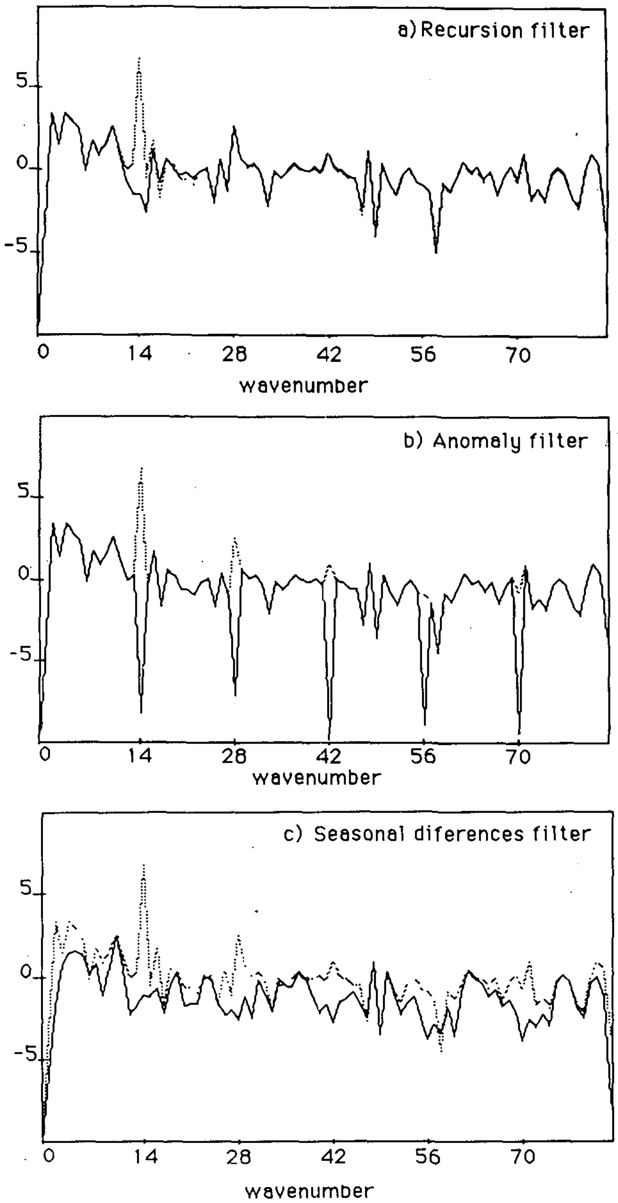


FIG. 8. Amplitude spectrum for the sea surface temperatures between 20° and 25°S along the coast of Chile for the period between 1970 to 1983 (gray line) and the spectra of the residual series after extracting the annual cycle with (a) the recursion filter, (b) the anomaly method and (c) seasonal differences.

anomaly and recursion filters where the feature is properly located. These and other discrepancies indicate that using the seasonal differences is not a good technique. The results of the anomaly and recursion filters are very similar in the broad features. For instance they show the relative intensities of the three El Niño events: a very strong case in 1982, followed by the 1972 event, and a rather weaker one in 1976. But they also show some differences in the finer structure of the events, particularly during the preceding year's summer and winter warmings.

When the Fourier spectra of input and outputs are considered the differences between filters is more easily appreciated. Figure 8 contains the amplitude squared

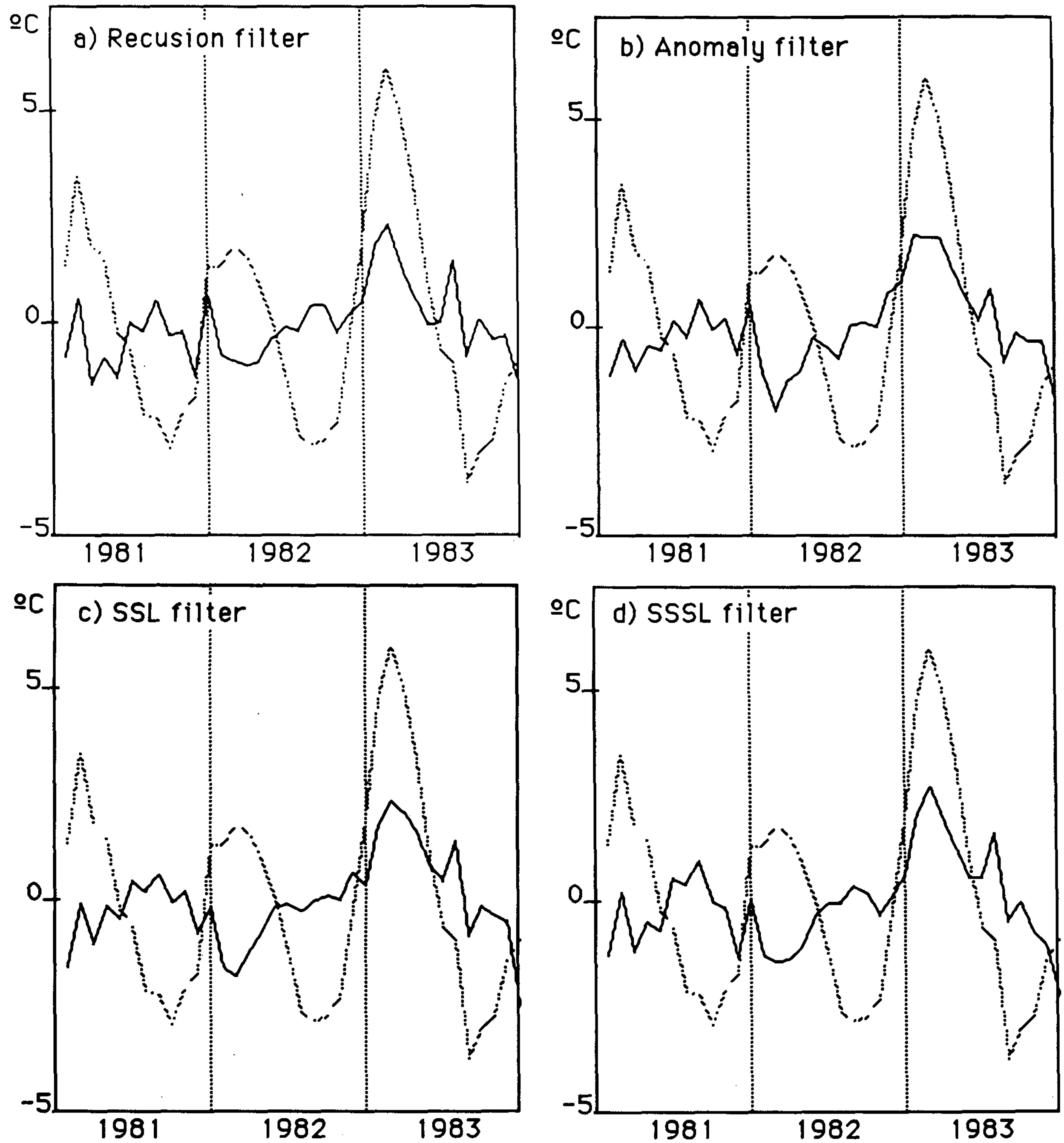


FIG. 9. The monthly mean sea surface temperatures between 20° and 25°S along the coast of Chile for the period between 1981 to 1983 (gray line) with the annual cycle suppressed through (a) recursion filter, (b) the anomaly filter, (c) SSL filter and (d) SSSL filter.

on a logarithmic scale with the input series spectrum as a background (dotted line) where the annual peak is clearly identified. The first part of the figure shows the output spectrum for the recursion filter. Except near the annual cycle, where the strong peak is absent, the rest of the spectrum is preserved. Note that the filter does not introduce discontinuities in the spectrum working in a similar fashion to the SSSL filter. The second part corresponds to the anomaly filter and here the annual frequency and its harmonics are absent. The suppression bands are narrow and deep so that most of the spectrum is preserved but severe discontinuities are introduced in the spectral representation. Finally, Fig. 8c shows the spectrum for the output of the seasonal differences filter. In this case the shape of the spectrum has been modified over most of the frequencies, particularly near the annual value, its harmonics and the slow trends. In summary, SSL, anomaly and recursion filters behave in a similar fashion with only minor discrepancies, but the recursion filter has some advantage over the anomaly method as far as the continuity of the spectrum is concerned and over the SSL filter from a computation effort point of view.

To test the short series case, the last three years of the input were used. Note that according to the spectra shown in Fig. 8 the annual cycle is heavily dominated by the fundamental component with a minor contribution from the first harmonic (semi-annual) and negligible contributions from the rest. This implies that the annual cycle approximates a sinusoid curve. Figure 9a shows the output from the recursion filter; Fig. 9b presents the output from the anomaly filter using the last three years. Figures 9c and 9d contain the outcome of the SSL and the SSSL filters, respectively. Again it can be seen that the outcome from the anomaly filter using only three years differs little from the SSL case and the recursion filter from the SSSL filter. Furthermore, the four compare rather well, a consequence of the sinusoidal character of the annual cycle. It is noteworthy that the recursion filter produces a result quite similar to the other filters in the last 24 months but differs to some extent in the first third of the series.

7. Conclusions

Prewhitening of environmental time series has traditionally used the anomaly technique. This method presents some drawbacks when the time series is short compared with the length of the variation to be suppressed, commonly the annual or daily cycles. Because this technique also eliminates higher harmonics of such cycles there is some risk of losing information other than the cycle to be suppressed and because of discontinuities in the frequency space some spurious oscillations might appear. Here two different procedures are presented. The first, which gives very good results with synthetic series, is applied through a Fourier analysis and a subsequent synthesis with the harmonic components to be suppressed replaced by the mean of neighbor components (SSL and SSSL filters). The second, which is a recursion type filter, is as simple as the anomaly evaluation from a computational point of view. This filter shows advantages over the anomaly and seasonal differences methods in the sense that is marginally affected by the length of the series and retains higher harmonics, suppressing only one frequency or a frequency band as wide as desired. The smoothness of the spectrum is as good as the SSSL filter. It also gives proper consideration to the spectral widening due to the finite length of the time series.

Table 1 is a summary of the most important features of the three prewhitening filters under comparison, including the expression for each filter, its frequency response, rejection bandwidth, phase and behavior with short time series. For short series the recursion filter seems the most recommendable. If the cycle to be suppressed is quasi-harmonic, such recommendation can be extended to any length series. If this is not the case, however, the filter has to be applied once for every important frequency component. The anomaly method is a second choice, although it introduces sharp variations in the spectrum of the output.

Acknowledgments. This research is a late outcome of Project E 1485 supported by the Departamento de Investigaciones y Bibliotecas de la Universidad de Chile. The authors would like to thank Dr. P. Aceituno for providing the sea surface temperature data and a careful revision of a draft version of the article.

APPENDIX

According to (3) the amplitude frequency response $|H|$ for one filter application can be written as

$$|H| = b_0(|z - z_0|\,|z - z_0^*|)(|z - z_p|\,|z - z_p^*|)^{-1},$$

where b_0 is a positive real number whose value is presently to be determined, z is a number located on the unit circle of the complex plane with a polar angle $\varphi = 2\pi f d$, where f is any frequency and d the sampling time interval, z_0 is another complex number on the unit circle with polar angle $\varphi_0 = 2\pi f_0 d$ representing a zero of H, z_p corresponds to a pole of H located at a distance c from the origin with the same polar angle φ_0 and the asterisk denotes complex conjugate. In the complex plane shown in Fig. A1, the point F corresponds to z, points Z and Z' to z_0 and its complex conjugate and points P and P' to z_p and its complex conjugate; then $|H| = b_0(FZ \cdot FZ')/(FP \cdot FP')$. Because $FZ = 2\sin[(\varphi - \varphi_0)/2]$, $FZ' = 2\sin[(\varphi + \varphi_0)/2]$, $FP = \{(1 - c)^2 + 4c\sin^2[(\varphi - \varphi_0)/2]\}^{1/2}$ and $FP' = \{(1 - c)^2 + 4c\sin^2[(\varphi + \varphi_0)/2]\}^{1/2}$,

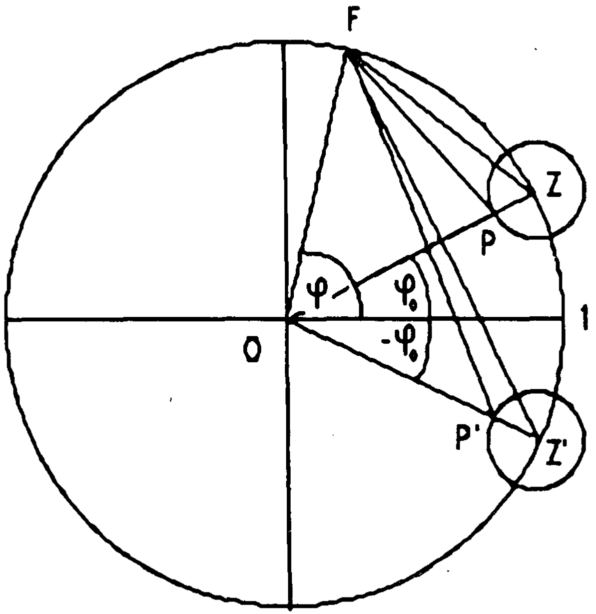


FIG. A1. Complex plane showing the frequency unit circle ($O1 = OZ = OZ' = OF = 1$). Zeros are located at Z and Z', poles at P and P' ($OP = OP' = c$) and the frequency point at F.

$$|H| = b_0\{c + (1-c)^2/4 \sin^2[(\varphi - \varphi_0)/2]\}^{-1/2}$$
$$\times \{c + (1-c)^2/4 \sin^2[(\varphi + \varphi_0)/2]\}^{-1/2}.$$

For c approaching 1 and away from the zeros Z and Z' to a good approximation $|H| = b_0/c$, and to have $|H| = 1$, a suitable normalization is $b_0 = c$.

When point F approaching Z comes to a distance $1 - c = 2 \sin[(\varphi - \varphi_0)/2]$ from it, it still is relatively far from Z', therefore for the frequency represented by this point $FZ'/FP' \approx 1$ and $FZ/FP \approx (c+1)^{-1}$, so that $|H| \approx b_0/(c+1)^{1/2}$. Using $b_0 \approx c \approx 1$, one obtains $|H| \approx 2^{-1/2}$. Considering that the filter has to be applied twice to get a zero phase, this point corresponds to the half power frequency for then $|H|^2 \approx 1/2$.

Matching the spectral widening $2\pi/N$ to the rejection bandwidth characterized by the half power points the relation $\varphi - \varphi_0 = 2\pi/N$ is obtained. Therefore $2 \sin\pi/N = 1 - c$ and the equation $c = 1 - 2 \sin\pi/N$ or $c \approx 1 - 2\pi/N$ for N sufficiently large follows.

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