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Concentrating standing waves for the fractional nonlinear Schrödinger equation

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Abstract

We consider the semilinear equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - u^p = 0, \quad u > 0,\ u \in H^{2s}\big(\mathbb{R}^N\big)$$

where $0 < s < 1$, $1 < p < \frac{N+2s}{N-2s}$, $V(x)$ is a sufficiently smooth potential with $\inf_{\mathbb{R}} V(x) > 0$, and $\varepsilon > 0$ is a small number. Letting w_λ be the radial ground state of $(-\Delta)^s w_\lambda + \lambda w_\lambda - w_\lambda^p = 0$ in $H^{2s}(\mathbb{R}^N)$, we build solutions of the form

$$u_\varepsilon(x) \sim \sum_{i=1}^{k} w_{\lambda_i}\big(\big(x - \xi_i^\varepsilon\big)/\varepsilon\big),$$

where $\lambda_i = V(\xi_i^\varepsilon)$ and the ξ_i^ε approach suitable critical points of V. Via a Lyapunov–Schmidt variational reduction, we recover various existence results already known for the case $s = 1$. In particular such a solution exists around k nondegenerate critical points of V. For $s = 1$ this corresponds to the classical results by Floer and Weinstein [13] and Oh [24,25].

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1. Introduction and main results

We consider the fractional nonlinear Schrödinger equation

$$i\hbar\psi_t = \hbar^{2s}(-\Delta)^s\psi + W(x)\psi - |\psi|^{p-1}\psi \tag{1.1}$$

where $(-\Delta)^s$, $0 < s < 1$, denotes the usual fractional Laplace operator, $W(x)$ is a bounded potential, $p > 1$ and $\hbar$ designates the usual Planck constant. Eq. (1.1) was introduced by Laskin [19] as an extension of the classical nonlinear Schrödinger equation $s = 1$ in which the Brownian motion of the quantum paths is replaced by a Lévy flight. Here $\psi = \psi(x,t)$ represents the quantum mechanical probability amplitude for a given unit-mass particle to have position x at time t (the corresponding probability density is $|\psi|^2$), under a confinement due to the potential W. We refer to [19–21] for detailed physical discussions and motivation of Eq. (1.1).

We are interested in the *semi-classical limit* regime, $0 < \varepsilon := \hbar \ll 1$. For small values of ε the wave function tends to concentrate as a material particle.

Our purpose is to find *standing-wave solutions* of (1.1), which are those of the form $\psi(x,t) = u(x)e^{iEt/\varepsilon}$ with $u(x)$ a real-valued function. Letting $V(x) = W(x) + E$, Eq. (1.1) becomes

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

We assume in what follows that V satisfies

$$V \in C^{1,\alpha}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \qquad \inf_{\mathbb{R}^N} V(x) > 0. \tag{1.3}$$

We are interested in finding solutions with a *spike pattern* concentrating around a finite number of points in space as $\varepsilon \to 0$. This has been the topic of many works in the standard case $s = 1$, relating the concentration points with critical points of the potential, starting in 1986 with the pioneering work by Floer and Weinstein [13], then continued by Oh [24,25]. The natural place to look for solutions to (1.2) that decay at infinity is the space $H^{2s}(\mathbb{R}^N)$, of all functions $u \in L^2(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \left(1 + |\xi|^{4s}\right)\left|\hat{u}(\xi)\right|^2 d\xi < +\infty,$$

where $\widehat{\ }$ denotes Fourier transform. The fractional Laplacian $(-\Delta)^s u$ of a function $u \in H^{2s}(\mathbb{R}^N)$ is defined in terms of its Fourier transform by the relation

$$\widehat{(-\Delta)^s u} = |\xi|^{2s}\hat{u} \in L^2(\mathbb{R}^N).$$

We will explain next what we mean by a *spike pattern* solution of Eq. (1.2). Let us consider the basic problem

$$(-\Delta)^s v + v - |v|^{p-1}v = 0, \quad v \in H^{2s}(\mathbb{R}^N). \tag{1.4}$$

We assume the following constraint in p,

$$1 < p < \begin{cases} \frac{N+2s}{N-2s} & \text{if } 2s < N, \\ +\infty & \text{if } 2s \geqslant N. \end{cases} \tag{1.5}$$

Under this condition it is known the existence of a positive, radial *least energy solution* $v = w(x)$, which gives the lowest possible value for the energy

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N} v(-\Delta)^s v + \frac{1}{2}\int_{\mathbb{R}^N} v^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} |v|^{p+1},$$

among all nontrivial solutions of (1.4). An important property, which has only been proven recently by Frank, Lenzmann and Silvestre [15] (see also [2,14]), is that there exists a radial least energy solution which is nondegenerate, in the sense that the space of solutions of the equation

$$(-\Delta)^s\phi + \phi - pw^{p-1}\phi = 0, \quad \phi \in H^{2s}(\mathbb{R}^n) \tag{1.6}$$

consists of exactly of the linear combinations of the translation-generators, $\frac{\partial w}{\partial x_j}$, $j = 1, \dots, N$.

It is easy to see that the function

$$w_\lambda(x) := \lambda^{\frac{1}{p-1}} w\big(\lambda^{\frac{1}{2s}} x\big)$$

satisfies the equation

$$(-\Delta)^s w_\lambda + \lambda w_\lambda - w_\lambda^p = 0 \quad \text{in } \mathbb{R}^N.$$

Therefore for any point $\xi \in \mathbb{R}^N$, taking $\lambda = V(\xi)$, the spike-shape function

$$u(x) = w_{V(\xi)}\left(\frac{x-\xi}{\varepsilon}\right) \tag{1.7}$$

satisfies

$$\varepsilon^{2s}(-\Delta)^s u + V(\xi)u - u^p = 0.$$

Since the ε-scaling makes it *concentrate* around ξ, this function constitutes a good positive approximate solution to Eq. (1.2), namely of

$$\begin{gathered} \varepsilon^{2s}(-\Delta)^s u + V(x)u - u^p = 0, \\ u > 0, \quad u \in H^{2s}\big(\mathbb{R}^N\big). \end{gathered} \tag{1.8}$$

We call a *k-spike pattern solution* of (1.8) one that looks approximately like a superposition of k spikes like (1.7), namely a solution u_ε of the form

$$u_\varepsilon(x) = \sum_{i=1}^k w_{V(\xi_i^\varepsilon)}\left(\frac{x-\xi_i^\varepsilon}{\varepsilon}\right) + o(1) \tag{1.9}$$

for points $\xi_1^\varepsilon, \dots, \xi_k^\varepsilon$, where $o(1) \to 0$ in $H^{2s}(\mathbb{R}^N)$ as $\varepsilon \to 0$.

In what follows we assume that p satisfies condition (1.5) and V condition (1.3).

Our first result concerns the existence of multiple spike solution at separate places in the case of stable critical points.

Theorem 1. *Let $\Lambda_i \subset \mathbb{R}^N$, $i = 1, \dots, k$, $k \geqslant 1$ be disjoint bounded open sets in $\mathbb{R}^N$. Assume that*

$$\deg(\nabla V, \Lambda_i, 0) \neq 0 \quad \textit{for all } i = 1, \dots, k.$$

Then for all sufficiently small ε, problem (1.8) has a solution of the form (1.9) where $\xi_i^\varepsilon \in \Lambda_i$ and

$$\nabla V\big(\xi_i^\varepsilon\big) \to 0 \quad \textit{as } \varepsilon \to 0.$$

An immediate consequence of Theorem 1 is the following.

Corollary 1.1. *Assume that V is of class C^2. Let $\xi_1^0, \dots, \xi_k^0$ be k nondegenerate critical points of V, namely*

$$\nabla V\big(\xi_i^0\big) = 0, \qquad D^2 V\big(\xi_i^0\big) \textit{ is invertible for all } i = 1, \dots k.$$

Then, a k-spike solution of (1.8) of the form (1.9) with $\xi_i^\varepsilon \to \xi_i^0$ exists.

When $s = 1$, the result of Corollary 1.1 is due to Floer and Weinstein [13] for $N = 1$ and $k = 1$ and to Oh [24,25] when $N \geqslant 1$, $k \geqslant 1$. Theorem 1 for $s = 1$ was proven by Yanyan Li [22].

Remark 1.1. As the proof will yield, Theorem 1 for $0 < s < 1$ holds true under the following, more general condition introduced in [22]. Let $\Lambda = \Lambda_1 \times \cdots \times \Lambda_k$ and assume that the function

$$\varphi(\xi_1, \dots, \xi_k) = \sum_{i=1}^k V(\xi_i)^\theta, \qquad \theta = \frac{p+1}{p-1} - \frac{N}{2s} > 0 \tag{1.10}$$

has a *stable critical point situation* in Λ: there is a number $\delta_0 > 0$ such that for each $g \in C^1(\bar{\Omega})$ with $\|g\|_{L^\infty(\Lambda)} + \|\nabla g\|_{L^\infty(\Lambda)} < \delta_0$, there is a $\xi_g \in \Lambda$ such that $\nabla\varphi(\xi_g) + \nabla g(\xi_g) = 0$. Then for all sufficiently small ε, problem (1.8) has a solution of the form (1.9) where $\xi^\varepsilon = (\xi_1^\varepsilon, \dots, \xi_k^\varepsilon) \in \Lambda$ and $\nabla\varphi(\xi^\varepsilon) \to 0$ as $\varepsilon \to 0$.

Theorem 2. *Let Λ be a bounded, open set with smooth boundary such that V is such that either*

$$c = \inf_\Lambda V < \inf_{\partial\Lambda} V \tag{1.11}$$

or

$$c = \sup_\Lambda V > \sup_{\partial\Lambda} V$$

or, there exist closed sets $B_0 \subset B \subset \Lambda$ such that

$$c = \inf_{\Phi\in\Gamma} \sup_{x\in B} V\big(\Phi(x)\big) > \sup_{B_0} V, \tag{1.12}$$

where $\Gamma = \{\Phi \in C(B, \bar{\Lambda}) \,/\, \Phi|_{B_0} = Id\}$ *and* $\nabla V(x) \cdot \tau \neq 0$ *for all* $x \in \partial\Lambda$ *with* $V(x) = c$ *and some tangent vector* τ *to* $\partial\Lambda$ *at* x.

Then, there exists a 1*-spike solution of* (1.8) *with* $\xi^\varepsilon \in \Lambda$ *with* $\nabla V(\xi_\varepsilon) \to 0$ *and* $V(\xi_i^\varepsilon) \to c$.

In the case $s = 1$, the above results were found by del Pino and Felmer [7,8]. The case of a (possibly degenerate) global minimizer was previously considered by Rabinowitz [26] and X. Wang [28]. An isolated maximum with a power type degeneracy appears in Ambrosetti, Badiale and Cingolani [1]. Condition (1.12) is called a *nontrivial linking situation* for V. The cases of k disjoint sets where (1.11) holds was treated in [9,17]. Multiple spikes for disjoint nontrivial linking regions were first considered in [10], see also [5,16] for other multiplicity results.

Our last result concerns the existence of multiple spikes *at the same point*.

Theorem 3. *Let* Λ *be a bounded, open set with smooth boundary such that* V *is such that*

$$\sup_{\Lambda} V > \sup_{\partial\Lambda} V.$$

Then for any positive integer k *there exists a* k*-spike solution of* (1.8) *with spikes* $\xi_j^\varepsilon \in \Lambda$ *satisfying* $V(\xi_j^\varepsilon) \to \max_\Lambda V$.

In the case $s = 1$, Theorem 3 was proved by Kang and Wei [18]. D'Aprile and Ruiz [6] have found a phenomenon of this type at a saddle point of V.

The rest of this paper will be devoted to the proofs of Theorems 1–3. The method of construction of a k-spike solution consists of a Lyapunov–Schmidt reduction in which the full problem is reduced to that of finding a critical point ξ^ε of a functional which is a small C^1-perturbation of φ in (1.10). In this reduction the nondegeneracy result in [15] is a key ingredient.

After this has been done, the results follow directly from standard degree theoretical or variational arguments. The Lyapunov–Schmidt reduction is a method widely used in elliptic singular perturbation problems. Some results of variational type for $0 < s < 1$ have been obtained for instance in [12] and [27]. We believe that the scheme of this paper may be generalized to concentration on higher dimensional regions, while that could be much more challenging. See [11, 23] for concentration along a curve in the plane and $s = 1$.

2. Generalities

Let $0 < s < 1$. Various definitions of the fractional Laplacian $(-\Delta)^s\phi$ of a function ϕ defined in $\mathbb{R}^N$ are available, depending on its regularity and growth properties.

As we have recalled in the introduction, for $\phi \in H^{2s}(\mathbb{R}^N)$ the standard definition is given via Fourier transform $\widehat{\ }$. $(-\Delta)^s\phi \in L^2(\mathbb{R}^N)$ is defined by the formula

$$|\xi|^{2s}\hat{\phi}(\xi) = \widehat{(-\Delta)^s\phi}. \tag{2.1}$$

When ϕ is assumed in addition sufficiently regular, we obtain the direct representation

$$(-\Delta)^s\phi(x) = d_{s,N} \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y)}{|x-y|^{N+2s}}\, dy \tag{2.2}$$

for a suitable constant $d_{s,N}$ and the integral is understood in a principal value sense. This integral makes sense directly when $s < \frac{1}{2}$ and $\phi \in C^{0,\alpha}(\mathbb{R}^N)$ with $\alpha > 2s$, or if $\phi \in C^{1,\alpha}(\mathbb{R}^N)$, $1 + \alpha > 2s$. In the latter case, we can desingularize the integral representing it in the form

$$(-\Delta)^s \phi(x) = d_{s,N} \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y) - \nabla\phi(x)(x-y)}{|x-y|^{N+2s}}\, dy.$$

Another useful (local) representation, found by Caffarelli and Silvestre [3], is via the following boundary value problem in the half space $\mathbb{R}^{N+1}_+ = \{(x, y) / x \in \mathbb{R}^N,\ y > 0\}$:

$$\begin{cases} \nabla \cdot \big(y^{1-2s} \nabla \tilde{\phi}\big) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \tilde{\phi}(x, 0) = \phi(x) & \text{on } \mathbb{R}^N. \end{cases}$$

Here $\tilde{\phi}$ is the s-harmonic extension of ϕ, explicitly given as a convolution integral with the s-Poisson kernel $p_s(x, y)$,

$$\tilde{\phi}(x, y) = \int_{\mathbb{R}^N} p_s(x-z, y)\phi(z)\, dz,$$

where

$$p_s(x, y) = c_{N,s} \frac{y^{4s-1}}{(|x|^2 + |y|^2)^{\frac{N-1+4s}{2}}}$$

and $c_{N,s}$ achieves $\int_{\mathbb{R}^N} p(x, y)\, dx = 1$. Then under suitable regularity, $(-\Delta)^s \phi$ is the Dirichlet-to-Neumann map for this problem, namely

$$(-\Delta)^s \phi(x) = \lim_{y \to 0^+} y^{1-2s} \partial_y \tilde{\phi}(x, y). \tag{2.3}$$

Characterizations (2.1), (2.2), (2.3) are all equivalent for instance in Schwartz's space of rapidly decreasing smooth functions.

Let us consider now for a number $m > 0$ and $g \in L^2(\mathbb{R}^N)$ the equation

$$(-\Delta)^s \phi + m\phi = g \quad \text{in } \mathbb{R}^N.$$

Then in terms of Fourier transform, this problem, for $\phi \in L^2$, reads

$$\big(|\xi|^{2s} + m\big)\hat{\phi} = \hat{g}$$

and has a unique solution $\phi \in H^{2s}(\mathbb{R}^N)$ given by the convolution

$$\phi(x) = T_m[g] := \int_{\mathbb{R}^N} k(x-z) g(z)\, dz, \tag{2.4}$$

where

$$\hat{k}(\xi) = \frac{1}{|\xi|^{2s} + m}.$$

Using the characterization (2.3) written in weak form, ϕ can then be characterized by $\phi(x) = \tilde{\phi}(x, 0)$ in trace sense, where $\tilde{\phi} \in H$ is the unique solution of

$$\iint_{\mathbb{R}^{N+1}_+} \nabla\tilde{\phi}\,\nabla\varphi\, y^{1-2s} + m \int_{\mathbb{R}^N} \phi\varphi = \int_{\mathbb{R}^N} g\varphi, \quad \text{for all } \varphi \in H, \tag{2.5}$$

where H is the Hilbert space of functions $\varphi \in H^1_{loc}(\mathbb{R}^{N+1}_+)$ such that

$$\|\varphi\|_H^2 := \iint_{\mathbb{R}^{N+1}_+} |\nabla\varphi|^2 y^{1-2s} + m \int_{\mathbb{R}^N} |\varphi|^2 < +\infty,$$

or equivalently the closure of the set of all functions in $C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ under this norm.

A useful fact for our purposes is the equivalence of the representations (2.4) and (2.5) for $g \in L^2(\mathbb{R}^N)$.

Lemma 2.1. *Let $g \in L^2(\mathbb{R}^N)$. Then the unique solution $\tilde{\phi} \in H$ of problem (2.5) is given by the s-harmonic extension of the function $\phi = T_m[g] = k * g$.*

Proof. Let us assume first that $\hat{g} \in C_c^\infty(\mathbb{R}^N)$. Then ϕ given by (2.4) belongs to $H^{2s}(\mathbb{R}^N)$. Take a test function $\psi \in C_c^\infty(\mathbb{R}^{N+1}_+)$. Then the well-known computation by Caffarelli and Silvestre shows that

$$\begin{aligned}\iint_{\mathbb{R}^{N+1}_+} \nabla\tilde{\phi}\nabla\psi\, y^{1-2s}\, dy\, dx &= \int_{\mathbb{R}^N} \lim_{y\to 0} y^{1-2s}\partial_y\tilde{\phi}(y,\cdot)\,\psi\, dx \\ &= \int_{\mathbb{R}^N} \psi(-\Delta)^s\phi\, dx = \int_{\mathbb{R}^N} (g - m\phi)\, dx.\end{aligned}$$

By taking $\psi = \tilde{\phi}\eta_R$ for a suitable sequence of smooth cut-off functions equal to one on expanding balls $B_R(0)$ in $\mathbb{R}^{N+1}_+$, and using the behavior at infinity of $\tilde{\phi}$ which resembles the Poisson kernel $p_s(x, y)$, we obtain

$$\iint_{\mathbb{R}^{N+1}_+} |\nabla\tilde{\phi}|^2 y^{1-2s}\, dy\, dx + m \int_{\mathbb{R}^N} |\phi|^2 = \int_{\mathbb{R}^N} g\phi$$

and hence $\|\tilde{\phi}\|_H \leqslant C\|g\|_{L^2}$ and satisfies (2.5). By density, this fact extends to all $g \in L^2(\mathbb{R}^N)$. The result follows since the solution of problem (2.5) in H is unique. □

Let us recall the main properties of the fundamental solution $k(x)$ in the representation (2.4), which are stated for instance in [15] or in [12].

We have that k is radially symmetric and positive, $k \in C^\infty(\mathbb{R}^N \setminus \{0\})$ satisfying

- $$\left|k(x)\right| + |x|\left|\nabla k(x)\right| \leqslant \frac{C}{|x|^{N-2s}} \quad \text{for all } |x| \leqslant 1,$$
- $$\lim_{|x|\to\infty} k(x)|x|^{N+2s} = \gamma > 0,$$
- $$|x|\left|\nabla k(x)\right| \leqslant \frac{C}{|x|^{N+2s}} \quad \text{for all } |x| \geqslant 1.$$

The operator T_m is not just defined on functions in L^2. For instance it acts nicely on bounded functions. The positive kernel k satisfies $\int_{\mathbb{R}^N} k = \frac{1}{m}$. We see that if $g \in L^\infty(\mathbb{R}^N)$ then

$$\left\|T_m[g]\right\|_\infty \leqslant \frac{1}{m}\|g\|_\infty.$$

We have indeed the validity of an estimate like this for L^∞ weighted norms as follows.

Lemma 2.2. *Let $0 \leqslant \mu < N + 2s$. Then there exists a $C > 0$ such that*

$$\left\|\left(1+|x|\right)^\mu T_m[g]\right\|_{L^\infty(\mathbb{R}^N)} \leqslant C\left\|\left(1+|x|\right)^\mu g\right\|_{L^\infty(\mathbb{R}^N)}.$$

Proof. Let us assume that $0 \leqslant \mu < N + 2s$ and let $\bar{g}(x) = \frac{1}{(1+|x|)^\mu}$. Then

$$T[\bar{g}](x) = \int_{|y-x|<\frac{1}{2}|x|} \frac{k(y)}{(1+|y-x|)^\mu}\,dy + \int_{|y-x|>\frac{1}{2}|x|} \frac{k(y)}{(1+|y-x|)^\mu}\,dy.$$

Then, as $|x| \to \infty$ we find

$$|x|^\mu \int_{|y-x|<\frac{1}{2}|x|} \frac{k(y)}{(1+|y-x|)^\mu}\,dy \sim |x|^{-2s} \to 0,$$

and since $k \in L^1(\mathbb{R}^N)$, by dominated convergence we find that as $|x| \to \infty$

$$\int_{|x-y|>\frac{1}{2}|x|} \frac{k(y)|x|^\mu}{(1+|x-y|)^\mu}\,dy \to \int_{\mathbb{R}^N} k(z)\,dz = \frac{1}{m}.$$

We conclude in particular that for a suitable constant $C > 0$, we have

$$T_m\left[\left(1+|x|\right)^{-\mu}\right] \leqslant C\left(1+|x|\right)^{-\mu}.$$

Now, we have that

$$\pm T_m[g] \leqslant \left\| (1+|x|)^{\mu} g \right\|_{L^\infty(\mathbb{R}^N)} T_m\left[(1+|x|)^{-\mu} \right],$$

and then

$$\left\| (1+|x|)^{\mu} T[g] \right\|_{L^\infty(\mathbb{R}^N)} \leqslant C \left\| (1+|x|)^{\mu} g \right\|_{L^\infty(\mathbb{R}^N)}$$

as desired. □

We also have the validity of the following useful estimate.

Lemma 2.3. *Assume that $g \in L^2 \cap L^\infty$. Then the following holds: if $\phi = T_m[g]$ then there is a $C > 0$ such that*

$$\sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x-y|^\alpha} \leqslant C \|g\|_{L^\infty(\mathbb{R}^N)} \tag{2.6}$$

where $\alpha = \min\{1, 2s\}$.

Proof. Since $\|T_m[g]\|_\infty \leqslant C\|g\|_\infty$, it suffices to establish (2.6) for $|x-y| < \frac{1}{3}$. We have

$$\left| \phi(x) - \phi(y) \right| \leqslant \int_{\mathbb{R}^N} \left| k(z+y-x) - k(z) \right| dz \, \|g\|_\infty.$$

Now, we decompose

$$\int_{\mathbb{R}^N} \left| k(z+y-x) - k(z) \right| dz = \int_{|z|>3|y-x|} \left| k(z+y-x) - k(z) \right| dz + \int_{|z|<3|y-x|} \left| k(z+y-x) - k(z) \right| dz.$$

We have

$$\int_{|z|>3|y-x|} \left| k\big(z+(y-x)\big) - k(z) \right| \leqslant \int_0^1 dt \int_{|z|>3|y-x|} \left| \nabla k\big(z + t(y-x)\big) \right| dz \, |y-x|,$$

and, since $3|y-x| < 1$,

$$\int_{|z|>3|y-x|} \left| \nabla k\big(z+t(y-x)\big) \right| dz \leqslant C \left(1 + \int_{1>|z|>3|y-x|} \frac{dz}{|z|^{N+1-2s}} \right) \leqslant C \big(1 + |y-x|^{2s-1} \big).$$

On the other hand

$$\int\limits_{|z|<3|y-x|} \left|k(z+y-x)-k(z)\right| dz \leqslant 2 \int\limits_{|z|<4|y-x|} \left|k(z)\right| dz \leqslant C|y-x|^{2s},$$

and (2.6) readily follows. □

Next we consider the more general problem

$$(-\Delta)^s\phi + W(x)\phi = g \quad \text{in } \mathbb{R}^N \tag{2.7}$$

where W is a bounded potential.

We start with a form of the weak maximum principle.

Lemma 2.4. *Let us assume that*

$$\inf_{x\in\mathbb{R}^N} W(x) =: m > 0$$

and that $\phi \in H^{2s}(\mathbb{R}^N)$ satisfies Eq. (2.7) with $g \geqslant 0$. Then $\phi \geqslant 0$ in $\mathbb{R}^N$.

Proof. We use the representation for ϕ as the trace of the unique solution $\tilde{\phi} \in H$ to the problem

$$\iint\limits_{\mathbb{R}^{N+1}_+} \nabla\tilde{\phi}\nabla\varphi y^{1-2s} + \int\limits_{\mathbb{R}^N} W\phi\varphi = \int\limits_{\mathbb{R}^N} g\varphi, \quad \text{for all } \varphi \in H.$$

It is easy to check that the test function $\varphi = \phi_- = \min\{\phi, 0\}$ does indeed belong to H. We readily obtain

$$\iint\limits_{\mathbb{R}^{N+1}_+} |\nabla\tilde{\phi}_-|^2 y^{1-2s} + \int\limits_{\mathbb{R}^N} W\phi_-^2 = \int\limits_{\mathbb{R}^N} g\phi_-.$$

Since $g \geqslant 0$ and $W \geqslant m$, we obtain that $\phi_- \equiv 0$, which means precisely $\phi \geqslant 0$, as desired. □

We want to obtain a priori estimates for problems of the type (2.7) when W is not necessarily positive. Let $\mu > \frac{N}{2}$, and let us assume that

$$\left\|\left(1+|x|^\mu\right)g\right\|_{L^\infty(\mathbb{R}^N)} < +\infty.$$

The assumption in μ implies that $g \in L^2(\mathbb{R}^N)$.

Below, and in all what follows, we will say that $\phi \in L^2(\mathbb{R}^N)$ solves Eq. (2.7) if and only if ϕ solves the linear problem

$$\phi = T_m\left((m-W)\phi + g\right).$$

Similarly, we will say that

$$(-\Delta)^s \phi + W(x)\phi \geqslant g \quad \text{in } \mathbb{R}^N$$

if for some $\tilde{g} \in L^2(\mathbb{R}^N)$ with $\tilde{g} \geqslant g$ we have

$$\phi = T_m\big((m - W)\phi + \tilde{g}\big).$$

The next lemma provides an a priori estimate for a solution $\phi \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ of (2.7).

Lemma 2.5. *Let W be a continuous function, such that for k points q_i $i = 1, \dots, k$ a number $R > 0$ and $B = \bigcup_{i=1}^k B_R(q_i)$ we have*

$$\inf_{x \in \mathbb{R}^N \setminus B} W(x) =: m > 0.$$

Then, given any number $\frac{N}{2} < \mu < N + 2s$ there exists a constant $C = C(\mu, k, R) > 0$ such that for any $\phi \in H^{2s} \cap L^\infty(\mathbb{R}^N)$ and g with

$$\|\rho^{-1} g\|_{L^\infty(\mathbb{R}^N)} < +\infty$$

that satisfy Eq. (2.7) we have the validity of the estimate

$$\|\rho^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} \leqslant C\big[\|\phi\|_{L^\infty(B)} + \|\rho^{-1} g\|_{L^\infty(\mathbb{R}^N)}\big].$$

Here

$$\rho(x) = \sum_{i=1}^k \frac{1}{(1 + |x - q_i|)^\mu}.$$

Proof. We start by noticing that ϕ satisfies the equation

$$(-\Delta)^s \phi + \hat{W}\phi = \hat{g}$$

where

$$\hat{g} = (m - W)\chi_B \phi, \qquad \hat{W} = m\chi_B + W(1 - \chi_B).$$

Observe that

$$|\hat{g}(x)| \leqslant M \sum_{i=1}^k (1 + |x - q_i|)^{-\mu}, \quad M = C\big(\|\phi\|_{L^\infty(B)} + \|\rho^{-1} g\|_{L^\infty(\mathbb{R}^N)}\big)$$

where C depends only on R, k and μ and

$$\inf_{x \in \mathbb{R}^N} \hat{W}(x) \geqslant m.$$

Now, from Lemma 2.2, since $0 < \mu < N + 2s$ we find a solution $\phi_0(x)$ to the problem

$$(-\Delta)^s \bar{\phi} + m\bar{\phi} = \big(1 + |x|\big)^{-\mu}$$

such that $\bar{\phi} = O(|x|^{-\mu})$ as $|x| \to \infty$. Then we have that

$$\big((-\Delta)^s + \hat{W}\big)(\bar{\phi}) \geqslant M \sum_{i=1}^{k} \big(1 + |x - q_i|\big)^{-\mu}$$

where

$$\bar{\phi}(x) = M \sum_{i=1}^{k} \phi_0(x - q_i).$$

Setting $\psi = (\phi - \bar{\phi})$ we get

$$(-\Delta)^s \psi + \hat{W}\psi = \tilde{g} \leqslant 0$$

with $\tilde{g} \in L^2$. Using Lemma 2.4 we obtain $\phi \leqslant \bar{\phi}$. Arguing similarly for $-\phi$, and using the form of $\bar{\phi}$ and M, the desired estimate immediately follows. □

Examining the proof above, we obtain immediately the following.

Corollary 2.1. *Let $\rho(x)$ be defined as in the previous lemma. Assume that $\phi \in H^{2s}(\mathbb{R}^N)$ satisfies Eq. (2.7) and that*

$$\inf_{x \in \mathbb{R}^N} W(x) =: m > 0.$$

Then we have that $\phi \in L^\infty(\mathbb{R}^N)$ and it satisfies

$$\big\|\rho^{-1}\phi\big\|_{L^\infty(\mathbb{R}^N)} \leqslant C \big\|\rho^{-1} g\big\|_{L^\infty(\mathbb{R}^N)}. \tag{2.8}$$

A last useful fact is that if $f, g \in L^2(\mathbb{R}^N)$ and $W = T(f)$, $Z = T(g)$ then the following holds:

$$\int_{\mathbb{R}^N} Z(-\Delta)^s W - \int_{\mathbb{R}^N} W(-\Delta)^s Z = \int_{\mathbb{R}^N} T_m[f]g - \int_{\mathbb{R}^N} T_m[g]f = 0,$$

the latter fact since the kernel k is radially symmetric.

3. Formulation of the problem: the ansatz

By a solution of the problem

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - u^p = 0 \quad \text{in } \mathbb{R}^N$$

we mean a $u \in H^{2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that the above equation is satisfied. Let us observe that it suffices to solve

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - u_+^p = 0 \quad \text{in } \mathbb{R}^N \tag{3.1}$$

where $u_+ = \max\{u, 0\}$. In fact, if u solves (3.1) then

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u \geqslant 0 \quad \text{in } \mathbb{R}^N$$

and, as a consequence to Lemma 2.4, $u \geqslant 0$.

After absorbing ε by scaling, the equation takes the form

$$(-\Delta)^s v + V(\varepsilon x)v - v_+^p = 0 \quad \text{in } \mathbb{R}^N. \tag{3.2}$$

Let us consider points $\xi_1, \dots, \xi_k \in \mathbb{R}^N$ and designate

$$q_i = \varepsilon^{-1}\xi_i, \qquad q = (q_1, \dots, q_k).$$

Given numbers $\delta > 0$ small and $R > 0$ large, we define the configuration space Γ for the points q_i as

$$\Gamma := \Big\{ q = (q_1, \dots, q_k) \,/\, R \leqslant \max_{i \neq j} |q_i - q_j|, \ \max_i |q_i| \leqslant \delta^{-1}\varepsilon^{-1} \Big\}. \tag{3.3}$$

We look for a solution with concentration behavior near each ξ_j. Letting $\tilde{v}(x) = v(x + \xi_j)$ translating the origin to q_j, Eq. (3.2) reads

$$(-\Delta)^s \tilde{v} + V(\xi_j + \varepsilon x)\tilde{v} - \tilde{v}_+^p = 0 \quad \text{in } \mathbb{R}^N.$$

Letting formally $\varepsilon \to 0$ we are left with the equation

$$(-\Delta)^s \tilde{v} + \lambda_j \tilde{v} - \tilde{v}_+^p = 0 \quad \text{in } \mathbb{R}^N, \ \lambda_j = V(\xi_j).$$

So we ask that $v(x) \approx w_{\lambda_j}(x - q_j)$ near q_j. We consider the sum of these functions as a first approximation. Thus, we look for a solution v of (3.2) of the form

$$v = W_q + \phi, \qquad W_q(x) = \sum_{j=1}^{k} w_j(x), \quad w_j(x) = w_{\lambda_j}(x - q_j), \ \lambda_j = V(\xi_j),$$

where ϕ is a small function, disappearing as $\varepsilon \to 0$. In terms of ϕ, Eq. (3.2) becomes

$$(-\Delta)^s\phi + V(\varepsilon x)\phi - pW_q^{p-1}\phi = E + N(\phi) \quad \text{in } \mathbb{R}^N \tag{3.4}$$

where

$$N(\phi) := (W_q + \phi)_+^p - pW_q^{p-1}\phi - W_q^p,$$
$$E := \sum_{j=1}^k \big(\lambda_j - V(\varepsilon x)\big)w_j + \left(\sum_{j=1}^k w_j\right)^p - \sum_{j=1}^k w_j^p. \tag{3.5}$$

Rather than solving problem (3.4) directly, we consider first a projected version of it. Let us consider the functions

$$Z_{ij}(x) := \partial_j w_i(x)$$

and the problem of finding $\phi \in H^{2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that for certain constants c_{ij}

$$(-\Delta)^s\phi + V(\varepsilon x)\phi - pW_q^{p-1}\phi = E + N(\phi) + \sum_{i=1}^k \sum_{j=1}^N c_{ij}Z_{ij}, \tag{3.6}$$

$$\int_{\mathbb{R}^N} \phi Z_{ij} = 0 \quad \text{for all } i, j. \tag{3.7}$$

Let $\mathcal{Z}$ be the linear space spanned by the functions Z_{ij}, so that Eq. (3.6) is equivalent to

$$(-\Delta)^s\phi + V(\varepsilon x)\phi - pW_q^{p-1}\phi - E - N(\phi) \in \mathcal{Z}.$$

On the other hand, for all ε sufficiently small, the functions Z_{ij} are linearly independent, hence the constants c_{ij} have unique, computable expressions in terms of ϕ. We will prove that problem (3.6)–(3.7) has a unique small solution $\phi = \Phi(q)$. In that way we will get a solution to the full problem (3.4) if we can find a value of q such that $c_{ij}(\Phi(q)) = 0$ for all i, j. In order to build $\Phi(q)$ we need a theory of solvability for associated linear operator in suitable spaces. This is what we develop in the next section.

4. Linear theory

We consider the linear problem of finding $\phi \in H^{2s}(\mathbb{R}^N)$ such that for certain constants c_{ij} we have

$$(-\Delta)^s\phi + V(\varepsilon x)\phi - pW_q^{p-1}(x)\phi + g(x) = \sum_{i=1}^N \sum_{i=1}^k c_{ij}Z_{ij}, \tag{4.1}$$

$$\int_{\mathbb{R}^N} \phi Z_{ij} = 0 \quad \text{for all } i, j. \tag{4.2}$$

The constants c_{ij} are uniquely determined in terms of ϕ and g when ε is sufficiently small, from the linear system

$$\sum_{i,j} c_{ij} \int_{\mathbb{R}^N} Z_{ij} Z_{lk} = \int_{\mathbb{R}^N} Z_{lk}\big[(-\Delta)^s \phi + V(\varepsilon x)\phi - pW_q^{p-1}(x)\phi + g\big]. \tag{4.3}$$

Taking into account that

$$\int_{\mathbb{R}^N} Z_{lk}(-\Delta)^s \phi = \int_{\mathbb{R}^N} \phi(-\Delta)^s Z_{lk} = \int_{\mathbb{R}^N} \big(pw_l^{p-1} - \lambda_l\big) Z_{lk}\phi,$$

we find

$$c_{ij} \int_{\mathbb{R}^N} Z_{ij} Z_{lk} = \int_{\mathbb{R}^N} gZ_{lk} + \big(pw_l^{p-1} - pW_q^{p-1} + V(\varepsilon x) - \lambda_l\big) Z_{lk}\phi. \tag{4.4}$$

On the other hand, we check that

$$\int_{\mathbb{R}^N} Z_{ij} Z_{lk} = \alpha_l \delta_{ijkl} + O\big(d^{-N}\big)$$

where the numbers α_l are positive, and independent of ε, and

$$d = \min\big\{|q_i - q_j| \,/\, i \neq j\big\} \gg 1.$$

Then, we see that relations (4.4) define a uniquely solvable (nearly diagonal) linear system, provided that ε is sufficiently small. We assume this last fact in what follows, and hence that the numbers $c_{ij} = c_{ij}(\phi, g)$ are defined by relations (4.4).

Moreover, we have that

$$\big|\big(pw_l^{p-1} - pW_q^{p-1} + V(\varepsilon x) - \lambda_l\big) Z_{lk}(x)\big| \leqslant C\big(R^{-N} + \varepsilon|x - q_j|\big)\big(1 + |x - q_j|\big)^{-N-s}$$

and then from expression (4.4) we obtain the following estimate.

Lemma 4.1. *The numbers c_{ij} in (4.1) satisfy*:

$$c_{ij} = \frac{1}{\alpha_i} \int_{\mathbb{R}^N} gZ_{ij} + \theta_{ij},$$

where

$$|\theta_{ij}| \leqslant C\big(\varepsilon + d^{-N}\big)\big[\|\phi\|_{L^2(\mathbb{R}^N)} + \|g\|_{L^2(\mathbb{R}^N)}\big].$$

In the rest of this section we shall build a solution to problem (4.1)–(4.2).

Proposition 4.1. *Given $k \geqslant 1$, $\frac{N}{2} < \mu < N + 2s$, $C > 0$, there exist positive numbers d_0, ε_0, C such that for any points $q_1, \dots, q_k$ and any ε with*

$$\sum_{i=1}^{k} |q_i| \leqslant \frac{C}{\varepsilon}, \qquad R := \min\{|q_i - q_j| / i \neq j\} > R_0, \quad 0 < \varepsilon < \varepsilon_0$$

there exists a solution $\phi = T[g]$ of (4.1)–(4.2) that defines a linear operator of g, provided that

$$\|\rho(x)^{-1} g\|_{L^\infty(\mathbb{R}^N)} < +\infty, \quad \rho(x) = \sum_{j=1}^{k} \frac{1}{(1 + |x - q_j|)^\mu}.$$

Besides

$$\|\rho(x)^{-1} \phi\|_{L^\infty(\mathbb{R}^N)} \leqslant C \|\rho(x)^{-1} g\|_{L^\infty(\mathbb{R}^N)}.$$

To prove this result we require several steps. We begin with corresponding a priori estimates.

Lemma 4.2. *Under the conditions of Proposition 4.1, there exists a $C > 0$ such that for any solution of (4.1)–(4.2) with $\|\rho(x)^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} < +\infty$ we have the validity of the a priori estimate*

$$\|\rho(x)^{-1} \phi\|_{L^\infty(\mathbb{R}^N)} \leqslant C \|\rho(x)^{-1} g\|_{L^\infty(\mathbb{R}^N)}.$$

Proof. Let us assume the a priori estimate does not hold, namely there are sequences $\varepsilon_n \to 0$, q_{jn}, $j = 1, \dots, k$, with

$$\min\{|q_{in} - q_{jn}| / i \neq j\} \to \infty$$

and ϕ_n, g_n with

$$\|\rho_n(x)^{-1} \phi_n\|_{L^\infty(\mathbb{R}^N)} = 1, \qquad \|\rho_n(x)^{-1} g_n\|_{L^\infty(\mathbb{R}^N)} \to 0,$$

where

$$\rho_n(x) = \sum_{j=1}^{k} \frac{1}{(1 + |x - q_{jn}|)^\mu},$$

with ϕ_n, g_n satisfying (4.1)–(4.2). We claim that for any fixed $R > 0$ we have that

$$\sum_{j=1}^{k} \|\phi_n\|_{L^\infty(B_R(q_{jn}))} \to 0. \tag{4.5}$$

Indeed, assume that for a fixed j we have that $\|\phi_n\|_{L^\infty(B_R(q_{jn}))} \geqslant \gamma > 0$. Let us set $\bar\phi_n(x) = \phi_n(q_{jn}+x)$. We also assume that $\lambda_j^n = V(q_{jn}) \to \bar\lambda > 0$ and

$$(-\Delta)^s \bar\phi_n + V(q_{jn}+\varepsilon_n x)\bar\phi_n + p\big(w_{\lambda_j^n}(x)+\theta_n(x)\big)^{p-1}\bar\phi_n = \bar g_n$$

where

$$\bar g_n(x) = g_n(q_{jn}+x) - \sum_{l=1}^{k}\sum_{i=1}^{n} c_{ln}^i \partial_i w_{\lambda_l^n}\big(q_{jn}-q'_{ln}+x\big).$$

We observe that $\bar g_n(x) \to 0$ uniformly on compact sets. From the uniform Hölder estimates (2.6), we also obtain equicontinuity of the sequence $\bar\phi_n$. Thus, passing to a subsequence, we may assume that $\bar\phi_n$ converges, uniformly on compact sets, to a bounded function $\bar\phi$ which satisfies $\|\bar\phi\|_{L^\infty(B_R(0))} \geqslant \gamma$. In addition, we have that

$$\big\|\big(1+|x|\big)^\mu \bar\phi\big\|_{L^\infty(\mathbb{R}^N)} \leqslant 1$$

and that $\bar\phi$ solves the equation

$$(-\Delta)^s\bar\phi + \bar\lambda\bar\phi + p w_{\bar\lambda}^{p-1}\bar\phi = 0.$$

Let us notice that $\bar\phi \in L^2(\mathbb{R}^N)$, and hence the nondegeneracy result in [15] applies to yield that $\bar\phi$ must be a linear combination of the partial derivatives $\partial_i w_{\bar\lambda}$. But the orthogonality conditions pass to the limit, and yield

$$\int_{\mathbb{R}^N} \partial_i w_{\bar\lambda}\bar\phi = 0 \quad \text{for all } i = 1,\dots,N.$$

Thus, necessarily $\bar\phi = 0$. We have obtained a contradiction that proves the validity of (4.5). This and the a priori estimate in Lemma 2.5 shows that also, $\|\rho_n(x)^{-1}\phi_n\|_{L^\infty(\mathbb{R}^N)} \to 0$, again a contradiction that proves the desired result. □

Next we construct a solution to problem (4.1)–(4.2). To do so, we consider first the auxiliary problem

$$(-\Delta)^s\phi + V\phi = g + \sum_{i=1}^{k}\sum_{j=1}^{N} c_{ij}Z_{ij}, \tag{4.6}$$

$$\int_{\mathbb{R}^N} \phi Z_{ij} = 0 \quad \text{for all } i, j, \tag{4.7}$$

where V is our bounded, continuous potential with

$$\inf_{\mathbb{R}^N} V = m > 0.$$

Lemma 4.3. *For each g with $\|\rho^{-1}g\|_\infty < +\infty$, there exists a unique solution of problem (4.1)–(4.2), $\phi =: A[g] \in H^{2s}(\mathbb{R}^N)$. This solution satisfies*

$$\|\rho^{-1}A[g]\|_{L^\infty(\mathbb{R}^N)} \leqslant C\|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)}. \tag{4.8}$$

Proof. First we write a variational formulation for this problem. Let X be the closed subspace of H defined as

$$X = \left\{\tilde{\phi} \in H \,/ \int_{\mathbb{R}^N} \phi Z_{ij} = 0 \text{ for all } i, j\right\}.$$

Then, given $g \in L^2$, we consider the problem of finding a $\tilde{\phi} \in X$ such that

$$\langle\tilde{\phi}, \tilde{\psi}\rangle := \iint_{\mathbb{R}^{N+1}_+} \nabla\tilde{\phi}\nabla\tilde{\psi} y^{1-2s} + \int_{\mathbb{R}^N} V\phi\psi = \int_{\mathbb{R}^N} g\psi \quad \text{for all } \psi \in X. \tag{4.9}$$

We observe that $\langle\cdot,\cdot\rangle$ defines an inner product in X equivalent to that of H. Thus existence and uniqueness of a solution follows from Riesz's theorem. Moreover, we see that

$$\|\phi\|_{L^2(\mathbb{R}^N)} \leqslant C\|g\|_{L^2(\mathbb{R}^N)}.$$

Next we check that this produces a solution in strong sense. Let $\mathcal{Z}$ be the space spanned by the functions Z_{ij}. We denote by $\Pi[g]$ the $L^2(\mathbb{R}^N)$ orthogonal projection of g onto $\mathcal{Z}$ and by $\tilde{\Pi}[g]$ its natural s-harmonic extension. For a function $\tilde{\varphi} \in H$ let us write

$$\tilde{\psi} = \tilde{\varphi} - \tilde{\Pi}[\varphi]$$

so that $\tilde{\psi} \in X$. Substituting this $\tilde{\psi}$ into (4.9) we obtain

$$\iint_{\mathbb{R}^{N+1}_+} \nabla\tilde{\phi}\nabla\tilde{\varphi} y^{1-2s} + \int_{\mathbb{R}^N} V\phi\varphi = \int_{\mathbb{R}^N} g\varphi + \int_{\mathbb{R}^N} [V\phi - g]\Pi[\varphi] + \int_{\mathbb{R}^N} \phi(-\Delta)^s\Pi[\varphi].$$

Here we have used that $\tilde{\Pi}[\varphi]$ is regular and

$$\iint_{\mathbb{R}^{N+1}_+} \nabla\phi\nabla\tilde{\Pi}[\varphi] y^{1-2s} = \int_{\mathbb{R}^N} \phi(-\Delta)^s\Pi[\varphi].$$

Let us observe that for $f \in L^2(\mathbb{R}^N)$ the functional

$$\ell(f) = \int_{\mathbb{R}^N} \phi(-\Delta)^s\Pi[f]$$

satisfies

$$\left|\ell(f)\right| \leqslant C\|\phi\|_{L^2(\mathbb{R}^N)}\|\psi\|_{L^2(\mathbb{R}^N)},$$

hence there is an $h(\phi) \in L^2(\mathbb{R}^N)$ such that

$$\ell(\psi) = \int_{\mathbb{R}^N} h\psi.$$

If ϕ was a priori known to be in $H^{2s}(\mathbb{R}^N)$ we would have precisely that

$$h(\phi) = \Pi\left[(-\Delta)^s\phi\right].$$

Since Π is a self-adjoint operator in $L^2(\mathbb{R}^N)$ we then find that

$$\iint_{\mathbb{R}^{N+1}_+} \nabla\tilde{\phi}\nabla\tilde{\varphi}y^{1-2s} + \int_{\mathbb{R}^N} V\phi\varphi = \int_{\mathbb{R}^N} \bar{g}\varphi$$

where

$$\bar{g} = g + \Pi[V\phi - g] + h(\phi).$$

Since $\bar{g} \in L^2(\mathbb{R}^N)$, it follows then that $\phi \in H^{2s}(\mathbb{R}^N)$ and it satisfies

$$(-\Delta)^s\phi + V\phi - g = \Pi\left[(-\Delta)^s\phi + V\phi - g\right] \in \mathcal{Z},$$

hence Eqs. (4.6)–(4.7) are satisfied. To establish estimate (4.8), we use just Corollary 2.1, observing that

$$\begin{aligned}\left\|\rho^{-1}\Pi\left[(-\Delta)^s\phi + V\phi - g\right]\right\|_{L^\infty(\mathbb{R}^N)} &\leqslant C\left(\|\phi\|_{L^2(\mathbb{R}^N)} + \|g\|_{L^2(\mathbb{R}^N)}\right)\\ &\leqslant C\|g\|_{L^2(\mathbb{R}^N)}\\ &\leqslant \left\|\rho^{-1}g\right\|_{L^\infty(\mathbb{R}^N)}.\end{aligned}$$

The proof is concluded. □

Proof of Proposition 4.1. Let us solve now problem (4.1)–(4.2). Let Y be the Banach space

$$Y := \left\{\phi \in C\left(\mathbb{R}^N\right) / \|\phi\|_Y := \left\|\rho^{-1}\phi\right\|_{L^\infty(\mathbb{R}^N)} < +\infty\right\}. \tag{4.10}$$

Let A be the operator defined in Lemma 4.3. Then we have a solution to problem (4.1)–(4.2) if we solve

$$\phi - A\left[pW_q^{p-1}\phi\right] = A[g], \quad \phi \in Y. \tag{4.11}$$

We claim that

$$B[\phi] := A\left[pW_q^{p-1}\phi\right]$$

defines a compact operator in Y. Indeed. Let us assume that ϕ_n is a bounded sequence in Y. We observe that for some $\sigma > 0$ we have

$$\left|W_q^{p-1}\phi_n\right| \leqslant C\|\phi_n\|_Y \rho^{1+\sigma}.$$

If σ is sufficiently small, it follows that $f_n := B[\phi_n]$ satisfies

$$\left|\rho^{-1} f_n\right| \leqslant C\rho^{\sigma}.$$

Besides, since $f_n = T_m((V-m)f_n + g_n)$ we use estimate (2.6) to get that for some $\alpha > 0$

$$\sup_{x\neq y} \frac{|f_n(x) - f_n(y)|}{|x-y|^{\alpha}} \leqslant C.$$

Arzela's theorem then yields the existence of a subsequence of f_n which we label the same way, that converges uniformly on compact sets to a continuous function f with

$$\left|\rho^{-1} f\right| \leqslant C\rho^{\sigma}.$$

Let $R > 0$ be a large number. Then we estimate

$$\left\|\rho^{-1}(f_n - f)\right\|_{L^\infty(\mathbb{R}^N)} \leqslant \left\|\rho^{-1}(f_n - f)\right\|_{L^\infty(B_R(0))} + C \max_{|x|>R} \rho^{\sigma}(x).$$

Since

$$\max_{|x|>R} \rho^{\sigma}(x) \to 0 \quad \text{as } R \to \infty$$

we conclude then that $\|f_n - f\|_\infty \to 0$ and the claim is proven.

Finally, the a priori estimate tells us that for $g = 0$, Eq. (4.11) has only the trivial solution. The desired result follows at once from Fredholm's alternative. □

We conclude this section by analyzing the differentiability with respect to the parameter q of the solution $\phi = T_q[g]$ of (4.1)–(4.2). As in the proof above we let Y be the space in (4.10), so that $T_q \in \mathcal{L}(Y)$

Lemma 4.4. *The map $q \mapsto T_q$ is continuously differentiable, and for some $C > 0$,*

$$\|\partial_q T_q\|_{\mathcal{L}(Y)} \leqslant C \tag{4.12}$$

for all q satisfying constraints (3.3).

Proof. Let us write $q = (q_1, \dots, q_k)$, $q_i = (q_{i1}, \dots, q_{iN})$, $\phi = T_q[g]$, and (formally)

$$\psi = \partial_{q_{ij}} T_q[g], \qquad d_{lk} = \partial_{q_{ij}} c_{lk}.$$

Then, by differentiation of Eqs. (4.1)–(4.2), we get

$$(-\Delta)^s \psi + V(\varepsilon x)\psi - pW_q^{p-1}\psi = p\partial_{q_{ij}} W_q^{p-1}\phi + \sum_{l,k} c_{lk}\partial_{q_{ij}} Z_{lk} + \sum_{l,k} d_{lk} Z_{lk}, \tag{4.13}$$

$$\int_{\mathbb{R}^N} \psi Z_{lk} = -\int_{\mathbb{R}^N} \phi \partial_{q_{ij}} Z_{lk} \quad \text{for all } l, k. \tag{4.14}$$

We let

$$\tilde{\psi} = \psi - \Pi[\psi]$$

where, as before, $\Pi[\psi]$ denotes the orthogonal projection of ψ onto the space spanned by the Z_{lk}. Writing

$$\Pi[\psi] = \sum_{l,k} \alpha_{lk} Z_{lk} \tag{4.15}$$

and relations (4.14) as

$$\int_{\mathbb{R}^N} \Pi[\psi] Z_{lk} = -\int_{\mathbb{R}^N} \phi \partial_{q_{ij}} Z_{lk} \quad \text{for all } l, k, \tag{4.16}$$

we get

$$|\alpha_{lk}| \leqslant C\|\phi\|_Y \leqslant C\|g\|_Y. \tag{4.17}$$

From (4.13) we have then that

$$(-\Delta)^s \tilde{\psi} + V(\varepsilon x)\tilde{\psi} - pW_q^{p-1}\tilde{\psi} = \tilde{g} + \sum_{l,k} d_{lk} Z_{lk}, \tag{4.18}$$

or $\tilde{\psi} = T_q[\tilde{g}]$ where

$$\tilde{g} = p\partial_{q_{ij}} W_q^{p-1}\phi + \sum_{l,k} c_{lk}\partial_{q_{ij}} Z_{lk} - \left[(-\Delta)^s + V(\varepsilon x) - pW_q^{p-1}\right]\Pi[\psi]. \tag{4.19}$$

Then we see that

$$\|\tilde{\psi}\|_Y \leqslant C\|\tilde{g}\|_Y.$$

Using (4.17) and Lemma 4.1, we see also that

$$\|\tilde{g}\|_Y \leqslant C\|g\|_Y, \qquad \big\|\Pi[\psi]\big\| \leqslant C\|g\|_Y$$

and thus

$$\|\psi\| \leqslant C\|g\|_Y. \tag{4.20}$$

Let us consider now, rigorously, the unique $\psi = \tilde{\psi} + \Pi[\psi]$ that satisfies Eqs. (4.14) and (4.19). We want to show that indeed

$$\psi = \partial_{q_{ij}} T_q[g].$$

To do so, $q_i^t = q_i + te_j$ where e_j is the j-th element of the canonical basis of $\mathbb{R}^N$, and set

$$q^t = \big(q_1, \ldots q_{i-1}, q_i^t, \ldots, q_k\big).$$

For a function $f(q)$ we denote

$$D_{ij}^t f = t^{-1}\big(f\big(q^t\big) - f(q)\big)$$

we also set

$$\phi^t := T_{q^t}[g], \qquad D_{ij}^t T_q[g] =: \psi^t = \tilde{\psi}^t + \Pi\big[\tilde{\psi}^t\big]$$

so that

$$(-\Delta)^s \tilde{\psi}^t + V(\varepsilon x)\tilde{\psi}^t - pW_q^{p-1}\tilde{\psi}^t = \tilde{g}^t + \sum_{l,k} d_{lk}^t Z_{lk},$$

where

$$\tilde{g}^t = pD_{ij}^t\big[W_q^{p-1}\big]\phi + \sum_{l,k} c_{lk} D_{ij}^t Z_{lk} - \big[(-\Delta)^s + V(\varepsilon x) - pW_q^{p-1}\big]\Pi\big[\psi^t\big],$$

$$d_{lk}^t = D_{ij}^t c_{lk}$$

and

$$\Pi\big[\psi^t\big] = \sum_{l,k} \alpha_{lk}^t Z_{lk},$$

where the constants α_{lk}^t are determined by the relations

$$\int_{\mathbb{R}^N} \Pi\big[\psi^t\big] Z_{lk} = -\int_{\mathbb{R}^N} \phi D_{ij}^t Z_{lk}.$$

Comparing these relations with (4.15), (4.16), (4.18) defining ψ, we obtain that

$$\lim_{t\to 0}\left\|\psi^t-\psi\right\|_Y=0$$

which by definition tells us $\psi=\partial_{q_{ij}}T_q[g]$. The continuous dependence in q is clear from that of the data in the definition of ψ. Estimate (4.12) follows from (4.20). The proof is concluded. □

5. Solving the nonlinear projected problem

In this section we solve the nonlinear projected problem

$$(-\Delta)^s\phi+V(\varepsilon x)\phi-pW_q^{p-1}\phi=E+N(\phi)+\sum_{i=1}^k\sum_{j=1}^N c_{ij}Z_{ij}, \tag{5.1}$$

$$\int_{\mathbb{R}^N}\phi Z_{ij}=0\quad\text{for all } i,\,j. \tag{5.2}$$

We have the following result.

Proposition 5.1. *Assuming that $\|E\|_Y$ is sufficiently small problem (5.1)–(5.2) has a unique small solution $\phi=\Phi(q)$ with*

$$\left\|\Phi(q)\right\|_Y\leqslant C\|E\|_Y.$$

The map $q\mapsto\Phi(q)$ is of class C^1, and for some $C>0$

$$\left\|\partial_q\Phi(q)\right\|_Y\leqslant C\left[\|E\|_Y+\|\partial_qE\|_Y\right], \tag{5.3}$$

for all q satisfying constraints (3.3).

Proof. Problem (5.1)–(5.2) can be written as the fixed point problem

$$\phi=T_q\big(E+N(\phi)\big)=:K_q(\phi),\qquad \phi\in Y. \tag{5.4}$$

Let

$$B=\left\{\phi\in Y\,/\,\|\phi\|_Y\leqslant\rho\right\}.$$

If $\phi\in B$ we have that

$$\left|N(\phi)\right|\leqslant C|\phi|^\beta,\qquad \beta=\min\{p,2\},$$

and hence

$$\left\|N(\phi)\right\|_Y\leqslant C\|\phi\|^2.$$

It follows that

$$\left\|K_q(\phi)\right\|_Y \leqslant C_0\left[\|E\| + \rho^2\right]$$

for a number C_0, uniform in q satisfying (3.3). Let us assume

$$\rho := 2C_0\|E\|, \quad \|E\| \leqslant \frac{1}{2C_0}.$$

Then

$$\left\|K_q(\phi)\right\|_Y \leqslant C_0\left[\frac{1}{2C_0}\rho + \rho^2\right] \leqslant \rho$$

so that $K_q(B) \subset B$. Now, we observe that

$$\left|N(\phi_1) - N(\phi_2)\right| \leqslant C\left[|\phi|^{\beta-1} + |\phi|^{\beta-1}\right]|\phi_1 - \phi_2|$$

and hence

$$\left\|N(\phi_1) - N(\phi_2)\right\|_Y \leqslant C\rho^{\beta-1}\|\phi_1 - \phi_2\|_Y$$

and

$$\left\|K_q(\phi_1) - K_q(\phi_2)\right\| \leqslant C\rho^{\beta-1}\|\phi_1 - \phi_2\|_Y.$$

Reducing ρ if necessary, we obtain that K_q is a contraction mapping and hence has a unique solution of Eq. (5.4) exists in B. We denote it as $\phi = \Phi(q)$. We prove next that Φ defines a C^1 function of q. Let

$$M(\phi, q) := \phi - T_q\left(E + N(\phi)\right).$$

Let $\phi_0 = \Phi(q_0)$. Then $M(\phi_0, q_0) = 0$. On the other hand,

$$\partial_\phi M(\phi, q)[\psi] = \psi - T_q\left(N'(\phi)\psi\right)$$

where $N'(\phi) = p[(W + \phi)^{p-1} - W^{p-1}]$, so that

$$\left\|N'(\phi)\psi\right\|_Y \leqslant C\rho^{\beta-1}\|\psi\|_Y.$$

If ρ is sufficiently small we have then that $D_\phi M(\phi_0, q_0)$ is an invertible operator, with uniformly bounded inverse. Besides

$$\partial_q M(\phi, q) = (\partial_q T_q)\left(E + N(\phi)\right) + T_q\left(\partial_q E + \partial_q N(\phi)\right).$$

Both partial derivatives are continuous in their arguments. The implicit function applies in a small neighborhood of (ϕ_0, q_0) to yield existence and uniqueness of a function $\phi = \phi(q)$ with $\phi(q_0) =$

ϕ_0 defined near q_0 with $M(\phi(q),q)=0$. Besides, $\phi(q)$ is of class C^1. But, by uniqueness, we must have $\phi(q)=\Phi(q)$. Finally, we see that

$$\partial_q\Phi(q) = -D_\phi M\big(\Phi(q),q\big)^{-1}\big[(\partial_q T_q)\big(E+N\big(\Phi(q)\big)\big)+T_q\big(\partial_q E+\partial_q N\big(\Phi(q)\big)\big)\big],$$
$$\partial_q N(\phi) = p\big[(W+\phi)^{p-1}-pW^{p-1}-(p-1)W^{p-2}\phi\big]\partial_q W$$

and hence

$$\big\|(\partial_q N)\big(\Phi(q)\big)\big\|_Y \leqslant C\big\|\Phi(q)\big\|_Y^\beta \leqslant C\|E\|_Y^\beta.$$

From here, the above expressions and the bound of Lemma 4.4 we finally get the validity of estimate (5.3). □

5.1. An estimate of the error

Here we provide an estimate of the error E defined in (3.5),

$$E := \sum_{j=1}^k\big(\lambda_j - V(\varepsilon x)\big)w_j + \left(\sum_{j=1}^k w_j\right)^p - \sum_{j=1}^k w_j^p$$

in the norm $\|\cdot\|_Y$. Here we need to take $\mu\in(\frac{N}{2},\frac{N+2s}{2})$. We denote

$$R = \min_{i\neq j}|q_i-q_j| \gg 1.$$

The first term in E can be easily estimated as

$$\left|\rho^{-1}(x)\sum_{j=1}^k\big(\lambda_j - V(\varepsilon x)\big)w_j\right| \leqslant C\varepsilon^{\min(2s,1)}.$$

To estimate the interaction term in E, we divide the $\mathbb{R}^N$ into the k sub-domains

$$\Omega_j = \{w_j \geqslant w_i,\ \forall i\neq j\},\quad j=1,\dots,k.$$

In Ω_j, we have

$$\begin{aligned}\left|\left(\sum_{j=1}^k w_j\right)^p - \sum_{j=1}^k w_j^p\right| &\leqslant Cw_j^{p-1}\sum_{i\neq j}\frac{1}{|x-q_i|^{N+2s}}\\ &\leqslant C\frac{1}{(1+|x-q_j|)^{(N+2s)(p-1)+\mu}}\sum_{i\neq j}\frac{1}{|q_j-q_i|^{N+2s-\mu}}\\ &\leqslant C\rho(x)R^{\mu-N-2s}.\end{aligned}$$

In summary, we conclude that

$$\|E\|_Y \leqslant C\varepsilon^{2s} + CR^{\mu-N-2s}. \tag{5.5}$$

As a consequence of Proposition 5.1 and the estimate (5.5), we obtain that

$$\|\Phi(q)\|_Y \leqslant C\varepsilon^{\min(2s,1)} + CR^{\mu-N-2s}.$$

Let us now take

$$\tau = C\varepsilon^{\min(2s,1)} + CR^{\mu-N-2s}.$$

6. The variational reduction

We will use the above introduced ingredients to find existence results for the equation

$$(-\Delta)^s v + V(\varepsilon x)v - v_+^p = 0. \tag{6.1}$$

An energy whose Euler–Lagrange equation corresponds formally to (6.1) is given by

$$J_\varepsilon(\tilde{v}) := \frac{1}{2}\int_{\mathbb{R}^N} v(-\Delta)^s v + V(\varepsilon x)v^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} V(\varepsilon x)v^2.$$

We want to find a solution of (6.1) with the form

$$v = v_q := W_q + \Phi(q)$$

where $\Phi(q)$ is the function in Proposition 5.1. We observe that

$$(-\Delta)^s v_q + V(\varepsilon x)v_q - (v_q)_+^p = \sum_{i,j} c_{ij} Z_{ij} \tag{6.2}$$

hence what we need is to find points q such that $c_{ij} = 0$ for all i, j. This problem can be formulated variationally as follows.

Lemma 6.1. *Let us consider the function of points $q = (q_1, \dots, q_k)$ given by*

$$I(q) := J_\varepsilon\big(W_q + \Phi(q)\big),$$

where $W_q + \Phi(q)$ is the unique s-harmonic extension of $W_q + \Phi(q)$. Then in (6.2), we have $c_{ij} = 0$ for all i, j if and only if

$$\partial_q I(q) = 0.$$

Proof. Let us write $v_q = W_q + \phi(q)$. We observe that

$$\partial_{q_{ij}} I(q) = \int_{\mathbb{R}_+^{N+1}} \nabla\tilde{v}_q \nabla(\partial_{q_{ij}}\tilde{v}_q) y^{1-2s} + \int_{\mathbb{R}^N} V(\varepsilon x) v_q \partial_{q_{ij}} v_q - \int_{\mathbb{R}^N} (v_q)_+^{p-1} \partial_{q_{ij}} v_q$$
$$= \int_{\mathbb{R}^N} \left[(-\Delta)^s v_q + V(\varepsilon x) v_q - (v_q)_+^p\right] \partial_{q_{ij}} v_q$$
$$= \sum_{k,l} c_{kl} \int_{\mathbb{R}^N} Z_{kl} \partial_{q_{ij}} v_q. \tag{6.3}$$

We observe that

$$\partial_{q_{ij}} v_q = -Z_{ij} + O(\varepsilon\rho) + \partial_{q_{ij}} \Phi(q).$$

Since, according to Proposition 5.1

$$\left\|\partial_q \Phi(q)\right\|_Y = O\left(\|E\|_Y + \|\partial_q E\|_Y\right)$$

and this quantity gets smaller as the number δ in (3.3) is reduced, and the functions Z_{kl} are linearly independent (in fact nearly orthogonal in L^2), it follows that the quantity in (6.3) equals zero for all i, j if and only if $c_{ij} = 0$ for all i, j. The proof is concluded. □

Our task is therefore to find critical points of the functional $I(q)$. Useful to this end is to achieve expansions of the energy in special situations.

Lemma 6.2. *Assume that the numbers δ and R in the definition of Γ in (3.3) is taken so small that*

$$\|E\|_Y + \|\partial_q E\| \leqslant \tau \ll 1.$$

Then

$$I_\varepsilon(q) = J_\varepsilon(W_q) + O\left(\tau^2\right)$$

and

$$\partial_q I_\varepsilon(q) = \partial_q J_\varepsilon(W_q) + O\left(\tau^2\right)$$

uniformly on points q in Γ.

Proof. Let us estimate

$$I(q) = J_\varepsilon(v_q), \quad v_q = W_q + \Phi(q).$$

We have that

$$I(\xi) = \frac{1}{2}\int_{\mathbb{R}^N} v_q(-\Delta)^s v_q + V v_q^2 - \frac{1}{p+1}\int v_q^{p+1}.$$

Thus we can expand

$$I(q) = J_\varepsilon(W_q) + \int_{\mathbb{R}^N} \Phi\big[(-\Delta)^s v_q + V v_q - v_q^p\big] + \frac{1}{2}\int_{\mathbb{R}^N} \Phi(-\Delta)^s\Phi + V\Phi^2$$
$$- \frac{1}{p+1}\int_{\mathbb{R}^N} \big[(W_q+\Phi)^{p+1} - W_q^{p+1} - (p+1)W_q^p\Phi\big].$$

Since, $\|E\|_Y \leqslant \tau$ then $\|\Phi\|_Y = O(\tau)$, and from the equation satisfied by Φ, also $\|(-\Delta)^s\Phi\|_Y = O(\tau)$. This implies

$$\left|\frac{1}{2}\int_{\mathbb{R}^N} \Phi(-\Delta)^s\Phi + V\Phi^2\right| \leqslant C\int_{\mathbb{R}^N} \rho^{2\mu}\tau^2 \leqslant C\tau^2$$

and

$$\left|\int_{\mathbb{R}^N} \big[(W_q+\Phi)^{p+1} - W_q^{p+1} - (p+1)W_q^p\Phi\big]\right| \leqslant C\int_{\mathbb{R}^N} \rho^{2\mu}\tau^2 \leqslant C\tau^2.$$

Here we have used the fact that $\mu \in (\frac{N}{2}, \frac{N+2s}{2})$.

On the other hand the second term in the above expansion equals 0, since by definition

$$(-\Delta)^s v_q + V v_q - v_q^p \in \mathcal{Z}$$

and Φ is L^2-orthogonal to that space. We arrive to the conclusion that

$$I(q) = J_\varepsilon(W_q) + O\big(\tau^2\big)$$

uniformly for q in a bounded set. By differentiation we also have that

$$\partial_q I(q) = \partial_q J_\varepsilon(W_q) + \int_{\mathbb{R}^N} \partial_q\Phi(-\Delta)^s\Phi + V\Phi\partial_q\Phi$$
$$+ \int_{\mathbb{R}^N} \big[(W_q+\Phi)^p - W_q^p - pW_q^{p-1}\Phi\big]\partial_q W_q + \big[(W_q+\Phi)^p - W_q^p\big]\partial_q\Phi.$$

Since we also have $\|\partial_q\Phi\|_Y = O(\tau)$, then the second and third term above are of size $O(\varepsilon^2)$. Thus,

$$\partial_q I(q) = \partial_q J_\varepsilon(W_q) + O\big(\rho^2\big)$$

uniformly on $q \in \Gamma$ and the proof is complete. □

Next we estimate $J_\varepsilon(W_q)$ and $\partial_q J_\varepsilon(W_q)$. We begin with the simpler case $k = 1$. Here it is always the case that

$$\|E\|_Y + \|\partial_q E\|_Y \leqslant \tau.$$

Let us also set $\xi = \varepsilon q$. We have now that

$$W_q(x) = w_\lambda(x - q), \quad \lambda = V(\xi).$$

We compute

$$J_\varepsilon(W_q) = J^\lambda(w_\lambda) + \frac{1}{2}\int_{\mathbb{R}^N} \big(V(\xi + \varepsilon x) - V(\xi)\big) w_\lambda^2(x)\, dx$$

where

$$J^\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N} v(-\Delta)^s v + \frac{\lambda}{2}\int_{\mathbb{R}^N} v^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} v^{p+1}.$$

We recall that

$$w_\lambda(x) := \lambda^{\frac{1}{p-1}} w\big(\lambda^{\frac{1}{2s}} x\big)$$

satisfies the equation

$$(-\Delta)^s w_\lambda + \lambda w_\lambda - w_\lambda^p = 0 \quad \text{in } \mathbb{R}^N,$$

where $w = w_1$ is the unique radial least energy solution of

$$(-\Delta)^s w + w - w^p = 0 \quad \text{in } \mathbb{R}^N.$$

Then, after a change of variables we find

$$J^\lambda(w_\lambda) = \frac{1}{2}\int_{\mathbb{R}^N} w_\lambda(-\Delta)^s w_\lambda + \frac{\lambda}{2}\int_{\mathbb{R}^N} w_\lambda^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} w_\lambda^{p+1} = \lambda^{\frac{p+1}{p-1} - \frac{N}{2s}} J^1(w).$$

Now since w is radial, we find

$$\int_{\mathbb{R}^N} x_i w_\lambda(x)\, dx = 0.$$

Thus,

$$\int_{\mathbb{R}^N} \big(V(\xi+\varepsilon x)-V(\xi)\big)w_\lambda^2(x)\,dx = \nabla V(\xi)\cdot \int_{\mathbb{R}^N} x w_\lambda + O\big(\varepsilon^2\big) = O\big(\varepsilon^2\big).$$

On the other hand

$$\begin{aligned}
&\partial_q \int_{\mathbb{R}^N} \big(V(\xi+\varepsilon x)-V(\xi)\big)w_\lambda^2(x)\,dx\\
&\quad = \varepsilon \int_{\mathbb{R}^N} \big(\nabla V(\xi+\varepsilon x)-\nabla V(\xi)\big)w_\lambda^2(x)\,dx + 2\int_{\mathbb{R}^N} \big(V(\xi+\varepsilon x)-V(\xi)\big)w_\lambda \partial_q w_\lambda\,dx\\
&\quad = O\big(\varepsilon^2\big).
\end{aligned}$$

Lemma 6.3. *Let $\theta = \frac{p+1}{p-1} - \frac{N}{2s}$, $c_* = J_1(w)$ and $k=1$. Then the following expansions hold*:

$$I(q) = c_* V^\theta(\xi) + O\big(\varepsilon^{\min(4s,2)}\big),$$
$$\nabla_q I(q) = c_*\varepsilon \nabla_\xi\big(V^\theta\big)(\xi) + O\big(\varepsilon^{\min(4s,2)}\big).$$

For the case $k>1$ and $\min_{i\neq j}|q_i-q_j|\geqslant R\gg 1$, we observe that, also, $\|E\|_Y = O(\tau)$ and hence we also have

$$I(q) = J_\varepsilon(W_q) + O\big(\tau^2\big), \qquad \partial_q I(q) = \partial_q J_\varepsilon(W_q) + O\big(\tau^2\big).$$

By expanding $I(q)$ we get the validity of the following estimate.

Lemma 6.4. *Letting $\xi = \varepsilon q$ we have that*

$$I(q) = c_*\sum_{i=1}^k V^\theta(\xi_i) - \sum_{i\neq j}\frac{c_{ij}}{|q_i-q_j|^{N+2s}} + O\left(\varepsilon^{\min(4s,2)} + \frac{1}{R^{2(N+2s-\mu)}}\right),$$

$$\nabla_q I(q) = c_*\varepsilon\nabla_\xi\left[\sum_{i=1}^k V^\theta(\xi_i) - \sum_{i\neq j}\frac{c_{ij}}{|q_i-q_j|^{N+2s}}\right] + O\left(\varepsilon^{\min(4s,2)} + \frac{1}{R^{2(N+2s-\mu)}}\right)$$

where c_* and $c_{ij} = c_0(V(\xi_i))^\alpha(V(\xi_j))^\beta$ are positive constants.

Proof. It suffices to expand $J_\varepsilon(W_q)$. We see that, denoting $w_i(x) := w_{\lambda_i}(x-q_i)$,

$$J_\varepsilon(W_q) = J_\varepsilon\left(\sum_{i=1}^k w_i\right)$$

$$= \sum_{i=1}^{k} J_\varepsilon(w_i) + \frac{1}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} w_i (-\Delta)^s w_j + \int_{\mathbb{R}^N} V(\varepsilon x) w_i w_j$$

$$- \frac{1}{p+1} \int_{\mathbb{R}^N} \left(\sum_{i=1}^{k} w_i \right)^{p+1} - \sum_{i=1}^{k} w_i^{p+1}. \tag{6.4}$$

We estimate for $i \neq j$,

$$\int_{\mathbb{R}^N} w_i (-\Delta)^s w_j + \int_{\mathbb{R}^N} V(\varepsilon x) w_i w_j = \int_{\mathbb{R}^N} w_i w_j^p + \int_{\mathbb{R}^N} \big(V(\varepsilon x) - \lambda_j\big) w_i w_j$$

$$= \big(c_{ij} + o(1)\big) \frac{1}{|q_i - q_j|^{N+2s}} + O\left(\frac{\varepsilon^{2s}}{R^{N+2s-\mu}} \right) \tag{6.5}$$

where $c_{ij} = c_0 (V(\xi_i))^\alpha ((V(\xi_j))^\beta)$ and c_0, α, β are constants depending on p, s and N only. Indeed,

$$w_i(x) = \lambda_i^{\frac{1}{p-1}} w\big(\lambda_i^{\frac{1}{2s}} (x - q_i)\big)$$

and it is known that

$$w(x) = \frac{c_0}{|x|^{N+2s}} \big(1 + o(1)\big) \quad \text{as } |x| \to \infty.$$

Then, we have

$$\int_{\mathbb{R}^N} w_j^p w_i \;=\; \lambda_i^{\frac{1}{p-1} - \frac{n+2s}{2s}} \lambda_j^{\frac{p}{p-1} - \frac{n}{2s}} \left(\int_{\mathbb{R}^N} w^p \right) \frac{c_0}{|q_i - q_j|^{N+2s}},$$

and hence

$$c_{ij} = c_0 \lambda_i^\alpha \lambda_j^\beta$$

where

$$\lambda_i = V(\xi_i), \qquad \lambda_j = V(\xi_j), \qquad \alpha = \frac{1}{p-1} - \frac{n+2s}{2s}, \qquad \beta = \frac{p}{p-1} - \frac{n}{2s}.$$

To estimate the last term we note that

$$\int_{\mathbb{R}^N} \left(\left(\sum_{i=1}^{k} w_i \right)^{p+1} - \sum_{i=1}^{k} w_i^{p+1} \right)^{p+1}$$

$$= \sum_{j=1}^{k} \int_{\Omega_j} \left(\left(\sum_{i=1}^{k} w_i \right)^{p+1} - \sum_{i=1}^{k} w_i^{p+1} \right)^{p+1}$$

$$= \sum_{j=1}^{k} \sum_{\Omega_j} \left((p+1) w_j^p \left(\sum_{i \neq j} w_i \right) + O\left(w_j^{\min(p-1,1)} \left(\sum_{i \neq j} w_i \right)^2 \right) \right)$$

$$= \sum_{j=1}^{K} \sum_{i \neq j} (p+1) \int_{\mathbb{R}^N} w_j^p w_i + O\left(\frac{1}{R^{2(N+2s-\mu)}} \right)$$

$$= \sum_{j=1}^{K} \sum_{i \neq j} (p+1) \frac{c_{ij} + o(1)}{|q_i - q_j|^{N+2s}} + O\left(\frac{1}{R^{2(N+2s-\mu)}} \right). \tag{6.6}$$

Substituting (6.5) and (6.6) into (6.4) and using the estimate of $J_\varepsilon(w_i)$ in the proof of Lemma 6.3, we have estimated $J_\varepsilon(w_i)$, and we have proven the lemma.

7. The proofs of Theorems 1–3

Based on the asymptotic expansions in Lemma 6.4, we present the proofs of Theorems 1–3.

Proof of Theorems 1 and 2. Let us consider the situation in Remark 1.1, which is more general than that of Theorem 1. Then, in the definition of the configuration space Γ (3.3), we can take a fixed δ and $R \sim \varepsilon^{-1}$ and achieve that $\Lambda \subset \varepsilon\Gamma$. Then we get

$$\|E\|_Y + \|\partial_q E\|_Y = O\left(\varepsilon^{\min\{2s,1\}}\right).$$

Letting

$$\tilde{I}(\xi) := I(\varepsilon q)$$

we need to find a critical point of $\tilde{I}$ inside Λ. By Lemma 6.4, we see then that

$$\tilde{I}(\xi) - c_*\varphi(\xi) = o(1), \qquad \nabla_\xi \tilde{I}(\xi) - c_*\nabla_\xi\varphi(\xi) = o(1),$$

uniformly in $\xi \in \Lambda$ as $\varepsilon \to 0$, where φ is the functional in (1.10). It follows, by the assumption on φ that for all ε sufficiently small there exists a $\xi^\varepsilon \in \Lambda$ such that $\nabla \tilde{I}(\xi^\varepsilon) = 0$, hence Lemma 6.1 applies and the desired result follows.

Theorem 2 follows in the same way. We just observe that because of the C^1-proximity, the same variational characterization of the numbers c, for the functional $\tilde{I}(\xi)$ holds. This means that the critical value predicted in that form is indeed close to c. The proof is complete. □

Proof of Theorem 3. Finally we prove Theorem 3. Following the argument in [18], we choose the following configuration space

$$\Lambda = \left\{ (\xi_1, \dots, \xi_k) \,/\, \xi_j \in \Gamma,\ \min_{i \neq j} |\xi_i - \xi_j| > \varepsilon^{1-\frac{s}{4}} \right\} \tag{7.1}$$

with Γ given by (3.3), and we prove the following Claim and then Theorem 3 follows from Lemma 6.1:

Claim. *Letting $\xi = \varepsilon q$, the problem*

$$\max_{(\xi_1,\dots,\xi_k)\in\Lambda} I(q) \tag{7.2}$$

admits a maximizer $(\xi_1^\varepsilon,\dots,\xi_k^\varepsilon)\in\Lambda$.

We shall prove this by contradiction. First, by continuity of $I(q)$, there is a maximizer $\xi^\varepsilon = (\xi_1^\varepsilon,\dots,\xi_k^\varepsilon)\in\bar{\Lambda}$. We need to prove that $\xi\in\Lambda$. Let us suppose, by contradiction, that $\xi^\varepsilon\notin\Lambda$, hence it lies on its boundary. Thus there are two possibilities: either there is an index i such that $\xi_i^k\in\partial\Gamma$, or there exist indices $i\neq j$ such that

$$\left|\xi_i^\varepsilon-\xi_j^\varepsilon\right| = \min_{i\neq j}|\xi_i-\xi_j| = \epsilon^{1-s}.$$

Denoting $q^\varepsilon = \frac{\xi^\varepsilon}{\varepsilon}$, and using Lemma 6.4, we have in the first case that

$$\begin{aligned} I\left(q^\varepsilon\right) &\leqslant c_* V^\theta\left(\xi_i^\varepsilon\right) + c_*\sum_{j\neq i} V^\theta\left(\xi_j^\varepsilon\right) + C\varepsilon^{2s} \\ &\leqslant c_* k \max_\Gamma V^\theta(x) + c_*\Big(\max_{\partial\Gamma} V^\theta(x) - \max_\Gamma V^\theta(x)\Big) + C\varepsilon^{2s}. \end{aligned} \tag{7.3}$$

In the second case, we invoke again Lemma 6.4 and obtain

$$I\left(q^\varepsilon\right) \leqslant c_* k \max_\Gamma V^\theta(x) - c_2\varepsilon^{\frac{s}{4}} + C\varepsilon^{2s}$$

for some $c_2>0$. On the other hand, we can get an upper bound for $I(q^\varepsilon)$ as follows. Let us choose a point ξ_0 such that $V(\xi_0)=\max_\Gamma V(x)$ and let

$$\xi_j = \xi_0 + \varepsilon^{1-\frac{1}{8}s}(1,0,\dots,0), \quad j=1,\dots,k.$$

It is easy to see that $(\xi_1,\dots,\xi_k)\in\Lambda$. Now, we compute by Lemma 6.4:

$$I\left(q^\varepsilon\right) = \max_\Lambda I(q) \geqslant c_* k \max_\Gamma V^\theta(x) - c_3\varepsilon^{\frac{s}{8}}. \tag{7.4}$$

For ε sufficiently small, a contradiction follows immediately from (7.3)–(7.4). □

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• After completion of this work we have learned about the paper [4] in which the result of Corollary 1.1 is found for $k = 1$ under further constraints in the space dimension N and the values of s and p.

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References

[1] A. Ambrosetti, M. Badiale, S. Cingolani, Semiclassical states of nonlinear Schrödinger equations, Arch. Ration. Mech. Anal. 140 (3) (1997) 285–300.
[2] C.J. Amick, J. Toland, Uniqueness and related analytic properties for the Benjamin–Ono equation—a nonlinear Neumann problem in the plane, Acta Math. 167 (1991) 107–126.
[3] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32 (7–9) (2007) 1245–1260.
[4] G. Chen, Y. Zheng, Concentration phenomenon for fractional nonlinear Schrödinger equations, preprint arXiv:1305.4426.
[5] S. Cingolani, M. Lazzo, Multiple semiclassical standing waves for a class of nonlinear Schrödinger equations, Topol. Methods Nonlinear Anal. 10 (1) (1997) 1–13.
[6] T. D'Aprile, D. Ruiz, Positive and sign-changing clusters around saddle points of the potential for nonlinear elliptic problems, Math. Z. 268 (3–4) (2011) 605–634.
[7] M. del Pino, P. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, Calc. Var. Partial Differential Equations 4 (2) (1996) 121–137.
[8] M. del Pino, P. Felmer, Semi-classical states for nonlinear Schrödinger equations, J. Funct. Anal. 149 (1) (1997) 245–265.
[9] M. del Pino, P. Felmer, Multi-peak bound states of nonlinear Schrödinger equations, Ann. Inst. H. Poincare Anal. Non Lineaire 15 (1998) 127–149.
[10] M. del Pino, P. Felmer, Semi-classical states of nonlinear Schrödinger equations: a variational reduction method, Math. Ann. 324 (2002) 1–32.
[11] M. del Pino, M. Kowalczyk, J. Wei, Concentration on curves for nonlinear Schrödinger equations, Comm. Pure Appl. Math. 60 (1) (2007) 113–146.
[12] P. Felmer, A. Quaas, J. Tan, Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian, Proc. Roy. Soc. Edinburgh Sect. A 142 (6) (2012) 1237–1262.
[13] A. Floer, M. Weinstein, Nonspreading wave packets for the cubic Schrödinger equations with a bounded potential, J. Funct. Anal. 69 (1986) 397–408.
[14] R. Frank, E. Lenzmann, Uniqueness and nondegeneracy of ground states for $(-\Delta)^s Q + Q - Q^{\alpha+1} = 0$ in $\mathbb{R}$, Acta Math. 210 (2) (2013) 261–318.
[15] R. Frank, E. Lenzmann, L. Silvestre, Uniqueness of radial solutions for the fractional Laplacian, preprint arXiv:1302.2652v1.
[16] M. Grossi, On the number of single-peak solutions of the nonlinear Schrödinger equations, Ann. Inst. H. Poincare Anal. Non Lineaire 19 (2002) 261–280.
[17] C. Gui, Existence of multi-bumps solutions for nonlinear Schrödinger equations via variational methods, Comm. Partial Differential Equations 21 (1996) 787–820.
[18] X. Kang, J. Wei, On interacting bumps of semi-classical states of nonlinear Schrödinger equations, Adv. Differential Equations 5 (7–9) (2000) 899–928.
[19] N. Laskin, Fractional quantum mechanics, Phys. Rev. E 62 (2000) 31–35.
[20] N. Laskin, Fractional quantum mechanics and Levy path integrals, Phys. Lett. A 268 (2000) 29–305.
[21] N. Laskin, Fractional Schrödinger equation, Phys. Rev. E 66 (2002) 056–108.
[22] Y.Y. Li, On a singularly perturbed elliptic equation, Adv. Differential Equations 2 (6) (1997) 955–980.
[23] F. Mahmoudi, A. Malchiodi, M. Montenegro, Solutions to the nonlinear Schrödinger equation carrying momentum along a curve, Comm. Pure Appl. Math. 62 (9) (2009) 1155–1264.

[24] Y.-G. Oh, Stability of semiclassical bound states of nonlinear Schrödinger equations with potentials, Comm. Math. Phys. 121 (1) (1989) 11–33.
[25] Y.-G. Oh, On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential, Comm. Math. Phys. 131 (2) (1990) 223–253.
[26] P. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. 43 (2) (1992) 270–291.
[27] S. Secchi, Ground state solutions for nonlinear fractional Schrödinger equations in $\mathbb{R}^N$, J. Math. Phys. 54 (2013) 031501.
[28] X.F. Wang, On concentration of positive bound states of nonlinear Schrödinger equations, Comm. Math. Phys. 153 (2) (1993) 229–244.