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Acyclicity and singleton cores in matching markets

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ABSTRACT

The absence of simultaneous cycles is a sufficient condition for the existence of singleton cores. Acyclicity in the preferences of either side of the market is a minimal condition that guarantees the existence of singleton cores.

1. Introduction

This paper shows that the absence of simultaneous cycles implies that the core is a singleton. Acyclicity in the preferences of either side of the market is a minimal condition that guarantees the set of stable matchings to be a singleton. The notion of acyclical preferences is a generalization of the notion of common preferences. Indeed, if the preferences of one side of the market are acyclical, the unique stable matching can be obtained through a "corrected" serial dictatorship.

The existence of singleton cores is relevant to the matching market literature. Sönmez (1999) shows that there exists an allocation rule that is Pareto efficient, individually rational and strategy-proof if and only if the core is single-valued. Ehlers and Massó (2007) explore the relationship between singleton cores and the existence of equilibrium in centralized matching markets with incomplete information. They show that truth-telling is an ordinal Bayesian Nash equilibrium of the revelation game induced by a common belief and a stable mechanism if and only if all profiles that are in the support of the common belief have singleton cores. Alternative conditions for the singleton core have been presented in the literature. In the marriage problem, Eeckhout (2000) identifies the Sequential Preference Condition (SPC) that is sufficient for the existence of singleton cores.[1]

Acyclicity has been extensively analyzed. Ergin (2002) introduces an alternative notion of acyclicity and shows that the worker-optimal stable matching is efficient if and only if the preferences of the men are acyclical.[2] Haeringer and Klijn (2009) show that Ergin's acyclicity is a necessary and sufficient condition for Nash implementation of the stable correspondence. Kesten (2006) shows that for some fixed priority profiles, the deferred acceptance rule and the top trading cycle rule are equivalent if and only if the preference profile is acyclic. Romero-Medina and Triossi (2012) prove that when the man-optimal stable rule is employed, acyclicity is the minimal condition that guarantees the stability of NE of the capacity manipulation games.

The structure of the paper is as follows. Section 2 presents the model and Section 3 presents its relationship to serial dictatorship.

2. The model

Let us consider a bilateral market with two finite disjoint sets $M = \{m_1, m_2, \ldots, m_k\}$ and $W = \{w_1, w_2, \ldots, w_t\}$, the sets

[1] Examples that the SPC is different from our condition are available upon request.

[2] Ergin's acyclicity is weaker than the concept of acyclicity used in this paper but it is independent of the notion of the absence of simultaneous cycles. Furthermore, Ergin's acyclicity does not guarantee that the set of stable matchings is a singleton.

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of men and women respectively. Here, m's preferences P_m are described by a linear order on $W \cup \{m\}$. Given two women $w_j, w_h \in W$, the expression $w_j P_m w_h$ means that m prefers to be matched to w_j rather than w_h; $mP_m w$ means that m prefers to stay single rather than being matched to w. Similarly, each woman w's preferences P_w are described by a linear order on $M \cup \{w\}$. For every $x \in M \cup W$ let R_x be x's weak preferences and let $A(x)$ be the set of x's acceptable agents which is $A(x) = \{y : yP_x x\}$. Let $P_M = \left(P_{m_1}, \ldots, P_{m_k}\right)$ be a list of men's preferences and let $P_W = \left(P_{w_1}, \ldots, P_{w_t}\right)$ be a list of women's preferences. The marriage problem is fully described by a triplet (M, W, P) where $P = (P_M, P_W)$, and it is called a *matching market*.[3]

A *matching* on (M, W) is a function $\mu : M \cup W \to M \cup W$ such that: (1) $(\mu(m) \notin W \Rightarrow \mu(m) = m)$, and $(\mu(w) \notin M \Rightarrow \mu(w) = w)$, and (2) $\mu(m) = w \Leftrightarrow \mu(w) = m$.

A matching μ is *blocked by an individual* m if and only if $mP_m\mu(m)$. A matching that cannot be blocked by any individual is called *individually rational*. A matching μ is *blocked by a pair* (m, w) if and only if $wP_m\mu(m)$ and $mP_w\mu(w)$. Any individually rational matching for (M, W, P) that cannot be blocked by pairs is said to be*stable* for this market. In this matching market the set of stable matchings coincides with the core. Otherwise, μ is *unstable*. $\Gamma(M, W, P)$ denotes the *set of stable matchings*, which is the set of matchings that are stable in market (M, W, P).

When there is no ambiguity, we use P_M and P_W to denote the following binary relations within the set of matchings: For every μ, ν matchings, let $\mu P_M \nu$ if and only if $\mu(m) R_m \nu(m)$ for all $m \in M$ and $\mu(m) P_m \nu(m)$ for at least one m. Let $\mu P_W \nu$ if and only if $\mu(w) R_w \nu(w)$ for all $w \in W$ and $\mu(w) P_w \nu(w)$ for at least one w.

3. Acyclicity and singleton cores

A cycle in the preferences of the men is a list of men and women "in a circle" in which every listed man prefers the woman on his clockwise side to the woman on his counterclockwise side and finds both acceptable. Formally we have the following.

Definition 1. A *cycle* (*of length* $T + 1$) *in the preferences of the men* is given by $m_0, m_1, \ldots, m_T$ such that $m_t \neq m_{t+1}$ for $t = 0, \ldots, T$ and distinct $w_0, w_1, \ldots, w_T$ such that

1. $w_T P_{m_T} w_{T-1} \cdots w_1 P_{m_1} w_0 P_{m_0} w_T$,
2. for every $t \geq 1$, $w_t \in A(m_t) \cap A(m_{t-1})$.[4]

Assume that a cycle exists. If every w_t is initially assigned to m_{t+1}, every man is willing to exchange his assigned woman with his predecessor. The notion of a cycle in the preference of the women is specular.

A simultaneous cycle arises when there is a list of men and women that are simultaneously a cycle for the preferences of men and women. Formally we have the following.

Definition 2. A *simultaneous cycle* of length $T + 1$ is a set of men $m_0, m_1, \ldots, m_T$ and women $w_0, w_1, \ldots, w_T$ forming a cycle both in the preferences of the men and of the women.[5]

If there are no simultaneous cycles, the set of stable matchings is a singleton.

Proposition 1. *Let* (M, W, P) *be a market without simultaneous cycles. Then, the set of stable matchings of* (M, W, P) *is a singleton.*

Proof. We show that if the set of stable matchings is not a singleton there exists a simultaneous cycle. Assume that the set of stable matchings is not a singleton. Then, there are two stable matchings μ and ν such that $\mu P_M \nu$ and $\nu P_W \mu$. Set $W' = \{w : \nu(w) P_w \mu(w)\} \neq \emptyset$. Let $m_0 \in \mu(W')$, where $\mu(W')$ denotes the set of men matched with the women in W'; then $\mu(m_0) P_{m_0} \nu(m_0)$. Let $w_0 \in \mu(m_0) \setminus \nu(m_0)$, $m_0 \in M' = \mu(W')$. For all $n \geq 0$ set $m_{n+1} = \nu(w_n)$ if $w_n \neq w_t$ for every $t < n$ and $w_t \in W'$. Set $m_{n+1} = m_n$ otherwise. Observe that $m_0 \neq m_1$. Let $w_n = \max_{P_{m_{n-1}}} \mu(m_{n-1}) \setminus (\nu(m_{n-1}) \cup \{w_1, \ldots, w_{n-1}\})$ if $\mu(m_{n-1}) \not\subseteq \nu(m_{n-1}) \cup \{w, \ldots, w_{n-1}\}$ and set $w_{n+1} = w_n$ otherwise. The sequence is stationary because W is finite, and it stops at some $\bar{n} > 1$ such that $m_{\bar{n}} = m_{\bar{n}+1}$. Let l be such that $m_l = m_{\bar{n}}$. Set $j_n = w_{n+l}$ and $r_n = m_{n+l}$ for every $n \leq \bar{n} - l$. The sequence comprises different women and two consecutive distinct men and satisfies $\mu(j_n) = r_n = \nu(j_{n+1})$ for $n \leq \bar{n} - l$, and $\nu(j_l) = r_0$. It follows that (i) $j_0 P_{r_0} j_k P_{r_{k-1}} j_{k-1} \cdots j_2 P_{r_2} j_1 P_{r_1} j_0$ and (ii) $r_0 P_{j_k} r_k P_{j_k} r_{k-1} \cdots P_{j_0} r_0$. Thus, $j_0, \ldots, j_l, r_0, \ldots, r_l$ is a simultaneous cycle. $\square$

From Proposition 1, it follows that the acyclicity in the preferences of either side of the market prevents the formation of simultaneous cycles, guaranteeing that the set of stable matchings is a singleton. From this perspective, the acyclicity condition is a minimal condition that guarantees that the set of stable matchings is a singleton.

Proposition 2. *If there is a cycle in* P_M *(respectively in* P_W*) there exists a profile of preferences,* P_W *(respectively for the men,* P_M*) such that the set* $\Gamma(M, W, P_M, P_W)$ *contains at least two matchings.*

Proof. Assume that there is a cycle in P_M. Let $m_0, \ldots, m_T$ and $w_0, \ldots, w_T$ be defined as they are in Definition 2. Let $M' = M \setminus \{m_0, \ldots, m_T\}$ and let $W' = W \setminus \{w_0, \ldots, w_T\}$. Let $P_{W'}$ be any vector of the preferences for the women in W' such that $A(w') \subset M'$ for all $w' \in W'$. Let $\bar{\mu}$ be any stable matching of $\Gamma(M', W', P_{M'}, P_{W'})$. Let $P_{w_i} : m_i, m_{i+1}$ for $i = 0, \ldots, T-1$ and $P_{w_T} : m_T, m_0$. Let $P_w : w$ if $w \notin \{w_0, \ldots, w_T\}$. Let $P_W = \left(P_{W'}, P_{w_0}, \ldots, P_{w_T}\right)$. Define the matchings μ and ν as follows: $\mu(w_i) = m_i$ and $\nu(w_i) = m_{i+1}$ for $i = 0, \ldots, T-1$, $v(w_T) = m_0$. Let $\mu(w) = \nu(w) = \bar{\mu}(w)$ if $w \in W'$. Both μ and ν are stable in $\Gamma(M, W, P_M, P_W)$, so the set of stable matchings of $\Gamma(M, W, P_M, P_W)$ contains at least two stable matchings, μ and ν.

The proof of the claim that at least two stable matchings exist when there is a cycle in P_W is identical and thus omitted. $\square$

3.1. Acyclicity and serial dictatorship

Next, we attempt to determine the restrictiveness of the acyclicity assumption. To this end, we first consider the case in which every woman (man) is acceptable to all men (women). In this case, the preferences are acyclical if and only if they are the same for every man (woman).

Proposition 3. *Assume that* $wP_m m$ $(mP_w w)$ *for every* $w \in W$ *and for every* $m \in M$*. Then the preferences of the men (women) are acyclical if and only if the men (women) have the same preferences for individual women (men).*

Proof. Assume that the preferences of the men are acyclical and that $wP_m m$ for every $w \in W$ and for every $m \in M$. Then, there is no cycle of length two, which implies that all men have the same preferences because all women are acceptable to every man. Next, we prove that if the men have the same preferences for individual women, then there is no cycle in the

[3] All results of the papers hold in many-to-one matching markets with responsive preferences.

[4] From this point forward, indices are considered modulo $T + 1$.

[5] The definition of a simultaneous cycle is equivalent to the definition of a ring of Definition 3 in Eeckhout (2000).

preference of the men. The proof is by contradiction. Assume that $w_0 P_{m_0} w_T P_{m_T} w_{T-1} \cdots w_1 P_{m_1} w_0$ for some T and some $w_0, \ldots, w_T, m_0, \ldots, m_T$. Because $P_{m_0} = P_{m_T}$, we have $w_0 P_{m_0} w_1$ and $w_1 P_{m_0} w_0$, which yields a contradiction.

The proof of the claim when the preferences of the women are acyclical and $m P_w w$ is identical and is thus omitted. □

The following example shows that the result does not hold when some women are not acceptable to every man.

Example 1. Let $M = \{m_1, m_2\}$, $W = \{w_1, w_2\}$. Let $P_{m_1} : w_1 m_1 w_2$, $P_{m_2} : w_1 w_2 m_2$. We have $w_2 \notin A(m_1)$, P_M is acyclic and $P_{m_1} \neq P_{m_2}$.

Under common preferences, the set of stable matchings can be generated by a serial dictatorship. This result still holds true when the preferences are acyclical. If the preferences of the men are acyclical, then there is an underlying order on the set of the women such that the unique stable matching is generated by a ''corrected'' serial dictatorship: the first woman chooses among the men for whom she is acceptable; woman t chooses from among the men for whom she is acceptable that remain unmatched; and so forth.

Proposition 4. *If the preferences of the men are acyclical, there is an ordering of the women $w_{i_1}, \ldots, w_{i_n}$ such that the unique stable matching μ is given by:*

$$\mu\left(w_{i_1}\right) = \max_{P_{w_{i_1}}} \left\{m \in M : w_{i_t} P_m m\right\}$$

$$\mu\left(w_{i_{t+1}}\right) = \max_{P_{w_{i_{t+1}}}} \{m \in M : w_{i_{t+1}} P_m m\} \setminus \{m \in M : |\{w_{i_s} : s \leq t, \mu\left(w_{i_s}\right) = m\}| = q_m\}$$

for all t, $1 \leq t \leq n-1$.

Proof. Assume that the preferences of the men are acyclical. Let $w_{i_1} \in W$ such that there are no $w \in W$ and $m \in M$ such that $w P_m w_{i_1} P_m m$. Such a w_{i_1} exists because P_M is acyclical. For $0 \leq t \leq n-1$, let $w_{i_{t+1}} \in W$ such that there are no $w \in W \setminus \{w_{i_1}, \ldots, w_{i_t}\}$ and $m \in M$ such that $w P_m w_{i_{t+1}} P_m m$. Such a $w_{i_{t+1}}$ exists because P_M is acyclical. To complete the proof, it suffices to show that the matching μ defined in the claim is stable. The proof is by contradiction. Let i_t be such that $\left(w_{i_t}, m\right)$ blocks the μ. Set $M_t = \left\{m \in M : w_{i_t} P_m m\right\} \setminus \{m \in M : |\{w_{i_s} : s \leq t, \mu(w_{i_s}) = m\}| = q_m\}$. We have $\mu\left(w_{i_t}\right) = \max_{P_{w_{i_t}}} M_t$. First, assume that $|\mu(m)| < q_m$. Then, $m \in M_t$, yields a contradiction. Second, consider the case in which $|\mu(m)| = q_m$. Because $\left(w_{i_t}, m\right)$ blocks μ, $w_{i_t} P_m w$ for some $w \in \mu(m)$. From the definition of the sequence $i_1, \ldots, i_n$, it follows that $w = w_{i_s}$ for some $s > t$. Thus, $m \in M_t$, which yields a contradiction. □

Corollary 1. *Let $\mu(P_M, P_W)$ be a stable matching from the market $\Gamma(M, W, P_M, P_W)$ for all P_M, P_W. Assume that P_M is acyclical. Then, for every $w \in W$ $\mu(P_M, P_W) R_w \mu(P_M, P'_w, P_{-w})$ for all P'_w.*

Analogous results to Proposition 4 and Corollary 1 hold when women's preferences are acyclical.

From Proposition 4, it follows that if the preferences of the men (women) are acyclical, then the unique stable matching is strongly efficient for the women (men). Analogous result holds when the preferences do not have simultaneous cycles.[6]

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References

Eeckhout, J., 2000. On the uniqueness of stable marriage matchings. Economics Letters 69, 1–8.

Ehlers, L., Massó, J., 2007. Incomplete information and singleton cores in matching markets. Journal of Economic Theory 136, 587–600.

Ergin, H., 2002. Efficient resource allocation on the basis of priorities. Econometrica 70, 2489–2497.

Haeringer, G., Klijn, F., 2009. Constrained school choice. Journal of Economic Theory 144, 1921–1947.

Kesten, O., 2006. On two competing mechanism for priority-based allocation problems. Journal of Economic Theory 127, 155–171.

Romero-Medina, A., Triossi, M., 2012. Games with capacity manipulation: incentives and Nash equilibria. Social Choice and Welfare http://dx.doi.org/10.1007/s00355-012-0703-1.

Sönmez, T., 1999. Strategy-proofness and essentially single-valued cores. Econometrica 67 (3), 677–689.

6 The proof is available upon request.