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Continuum Statistics of the Airy_2 Process

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Abstract: We develop an exact determinantal formula for the probability that the Airy_2 process is bounded by a function g on a finite interval. As an application, we provide a direct proof that $\sup(\mathcal{A}_2(x) - x^2)$ is distributed as a GOE random variable. Both the continuum formula and the GOE result have applications in the study of the end point of an unconstrained directed polymer in a disordered environment. We explain Johansson's (Commun. Math. Phys. 242(1–2):277–329, 2003) observation that the GOE result follows from this polymer interpretation and exact results within that field. In a companion paper (Moreno Flores et al. in Commun. Math. Phys. 2012) these continuum statistics are used to compute the distribution of the endpoint of directed polymers.

1. Introduction

The Airy_2 process $\mathcal{A}_2$ was introduced in [PS02] in the study of the scaling limit of a discrete polynuclear growth (PNG) model. It is expected to govern the asymptotic spatial fluctuations in a wide variety of random growth models on a one dimensional substrate with curved initial conditions, and the point-to-point free energies of directed random polymers in 1 + 1 dimensions (the KPZ universality class). It also arises as the scaling limit of the top eigenvalue in Dyson's Brownian motion [Dys62] for the Gaussian Unitary Ensemble (GUE) of random matrix theory (see [AGZ10] for more details).

$\mathcal{A}_2$ is defined through its finite-dimensional distributions, which are given by a Fredholm determinant formula: given $x_0, \dots, x_n \in \mathbb{R}$ and $t_0 < \cdots < t_n$ in $\mathbb{R}$,

$$\mathbb{P}(\mathcal{A}_2(t_0) \le x_0, \dots, \mathcal{A}_2(t_n) \le x_n) = \det(I - \mathrm{f}^{1/2} K_{\mathrm{ext}} \mathrm{f}^{1/2})_{L^2(\{t_0,\dots,t_n\}\times\mathbb{R})}, \quad (1.1)$$

where we have counting measure on $\{t_0, \dots, t_n\}$ and Lebesgue measure on $\mathbb{R}$, f is defined on $\{t_0, \dots, t_n\} \times \mathbb{R}$ by $\mathrm{f}(t_j, x) = \mathbf{1}_{x \in (x_j,\infty)}$, and the *extended Airy kernel* [PS02,FNH99, Mac94] is defined by

$$K_{\mathrm{ext}}(t, \xi; t', \xi') = \begin{cases} \int_0^\infty d\lambda\, e^{-\lambda(t-t')} \operatorname{Ai}(\xi + \lambda) \operatorname{Ai}(\xi' + \lambda), & \text{if } t \ge t' \\ -\int_{-\infty}^0 d\lambda\, e^{-\lambda(t-t')} \operatorname{Ai}(\xi + \lambda) \operatorname{Ai}(\xi' + \lambda), & \text{if } t < t', \end{cases}$$

where $\mathrm{Ai}(\cdot)$ is the Airy function. In particular, the one point distribution of $\mathcal{A}_2$ is given by the Tracy-Widom largest eigenvalue distribution for GUE.

K. Johansson [Joh03] proved the remarkable fact that

Theorem 1. *For every* $m \in \mathbb{R}$,[1]

$$\mathbb{P}\left(\sup_{t\in\mathbb{R}}\left(\mathcal{A}_2(t) - t^2\right) \le m\right) = F_{\mathrm{GOE}}(4^{1/3}m).$$

Here F_{GOE} denotes the Tracy-Widom largest eigenvalue distribution for the Gaussian Orthogonal Ensemble (GOE) [TW96]. It also arises as the one point distribution of the Airy_1 process, which governs the asymptotic spatial fluctuations in one dimensional random growth models with flat initial conditions, and the point-to-line free energies of directed random polymers in 1 + 1 dimensions.

The proof of Theorem 1 in [Joh03] is indirect, using a functional limit theorem for the convergence of the PNG model to the Airy_2 process, together with the connection between the PNG process and a certain last passage percolation model for which [BR01] had proved the connection with GOE. In this article we develop a method to compute continuum probabilities for the Airy_2 process–which is to say, compute the probability that the sample paths of the Airy_2 process lie below a given function on any finite interval. This is then used to provide a direct proof of Theorem 1 starting only from determinantal formulas.

Theorem 1 reflects a universal behaviour seen in a large class of one dimensional systems (the KPZ universality class starting with flat initial conditions) and therefore has attracted quite a bit of interest at the physical level. Much of the recent work is on finite systems of N nonintersecting random walks, the so-called vicious walkers [Fis84]. [Fei09, RS10, RS11] obtain various expressions for the maximum and position of the maximum at the finite N level. [FMS11] uses non-rigorous methods from gauge theory to obtain the GOE distribution in the large N limit, and furthermore connect the problem to Yang-Mills theory.

Our computation of continuum probabilities starts with the following (earlier) variant of (1.1) due to [PS02],

$$\begin{aligned}&\mathbb{P}(\mathcal{A}_2(t_0) \le x_0, \ldots, \mathcal{A}_2(t_n) \le x_n)\\ &\quad = \det\left(I - K_{\mathrm{Ai}} + \bar{P}_{x_0} e^{(t_0-t_1)H}\, \bar{P}_{x_1} e^{(t_1-t_2)H} \cdots \bar{P}_{x_n} e^{(t_n-t_0)H} K_{\mathrm{Ai}}\right),\end{aligned} \tag{1.2}$$

where K_{Ai} is the *Airy kernel*

$$K_{\mathrm{Ai}}(x, y) = \int_{-\infty}^{0} d\lambda\, \mathrm{Ai}(x - \lambda)\, \mathrm{Ai}(y - \lambda),$$

H is the *Airy Hamiltonian* $H = -\partial_x^2 + x$ and $\bar{P}_a$ denotes the projection onto the interval $(-\infty, a]$. Here, and in everything that follows, the determinant means the Fredholm determinant in the Hilbert space $L^2(\mathbb{R})$. The equivalence of (1.1) and (1.2) was derived formally in [PS02] and [PS11]. In fact there are some subtleties, because, for example, it is not apriori obvious that for $s, t > 0$, e^{-sH} can be applied to the image of $\bar{P}_a e^{-tH}$. See [QR12] for a discussion of the technical details.

[1] The factor $4^{1/3}$ corrects a minor mistake in Johansson's statement. See Sect. 2 for a discussion.

Remark 1.1. The shifted Airy functions are the generalized eigenfunctions of the Airy Hamiltonian, as $H\mathrm{Ai}(x-\lambda) = \lambda\mathrm{Ai}(x-\lambda)$. The Airy kernel K_{Ai} is the projection of H onto its negative generalized eigenspace. This is seen by observing that if we define the operator A to be the *Airy transform*, $Af(x) := \int_{-\infty}^{\infty} dz\,\mathrm{Ai}(x-z)f(z)$, then $K_{\mathrm{Ai}} = A\bar{P}_0 A^*$.

Fix $\ell < r$. Given $g \in H^1([\ell,r])$ (i.e. both g and its derivative are in $L^2([\ell,r])$), define an operator $\Theta^g_{[\ell,r]}$ acting on $L^2(\mathbb{R})$ as follows: $\Theta^g_{[\ell,r]}f(\cdot) = u(r,\cdot)$, where $u(r,\cdot)$ is the solution at time r of the boundary value problem

$$\begin{aligned} \partial_t u + Hu &= 0 \quad \text{for } x < g(t),\ t \in (\ell,r),\\ u(\ell,x) &= f(x)\mathbf{1}_{x<g(\ell)},\\ u(t,x) &= 0 \quad \text{for } x \geq g(t). \end{aligned}$$

The fact that this problem makes sense for $g \in H^1([\ell,r])$ is easy to prove and can be seen from the proof of Proposition 3.2 below. By taking a fine mesh in t we obtain a continuum version of (1.2):

Theorem 2.

$$\mathbb{P}(\mathcal{A}_2(t) \leq g(t)\, for\, t \in [\ell,r]) = \det\Big(I - K_{\mathrm{Ai}} + \Theta^g_{[\ell,r]}e^{(r-\ell)H}K_{\mathrm{Ai}}\Big). \tag{1.3}$$

An expression in terms of determinants of solution operators of boundary value problems may not seem very practical. But in fact one can give an explicit expression for the kernel of the operator $\Theta^g_{[\ell,r]}$ in terms of Brownian motion. Let $b(s)$ denote a Brownian motion with diffusion coefficient 2. By the Feynman-Kac formula,

$$u(r,x) = \mathbb{E}_{b(\ell)=x}\Big(f(b(r))e^{-\int_\ell^r b(s)ds}\mathbf{1}_{b(s)\leq g(s) \text{ on } [\ell,r]}\Big).$$

The linear potential is removed by a parabolic shift,

$$\begin{aligned} \Theta^g_{[\ell,r]}f(x) &= \mathbb{E}_{b(\ell)=x}\Big(f(b(r))e^{-\int_\ell^r b(s)ds}\mathbf{1}_{b(s)\leq g(s) \text{ on } [\ell,r]}\Big)\\ &= \mathbb{E}_{b(\ell)=x}\Big(f(b(r))e^{\ell b(\ell)-rb(r)+(r^3-\ell^3)/3+\int_\ell^r s\,db(s)-\int_\ell^r s^2 ds}\mathbf{1}_{b(s)\leq g(s) \text{ on } [\ell,r]}\Big)\\ &= \mathbb{E}_{b(\ell)=x-\ell^2}\Big(f(b(r)+r^2)e^{\ell(b(\ell)+\ell^2)-r(b(r)+r^2)+(r^3-\ell^3)/3}\mathbf{1}_{b(s)+s^2\leq g(s) \text{ on } [\ell,r]}\Big), \end{aligned}$$

where in the second equality we used integration by parts and added and subtracted $(r^3-\ell^3)/3$ and in the third one we used the Cameron-Martin-Girsanov formula. This gives

Theorem 3. *Let $\Theta^g_{[\ell,r]}(x,y)$ denote the integral kernel of $\Theta^g_{[\ell,r]}$. Then*

$$\begin{aligned} \Theta^g_{[\ell,r]}(x,y) = e^{\ell x - ry + (r^3-\ell^3)/3}\,&\frac{e^{-(x-y)^2/4(r-\ell)}}{\sqrt{4\pi(r-\ell)}}\\ &\cdot \mathbb{P}_{\hat{b}(\ell)=x-\ell^2,\hat{b}(r)=y-r^2}\Big(\hat{b}(s) \leq g(s) - s^2 \text{ on } [\ell,r]\Big), \end{aligned} \tag{1.4}$$

where the probability is computed with respect to a Brownian bridge $\hat{b}(s)$ from $x-\ell^2$ at time ℓ to $y-r^2$ at time r and with diffusion coefficient 2.

This gives a formula which can be used in applications. The obvious one is the case $g(t) = t^2 + m$, in which the probability can easily be computed by the reflection principle (method of images). A second one is the computation of the joint distribution of the max and argmax of the Airy$_2$ process minus a parabola, which appears in a companion paper [MQR11]. The simple result in the case $g(t) = t^2 + m$, setting $-\ell = r = L$, is that

$$\Theta_L := \Theta_{[-L,L]}^{g(t)=t^2+m} = \bar{P}_{m+L^2} e^{-2LH} \bar{P}_{m+L^2} - \bar{P}_{m+L^2} R_L \bar{P}_{m+L^2}, \tag{1.5}$$

where R_L is the reflection term

$$R_L(x, y) = \frac{1}{\sqrt{8\pi L}} e^{-(x+y-2m-2L^2)^2/8L-(x+y)L+2L^3/3}. \tag{1.6}$$

The first term in Θ_L has been reexpressed in terms of the Airy Hamiltonian by reversing the use of the Cameron-Martin-Girsanov and Feynman-Kac formulas.

To obtain the $L \to \infty$ asymptotics, decompose Θ_L so as to expose the two limiting terms, as well as a remainder term Ω_L:

$$\Theta_L = e^{-2LH} - R_L + \Omega_L, \tag{1.7}$$

where $\Omega_L = \left(R_L - \bar{P}_{m+L^2} R_L \bar{P}_{m+L^2}\right) - \left(e^{-2LH} - \bar{P}_{m+L^2} e^{-2LH} \bar{P}_{m+L^2}\right)$. In Sect. 5 we will show that

Lemma 1.2. *As L goes to infinity,*

$$\widetilde{\Omega}_L := e^{LH} K_{\mathrm{Ai}} \Omega_L e^{LH} K_{\mathrm{Ai}} \to 0$$

in trace norm.

Referring to (1.3), we have by the cyclic property of determinants and the identity $e^{2LH} K_{\mathrm{Ai}} = (e^{LH} K_{\mathrm{Ai}})^2$ that

$$\mathbb{P}(\mathcal{A}_2(t) \le g(t) \text{ for } t \in [-L, L]) = \det\Big(I - K_{\mathrm{Ai}} + e^{LH} K_{\mathrm{Ai}} \Theta_L e^{LH} K_{\mathrm{Ai}}\Big). \tag{1.8}$$

Since $e^{LH} K_{\mathrm{Ai}} e^{-2LH} e^{LH} K_{\mathrm{Ai}} = K_{\mathrm{Ai}}$ and due to Lemma 1.2, one sees that the key point is the limiting behaviour in L of $e^{LH} K_{\mathrm{Ai}} R_L e^{LH} K_{\mathrm{Ai}}$. Remarkably, it does not depend on L and gives the kernel of F_{GOE}, thus providing a proof of Theorem 1.

Proposition 1.3. *For all* $L > 0$,

$$e^{LH} K_{\mathrm{Ai}} R_L e^{LH} K_{\mathrm{Ai}} = A \bar{P}_0 \hat{R} \bar{P}_0 A^*,$$

where the A is the Airy transform *(see Remark 1.1), and*

$$\hat{R}(\lambda, \tilde{\lambda}) := 2^{-1/3} \operatorname{Ai}(2^{-1/3}(2m - \lambda - \tilde{\lambda})).$$

Furthermore,

$$\det\Big(I - A \bar{P}_0 \hat{R} \bar{P}_0 A^*\Big) = F_{\mathrm{GOE}}(4^{1/3} m). \tag{1.9}$$

The last equality is a version of the determinantal formula for F_{GOE} proved by [FS05], and which essentially goes back to [Sas05]:

$$F_{\mathrm{GOE}}(m) = \det(I - P_0 B_m P_0), \quad \text{where} \quad B_m(x, y) = \mathrm{Ai}(x + y + m).$$

This can be seen as follows. Using the cyclic property of the determinant and the *reflection operator* $\sigma f(x) = f(-x)$ we may rewrite the determinant in (1.9) as

$$\det\Big(I - \bar{P}_0 \hat{R} \bar{P}_0\Big) = \det\Big(I - \sigma \bar{P}_0 \hat{R} \bar{P}_0 \sigma\Big) = \det\Big(I - P_0 \sigma \hat{R} \sigma P_0\Big), \tag{1.10}$$

where we have used that $AA^* = \sigma^2 = I$. On the other hand we have $\sigma \hat{R} \sigma(\lambda, \tilde{\lambda}) = 2^{-1/3}\,\mathrm{Ai}(2^{-1/3}(\lambda + \tilde{\lambda} + 2m))$. Performing the change of variables $\lambda \mapsto 2^{1/3}\lambda$, $\tilde{\lambda} \mapsto 2^{1/3}\tilde{\lambda}$ in the Fredholm determinant shows that the determinants in (1.10) equal $\det(I - P_0 B_{4^{1/3}m} P_0)$.

The rest of the paper is organized as follows. In Sect. 2 we give an overview of the approach of [Joh03] explaining how Theorem 1 can be obtained indirectly using the connection of the Airy$_2$ process with last passage percolation. Sect. 3 contains a brief introduction to relevant ideas of Fredholm determinants and then provides a proof of Theorem 2. Sect. 4 provides a short proof of Proposition 1.3. Finally, Sect. 5 is devoted to the proof of Lemma 1.2, which essentially amounts to asymptotic analysis involving the Airy function.

2. Indirect Derivation of Theorem 1 Through Last Passage Percolation

As we mentioned in the Introduction, [Joh03] presented an indirect proof of Theorem 1 by way of the PNG model. His idea was entirely correct, but in the process of translating between the available results at the time, a factor of $4^{1/3}$ was lost. The purpose of this section is to explain Johansson's approach and account for the missing $4^{1/3}$.

We consider the PNG model (which we define below) with two types of initial conditions (droplet and flat), and show that by coupling them to the same Poisson point process environment we can represent the one-point distribution for the flat case as the maximum of the interface in the droplet case. Asymptotics of this relationship leads to the identity in Theorem 1.

Consider a space-time Poisson point process P of intensity 2. Define a height function above x at time t as

$$h_g(x, t) = \max_{\pi : g \to (x,t)} T(\pi),$$

where g represents a space-time curve $(g(x), x)_{x \in \mathbb{R}}$, π is a Lipschitz 1 function of time (i.e., $|\pi(s) - \pi(s')| \leq |s - s'|$ for all s, s'), $\pi : g \to (x, t)$ means that π starts at a point of the form $(g(x), x)$ and ends at the point (x, t), and $T(\pi)$ represents the sum of the number of Poisson points that π touches. We will specialize this definition to two cases. In the *droplet* geometry (for which we write h^{droplet}) we take $g = |x|$, hence we only consider paths originating along a wedge. As a result the maximal path will always originate at the origin $(0, 0)$. In the *flat* geometry (for which we write h^{flat}) we take $g \equiv 0$, hence we consider Lipschitz paths starting in any spatial location at time 0 and ending at x at time t. This are illustrated in Fig. 1.

Couple P to another Poisson point process $\tilde{P}$ via $\tilde{P}(A) = P(\tau_t A)$, where for any Borel set $A \in \mathbb{R}^2$, $(y, s) \in \tau_t A$ if and only if $(-y, t - s) \in A$ (one should think of this

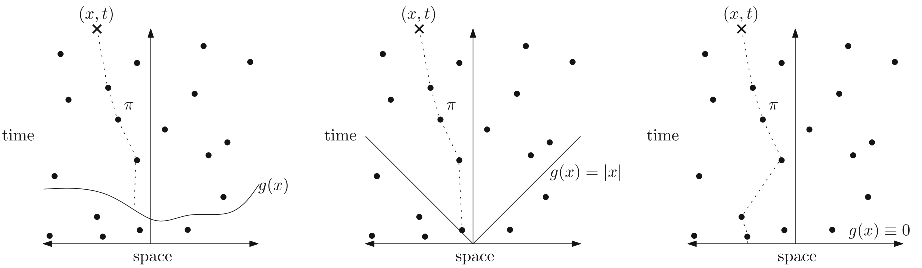

Fig. 1. The maximization problems coupled to the same Poissonian environment. Paths π must be Lipschitz 1 functions of time and $T(\pi)$ represents a count of the number of Poisson points encountered by π. *Left*: A general function $g(x)$ represents the possible starting space-time starting location. *Middle*: The *droplet* geometry in which $g(x) = |x|$. *Right*: The *flat* geometry in which $g(x) \equiv 0$

as a time-reversal of the Poisson point process where $s \mapsto t - s$ and $x \mapsto -x$). Let $\tilde{h}^{\text{flat}}$ represent the flat geometry height function built on the $\tilde{P}$ Poisson point process. Then the following relation holds

$$\tilde{h}^{\text{flat}}(t, 0) = \max_{x \in \mathbb{R}} h^{\text{droplet}}(t, x).$$

Asymptotic fluctuation statistics have been derived for both the droplet and flat geometries and (up to justification of taking the limit inside the maximum, as done in [Joh03] for a related model) the limiting statistics also respect the same relationship above. Specifically [PS02] (see also [BFP08] for the specific choices of scaling used below) shows that

$$\lim_{t \to \infty} \frac{h^{\text{droplet}}(t, t^{2/3}x) - 2t}{t^{1/3}} = \mathcal{A}_2(x) - x^2.$$

This implies that (up to the justifications mentioned above)

$$\lim_{t \to \infty} \frac{\tilde{h}^{\text{flat}}(t, 0) - 2t}{t^{1/3}} = \max_{x \in \mathbb{R}} \left(\mathcal{A}_2(x) - x^2 \right).$$

On the other hand, [BFS08] shows that

$$\lim_{t \to \infty} \frac{\tilde{h}^{\text{flat}}(t, 0) - 2t}{t^{1/3}} = 2^{1/3} \mathcal{A}_1(0),$$

where $\mathcal{A}_1$ is the Airy$_1$ process. Combining these two identities shows that

$$\mathbb{P}\big(\max_{x \in \mathbb{R}} (\mathcal{A}_2(x) - x^2) \le m \big) = \mathbb{P}\big(\mathcal{A}_1(0) \le 2^{-1/3} m \big) = F_{\text{GOE}}(4^{1/3} m),$$

where the last equality follows from work of Ferrari and Spohn [FS05] which shows that $\mathbb{P}(\mathcal{A}_1(0) < m) = F_{\text{GOE}}(2m)$.

3. Proof of Theorem 2

The operator in (1.2) should be seen as a discrete version of the boundary value problem operator $\Theta^{\hat{g}}_{[\ell,r]}$, where $\hat{g}(t) = g(\ell + r - t)$. In particular, for $n > 0$ let $t_i = \ell + i(r - \ell)/n$, $i = 0, \ldots, n$, and define the discrete time boundary value problem operator

$$\Theta^{g}_{n,[\ell,r]} = \bar{P}_{g(t_0)} e^{(t_0 - t_1)H} \bar{P}_{g(t_1)} e^{(t_1 - t_2)H} \cdots e^{(t_{n-1} - t_n)H} \bar{P}_{g(t_n)}.$$

The proof of Theorem 2 amounts to showing that, as n goes to infinity, the discrete operator converges to the limiting operator $\Theta^{\hat{g}}_{[\ell,r]}$ (note the order in which the points $g(t_i)$ appear in (3.1), which explains why we need to introduce $\hat{g}$). This convergence must be in a suitably strong sense to ensure the convergence of the Fredholm determinants. Therefore, before turning to the proof of Theorem 2, let us briefly review some facts about Fredholm determinants, trace class operators and Hilbert-Schmidt operators (see Sect. 2.3 in [ACQ11] for more details, a complete treatment can be found in [Sim05]).

Consider a separable Hilbert space $\mathcal{H}$ and let A be a bounded linear operator acting on $\mathcal{H}$. Let $|A| = \sqrt{A^*A}$ be the unique positive square root of the operator A^*A. The *trace norm* of A is defined as $\|A\|_1 = \sum_{n=1}^{\infty} \langle e_n, |A| e_n \rangle$, where $\{e_n\}_{n\geq 1}$ is any orthonormal basis of $\mathcal{H}$. We say that $A \in \mathcal{B}_1(\mathcal{H})$, the family of *trace class operators*, if $\|A\|_1 < \infty$. For $A \in \mathcal{B}_1(\mathcal{H})$, one can define the trace $\operatorname{tr}(A) = \sum_{n=1}^{\infty} \langle e_n, Ae_n \rangle$ and then the *Hilbert-Schmidt norm* $\|A\|_2 = \sqrt{\operatorname{tr}(|A|^2)}$. We say that $A \in \mathcal{B}_2(\mathcal{H})$, the family *Hilbert-Schmidt operators*, if $\|A\|_2 < \infty$. The following lemma collects some results which we will need in the sequel, they can be found in Chaps. 1–3 of [Sim05]:

Lemma 3.1. *(a) $A \mapsto \det(I + A)$ is a continuous function on $\mathcal{B}_1(\mathcal{H})$. Explicitly,*

$$|\det(I + A) - \det(I + B)| \leq \|A - B\|_1 \exp(\|A\|_1 + \|B\|_1 + 1).$$

(b) If $A \in \mathcal{B}_1(\mathcal{H})$ and $A = BC$ with $B, C \in \mathcal{B}_2(\mathcal{H})$, then $\|A\|_1 \leq \|B\|_2 \|C\|_2$.

(c) If $\|A\|_{\mathrm{op}}$ denotes the operator norm of A in $\mathcal{H}$, then $\|A\|_{\mathrm{op}} \leq \|A\|_2 \leq \|A\|_1$, $\|AB\|_1 \leq \|A\|_{\mathrm{op}} \|B\|_1$ and $\|AB\|_2 \leq \|A\|_{\mathrm{op}} \|B\|_2$.

(d) If $A \in \mathcal{B}_2(\mathcal{H})$, then $\|A^*\|_2 = \|A\|_2$. If A has integral kernel $A(x, y)$, then

$$\|A\|_2 = \left(\int dx\, dy\, |A(x, y)|^2 \right)^{1/2}.$$

The proof of the continuum limit of (1.2) will follow easily from the next proposition.

Proposition 3.2. *Assume $g \in H^1([\ell, r])$ and let $\hat{g}(t) = g(\ell + r - t)$. Then the operators $K_{\mathrm{Ai}} - \Theta^{g}_{n,[\ell,r]} e^{(r-\ell)H} K_{\mathrm{Ai}}$ and $K_{\mathrm{Ai}} - \Theta^{\hat{g}}_{[\ell,r]} e^{(r-\ell)H} K_{\mathrm{Ai}}$ are in $\mathcal{B}_1(L^2(\mathbb{R}))$, with $\|K_{\mathrm{Ai}} - \Theta^{g}_{n,[\ell,r]} e^{(r-\ell)H} K_{\mathrm{Ai}}\|_1$ bounded uniformly in n. Furthermore, for any fixed $\ell < r$ we have, writing $n_k = 2^k$,*

$$\lim_{k\to\infty} \|(K_{\mathrm{Ai}} - \Theta^{g}_{n_k,[\ell,r]} e^{(r-\ell)H} K_{\mathrm{Ai}}) - (K_{\mathrm{Ai}} - \Theta^{\hat{g}}_{[\ell,r]} e^{(r-\ell)H} K_{\mathrm{Ai}})\|_1 = 0. \quad (3.1)$$

The idea of the proof is the following. Just as done in the Introduction for $\Theta^{g}_{[\ell,r]}$, it is possible to use the Feynman-Kac and Cameron-Martin-Girsanov formulas to write a formula for the kernel of $\Theta^{g}_{n_k,[\ell,r]}$ in terms of a path integral with a killing potential enforced only at the dyadic mesh of times $\{t_i\}_{i=1}^{n_k}$ (as opposed to being enforced at all times in $[\ell, r]$). If one considers a parabolic barrier g then the kernel for $\Theta^{g}_{n_k,[\ell,r]}$ is given in terms of the probability of a Brownian bridge exceeding a fixed value at some time $\{t_i\}_{i=1}^{n_k}$. This is compared to the analogous kernel for $\Theta^{\hat{g}}_{[\ell,r]}$ given in terms of the probability of a Brownian bridge exceeding a fixed valued at any time $t \in [\ell, r]$. As the mesh goes to zero, these two probabilities converge and hence so do the kernels. This proves the proposition for parabolic g, and the extension to $g \in H^1([\ell, r])$ then follows readily since H^1 is the Cameron-Martin space for Brownian motion.

Proof of Proposition 3.2. We will first prove the result assuming that $g(s) = (s - \frac{1}{2}(\ell + r))^2$. Let $\varphi(x) = (1+x^2)^{1/2}$ and define the multiplication operator $Mf(x) = \varphi(x)f(x)$ (note that the choice of φ is not particularly important and any strictly positive, polynomially growing function would do). To estimate the trace norm of $K_{\rm Ai} - \Theta^{\hat{g}}_{[\ell,r]} e^{(r-\ell)H} K_{\rm Ai}$ we use Lemma 3.1 to write

$$\big\| K_{\rm Ai} - \Theta^{\hat{g}}_{[\ell,r]} e^{(r-\ell)H} K_{\rm Ai} \big\|_1 \leq \big\| (e^{-(r-\ell)H} - \Theta^{\hat{g}}_{[\ell,r]}) M \big\|_2 \| M^{-1} e^{(r-\ell)H} K_{\rm Ai} \big\|_2. \tag{3.2}$$

For the second Hilbert-Schmidt norm above we have by (4.1) that

$$\begin{aligned} \|M^{-1}e^{(r-\ell)H}K_{\rm Ai}\|_2^2 &= \int_{\mathbb{R}^2} dx\,dy \int_{(-\infty,0]^2} d\lambda\,d\tilde{\lambda}\,\varphi(x)^{-2} e^{(\lambda+\tilde{\lambda})(r-\ell)} \\ &\quad \cdot \operatorname{Ai}(x-\lambda)\operatorname{Ai}(y-\lambda)\cdot\operatorname{Ai}(x-\tilde{\lambda})\operatorname{Ai}(y-\tilde{\lambda}) \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{0} d\lambda\,\varphi(x)^{-2} e^{2\lambda(r-\ell)} \operatorname{Ai}(x-\lambda)^2 \\ &\leq c\,(2(r-\ell))^{-1}\|\varphi^{-1}\|_2^2, \end{aligned} \tag{3.3}$$

where $c = \max_{x\in\mathbb{R}} \operatorname{Ai}(x)^2 < \infty$.

Now we consider the first norm on the right side of (3.2). Shifting time by $-(\ell+r)/2$ in the definition of $\Theta^{\hat{g}}_{[\ell,r]}$ it is clear that this operator equals $\Theta^{\tilde{g}}_{[-L,L]}$, where $L = (r-\ell)/2$ and $\tilde{g}(s) = s^2$. Using the formula for $\Theta^{\tilde{g}}_{[-L,L]}(x,y)$ given in Theorem 3 we get

$$\Theta^{\hat{g}}_{[\ell,r]}(x,y) = \frac{e^{-(x-y)^2/8L-(x+y)L+2L^3/3}}{\sqrt{8\pi L}} \mathbb{P}_{\hat{b}(-L)=x-L^2,\hat{b}(L)=y-L^2}\Big(\hat{b}(s) \leq 0 \text{ on } [-L,L]\Big).$$

Similarly, the kernel of $e^{-(r-\ell)H} = e^{-2LH}$ equals the above one with the probability replaced by 1, and hence

$$\begin{aligned} \big(e^{-(r-\ell)H} - \Theta^{\hat{g}}_{[\ell,r]}\big)M(x,y) &= \frac{e^{-(x-y)^2/8L-(x+y)L+2L^3/3}}{\sqrt{8\pi L}}\varphi(y) \\ &\quad \cdot \mathbb{P}_{\hat{b}(-L)=x-L^2,\hat{b}(L)=y-L^2}\Big(\hat{b}(s) \geq 0 \text{ for some } s \in [-L,L]\Big). \end{aligned} \tag{3.4}$$

Using a known Brownian bridge formula (see for example p. 67 in [BS02]), the latter crossing probability equals $e^{-(x-L^2)(y-L^2)/2L}$ if $x \leq L^2$, $y \leq L^2$ and 1 otherwise, and therefore

$$\begin{aligned} &\|(e^{-(r-\ell)H} - \Theta^{\hat{g}}_{[\ell,r]})M\|_2^2 \\ &\quad = \frac{1}{8\pi L}\int_{\mathbb{R}^2\setminus(-\infty,0]^2} dx\,dy\,\big[e^{-(x-y)^2/8L-(x+y)L-4L^3/3}\big]^2 \varphi(y+L^2)^2 \\ &\qquad + \frac{1}{8\pi L}\int_{(-\infty,0]^2} dx\,dy\,\big[e^{-(x+y)^2/8L-(x+y)L-4L^3/3}\big]^2 \varphi(y+L^2)^2, \end{aligned} \tag{3.5}$$

where we have performed the change of variables $x \mapsto x+L^2$, $y \mapsto y+L^2$. Both Gaussian integrals can be easily seen to be finite, so we have shown that $(e^{-(r-\ell)H} - \Theta^{\hat{g}}_{[\ell,r]})M \in \mathcal{B}_2(L^2(\mathbb{R}))$. Using this with (3.2) and (3.3) it follows that $K_{\rm Ai} - \Theta^{\hat{g}}_{[\ell,r]} e^{(r-\ell)H} K_{\rm Ai}$ is in $\mathcal{B}_1(L^2(\mathbb{R}))$.

Next we observe that we can shift time and apply the Feynman-Kac and Cameron-Martin-Girsanov formulas directly on $\Theta^g_{n,[\ell,r]}$ (n times) exactly as we did for $\Theta^g_{[\ell,r]}$, and it is not hard to check that we get a formula analogous to (3.4):

$$\big(e^{-(r-\ell)H} - \Theta^g_{n,[\ell,r]}\big)M(x,y) = \frac{1}{\sqrt{8\pi L}}e^{-(x-y)^2/8L-(x+y)L+2L^3/3}\varphi(y)$$
$$\cdot \mathbb{P}_{\hat{b}^n(-L)=x-L^2,\hat{b}^n(L)=y-L^2}\Big(\hat{b}^n(t^n_i) \geq 0 \text{ for some } i \in \{0,\dots,n\}\Big),$$

where $\hat{b}^n$ is now a discrete time random walk with Gaussian jumps with mean 0 and variance $2L/n$, started at time $-L$ at $x-L^2$, conditioned to hit $y-L^2$ at time L, and jumping at times $t^n_i = -L+2iL/n$, $i \geq 0$. A simple coupling argument (see the next paragraph) shows that the last probability is less than the corresponding one for the Brownian bridge, and thus we obtain for $\|K_{\mathrm{Ai}} - \Theta^g_{n,[\ell,r]}e^{(r-\ell)H}K_{\mathrm{Ai}}\|_1$ the same bound as the one we get for $\|K_{\mathrm{Ai}} - \Theta^{\hat{g}}_{[\ell,r]}e^{(r-\ell)H}K_{\mathrm{Ai}}\|_1$ from (3.5). This bound is, in particular, independent of n.

Finally, in order to prove (3.1) we couple the Brownian bridge $\hat{b}$ and the conditioned random walk $\hat{b}^{n_k}$ by simply letting $\hat{b}^{n_k}(t^{n_k}_i) = \hat{b}(t^{n_k}_i)$ for each $i = 0,\dots,n_k$. Since the Brownian bridge hits the positive half-line whenever the conditioned random walk does, it is clear that

$$\begin{aligned}&\Big|\big(e^{-(r-\ell)H} - \Theta^{\hat{g}}_{[\ell,r]}\big)M - \big(e^{-(r-\ell)H} - \Theta^g_{n_k,[\ell,r]}\big)M\Big|(x,y)\\ &\quad= \frac{e^{-(x-y)^2/8L-(x+y)L+2L^3/3}}{\sqrt{8\pi L}}\varphi(y)q_{n_k}(x,y),\end{aligned} \tag{3.6}$$

where $q_{n_k}(x,y)$ is the probability that the Brownian bridge $\hat{b}(s)$ hits the positive half-line for $s \in [-L,L]$ but not for any $s \in \{t^{n_k}_0,\dots,t^{n_k}_{2n_k}\}$. Since every point is regular for one-dimensional Brownian motion, $q_{n_k}(x,y) \searrow 0$ as $k \to \infty$ for every fixed x, y, and thus by the monotone convergence theorem we deduce that $\|(e^{-(r-\ell)H} - \Theta^{\hat{g}}_{[\ell,r]})M - (e^{-(r-\ell)H} - \Theta^{\hat{g}}_{n_k,[\ell,r]})M\|_2 \to 0$ as $k \to \infty$. Using (3.3) and a decomposition analogous to (3.2) yields (3.1).

To extend the result to $g \in H^1([\ell,r])$ we note that everything in the above argument deals with properties of a Brownian motion $b(s)$ killed at the positive half-line. In the general case we will have by Theorem 3 a Brownian motion $b(s)$ killed at the boundary $\hat{g}(s)-s^2$ or, equivalently, a process $\tilde{b}(s) = b(s)-\hat{g}(s)+s^2$ killed at the positive half-line. Using the Cameron-Martin-Girsanov Theorem we can rewrite the probabilities for $\tilde{b}(s)$ in terms of probabilities for $b(s)$. Since $\hat{g}(s)$ is a deterministic function in $H^1([\ell,r])$, the Radon-Nikodym derivative of $\tilde{b}(s)$ with respect to $b(s)$ has finite second moment, and thus by using the Cauchy-Schwarz inequality we get the first two statements in the result from the above arguments. The convergence in (3.1) follows as well from the above arguments because it only depends on almost sure properties of the corresponding Brownian motion. □

Proof of Theorem 2. Using the time reversal invariance of the Airy$_2$ process and the notation introduced before Proposition 3.2 we have

$$\begin{aligned}\mathbb{P}\big(\mathcal{A}_2(t_0) \leq g(t_0),\dots,\mathcal{A}_2(t_n) \leq g(t_{n_k})\big) &= \mathbb{P}\big(\mathcal{A}_2(t_0) \leq \hat{g}(t_0),\dots,\mathcal{A}_2(t_n) \leq \hat{g}(t_{n_k})\big)\\ &= \det\Big(I - K_{\mathrm{Ai}} + \Theta^{\hat{g}}_{n_k,[\ell,r]}e^{(r-\ell)H}K_{\mathrm{Ai}}\Big)\end{aligned}$$

where $n_k = 2^k$. Since the Airy$_2$ process has a version with continuous paths (see [Joh03,CH11,QR12]), the probability above converges to $\mathbb{P}(\mathcal{A}_2(t) \le g(t) \text{ for } t \in [\ell, r])$ as $k \to \infty$. The result now follows from Proposition 3.2 and Lemma 3.1, which imply that

$$\lim_{k\to\infty} \det\big(I - K_{\mathrm{Ai}} + \Theta^{\hat{g}}_{n_k,[\ell,r]} e^{(r-\ell)H} K_{\mathrm{Ai}}\big) = \det\big(I - K_{\mathrm{Ai}} + \Theta^{g}_{[\ell,r]} e^{(r-\ell)H} K_{\mathrm{Ai}}\big).$$

□

4. Proof of Proposition 1.3

Since K_{Ai} is the projection onto the negative (generalized) eigenspace of H (see Remark 1.1), we have

$$e^{LH} K_{\mathrm{Ai}}(x, z) = \int_{-\infty}^{0} d\lambda\, e^{\lambda L} \operatorname{Ai}(x - \lambda) \operatorname{Ai}(z - \lambda)\, d\lambda. \tag{4.1}$$

Then, recalling that the Airy transform is given by $Af(x) = \int_{-\infty}^{\infty} d\lambda \operatorname{Ai}(x - \lambda) f(\lambda)$, we can write

$$e^{LH} K_{\mathrm{Ai}} R_L e^{LH} K_{\mathrm{Ai}} = A \bar{P}_0 \hat{R}_L \bar{P}_0 A^*, \tag{4.2}$$

where

$$\hat{R}_L(\lambda, \tilde{\lambda}) = \frac{1}{\sqrt{8\pi L}} \int_{\mathbb{R}^2} d\tilde{z}\, dz\, e^{-(z+\tilde{z}-2m-2L^2)^2/8L-(z+\tilde{z})L+(\lambda+\tilde{\lambda})L+2L^3/3} \cdot \operatorname{Ai}(z - \lambda) \operatorname{Ai}(\tilde{z} - \tilde{\lambda}).$$

Applying the change of variables $2u = z + \tilde{z}$, $2v = z - \tilde{z}$, we get

$$\hat{R}_L(\lambda, \tilde{\lambda}) = \frac{1}{\sqrt{2\pi L}} \int_{\mathbb{R}^2} du dv\, e^{-\frac{(u-m-L^2)^2}{2L}-2uL+(\lambda+\tilde{\lambda})L+\frac{2}{3}L^3} \operatorname{Ai}(u+v-\lambda) \operatorname{Ai}(u-v-\tilde{\lambda}).$$

Using the formula

$$\int_{-\infty}^{\infty} dx \operatorname{Ai}(a + x) \operatorname{Ai}(b - x) = 2^{-1/3} \operatorname{Ai}(2^{-1/3}(a + b))$$

(see, for example, (3.108) in [VS10]), the v integral equals $2^{-1/3} \operatorname{Ai}(2^{-1/3}(2u - \lambda - \tilde{\lambda}))$. Therefore

$$\begin{aligned} \hat{R}_L(\lambda, \tilde{\lambda}) &= \frac{2^{-1/3}}{\sqrt{2\pi L}} \int_{-\infty}^{\infty} du\, e^{-(u-m-L^2)^2/2L-2uL+(\lambda+\tilde{\lambda})L+2L^3/3} \operatorname{Ai}(2^{-1/3}(2u - \lambda - \tilde{\lambda})) \\ &= \frac{2^{-1/3}}{\sqrt{2\pi L}} \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} dt \int_{-\infty}^{\infty} du\, e^{-(u-m-L^2)^2/2L-2uL+(\lambda+\tilde{\lambda})L+2L^3/3+t^3/3-2^{-1/3}t(\lambda+\tilde{\lambda}-2u)}, \end{aligned}$$

where in the second equality we have used the contour integral representation of the Airy function, $\operatorname{Ai}(x) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} dt\, e^{t^3/3-tx}$ with $\Gamma = \{c + \mathrm{i}s : s \in \mathbb{R}\}$ and c any positive real number. The u integral is just a Gaussian integral, and computing it we get

$$\hat{R}_L(\lambda, \tilde{\lambda}) = \frac{2^{-1/3}}{2\pi \mathrm{i}} \int_{\Gamma} dt\, e^{t^3/3+2^{1/3}Lt^2+[4^{1/3}L^2+2^{-1/3}(\lambda+\tilde{\lambda}-2m)]t+2L^3/3+(\lambda+\tilde{\lambda})L}.$$

Now we perform the change of variables $t = s - 2^{1/3}L$ to obtain

$$\hat{R}_L(\lambda, \tilde{\lambda}) = \frac{2^{-1/3}}{2\pi\mathrm{i}} \int_{\Gamma'} ds\, e^{s^3/3 - 2^{-1/3}(2m-\lambda-\tilde{\lambda})s} = 2^{-1/3}\,\mathrm{Ai}(2^{-1/3}(2m - \lambda - \tilde{\lambda}))$$

(here the contour Γ' is simply Γ shifted by $2^{1/3}L$, so the integral still gives an Airy function). Note how all the terms involving L have canceled.

5. Proof of Lemma 1.2

The proof of this result amounts to asymptotic analysis of integrals involving the Airy function. The following well-known estimates for the Airy function (see (10.4.59-60) in [AS64]) go a long way in the proof:

$$|\mathrm{Ai}(x)| \le Ce^{-\frac{2}{3}x^{3/2}} \quad \text{for } x > 0, \qquad |\mathrm{Ai}(x)| \le C \quad \text{for } x \le 0. \tag{5.1}$$

However, at one important point the above bounds for $x \le 0$ will prove insufficient, and it will become necessary to utilize a representation (5.6) for $\mathrm{Ai}(x)$ which splits it into complex oscillations of opposite phase. Then, following standard methods of asymptotics for oscillatory integrals (i.e., shifting real contours up and down to turn oscillations into exponential decay), we will achieve our bounds needed to complete the proof of this lemma.

We will use the following version of Laplace's method, which we state without proof (see, for instance, [Erd56]):

Lemma 5.1. *Let*

$$I(M) = \int_\Omega dx\, f(x) e^{\varphi(x)M},$$

where $\Omega \subseteq \mathbb{R}^n$ is a (possibly unbounded) open polygonal domain and f and φ are smooth functions defined on $\overline{\Omega}$. Assume that the local maxima of φ are attained at a finite subset $\{x_1, \ldots, x_n\}$ of $\overline{\Omega}$. Then there is a constant $C > 0$ such that

$$|I(M)| \le C \sum_{k=1}^{n} M^{-\kappa_i} |f(x_i)| e^{\varphi(x_i)M}$$

for large enough M, where $\kappa_i = (n+1)/2$ if $x_i \in \partial\Omega$ and $\kappa_i = n/2$ if $x_i \in \Omega$.

We write $\widetilde{\Omega}_L = \widetilde{\Omega}_L^1 - \widetilde{\Omega}_L^2$, where

$$\widetilde{\Omega}_L^1 = e^{LH} K_{\mathrm{Ai}} \left(R_L - \bar{P}_{m+L^2} R_L \bar{P}_{m+L^2} \right) e^{LH} K_{\mathrm{Ai}},$$
$$\widetilde{\Omega}_L^2 = e^{LH} K_{\mathrm{Ai}} \left(e^{-2LH} - \bar{P}_{m+L^2} e^{-2LH} \bar{P}_{m+L^2} \right) e^{LH} K_{\mathrm{Ai}}.$$

The proof of Lemma 1.2 is contained in the next two lemmas.

Lemma 5.2.

$$\|\widetilde{\Omega}_L^1\|_1 \xrightarrow[L\to\infty]{} 0.$$

Proof. We proceed as in (4.2) and factorize $\widetilde{\Omega}_L^1$ as

$$\widetilde{\Omega}_L^1 = A \hat{\Omega}_L^1 A^*,$$

where

$$\hat{\Omega}^1_L(\lambda,\tilde{\lambda}) = \frac{1}{\sqrt{8\pi L}}\int_{\widetilde{D}} dz\, d\tilde{z}\, e^{-(z+\tilde{z}-2m-2L^2)^2/8L-(z+\tilde{z})L+2L^3/3+(\lambda+\tilde{\lambda})L} \cdot \operatorname{Ai}(z-\lambda)\operatorname{Ai}(\tilde{z}-\tilde{\lambda})\mathbf{1}_{\lambda,\tilde{\lambda}\le 0}, \tag{5.2}$$

with $\widetilde{D} = \mathbb{R}^2 \setminus (-\infty, m+L^2]^2$. Using the Plancherel formula for the Airy transform $\int f^2 = \int (Af)^2$, we have $\|A\|_{\mathrm{op}} = \|A^*\|_{\mathrm{op}} = 1$, so by Lemma 3.1 it will be enough to show that

$$\|\hat{\Omega}^1_L\|_1 \xrightarrow[L\to\infty]{} 0. \tag{5.3}$$

Performing the change of variables $z = L^2(1-w)+m$, $\tilde{z} = L^2(1-\tilde{w})+m$ in (5.2) the kernel becomes

$$\hat{\Omega}^1_L(\lambda,\tilde{\lambda}) = \frac{L^{7/2}e^{(\lambda+\tilde{\lambda})L-2mL}}{\sqrt{8\pi}}\int_D dw\, d\tilde{w}\, e^{L^3 f(w,\tilde{w})}\operatorname{Ai}(L^2(1-w)+m-\lambda) \cdot \operatorname{Ai}(L^2(1-\tilde{w})+m-\tilde{\lambda})\mathbf{1}_{\lambda,\tilde{\lambda}\le 0}, \tag{5.4}$$

where $D = \mathbb{R}^2 \setminus [0,\infty)^2$ and $f(w,\tilde{w}) = \frac{-(w+\tilde{w})^2}{8} + (w+\tilde{w}) - \frac{4}{3}$.

We split the region D into the union of three disjoint regions of pairs $(w,\tilde{w})$: $D_1 = \{w \le 1, \tilde{w} \le 1\} - \{0 \le w \le 1, 0 \le \tilde{w} \le 1\}$, $D_2 = \{w \le 0, \tilde{w} \ge 1\}$ and $D_2' = \{w \ge 1, \tilde{w} \le 0\}$. By the triangle inequality we can bound $\|\hat{\Omega}^1_L(\lambda,\tilde{\lambda})\|_1$ by the sum of the trace norms of the operators obtained by restricting the integral in (5.4) to each of the regions D_1, D_2 and D_2'. We will write $\hat{\Omega}^1_L\mathbf{1}_{D_1}$ for the operator restricted to D_1, with the analogous notation for the other regions. Notice that, due to the symmetry of our formula, we do not need to bound $\|\hat{\Omega}^1_L\mathbf{1}_{D_2'}\|_1$, as it satisfies the same bound as $\|\hat{\Omega}^1_L\mathbf{1}_{D_2}\|_1$.

Let us focus first on the operator restricted to D_1, which is the simplest case because $1-w \ge 0$ and $1-\tilde{w} \ge 0$. We write $\hat{\Omega}^1_L = G_1G_2G_3$ with

$$G_1(\lambda,w) = e^{(\lambda-m)L}\operatorname{Ai}(L^2(1-w)+m-\lambda)\mathbf{1}_{\lambda\le 0, w\le 1},$$
$$G_2(w,\tilde{w}) = \frac{L^{7/2}}{\sqrt{8\pi}}e^{L^3 f(w,\tilde{w})}\mathbf{1}_{(w,\tilde{w})\in D_1},$$
$$G_3(\tilde{w},\tilde{\lambda}) = e^{(\tilde{\lambda}-m)L}\operatorname{Ai}(L^2(1-\tilde{w})+m-\tilde{\lambda})\mathbf{1}_{\tilde{\lambda}\le 0, \tilde{w}\le 1}$$

and use Lemma 3.1 to estimate $\|\hat{\Omega}^1_L\|_1 \le \|G_1\|_2\|G_2\|_2\|G_3\|_2$. We begin with G_1 and assume first that $w \le 1 + mL^{-2}$, so that using (5.1) have

$$\|G_1\bar{P}_{1+mL^{-2}}\|_2^2 = \int_{-\infty}^0 d\lambda \int_{-\infty}^{\min\{1,1+mL^{-2}\}} dw\, e^{2(\lambda-m)L}\operatorname{Ai}(L^2(1-w)+m-\lambda)^2 \le Ce^{-2mL}\int_0^\infty d\lambda\int_0^\infty dw\, e^{-2\lambda-4/3(L^2w+\lambda)^{3/2}} \le C'e^{-2mL}.$$

Likewise, using the other bound in (5.1) it is easy to get $\|G_1P_{1+mL^{-2}}\|_2^2 \le C|m|e^{-2mL}$. Therefore $\|G_1\|_2 \le Ce^{-mL}$, and of course the same bound works for G_3. On the other hand, f attains its maximum on D_1 at the points $(1,0)$ and $(0,1)$, where its value is $-\frac{11}{24}$. Lemma 5.1 then allows to conclude that

$$\|G_2\|_2^2 = \int_{D_1} dw\, d\tilde{w}\, \frac{L^7}{8\pi} e^{2L^3 f(w,\tilde{w})} \le CL^7 (L^3)^{-3/2} e^{-11L^3/12}.$$

Putting the three bounds together we deduce that

$$\|\hat{\Omega}_L^1 \mathbf{1}_{D_1}\|_1 \le e^{-CL^3} \tag{5.5}$$

for some $C > 0$.

Let us now turn to the trace norm of the operator restricted to D_2 (and hence also to D_2'). This bound is slightly harder owing to the fact that one of the Airy functions is oscillatory (rather than rapidly decaying) in this region. As readily derived from the contour integral representation of the Airy function by deforming the contour and performing a change of variables, $\mathrm{Ai}(\cdot)$ may alternatively be expressed as

$$\mathrm{Ai}(x) = \mathrm{Re}\left[\frac{\sqrt{-x}}{2\pi \mathrm{i}} \int_\Gamma ds\, \exp(\mathrm{i}(-x)^{3/2}(-s + s^3/3))\right],$$

where Γ is the contour $\{s = a + b(a)\mathrm{i} : a > 0\}$ with $b(a) = (a-1)\sqrt{\frac{a+2}{3a}}$. This contour is the steepest descent contour for $f(s) = \mathrm{i}(-s + s^3/3)$ and has the property that $\mathrm{Im}\, f(s) = \mathrm{Im}\, f(s_0) = -2/3$, where $s_0 = 1$ is a critical point of f. Along Γ we can write $f(s) = -2/3\mathrm{i} + g(s)$, where $g(s)$ is real valued, $g(s_0) = 0$ and $g(s)$ decays to $-\infty$ monotonically and quadratically with respect to $|s - s_0|$. Thus we may also write

$$\mathrm{Ai}(x) = \frac{1}{2}(G(-x) + \overline{G(-x)}), \tag{5.6}$$

where

$$G(x) = \exp(-\frac{2}{3}x^{3/2}\mathrm{i})\frac{\sqrt{x}}{2\pi \mathrm{i}} \int_\Gamma ds\, \exp(x^{3/2} g(s)).$$

This expansion of the Airy function is the key to our oscillatory asymptotics.

By applying the change of variables $w = 0 + L^{-3/2}v$ and $\tilde{w} = 4 + L^{-3/2}\tilde{v}$ the integral we wish to bound is given by

$$\begin{aligned}\hat{\Omega}^1(\lambda, \tilde{\lambda}) = {} & \frac{L^{1/2} e^{(\lambda+\tilde{\lambda})L}}{\sqrt{8\pi}} \int_{-\infty}^0 dv\, h_L(v) \\ & \times \int_{-3L^{3/2}}^\infty d\tilde{v}\, e^{-\frac{(v+\tilde{v})^2}{8}} \mathrm{Ai}(-3L^2 - L^{1/2}\tilde{v} + m - \tilde{\lambda}) \mathbf{1}_{\lambda,\tilde{\lambda} \le 0},\end{aligned} \tag{5.7}$$

where

$$h_L(v) = e^{\frac{2}{3}L^3} \mathrm{Ai}(L^2 - L^{1/2}v + m - \lambda).$$

We rewrite this as

$$\hat{\Omega}_L^1 = H_1 H_2 \tag{5.8}$$

with

$$H_1(\lambda, v) = \frac{L^{1/2}}{\sqrt{8\pi}} e^{\lambda L - v^2/16} h_L(v) \mathbf{1}_{\lambda, v \le 0},$$

$$H_2(v, \tilde{\lambda}) = e^{\tilde{\lambda}L - v^2/16} \int_{-3L^{3/2}}^\infty d\tilde{v}\, e^{-\frac{2v\tilde{v}+\tilde{v}^2}{8}} \mathrm{Ai}(-3L^2 - L^{1/2}\tilde{v} + m - \tilde{\lambda}) \mathbf{1}_{v,\tilde{\lambda} \le 0}.$$

We will focus on the last integral in $\tilde{v}$ and prove that it is bounded by e^{-CL^3} for some $C > 0$. As the Airy function is bounded on the real axis, we readily find that due to the Gaussian term, we may cut our integral outside of a region $R_\delta = (-\delta L^{3/2}, \delta L^{3/2})$ by introducing an error of order $e^{-C'L^3}$. Thus we may restrict our attention to R_δ.

Using the expansion given by Eq. (5.6), the $\tilde{v}$ integral may be written as $\frac{1}{2}(I_L + \overline{I_L})$ with

$$I_L = \int_{-\delta L^{3/2}}^{\delta L^{3/2}} d\tilde{v}\, e^{-\frac{2v\tilde{v}+\tilde{v}^2}{8}} G(3L^2 + L^{1/2}\tilde{v} - m + \tilde{\lambda}).$$

We wish to show that $|I_L| = |\overline{I_L}| \leq e^{-CL^3}$. For simplicity we set $v = \tilde{\lambda} = m = 0$, though the argument below does not rely on this assumption and applies equally well for all $\tilde{\lambda} \leq 0$, $v \leq 0$ and $m \neq 0$ as necessary. Under this simplification, and performing a change of variables from $\tilde{v}$ to r by setting

$$L^3 r = (3L^2 + L^{1/2}\tilde{v})^{3/2},$$

we obtain

$$I_L = \frac{2}{3} L^{3/2} \int_{3^{3/2}-\delta'}^{3^{3/2}+\delta'} dr\, r^{-1/3} e^{-\frac{L^3}{8}(r^{2/3}-3)^2} G((L^3 r)^{2/3}).$$

Since we can consider an arbitrary δ before the change of variables, we can likewise consider an arbitrary $\delta' > 0$ for which to bound I_L. Plugging in the expression for G we get

$$I_L = \frac{L^{5/2}}{3\pi} \int_{3^{3/2}-\delta'}^{3^{3/2}+\delta'} dr\, e^{-L^3\left[\frac{(r^{2/3}-3)^2}{8} - \frac{2}{3} r\mathrm{i}\right]} \int_\Gamma \exp(L^3 r g(s)) ds.$$

Observe that this integrand is analytic in r. Thus by Cauchy's theorem, rather than integrating from $3^{3/2} - \delta'$ to $3^{3/2} + \delta'$ along the real axis, we may do so along any other curve between these points. Due to the properties of $g(s)$ along Γ, as long as $\mathrm{Re}(r) > 0$ we have that

$$\left| \int_\Gamma ds \exp(L^3 r g(s)) \right| \leq \int_\Gamma ds \exp(L^3 \,\mathrm{Re}(r) g(s)),$$

which is certainly a bounded function of r for $\mathrm{Re}(r) > 0$. The decay of the integrand is thus controlled by

$$\mathrm{Re}\left(-\left[\frac{1}{8}(r^{2/3} - 3)^2 - \frac{2}{3} r\mathrm{i} \right] \right). \tag{5.9}$$

Informed by this we may deform the r integration contour to the contour $B = B_1 \cup B_2 \cup B_3$, where $B_1 = \{3^{3/2} - \delta' + \mathrm{i}y : y \in [0, \eta]\}$, $B_2 = \{x + \mathrm{i}\eta : x \in [3^{3/2} - \delta', 3^{3/2} + \delta']\}$ and $B_3 = \{3^{3/2} + \delta' + \mathrm{i}y : y \in [0, \eta]\}$. It is an exercise in basic complex analysis to see that one can choose η in such a way that, along the contour B, (5.9) stays bounded below a constant $-C$ for $C > 0$. This implies that the exponential is bounded by e^{-CL^3} along that curve and hence for the entire integral we get $|I_L| \leq e^{-CL^3}$ for some $C > 0$. Going back to the definition of H_2 this implies that

$$\|H_2\|_2^2 = \frac{1}{4} \int_{-(\infty,0]^2} dv\, d\tilde{\lambda}\, e^{2\tilde{\lambda}L - v^2/8} (I_L + \overline{I_L})^2 \leq e^{-CL^3}.$$

Returning to (5.8), it now suffices by Lemma 3.1 to prove that $\|H_1\|_2$ does not grow like e^{CL^3}. This follows readily by integration using (5.1), which implies that

$$\log(h_L(v)) \approx \frac{2}{3}L^3 - \frac{2}{3}(L^2 - L^{1/2}v + \lambda)^{3/2} \approx CL^{3/2}v \tag{5.10}$$

for some fixed $C > 0$. Note how the L^3 terms perfectly cancel. This finishes showing that $\|\hat{\Omega}_L^1 \mathbf{1}_{D_2}\|_1 \leq e^{-CL^3}$. As noted before, we may likewise develop a bound for $\|\hat{\Omega}_L^1 \mathbf{1}_{D_2'}\|_1$. Putting this together with (5.5) gives (5.3), which finishes the proof. □

Lemma 5.3.

$$\|\widetilde{\Omega}_L^2\|_1 \xrightarrow[L\to\infty]{} 0.$$

Proof. The proof of this result is the same as that of the previous lemma. Using the definition of $\widetilde{\Omega}_L^2$ and factorizing as in the above proof we get

$$\widetilde{\Omega}_L^2 = A\hat{\Omega}_L^2 A^*$$

with

$$\hat{\Omega}_L^2(\lambda, \tilde{\lambda}) = \frac{e^{-2mL+(\lambda+\tilde{\lambda})L}}{\sqrt{8\pi L}} \int_D d\tilde{w}\, dw\, e^{-(w-\tilde{w})^2/8L+(w+\tilde{w})L-4L^3/3} \cdot \mathrm{Ai}(-w + L^2 - \lambda + m)\, \mathrm{Ai}(-\tilde{w} + L^2 - \tilde{\lambda} + m)\mathbf{1}_{\lambda,\tilde{\lambda}\leq 0}.$$

Applying the change of variables $w \mapsto L^2 w$ and $\tilde{w} \mapsto L^2\tilde{w}$ the kernel becomes

$$\hat{\Omega}_L^2(\lambda, \tilde{\lambda}) = \frac{L^{7/2}e^{(\lambda+\tilde{\lambda})L/2-2mL}}{\sqrt{8\pi}} \int_D dw\, d\tilde{w}\, e^{L^3\tilde{f}(w,\tilde{w})}\, \mathrm{Ai}(L^2(1-w) + m - \lambda) \cdot \mathrm{Ai}(L^2(1-\tilde{w}) + m - \tilde{\lambda})\mathbf{1}_{\lambda,\tilde{\lambda}\leq 0},$$

where $\tilde{f}(w, \tilde{w}) = \frac{-(w-\tilde{w})^2}{8} + (w + \tilde{w}) - \frac{4}{3}$. Note the similarity with (5.4), the only difference being that in $\tilde{f}$ we have a term $-(w - \tilde{w})^2/8$ instead of $-(w + \tilde{w})^2/8$.

As in the above proof we need to bound $\|\hat{\Omega}_L^2\|_1$, and to that end we split D into the same three regions D_1, D_2 and D_2'. The operator restricted to D_1 is easy to bound, exactly as before. On D_2 (and thus also on D_2') we can repeat the same argument as before. The only diference is that, when we apply the change of variables $w = 0 + L^{-3/2}v$ and $\tilde{w} = 4 + L^{-3/2}\tilde{v}$, the function $h_L(v)$ in the resulting integral in (5.7) is now multiplied by $e^{2vL^{3/2}}$, coming from the difference between $\tilde{f}$ and the function f defined after (5.4). This change does not affect the bound on the $\tilde{v}$ integral (I_L and $\overline{I_L}$ in the above proof). It is straightforward to check that the rest of the proof is not affected either (note in fact that the only place where the definition of $h_L(v)$ is used is (5.10), and the approximation there is still valid). □

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