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FAST PROPAGATION FOR FRACTIONAL KPP EQUATIONS WITH SLOWLY DECAYING INITIAL CONDITIONS*

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Abstract. In this paper we study the large-time behavior of solutions of one-dimensional fractional Fisher-KPP reaction-diffusion equations, when the initial condition is asymptotically front-like and it decays at infinity more slowly than a power x^{-b}, where $b < 2\alpha$ and $\alpha \in (0,1)$ is the order of the fractional Laplacian. We prove that the level sets of the solutions move exponentially fast as time goes to infinity. Moreover, a quantitative estimate of motion of the level sets is obtained in terms of the decay of the initial condition.

Key words. fractional reaction-diffusion, KPP, asymptotic behavior

AMS subject classifications. 35K57, 47G20, 35B40

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1. Introduction.

In this paper we study the large-time behavior of the solution of the Cauchy problem for fractional reaction-diffusion equations

$$u_t + (-\Delta)^\alpha u = f(u), \tag{1.1}$$

$$u(0,x) = u_0(x) \tag{1.2}$$

with $\alpha \in (0,1)$ in one spatial dimension. The nonlinearity f is assumed to be in the Fisher-KPP class, and the initial condition is front-like, decaying to zero at infinity. More precisely, the nonlinearity is assumed to have two zeros, an unstable one at $u = 0$ and a stable one at $u = 1$, while the initial condition u_0 is assumed to decay slower than a power x^{-b}, where $b < 2\alpha$ as $x \to \infty$.

When $\alpha = 1$, the reaction-diffusion equation with Fisher-KPP nonlinearity has been the subject of intense research since the seminal work by Kolmogorov, Petrovskii, and Piskunov [12]. Of particular interest are the results of Aronson and Weinberger [1] which describe the evolution of compactly supported data. They showed that there is a critical number $c^* = 2\sqrt{f'(0)}$ such that for a compactly supported initial value u_0, such that $0 \le u_0 \le 1$, if $c > c^*$, then $u(t,x) \to 0$ uniformly in $\{|x| \ge ct\}$ as $t \to \infty$, and if $c < c^*$, then $u(t,x) \to 1$ uniformly in $\{|x| \le ct\}$ as $t \to \infty$. In addition, (1.1)–(1.2) admits planar traveling wave solutions connecting 0 and 1, that is, solutions of the form $u(t,x) = \phi(x - ct)$ with

$$-\phi'' + c\phi' = f(\phi) \quad \text{in } \mathbb{R}, \qquad \phi(-\infty) = 1,\ \phi(+\infty) = 0,$$

whenever $c \ge c^*$. Many papers have been concerned with the large-time behavior of solutions of (1.1) or more general reaction-diffusion equations with exponentially decaying initial conditions, leading to finite propagation speeds; see, for example, [3], [11], [13], [15], and the references in [10].

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In contrast with the results just mentioned, where finite speed of propagation is obtained whenever the initial value decays faster than an exponential, it is shown by Hamel and Roques [10] that when the initial condition is globally front-like and decays more slowly than any exponential, then the asymptotic behavior of the front exhibits infinite speed and a very precise estimate can be obtained for the propagation of the level sets of the front in terms of the initial value, giving a precise superlinear behavior.

In this paper we are interested in understanding the phenomena described above in the case of nonlocal diffusion, that is, when $\alpha \in (0,1)$ in (1.1)–(1.2). In particular we want to study the asymptotic behavior of solutions with slowly decaying, globally front-like initial value. Reaction-diffusion equations with fractional Laplacian appear in physical models when the diffusive phenomena are better described by Lévy processes allowing long jumps than by Brownian processes; see, for example, [14] for a description of some of these models. The Lévy processes occur widely in physics, chemistry, and biology, giving rise to equations with the fractional Laplacian like (1.1)–(1.2), and have attracted much interest in recent years.

Regarding (1.1)–(1.2) with Fisher-KPP nonlinearity, in connection with the above discussion, in the recent paper [5] announced in [4], Cabré and Roquejoffre show that for compactly supported initial value or, more generally, for initial values decaying like $|x|^{-N-2\alpha}$, where N is the dimension of the spatial variable, the speed of propagation becomes exponential; they also show that no traveling waves exist for this equation, all results that are in great contrast with the case $\alpha = 1$. Here we recall the earlier work in the case $\alpha \in (0,1)$ by Berestycki, Roquejoffre, and Rossi [2], where it is proved that there is invasion of the unstable state by the stable one. For a large class of nonlinearities, Engler [7] has proved that the invasion has unbounded speed. For another type of integro-differential equations, Garnier [9] also establishes that the position of the level sets moves exponentially in time for algebraically decaying dispersal kernels.

In the case of propagation of front-like initial values, Cabré and Roquejoffre [5, Theorem 1.5] prove that for $c_* = \frac{f'(0)}{2\alpha}$ and for initial value u_0 measurable and nonincreasing, $u_0 \not\equiv 0$, $0 \le u_0 \le 1$, and

$$u_0(x) \le C(x)^{-2\alpha} \qquad \text{if } x > 0, \tag{1.3}$$

the solution u of (1.1)–(1.2) satisfies the following:

(a) If $c > c_*$, $u(t,x) \to 0$ uniformly in $\{x \ge e^{ct}\}$ as $t \to \infty$.

(b) If $c < c_*$, $u(t,x) \to 1$ uniformly in $\{x \le e^{ct}\}$ as $t \to \infty$.

In view of this result for initial values decaying fast enough and having in mind the conclusion obtained by Hamel and Roques for slowly decaying front-like initial values in [10] in the case $\alpha = 1$, a natural question is, What kind of asymptotic behavior does a solution of (1.1)–(1.2) with initial value decaying slower than a power $|x|^{-b}$ have? It is the purpose of this paper to answer this question for the case of fractional Laplacians, with $\alpha \in (0,1)$ in the one-dimensional case. Our main result states that, for $b < 2\alpha$, the central part of the solution moves to the right at exponential speed $f'(0)/b$, which is faster than c_*, the exponential speed for solutions with initial values decaying faster than $x^{-2\alpha}$. Thus we show that the exponent 2α is critical regarding the speed of propagation of the solution; see the discussion at the end of the introduction. Furthermore, we prove that the initial condition u_0 can be chosen so that the location of the solution u is asymptotically larger than any prescribed real-valued function.

Let us now provide a precise description of our assumptions and results. We assume that the nonlinearity in (1.1) is of Fisher-KPP type; that is, $f : [0,1] \to \mathbb{R}$ is of class C^1, is concave, and satisfies

$$f(0) = f(1) = 0, \quad f'(1) < 0 < f'(0). \tag{1.4}$$

These properties mean that the growth rate $\frac{f(s)}{s}$ is maximal at $s = 0$.

We assume that the initial condition $u_0 : \mathbb{R} \to [0,1]$ is continuous and satisfies

$$u_0 > 0 \quad \text{in } \mathbb{R}, \quad \lim_{x\to-\infty} u_0(x) > 0 \quad \text{and} \quad \lim_{x\to\infty} u_0(x) = 0. \tag{1.5}$$

Furthermore, we assume that

$$\text{there exists } \ \xi_0 \in \mathbb{R}, \text{ such that } u_0 \text{ is nonincreasing in } [\xi_0, \infty). \tag{1.6}$$

When u_0 satisfies the earlier conditions we say that u_0 is asymptotically front-like. Before stating our main results we introduce some notation. For any $\lambda \in (0,1)$ and $t \geq 0$, we denote by

$$E_\lambda(t) = \{x \in \mathbb{R} : u(t,x) = \lambda\}$$

the level set of u of value λ at time t. For any subset $A \subset (0,1]$, we set

$$u_0^{-1}(A) = \{x \in \mathbb{R} : u_0(x) \subset A\},$$

the inverse image of A by u_0. Our first result provides basic properties of the solutions of (1.1)–(1.2) and states that the level sets $E_\lambda(t)$ move at least exponentially fast, as $t \to \infty$.

Theorem 1.1. *Let $\alpha \in (0,1)$ and $c_* = \frac{f'(0)}{2\alpha}$, and let u be the solution of* (1.1)–(1.2)*, where f satisfies* (1.4) *and the initial condition $u_0 : \mathbb{R} \to [0,1]$ satisfies* (1.5) *and* (1.6)*. Then u satisfies the following:*

(a) *$0 \leq u(t,x) \leq 1$ for all $(t,x) \in (0,\infty) \times \mathbb{R}$ and*

$$\lim_{x\to+\infty} u(t,x) = 0 \ \ \forall t \geq 0 \qquad \textit{and} \qquad \lim_{t\to+\infty} \inf_{x \leq e^{ct}} u(t,x) = 1 \quad \forall c < c_*.$$

(b) *For any given $\lambda \in (0,1)$, there is a real number $t_\lambda > 1$ such that $E_\lambda(t)$ is compact and nonempty for all $t \geq t_\lambda$.*

As a direct consequence we see that

$$\lim_{t\to\infty} \frac{E_\lambda(t)}{e^{ct}} = \infty \quad \forall c < c_*. \tag{1.7}$$

The purpose of our main theorem is to obtain a more accurate understanding of the behavior of $E_\lambda(t)$. Actually, we express the motion of $E_\lambda(t)$ in terms of the behavior of the initial value u_0, and we improve the estimate for c in (1.7). To do this we need some additional hypotheses that express the slow decay of the initial values:

(H1) There exists $b < 2\alpha$ such that $u_0(x) \geq x^{-b}$ for all $x \geq \xi_0$.

(H2) There exist $\rho > 1$ and $k > 0$ such that

$$\frac{u_0(\rho x)}{u_0(x)} \geq k \qquad \text{for } x \geq \xi_0.$$

Now we are in a position to state our main theorem.

THEOREM 1.2. *Let $\alpha \in (0,1)$, $\lambda \in (0,1)$, and let u be the solution of* (1.1)–(1.2), *where f satisfies* (1.4) *and the initial condition $u_0 : \mathbb{R} \to [0,1]$ satisfies* (1.5), (1.6), *and hypotheses* (H1) *and* (H2).

Then, for any $\Gamma > 0$, $\gamma > 0$, and $\delta \in (0, f'(0))$, there exist $\tau = \tau(\lambda, \Gamma, \gamma, \delta, b) \geq t_\lambda$ such that

$$E_\lambda(t) \subset u_0^{-1}\{[\gamma e^{-(f'(0)+\delta)t}, \Gamma e^{-(f'(0)-\delta)t}]\} \quad \forall t \geq \tau,$$

where t_λ was given in Theorem 1.1.

As a corollary of this theorem, we see that by choosing the initial condition appropriately, we are able to obtain any fast behavior of the set $E_\lambda(t)$. In precise terms we have the following corollary.

COROLLARY 1.1. *Under the assumptions of Theorem* 1.1, *given any function $\chi : [0,\infty) \to \mathbb{R}$ which is locally bounded, there are initial conditions u_0 such that, for any given $\lambda \in (0,1)$,*

$$\min E_\lambda(t) \geq \chi(t)$$

for all t large enough.

The proof of Theorem 1.2 is inspired in the work by Hamel and Roques [10], by basically making two estimates to capture the set $u_0(E_\lambda(t))$, with appropriate super- and subsolutions. However, the nonlocal character of the differential operator introduces a series of difficulties that were not present in the local case. This is especially so in the proof of Proposition 3.1, where we have to introduce a staggered subsolution to gain a global control in time. Moreover, the choice of ω in (3.8) is not obvious and the estimates are much more involved. It is important to mention that to obtain the lower estimate we only need to assume that the initial condition u_0 satisfies (1.5) and (1.6); see Propositions 3.1 and 4.1. Finally, we observe that, since there are no traveling waves for the fractional problem, as proved in [5], various other arguments given in [10] need to be changed in our case.

Now we would like to make some comments on hypothesis (H2). This condition complements hypothesis (H1), and it also expresses the slow decay of u_0. Actually, we observe that any power $u_0(x) = x^{-b}$ also satisfies (H2). More generally, any function $u_0 \in C^1([\xi_0, \infty))$, decreasing and convex in $[\xi_0, \infty)$, such that

$$\left|\frac{u_0'(x)}{u_0(x)}\right| = O\left(\frac{1}{x}\right) \quad \text{as} \quad x \to \infty,$$

satisfies hypothesis (H2).

Theorem 1.2 and Corollary 1.1 complement the results by Cabré and Roquejoffre [4], [5], where they estimate the asymptotic behavior of solutions with front-like initial values which decay faster than $x^{-2\alpha}$ as $x \to \infty$. In our case we assume the initial value decays slower than a power x^{-b}, with $b < 2\alpha$, the complementary exponents. In a sense we generalize to the case $\alpha \in (0,1)$ results proved by Hamel and Roques in [10], replacing the Laplacian by the fractional Laplacian.

Let us assume that the initial value is a pure power, that is, $u_0(x) = x^{-b}$, with $b < 2\alpha$, for x large. In this case we see that Theorem 1.2 implies that for all c_1 and c_2 such that

$$c_* = \frac{f'(0)}{2\alpha} < c_1 < \frac{f'(0)}{b} < c_2,$$

there is τ such that for all $x_\lambda(t) \in E_\lambda(t)$ we have

$$e^{c_1 t} \le x_\lambda(t) \le e^{c_2 t} \quad \forall t \ge \tau.$$

These observations are in contrast with the results of Cabré and Roquejoffre [5], who showed that all solutions with front-like initial conditions decaying slower than $x^{-2\alpha}$ spread at an exponential speed c_* independent of further properties of u_0. In our case, using the comparison principle and the discussion above, we see that solutions with front-like initial conditions decaying slower than x^{-b}, with $b < 2\alpha$, spread at an exponential speed $f'(0)/b$, which is larger than c_* and depends explicitly on the exponent b.

In this sense, our results show that the exponent 2α is a critical exponent. If the initial value decays faster than $x^{-2\alpha}$, then the exponential speed is c_*, and if the initial value decays slower than x^{-b}, with $b < 2\alpha$, then the exponential speed is $f'(0)/b$ or larger. Above the exponent 2α, the solution's speed of propagation starts getting influenced by the initial value, propagating faster the slower the decay is.

This paper is organized as follows. In section 2 we present some preliminaries, reviewing the notion of mild solution and the comparison principle. Then we prove Theorem 1.1. Section 3 is devoted to the proof of the upper estimate in Theorem 1.2 and Corollary 1.1. In section 4 we prove the lower estimate and then offer a conclusion.

2. Preliminaries and proof of Theorem 1.1.

In this section we first recall the notion of a mild solution that is suited to our problem, and we state the comparison principle, which will be a crucial tool in our analysis. Then we present the proof of Theorem 1.1, which follows the line of the corresponding result in [5] and [4].

In studying the existence of solution of (1.1)–(1.2) we first consider the linear heat equation for the fractional Laplacian,

$$u_t + (-\Delta)^\alpha u = f_0(t, x), \tag{2.1}$$

$$u(0, x) = u_0(x), \tag{2.2}$$

whose solution may be obtained by the formula of variation of parameters or the Duhamel formula

$$u(t, x) = p(t, x) * u_0(x) + \int_0^t p(t - s, x) * f(s, x) ds, \tag{2.3}$$

where the convolution is taken in the variable x. Here the kernel p is given by $p(t, x) = t^{-\frac{1}{2\alpha}} p_\alpha(t^{-\frac{1}{2\alpha}} x)$, where

$$p_\alpha(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi - |\xi|^{2\alpha}} d\xi,$$

and it satisfies the following properties:

1. $p \in C((0, +\infty), \mathbb{R})$.
2. $p(t, x) \ge 0$ and $\int_{\mathbb{R}} p(t, x) dx = 1$ for all $t > 0$.
3. $p(t, \cdot) * p(s, \cdot) = p(t + s, \cdot)$ for all $t, s \in \mathbb{R}_+$.
4. There exists $B > 1$ such that, for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\frac{B^{-1}}{t^{\frac{1}{2\alpha}}(1 + |\, x t^{-\frac{1}{2\alpha}} \,|^{1+2\alpha})} \le p(t, x) \le \frac{B}{t^{\frac{1}{2\alpha}}(1 + |\, x t^{-\frac{1}{2\alpha}} \,|^{1+2\alpha})}. \tag{2.4}$$

Now we consider the Banach space

$$\mathcal{C}_{lim} = \{w \in C(\mathbb{R}) \mid \lim_{x\to-\infty} w(x) \quad \text{and} \qquad \lim_{x\to+\infty} w(x) \;\; \text{exist}\},$$

equipped with the supremum norm. Given $u_0 \in \mathcal{C}_{lim}$, equations (1.1)–(1.2), with Fisher-KPP nonlinearities f and initial condition u_0, have a unique solution u that exists for all $x \in \mathbb{R}$ and $t \geq 0$; moreover, $u(t,\cdot) \in C([0,\infty), \mathcal{C}_{lim})$. This solution u can be obtained as the limit of the iteration scheme

$$u_{n+1}(t,x) = p(t,x) * u_0(x) + \int_0^t p(t-s,x) * f(s,u_n(s,x))ds, \tag{2.5}$$

with $u_0(t,x) = p(t,x) * u_0(x)$. The limit is uniform in x and local in time; see [1] and [5] for details. The solution obtained in this way is called a mild solution and in this paper will be the notion of solution that we consider in all our statements.

To continue we recall the comparison principle, which will be frequently used in this paper. For the proof of this result we refer the reader to [5] or [7].

THEOREM 2.1 (comparison principle). *Let* $u, v \in C([0,T], \mathcal{C}_{lim})$ *be mild solutions of the equations*

$$u_t + (-\triangle)^\alpha u = g(u), \qquad v_t + (-\triangle)^\alpha v = h(v),$$

where $g, h : \mathbb{R} \to \mathbb{R}$ *are locally Lipschitz continuous. If*

$$g(\zeta) \leq h(\zeta) \quad \forall \zeta \in \mathbb{R}$$

and

$$u(0,x) \leq v(0,x) \quad \forall x \in \mathbb{R},$$

then

$$u(t,x) \leq v(t,x) \quad \forall (t,x) \in [0,T] \times \mathbb{R}.$$

Now we are able to give the following proof.

Proof of Theorem 1.1, *part* (a) We start by using the comparison principle, recalling that $0 \leq u_0(x) \leq 1$, to obtain that the solution u of (1.1)–(1.2) satisfies

$$0 < u(t,x) \leq 1 \quad \forall (t,x) \in [0,\infty) \times \mathbb{R}.$$

Next we analyze the limit of $u(t,x)$ as $x \to -\infty$. To do this, let us first note that the function

$$\overline{u}(t,x) = e^{f'(0)t} \int_{\mathbb{R}} p(t,x-y) u_0(y) dy$$

is the solution of the equation

$$\begin{aligned} \overline{u}_t + (-\Delta)^\alpha \overline{u} &= f'(0)\overline{u}, \\ \overline{u}(0,x) &= u_0(x). \end{aligned}$$

But, since f satisfies (1.4) and is concave and of class C^1, we have that $0 < f(s) \leq f'(0)s$ for all $s \geq 0$; therefore, we conclude that $\overline{u}$ is a supersolution of (1.1)–(1.2), and then the comparison principle implies that $u(t,x) \leq \overline{u}(t,x)$ for all $t \geq 0$ and $x \in \mathbb{R}$. To continue, let us define

$$C_\alpha := \int_{-\infty}^{\infty} \frac{1}{1+\mid s \mid^{1+2\alpha}} ds \tag{2.6}$$

and note that $C_\alpha > 1$ for all $\alpha \in (0,1)$. We may assume without loss of generality that $C_\alpha \leq B$, where B is given in (2.4).

We observe that the property is true for $t = 0$ by hypothesis on u_0. For $t > 0$, we consider $\varepsilon > 0$, and we find $M_t > 0$ such that, for each $x \geq M_t$, we have $u(\bar{t}, x) < \varepsilon$. Let us start by considering $\sigma > 0$ small enough such that $C_\alpha B e^{f'(0)t}\sigma < \frac{\varepsilon}{2}$, and let $\xi_1 \in [\xi_0, \infty)$ and $\xi > 0$ be such that

$$u_0(z) \leq \sigma \quad \forall z \geq \xi_1 \quad \text{and} \quad \int_{\xi}^{\infty} \frac{1}{1+s^{1+2\alpha}} ds < \sigma. \tag{2.7}$$

Then let us take

$$M_t := \xi_1 + \xi t^{\frac{1}{2\alpha}}$$

and consider $x \geq M_t$. Then we use the definition of $\bar{u}$ and (2.4) to find that

$$\begin{aligned}
\overline{u}(t,x) &\leq B\frac{e^{f'(0)t}}{t^{\frac{1}{2\alpha}}} \int_{-\infty}^{\infty} \frac{u_0(y)}{1+\mid (x-y)t^{-\frac{1}{2\alpha}} \mid^{1+2\alpha}} dy = B\frac{e^{f'(0)t}}{t^{\frac{1}{2\alpha}}} \int_{-\infty}^{\infty} \frac{u_0(x-r)}{1+\mid rt^{-\frac{1}{2\alpha}} \mid^{1+2\alpha}} dr \\
&= Be^{f'(0)t} \int_{-\infty}^{\infty} \frac{u_0(x - st^{\frac{1}{2\alpha}})}{1+\mid s \mid^{1+2\alpha}} ds \\
&= Be^{f'(0)t} \left(\int_{-\infty}^{\xi} \frac{u_0(x - st^{\frac{1}{2\alpha}})}{1+\mid s \mid^{1+2\alpha}} ds + \int_{\xi}^{\infty} \frac{u_0(x - st^{\frac{1}{2\alpha}})}{1+\mid s \mid^{1+2\alpha}} ds \right).
\end{aligned}$$

If $s \leq \xi$, we have $x - st^{\frac{1}{2\alpha}} \geq x - \xi t^{\frac{1}{2\alpha}} \geq \xi_1$. Then, using (2.7) we find

$$\begin{aligned}
\bar{u}(t,x) &\leq Be^{f'(0)t} \left(\int_{-\infty}^{\xi} \frac{\sigma}{1+\mid s \mid^{1+2\alpha}} ds + \int_{\xi}^{\infty} \frac{1}{1+\mid s \mid^{1+2\alpha}} ds \right) \\
&\leq Be^{f'(0)t} (\sigma C_\alpha + \sigma) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence $0 \leq u(t,x) < \varepsilon$. Thus we have proved that $\lim_{x\to\infty} u(t,x) = 0$.

Now we study $u(t,x)$ when $t \to \infty$. From (1.5), we may find a continuous nonincreasing function $v_0 : \mathbb{R} \to [0,1]$ in $\mathcal{C}_{lim}$, such that $\mathbb{R}_+ \cap \operatorname{supp}(v_0)$ is compact and satisfies $0 \leq v_0(x) \leq u_0(x)$. Denote by v the solution of the Cauchy problem (1.1)–(1.2) with initial condition v_0; then by the comparison principle we find $v(t,x) \leq u(t,x) \leq 1$ for all $t \geq 0$ and $x \in \mathbb{R}$. Then we can use Theorem 1.5 of [5] to conclude that

$$\lim_{t\to\infty} \inf_{x \leq e^{ct}} v(t,x) = 1$$

for all $c < c_*$. Then our result follows.

Part (b). From part (a) it follows that, given $d \in (0, c_*)$ and any $\lambda \in (0,1)$, there exists $t_\lambda \geq 1$ such that

$$\inf_{x \leq e^{dt}} u(t,x) > \lambda > 0 = u(t, +\infty)$$

for all $t \geq t_\lambda$. By continuity of $x \mapsto u(t,x)$ we conclude that $E_\lambda(t)$ is a nonempty compact set for all $t \geq t_\lambda$. ◻

3. The lower estimate. In order to prove Theorem 1.2 we need to obtain an upper and a lower estimate for the set $E_\lambda(t)$ for t large. In this section we obtain the lower estimate. It is important to note that in getting the lower estimate we do not require the initial condition to satisfy hypotheses (H1) and (H2), but only (1.5) and (1.6).

PROPOSITION 3.1. *Let $\alpha \in (0,1)$, and let u be the solution of* (1.1)–(1.2), *where f satisfies* (1.4) *and the initial condition u_0 satisfies* (1.5) *and* (1.6).

Then, for any $\Gamma > 0$, $\lambda \in (0,1)$, and $\delta \in (0, f'(0))$, there exists a time $\tau_u = \tau_u(\lambda, \Gamma, \delta) \geq t_\lambda$ such that

$$E_\lambda(t) \subset u_0^{-1}\{(0, \Gamma e^{-(f'(0)-\delta)t}]\} \quad \forall t \geq \tau_u. \tag{3.1}$$

For proving this proposition, we first prove a lemma where we construct an appropriate subsolution of (1.1)–(1.2) which will enable us to prove the lower bound for small values of λ. Then we will show that such an estimate can also be done for the remaining values of $\lambda \in (0,1)$.

Let us start by setting up some notation. Given $\delta \in (0, f'(0))$, we let d and δ' be such that

$$d \in \left(1, \frac{f'(0)}{f'(0)-\delta}\right) \quad \text{and} \quad \delta' = f'(0) - d(f'(0)-\delta).$$

We notice that $\delta' \in (0, f'(0))$, so we may choose ρ such that

$$f'(0) - \delta' < \rho < f'(0).$$

Next we let $s_0 \in (0,1)$ be such that $f(s_0) = \rho s_0$, and we choose $\tau > 0$ such that $\xi_* \in \mathbb{R}$, $u_0(\xi_*) = e^{-\rho\tau} s_0$ implies $\xi_* \geq \xi_0$ and $\xi_* \geq 0$, and

$$e^{(1-\frac{1}{d})\rho\tau} C_\alpha > 2B. \tag{3.2}$$

Now we state a lemma on the existence of a small subsolution.

LEMMA 3.1. *There are $T > \tau + 1$ and a sequence of continuous functions $\underline{u}_n : [(n-1)T, nT] \to [0, s_0]$, for $n \geq 1$, such that*

$$\underline{u}_1(0,x) \leq u_0(x) \quad \forall x \in \mathbb{R}, \tag{3.3}$$

$$(\underline{u}_n)_t + (-\Delta)^\alpha \underline{u}_n = \rho \underline{u}_n \quad \textit{in } ((n-1)T, nT) \times \mathbb{R}, \tag{3.4}$$

$$\underline{u}_{n+1}(nT, x) \leq \underline{u}_n(nT, x) \quad \forall x \in \mathbb{R}, \tag{3.5}$$

$$\lim_{x \to -\infty} \underline{u}_n(nT, x) = s_0, \tag{3.6}$$

and $\underline{u}_n(t,x)$ is nonincreasing in $x \in \mathbb{R}$ for all $t \in [(n-1)T, nT]$.

Proof. Let $\varepsilon > 0$ be such that $\varepsilon < \inf_{(-\infty,\xi_0)} u_0$, and let $\xi \in \mathbb{R}$ be such that $u_0(\xi) = \varepsilon$ and $u_0(x) < \varepsilon$ for all $x > \xi$. By making ε smaller if necessary, we may assume that $\xi > 0$, and we can choose $T > \tau + 1$ such that $\varepsilon = e^{-\rho T} s_0$. Then let us define $\underline{u}_0(x) = \inf(u_0(x), \varepsilon)$. We let $\underline{u}_1$ be the solution of the equation

$$\begin{aligned} (\underline{u}_1)_t + (-\Delta)^\alpha \underline{u}_1 &= \rho \underline{u}_1, \\ \underline{u}_1(0,x) &= \underline{u}_0(x). \end{aligned}$$

This solution is given by

$$\underline{u}_1(t,x) = e^{\rho t} \int_{\mathbb{R}} p(t, x-y) \underline{u}_0(y) dy,$$

so that, by the choice of ε and T, we have that $\underline{u}_1(t,x) \le s_0$ for all $(t,x) \in [0,T] \times \mathbb{R}$. Moreover, we have

$$\lim_{x\to-\infty} \underline{u}_1(T,x) = \lim_{x\to-\infty} e^{\rho T} \int_{\mathbb{R}} p(t,z)\underline{u}_0(x-z)dz = e^{\rho T}\varepsilon = s_0.$$

Furthermore, since $\underline{u}_0$ is nonincreasing, we see that for $x_1 \le x_2$, we have

$$\begin{aligned}\underline{u}_1(t,x_1) &= e^{\rho t}\int_{\mathbb{R}} p(t,z)\underline{u}_0(x_1-z)dz \\ &\ge e^{\rho t}\int_{\mathbb{R}} p(t,z)\underline{u}_0(x_2-z)dz = \underline{u}_1(t,x_2)\end{aligned}$$

for all $t \in [0,T]$. Thus $\underline{u}_1(t,x)$ is nonincreasing in x for all $t \in [0,T]$. Now we perform a recursive process to define $\underline{u}_n$, given $\underline{u}_{n-1}$, for all $n \ge 2$. We let

$$\underline{u}_{0,n-1}(x) = \inf(\underline{u}_{n-1}((n-1)T,x),\varepsilon),$$

where $\underline{u}_{n-1}((n-1)T,\cdot)$ is nonincreasing and $\underline{u}_{n-1}((n-1)T,-\infty) = s_0$. Then we define $\underline{u}_n$ as the solution of

$$\begin{aligned}(\underline{u}_n)_t + (-\Delta)^\alpha \underline{u}_n &= \rho\underline{u}_n, \\ \underline{u}_n((n-1)T,x) &= \underline{u}_{0,n-1}(x)\end{aligned}$$

for $(t,x) \in [(n-1)T,nT] \times \mathbb{R}$. This solution may be written as

$$\underline{u}_n(t,x) = e^{\rho(t-(n-1)T)}\int_{\mathbb{R}} p(t-(n-1)T,x-y)\underline{u}_{0,n-1}(y)dy,$$

so that, by the choice of ε and T, we have that $\underline{u}_n(t,x) \le s_0$ for all $(t,x) \in [(n-1)T,nT] \times \mathbb{R}$. Moreover, by definition

$$\underline{u}_n((n-1)T,x) = \underline{u}_{0,n-1}(x) \le \underline{u}_{n-1}((n-1)T,x) \quad \forall x \in \mathbb{R}.$$

We also have

$$\lim_{x\to-\infty} \underline{u}_{0,n-1}(nT,x) = \lim_{x\to-\infty} e^{\rho T}\int_{\mathbb{R}} p(t,z)\underline{u}_{0,n-1}(x-z)dz = e^{\rho T}\varepsilon = s_0,$$

and, since $\underline{u}_{n-1}((n-1)T,\cdot)$ is nonincreasing, for $x_1 \le x_2$ we obtain

$$\begin{aligned}\underline{u}_n(t,x_1) &= e^{\rho(t-(n-1)T)}\int_{\mathbb{R}} p(t-(n-1)T,z)\underline{u}_{0,n-1}(x_1-z)dz \\ &\ge e^{\rho(t-(n-1)T)}\int_{\mathbb{R}} p(t-(n-1)T,z)\underline{u}_{0,n-1}(x_2-z)dz = \underline{u}_n(t,x_2).\end{aligned}$$

Thus, $\underline{u}_n$ is nonincreasing in $x \in \mathbb{R}$ for all $t \in [(n-1)T,nT]$. ∎

REMARK 3.1. *We may define the function* $\underline{u} : [0,+\infty) \times \mathbb{R} \to [0,s_0]$ *in such a way that, for all integers* $n \ge 1$,

$$\underline{u}(t,x) = \underline{u}_n(t,x) \quad \textit{for } (t,x) \in [(n-1)T,nT) \times \mathbb{R}.$$

Since, for all integers $n \ge 1$, *the function* u_n *satisfies*

$$(\underline{u}_n)_t + (-\Delta)^\alpha \underline{u}_n \le f(\underline{u}_n) \quad \textit{in } ((n-1)T,nT) \times \mathbb{R},$$

we may use the comparison principle to find that

$$\underline{u}(t,x) \le u(t,x) \quad \forall (t,x) \in [0,\infty)\times\mathbb{R}. \tag{3.7}$$

We finally observe that, from the monotonicity property of the functions $\underline{u}_n$, *the function* $\underline{u}$ *is nonincreasing in* $x \in \mathbb{R}$ *for all* $t \ge 0$.

We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. Using the same notation as in the last proof, we let ω be defined as

$$\omega = \frac{e^{\frac{\rho}{d}\tau}}{B}\int_0^\infty \frac{\underline{u}_0(\xi - s\tau^{\frac{1}{2\alpha}})}{1+\mid s\mid^{1+2\alpha}}ds. \tag{3.8}$$

We observe that ω does not depend on λ or Γ and that $0 < \omega < s_0$. In order to see this last fact we recall that $\tau < T$, $d > 1$, and $C_\alpha < 2B$, where C_α is as defined in (2.6) and B is given in (2.4). Then

$$\omega = \frac{e^{\frac{\rho}{d}\tau}}{B}\int_0^\infty \frac{\underline{u}_0(\xi - s\tau^{\frac{1}{2\alpha}})}{1+\mid s\mid^{1+2\alpha}}ds \le \frac{e^{\frac{\rho}{d}\tau}C_\alpha}{2B}\varepsilon < \frac{C_\alpha}{2B}s_0 < s_0. \tag{3.9}$$

Next, for each $t \in [\tau,\infty)$, we consider the equation for $y \in [\xi,\infty)$:

$$\frac{e^{\frac{\rho}{d}t}}{B}\int_0^\infty \frac{\underline{u}_0(y - s\tau^{\frac{1}{2\alpha}})}{1+\mid s\mid^{1+2\alpha}}ds = \omega. \tag{3.10}$$

The function $G : [\xi,\infty) \to \mathbb{R}$ given by

$$G(y) = \int_0^\infty \frac{\underline{u}_0(y - s\tau^{\frac{1}{2\alpha}})}{1+\mid s\mid^{1+2\alpha}}ds \quad \forall y \in [\xi,\infty)$$

is clearly continuous and nonincreasing, since $\underline{u}_0$ is continuous and nonincreasing. Moreover, by definition of ξ we see that G is decreasing in $[\xi,\infty)$. Consequently, for every $t \in [\tau,\infty)$, equation (3.10) has a unique solution that we call $y_\omega(t)$, defining a continuous function $y_\omega : [\tau,\infty) \to [\xi,\infty)$. We see that y_ω satisfies $y_\omega(\tau) = \xi$ and is increasing.

Now we consider the open set Ω defined by

$$\Omega = \{(t,x) \in (\tau,\infty)\times\mathbb{R} \mid x < y_\omega(t)\},$$

and we claim that $\inf_{\overline{\Omega}} u > 0$. To prove the claim, we first look at $\partial\Omega$, which consists of two parts: the set of all points (t,x) for which $t \in (\tau,\infty)$ and $x = y_\omega(t)$, and the set $\{\tau\}\times(-\infty, y_\omega(\tau)]$.

(i) In the first case, when $t \in (\tau,\infty)$ and $x = y_\omega(t)$, there exists $n \in \mathbb{N}$ such that $t \in [(n-1)T, nT)$. Since $\underline{u}_{0,n-1}(x)$ in Lemma 3.1 is nonincreasing, we have that

$$\begin{aligned}\underline{u}(t,x) = u_n(t,x) &= e^{\rho(t-(n-1)T)}\int_{\mathbb{R}} p(t-(n-1)T, x-y)\underline{u}_{0,n-1}(y)dy\\ &\ge e^{\rho(t-(n-1)T)}\int_0^\infty p(t-(n-1)T, z)\underline{u}_{0,n-1}(x-z)dz\\ &\ge \frac{C_\alpha e^{\rho(t-(n-1)T)}}{2B}\underline{u}_{0,n-1}(x).\end{aligned}$$

In the case when $\underline{u}_{0,n-1}(x) = \varepsilon$, we conclude that

$$\underline{u}(t,x) \geq \frac{C_\alpha e^{\rho(t-(n-1)T)}}{2B}\varepsilon \geq \frac{C_\alpha \varepsilon}{2B}.$$

Otherwise, we have that $\underline{u}_{0,n-1}(x) = \underline{u}_{n-1}((n-1)T,x)$, and then, as before, we obtain that

$$\begin{aligned}\underline{u}(t,x) &\geq \frac{C_\alpha e^{\rho(t-(n-1)T)}}{2B}\underline{u}_{n-1}((n-1)T,x)\\ &\geq \frac{C_\alpha e^{\rho(t-(n-2)T)}}{2B}\int_0^\infty p(T,z)\underline{u}_{0,n-2}(x-z)dz\\ &\geq \left(\frac{C_\alpha}{2B}\right)^2 e^{\rho(t-(n-2)T)}\underline{u}_{0,n-2}(x).\end{aligned}$$

Again, we have two cases. If $\underline{u}_{0,n-2}(x) = \varepsilon$, then we conclude that

$$\begin{aligned}\underline{u}(t,x) &\geq \left(\frac{C_\alpha}{2B}\right)^2 e^{\rho(t-(n-2)T)}\varepsilon \geq \left(\frac{C_\alpha}{2B}\right)^2 e^{\rho T}\varepsilon\\ &\geq \left(\frac{C_\alpha e^{(1-\frac{1}{d})\rho\tau}}{2B}\right)\left(\frac{C_\alpha \varepsilon}{2B}\right) \geq \frac{C_\alpha\varepsilon}{2B},\end{aligned} \tag{3.11}$$

where we have used (3.2). Otherwise, we have that $\underline{u}_{0,n-2}(x) = \underline{u}_{n-2}((n-2)T,x)$, and then, as before, we have

$$\underline{u}(t,x) \geq \left(\frac{C_\alpha}{2B}\right)^2 e^{\rho(t-(n-2)T)}\underline{u}_{n-2}((n-2)T,x).$$

Repeating this procedure, we will either reach

$$\underline{u}(t,x) \geq \frac{C_\alpha \varepsilon}{2B}$$

as in (3.11), or we will have that x satisfies $\underline{u}_{0,m}(x) \neq \varepsilon$ for all $m \in \{1,2,3,\ldots,n-1\}$. In the latter case we have that $\underline{u}_{0,1}(x) = \underline{u}_1(T,x)$, and then

$$\begin{aligned}\underline{u}(t,x) &\geq \left(\frac{C_\alpha}{2B}\right)^{n-1} e^{\rho(t-T)}\underline{u}_1(T,x)\\ &\geq \left(\frac{C_\alpha}{2B}\right)^{n-1}\frac{e^{\rho t}}{B}\int_0^\infty \frac{\underline{u}_0(x-sT^{\frac{1}{2\alpha}})}{1+\mid s\mid^{1+2\alpha}}ds\\ &\geq \left(\frac{C_\alpha}{2B}\right)^{n-1} e^{(1-\frac{1}{d})\rho(n-1)\tau}\frac{e^{\frac{\rho}{d}t}}{B}\int_0^\infty \frac{\underline{u}_0(y_\omega(t)-s\tau^{\frac{1}{2\alpha}})}{1+\mid s\mid^{1+2\alpha}}ds\\ &= \left(\frac{C_\alpha e^{(1-\frac{1}{d})\rho\tau)}}{2B}\right)^{n-1}\omega > \omega.\end{aligned}$$

Summarizing, we have obtained that

$$\underline{u}(t,y_\omega(t)) \geq \min\left(\frac{C_\alpha\varepsilon}{2B},\omega\right) \quad \forall t \geq \tau.$$

(ii) In the second case, that is, when $t = \tau$ and $x \in (-\infty, y_\omega(\tau)]$, we have that $x - s\tau^{\frac{1}{2\alpha}} \le y_\omega(\tau) - s\tau^{\frac{1}{2\alpha}} = \xi - s\tau^{\frac{1}{2\alpha}}$; hence

$$\begin{aligned}\underline{u}(\tau, x) = \underline{u}_1(\tau, x) &\ge \frac{e^{\rho\tau}}{B}\int_{-\infty}^{\infty} \frac{\underline{u}_0(x - s\tau^{\frac{1}{2\alpha}})}{1+\mid s\mid^{1+2\alpha}} ds \\ &\ge \frac{e^{\frac{\rho}{d}\tau}}{B}\int_0^{\infty} \frac{\underline{u}_0(\xi - s\tau^{\frac{1}{2\alpha}})}{1+\mid s\mid^{1+2\alpha}} ds = \omega > 0.\end{aligned}$$

This completes the analysis on the boundary of Ω. To complete the proof we consider $(t, x) \in \Omega$, that is, $t > \tau$ and $x < y_\omega(t)$. Since $\underline{u}(t, \cdot)$ is nonincreasing for each $t \ge \tau$, from (i) we deduce that

$$\underline{u}(t, x) \ge \underline{u}(t, y_\omega(t)) \ge \min\left(\frac{C_\alpha \varepsilon}{2B}, \omega\right).$$

Thus, we have found $\theta > 0$ such that

$$u(t, x) \ge \underline{u}(t, x) \ge \theta \qquad \forall (t, x) \in \overline{\Omega}. \tag{3.12}$$

Now we can get the upper estimate for $\lambda \in (0, \theta)$. Let $x \in E_\lambda(t)$ for $t \ge \max(\tau, t_\lambda)$; then we have

$$x > y_\omega(t) \ge \xi. \tag{3.13}$$

In fact, let us assume that $x \le y_\omega(t)$; then $(t, x) \in \bar{\Omega}$, and, by our estimate above, we have that $u(t, x) \ge \theta$. On the other hand, by definition of $E_\lambda(t)$ we have $u(t, x) = \lambda$. Since $\lambda < \theta$, we obtain a contradiction.

Thus, from (3.13) we have that, for all $t \ge \max(\tau, t_\lambda)$ and $x \in E_\lambda(t)$,

$$\begin{aligned}B\omega e^{-\frac{\rho}{d}t} &= \int_0^{\infty} \frac{\underline{u}_0(y_\omega(t) - s\tau^{\frac{1}{2\alpha}})}{1+\mid s\mid^{1+2\alpha}} ds \ge \int_0^{\infty} \frac{\underline{u}_0(x - s\tau^{\frac{1}{2\alpha}})}{1+\mid s\mid^{1+2\alpha}} ds \\ &\ge \int_0^{\infty} \frac{\underline{u}_0(x)}{1+\mid s\mid^{1+2\alpha}} ds = \frac{C_\alpha}{2}\underline{u}_0(x) = \frac{C_\alpha}{2}u_0(x),\end{aligned} \tag{3.14}$$

where the last equality holds since $x > \xi$. From (3.14) and since $\Gamma > 0$ and $\rho > f'(0) - \delta'$, there exists $\tau_1(\lambda, \Gamma, \delta) \ge \max(\tau, t_\lambda)$ such that for all $t \ge \tau_1(\lambda, \Gamma, \delta)$ and $x \in E_\lambda(t)$,

$$u_0(x) \le \frac{2B}{C_\alpha}\omega e^{-\frac{\rho}{d}t} \le s_0 e^{-\frac{\rho}{d}t} \le \Gamma e^{-\frac{f'(0)-\delta'}{d}t}.$$

Here we used (3.9). But, by definition of δ' we have $\frac{f'(0)-\delta'}{d} = f'(0) - \delta$, so we conclude that for all $t \ge \tau_1(\lambda, \Gamma, \delta)$ and $x \in E_\lambda(t)$,

$$u_0(x) \le \Gamma e^{-(f'(0)-\delta)t}.$$

In order to complete the proof of the proposition, let us now consider $\lambda \in [\theta, 1)$. Let $\underline{u}_{\theta,0}$ be the function defined by

$$\underline{u}_{\theta,0}(z) = \begin{cases} \theta & \text{if } z \le 0, \\ \theta(1-z) & \text{if } 0 < z < 1, \\ 0 & \text{if } z \ge 1, \end{cases}$$

and denote by $\underline{u}_\theta$ the solution of the Cauchy problem (1.1)–(1.2) with initial condition $\underline{u}_{\theta,0}$. It follows from (3.12) that

$$u(s,x) \geq \underline{u}_{\theta,0}(x - y_\omega(s) + 1) \quad \forall (s,x) \in [\tau,\infty) \times \mathbb{R},$$

and then, using the comparison principle, we obtain

$$u(s+t,x) \geq \underline{u}_\theta(t, x - y_\omega(s) + 1) \quad \forall (s,x) \in [\tau,\infty) \times \mathbb{R} \text{ and } t \geq 0.$$

Now we consider $0 < c < c_* = \frac{f'(0)}{2\alpha}$, and we use Theorem 1.5 of [5] to find $T_\lambda > 0$ such that

$$\underline{u}_\theta(T_\lambda, z) > \lambda \quad \forall z \leq e^{cT_\lambda}.$$

We observe that T_λ may depend on θ, and thus on ε, but does not depend on s. Directly from the last two inequalities we get

$$u(s+T_\lambda, x) > \lambda \quad \forall s \in [\tau,\infty) \text{ and } x \leq e^{cT_\lambda} + y_\omega(s) - 1.$$

As a consequence we have that for all $t \geq \max(\tau + T_\lambda, t_\lambda)$ and $x \in E_\lambda(t)$, we obtain $x - y_\omega(t - T_\lambda) + 1 > e^{cT_\lambda}$. In fact, if $x - y_\omega(t-T_\lambda) + 1 \leq e^{cT_\lambda}$, using that $\tau \leq t - T_\lambda$, we see that $\lambda = u(t,x) = u((t-T_\lambda) + T_\lambda, x) > \lambda$, which is a contradiction. Thus, for such t and x, we have $x - y_\omega(t-T_\lambda) > e^{cT_\lambda} - 1 > 0$ and hence $x > y_\omega(t - T_\lambda)$. As a consequence,

$$\begin{aligned} B\omega e^{-\frac{\rho}{d}(t-T_\lambda)} &= \int_0^\infty \frac{\underline{u}_0(y_\omega(t-T_\lambda) - s\tau^{\frac{1}{2\alpha}})}{1+\mid s\mid^{1+2\alpha}} ds \geq \int_0^\infty \frac{\underline{u}_0(x - s\tau^{\frac{1}{2\alpha}})}{1+\mid s\mid^{1+2\alpha}} ds \\ &\geq \int_0^\infty \frac{\underline{u}_0(x)}{1+\mid s\mid^{1+2\alpha}} ds = \frac{C_\alpha}{2}\underline{u}_0(x) = \frac{C_\alpha}{2} u_0(x). \end{aligned}$$

Here, the last equality is satisfied because $x > \xi$. Now we conclude as in the other case, since $\Gamma > 0$ and $\rho > f'(0) - \delta'$, that there exist $\tau_2(\lambda,\Gamma,\delta) \geq \max(\tau + T_\lambda, t_\lambda)$ such that for all $t \geq \tau_2(\lambda,\Gamma,\delta)$ and $x \in E_\lambda(t)$,

$$u_0(x) \leq \frac{2B}{C_\alpha}\omega e^{-\frac{\rho}{d}(t-T_\lambda)} < \Gamma e^{-\frac{f'(0)-\delta'}{d}t} = \Gamma e^{-(f'(0)-\delta)t}.$$

We complete the proof of the proposition, choosing

$$\tau_u(\lambda,\Gamma,\delta) = \max(\tau_1(\lambda,\Gamma,\delta), \tau_2(\lambda,\Gamma,\delta)). \qquad \square$$

The proof of Corollary 1.1 follows from Proposition 3.1, finding a suitable initial condition that satisfies (1.5) and (1.6). This proof closely follows the ideas of Hamel and Roques in [10].

Proof of Corollary 1.1. Let $\chi : [0,\infty) \to \mathbb{R}$ be any locally bounded function; then there exists a continuous, increasing function $g : [0,\infty) \to \mathbb{R}$ such that

$$g(z) \geq \chi(2z) \quad \forall z \geq 0.$$

Denoting by $g^{-1} : [g(0),\infty) \to [0,\infty)$ the inverse of g, we define $u_0 : \mathbb{R} \to (0,1]$ as

$$u_0(x) = e^{-f'(0)g^{-1}(x)} \quad \forall x \geq g(0)$$

and extended by one, to the left of $g(0)$. We easily see that $u_0 \in \mathcal{C}_{lim}$ is decreasing and that it satisfies (1.5) and (1.6).

Let u be the solution of Cauchy problem (1.1)–(1.2) with initial condition u_0, and let $\lambda \in (0,1)$ and $\delta \in (0, \frac{f'(0)}{2})$. Moreover, let us consider $\tau_1 > 0$ large so that

$$e^{-(f'(0)-\delta)\tau_1} \le u_0(g(0)). \tag{3.15}$$

It follows from Proposition 3.1 with $\Gamma = 1$ that there exists $\tau_2 \ge \max(\tau_1, t_\lambda)$ such that

$$y \ge u_0^{-1}(e^{-(f'(0)-\delta)t}) \quad \forall t \ge \tau_2,\ \forall y \in E_\lambda(t).$$

Therefore, from (3.15) we conclude that, for each $t \ge \tau_2$,

$$\min E_\lambda(t) \ge g\left(\frac{f'(0)-\delta}{f'(0)}t\right) \ge g\left(\frac{t}{2}\right) \ge \chi(t). \qquad \square$$

4. Proof of the upper bound and conclusion. In this section we prove the upper bound for the set $E_\lambda(t)$, and we complete the proof of Theorem 1.2. The proof of the upper bound is obtained by constructing an appropriate supersolution of (1.1)–(1.2). The construction of such a supersolution relies strongly on the hypotheses (H1) and (H2). Specifically we prove the following proposition.

PROPOSITION 4.1. *Let $\alpha \in (0,1)$, and let u be the solution of* (1.1)–(1.2), *where f satisfies* (1.4) *and the initial condition u_0 satisfies* (1.5), (1.6), (H1), *and* (H2).

Then, for any $\gamma > 0$, $\lambda \in (0,1)$, and $\delta \in (0, f'(0))$, there exists a time $\tau_\ell = \tau_\ell(\lambda, \gamma, \delta, b) \ge t_\lambda$ such that

$$E_\lambda(t) \subset u_0^{-1}\{[\gamma e^{-(f'(0)+\delta)t}, 1]\} \quad \forall t \ge \tau_\ell.$$

Proof. Let $\overline{u}$ be the solution of the problem

$$\overline{u}_t + (-\Delta)^\alpha \overline{u} = f'(0)\overline{u},$$
$$\overline{u}(0,x) = u_0(x),$$

which can be expressed as

$$\overline{u}(t,x) = e^{f'(0)t} \int_{\mathbb{R}} p(t, x-y) u_0(y) dy. \tag{4.1}$$

By the assumptions on f we see that $\overline{u}$ is a supersolution for (1.1)–(1.2), and then the comparison principle implies that

$$0 < u(t,x) \le \overline{u}(t,x) \qquad \forall (t,x) \in [0,\infty) \times \mathbb{R}.$$

Since $b < 2\alpha$, there exists $a > 0$ small enough such that $a + b \le 2\alpha$. Let $\lambda \in (0,1)$ and $\delta \in (0, f'(0))$; then there exists $\tau_1 = \tau_1(\lambda, \delta, b) \ge t_\lambda > 1$ such that for all $t \ge \tau_1$ we have

$$\frac{\lambda}{8B^2} e^{-f'(0)t} < \inf_{x \in (-\infty, \xi_0)} \{u_0(x), \xi_0^{-b}\}, \qquad \frac{8B^2}{\lambda} < e^{\frac{\delta}{2}t}, \tag{4.2}$$

and

$$\left(\frac{8B^2}{\lambda}\right)^{\frac{a}{2\alpha b}} e^{\frac{a f'(0)}{2\alpha b}t} > \frac{1}{\rho - 1}\left(\frac{t}{\alpha B}\right)^{\frac{1}{2\alpha}}. \tag{4.3}$$

Here B is the constant appearing in (2.4), and ρ is given in (H2). Let $t \geq \tau_1$, and let us denote by $\varepsilon > 0$ the number such that

$$4B^2 e^{f'(0)t}\varepsilon = \frac{\lambda}{2}. \tag{4.4}$$

By (4.2) it is possible to find $\xi_1 \in [\xi_0, \infty)$ such that $u_0(\xi_1) = \varepsilon$; we assume that ξ_1 is the largest number with that property. Then, we choose $\xi > 0$ large enough such that

$$\int_{\xi}^{\infty} \frac{1}{s^{1+2\alpha}} ds = \frac{B\varepsilon}{2},$$

and we define $\xi_t := \xi_1 + \xi t^{\frac{1}{2\alpha}}$. For $x \geq \xi_t$ we estimate the values of $\overline{u}(t,x)$. From (4.1) we have that

$$\overline{u}(t,x) \leq B\frac{e^{f'(0)t}}{t^{\frac{1}{2\alpha}}} \int_{-\infty}^{\infty} \frac{u_0(y)}{1+\mid (x-y)t^{-\frac{1}{2\alpha}} \mid^{1+2\alpha}} dy \leq Be^{f'(0)t}(I1 + I2),$$

where

$$I1 = \frac{1}{t^{\frac{1}{2\alpha}}} \int_{-\infty}^{\infty} \frac{u_0(x)}{1+\mid (x-y)t^{-\frac{1}{2\alpha}} \mid^{1+2\alpha}} dy$$

and

$$I2 = \frac{1}{t^{\frac{1}{2\alpha}}} \int_{-\infty}^{\infty} \frac{\mid u_0(y) - u_0(x) \mid}{1+\mid (x-y)t^{-\frac{1}{2\alpha}} \mid^{1+2\alpha}} dy.$$

Calculating the integrals separately, we have

$$I1 = u_0(x) \int_{-\infty}^{\infty} \frac{1}{1+\mid r \mid^{1+2\alpha}} dr = C_\alpha u_0(x) \leq Bu_0(x).$$

For $I2$ we recall the definition of ξ_1, ξ, ε, and we notice that $x - st^{\frac{1}{2\alpha}} \geq \xi_t - \xi t^{\frac{1}{2\alpha}} = \xi_1$ for all $s \in (-\infty, \xi]$. With a change of variables we then have

$$\begin{aligned} I2 &= \int_{-\infty}^{\xi} \frac{\mid u_0(x - st^{\frac{1}{2\alpha}}) - u_0(x) \mid}{1+\mid s \mid^{1+2\alpha}} ds + \int_{\xi}^{\infty} \frac{\mid u_0(x - st^{\frac{1}{2\alpha}}) - u_0(x) \mid}{1+\mid s \mid^{1+2\alpha}} ds \\ &\leq 2\varepsilon \int_{-\infty}^{\xi} \frac{1}{1+\mid s \mid^{1+2\alpha}} ds + 2\int_{\xi}^{\infty} \frac{1}{1+\mid s \mid^{1+2\alpha}} ds \\ &< 2\varepsilon B + \varepsilon B = 3\varepsilon B. \end{aligned}$$

Therefore,

$$u(t,x) < B^2 e^{f'(0)t}(u_0(x) + 3\varepsilon) \qquad \forall x \geq \xi_t.$$

From here we obtain that $y < \xi_t$ for all $y \in E_\lambda(t)$. In fact, if $y \geq \xi_t$ and $y \in E_\lambda(t)$, then $y \geq \xi_t > \xi_1$ and

$$\lambda = u(t,y) < B^2 e^{f'(0)t}(u_0(y) + 3\varepsilon) \leq B^2 e^{f'(0)t}(u_0(\xi_1) + 3\varepsilon) = B^2 e^{f'(0)t} 4\varepsilon = \frac{\lambda}{2},$$

which is a contradiction. Since $u_0(\xi_1) = \varepsilon$, from (H1), (4.3), and (4.4) we see that

$$\xi_1^{\frac{a}{2\alpha}} \geq \varepsilon^{-\frac{a}{2\alpha b}} = \left(\frac{8B^2}{\lambda}\right)^{\frac{a}{2\alpha b}} e^{\frac{af'(0)}{2\alpha b}t} \geq \frac{1}{(\rho - 1)} \left(\frac{t}{\alpha B}\right)^{\frac{1}{2\alpha}}.$$

From here, since $\xi_1^{-b} \le u_0(\xi_1) = \varepsilon$ and by the choice of τ_1 and ξ, we conclude that

$$\xi_t = \xi_1 + \xi t^{\frac{1}{2\alpha}} = \xi_1 + \left(\frac{t}{\alpha B}\right)^{\frac{1}{2\alpha}} (u_0(\xi_1))^{-\frac{1}{2\alpha}}$$
$$\le \xi_1 + \left(\frac{1}{\alpha B}\right)^{\frac{1}{2\alpha}} t^{\frac{1}{2\alpha}} \xi_1^{\frac{b}{2\alpha}} \le \xi_1 + (1-\rho)\xi_1^{\frac{a}{2\alpha}} \xi_1^{\frac{b}{2\alpha}} \le \rho \xi_1.$$

Now, if ξ_2 is such that $u_0(\xi_2) = e^{-(f'(0)+\frac{\delta}{2})t}$, then since $e^{-(f'(0)+\frac{\delta}{2})t} < \varepsilon$ by (4.2) and (4.4), we have that $\xi_1 < \xi_2$. Therefore, for each $y \in E_\lambda(t)$

$$u_0(y) \ge u_0(\xi_t) \ge u_0(\rho\xi_2) \ge k u_0(\xi_2) = k e^{-(f'(0)+\frac{\delta}{2})t}.$$

Finally, making $\tau_\ell = \tau_\ell(\lambda, \gamma, \delta, b) \ge \tau_1(\lambda, \delta, b)$ larger if necessary, we may assume that if $t \ge \tau_\ell$, then $e^{\frac{\delta}{2}t} \ge \frac{\gamma}{k}$ and we find that

$$\{[\gamma e^{-(f'(0)+\delta)t}, 1]\} \quad \forall t \ge \tau_\ell. \qquad \square$$

Thanks to Propositions 3.1 and 4.1, we can prove Theorem 1.2 on the behavior of level sets for large times, expressed in terms of the decay of the initial condition.

Proof of Theorem 1.2. The proof follows directly from Propositions 3.1 and 4.1, taking $\tau = \tau(\lambda, \Gamma, \gamma, \delta, b) = \max(\tau_u, \tau_\ell)$. $\square$

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