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High energy rotation type solutions of the forced pendulum equation

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Abstract
In this article we study the existence and asymptotic profiles of high-energy rotation type solutions of the singularly perturbed forced pendulum equation

$$\varepsilon^2 u_\varepsilon'' + \sin u_\varepsilon = \varepsilon^2 \alpha(t) u_\varepsilon \qquad \text{in } (-L, L).$$

We prove that the asymptotic profile of these solutions is described in terms of an energy function which satisfy an integro-differential equation. Also we show that given a suitable energy function E satisfying the integro-differential equation, a family of solutions of the pendulum equation above exists having E as the asymptotic profile, when $\varepsilon \to 0$.

Mathematics Subject Classification: 34D15, 34B15, 35B25

(Some figures may appear in colour only in the online journal)

1. Introduction

In this article we will study the following equation describing the motion of a singularly perturbed pendulum with external forcing

$$\varepsilon^2 u_\varepsilon'' + \sin u_\varepsilon = \varepsilon^2 \alpha(t) u_\varepsilon \qquad \text{in } (-L, L), \tag{1.1}$$

where the function u_ε stands for the angle of the pendulum and the external forcing is of the form $\varepsilon^2 \alpha(t) u_\varepsilon$, which is not 2π-periodic. For the coefficient $\alpha(t) : [-L, L] \to \mathbb{R}$ we assume from now on that it satisfies

$$\alpha(t) \in C^2([-L, L], \mathbb{R}). \tag{1.2}$$

We will say that the solution u_ε has one positive (negative respectively) rotation in the time interval $[t, s]$, if for $t < s$ we have $u_\varepsilon(t) = 2N\pi$, $u_\varepsilon(s) = 2(N+1)\pi$ $(2(N-1)\pi$ respectively), with $N \in \mathbb{Z}$, and $u_\varepsilon(\tau) \notin 2\pi\mathbb{Z}$ for all $\tau \in (t, s)$. The type of solutions that we study rotate N_ε times in the interval $(-L, L)$, that is there exists $t_0 < \cdots < t_{N_\varepsilon} \in (-L, L)$ such that u_ε has one rotation in each interval $[t_{i-1}, t_i]$ for $i = 1, ..., N_\varepsilon$, and $u_\varepsilon(t) \notin 2\pi\mathbb{Z}$ for all $t \in [-L, L] \setminus [t_0, t_{N_\varepsilon}]$.

We are interested in the case where the number of rotations N_ε satisfies $2\pi\varepsilon N_\varepsilon \to R$ as $\varepsilon \to 0$. Since each time that u_ε rotates it increases by 2π (or decreases by 2π) we will be dealing with solutions of (1.1) which are of order of ε^{-1} as $\varepsilon \to 0$. More precisely, the solutions that we construct satisfy the followings bounds

$$u_\varepsilon(t) \in [R_1/\varepsilon, R_2/\varepsilon], \qquad \text{with } 0 < R_1 < R_2 \tag{1.3}$$

for all $t \in [-L, L]$. We observe that as a consequence of (1.3), the forcing term in (1.1) turns out to be $O(\varepsilon)$.

There is a large amount of works concerning the pendulum equation, see for instance the review of Mawhin [6]. In particular, concerning solutions having a prescribed number of rotations, we have the earlier work of Wiggins [8], and the articles of Hastings and McLeod [4, 5], in which solutions having a prescribed number of rotations (clockwise or counterclockwise) of a pendulum equation similar to (1.1) are constructed. It is important to note that in these papers the rotations are prescribed in a sequence of integers which is independent of $\varepsilon > 0$, even though the solutions constructed may have infinitely many rotations in $\mathbb{R}$, in each fixed time interval the number is finite and independent of ε. It may be further noted that the solutions constructed in [4, 5] are close to heteroclinic solutions of the pendulum equation and that the rotation occurs at certain points that depend on the forcing.

The solutions that we construct in this article exhibit clusters, each of which has a number of rotations of the order of $O(\varepsilon^{-1})$. Moreover, as $\varepsilon \to 0$ the set of rotation points $\{t_0, ..., t_{N_\varepsilon}\}$ described above become dense in subintervals of $(-L, L)$, thus clusters accumulate in open sets of $\mathbb{R}$ rather than points.

In order to provide a precise description of our rotation type solutions we introduce the approximate energy function:

$$E_\varepsilon(t) = \frac{1}{2}(\varepsilon u_\varepsilon')^2 - (\cos u_\varepsilon + 1). \tag{1.4}$$

The role of E_ε is made clear once we consider the rescaled version of (1.1). Set $t_\varepsilon \in (-L, L)$ such that $u_\varepsilon(t_\varepsilon) = 2\pi n_\varepsilon$. Except possibly for a subsequence, we have that $\varepsilon u_\varepsilon(t_\varepsilon) \to R_0$ with $R_0 > 0$ and $t_\varepsilon \to t_0 \in (-L, L)$. We will prove in section 2 that

$$v_\varepsilon(s) = u_\varepsilon(t_\varepsilon + \varepsilon s) - 2\pi n_\varepsilon$$

converges to v, a solution of the conservative pendulum equation

$$v'' + \sin v = 0. \tag{1.5}$$

with energy

$$E = \frac{1}{2}(v')^2 - (\cos v + 1), \tag{1.6}$$

with $E = \lim_{\varepsilon\to 0} E_\varepsilon(t_\varepsilon)$ and $v(0) = 0$. The energy characterizes the behaviour of the solutions of (1.5), particularly if $E > 0$ the solution v is of rotation type, if $E \in (-2, 0)$ the solution v is periodic, while if $E = 0$ the solution v is a heteroclinic solution with $v(0) = 0$. It will be shown in section 2, that under our assumptions, up to a subsequence, the function $E_\varepsilon(t)$ converges uniformly to a function $E(t) \geqslant 0$ in $[-L, L]$. The solutions we construct have high energy, that is, $E(t) \not\equiv 0$ and the clustering occurs in the intervals where $E > 0$. Now, the

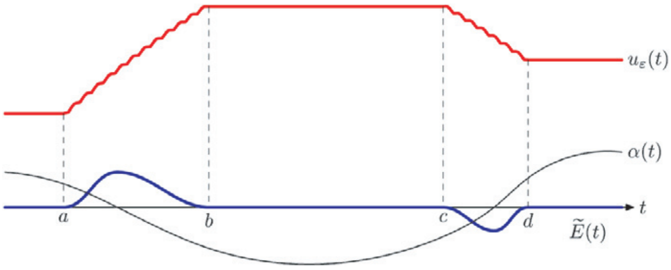

Figure 1. $\tilde{E}$ and clusters.

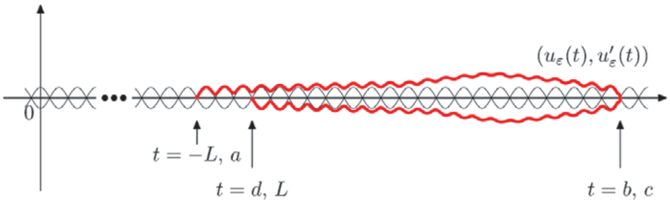

Figure 2. Phase plane.

rotation may be clockwise or counterclockwise in the intervals where $E > 0$, to account for this we define $\tilde{E}$ as E when the rotation is counterclockwise (i.e. u_ε increases), and as $-E$ when it is clockwise (i.e. u_ε decreases). The profile of the solutions that we construct is shown in figure 1.

We observe that, as is shown in figure 2 if $\tilde{E}(a) = \tilde{E}(b) = 0$ and $\tilde{E} > 0$ in (a, b), then after rescaling, the solution is close to an heteroclinic solution of (1.5) close to a and b, while it is close to a rotating solution of (1.5) with energy E in (a, b).

We should stress that although the solutions that we construct rotate clockwise or counterclockwise, they satisfy (1.3) and thus $\varepsilon u_\varepsilon$ is always bounded away from 0 in $[-L, L]$ as $\varepsilon \to \infty$.

In this paper we address two basic issues concerning existence and profiles of the type of clustering solutions described above:

(Q1) If the u_ε has N_ε rotations in $(-L, L)$, with $\varepsilon N_\varepsilon \to R > 0$, and $E_\varepsilon \to E$, what is the equation satisfied by E? Is the limit function E unique?

(Q2) If we have an admissible energy profile, i.e. E is a solution of the equation above in $(-L, L)$, can we find a family $\{u_\varepsilon\}$ of solutions of (1.1) such that $E_\varepsilon \to E$?

These type of questions have been addressed by Felmer, Martínez and Tanaka for the Schrödinger equation [1] and the Allen–Cahn equation [2, 3]. In these works the problem was to show the existence of highly oscillatory solutions for a singularly perturbed equation with time dependent potential. The asymptotic profile of these solutions was characterized in terms of an adiabatic invariant, which was constant in the intervals where the solutions oscillates. The existence of highly oscillatory solutions was also established and it was proved that for any admissible adiabatic profile there exists a family of solutions whose asymptotic behaviour was characterized by it. We should point out that the methods developed in these papers can be applied to construct highly oscillatory solutions of pendulum type equations (for instance with periodic forcing), which have suitable a priori bounds independent of ε. The unboundedness of rotating solutions, and the nature of the forcing considered in (1.1), which depends on the number of rotations measured by $\varepsilon u_\varepsilon$, are strong differences between this problem and those considered before.

The question (Q1) is addressed partially in section 2 (proposition 2.2) in which the following equation is derived by E assuming that the limiting energy E is nonzero (for simplicity, we suppose that all rotations are counterclockwise):

$$E'(t) = (2\pi)^2\alpha(t)\frac{1}{T(E(t))}\left(R + \int_{-L}^{t}\frac{\mathrm{d}s}{T(E(s))}\right) \qquad \text{in} \quad (-L, L). \tag{1.7}$$

In this equation $R = \lim_{\varepsilon\to 0}\varepsilon u_\varepsilon(-L)$ (initial angle) and $T(E)$ denotes the time of one rotation of a solution of (1.5) with energy E (see section 2). We write $1/T(E(t)) = 0$ whenever $E(t) = 0$. The equation (1.7) is nonlocal, due to the fact that the forcing considered depends on the effect of the number of rotations between $-L$ and t measured by the integral term. To understand the integral term, we prove in section 2 that if the family of solutions $\{u_\varepsilon\}$ of (1.1) having energy $E_\varepsilon \to E$ oscillates n_ε times in the interval $(-L, t)$ then

$$2\pi n_\varepsilon \to \int_{-L}^{t}\frac{\mathrm{d}s}{T(E(s))} \qquad \text{as } \varepsilon \to 0.$$

The sign of α is key in the behaviour of the solutions since it gives the sign of E', for instance, if a solution of (1.7) is positive in $(a, b) \subset (-L, L)$ and $E(a) = E(b) = 0$, we must have that $\alpha(a) \geqslant 0$ and $\alpha(b) \leqslant 0$. The key question regarding (1.7) is that if R and $\int_{-L}^{L}\frac{\mathrm{d}t}{T(E(t))}$ are given (that is if the initial angle and the number of rotations are prescribed) do solutions of (1.7) with these values exist and if so are they unique? The existence part of this question is provided in proposition 4.1, and the uniqueness is proved in proposition 4.2 under additional assumptions on α, and assuming that the support of the solution E is a single interval. We should point out that a difficulty in proving such results arises because the right hand side of (1.7) is non Lipchitz when $E = 0$, and that the question is basically a boundary value problem. Also, this equation may have solutions with support consisting of disjoint intervals, hence we do not expect uniqueness in the general case.

To answer (Q2) we need to construct a family of solutions of (1.1) having asymptotic energy E, a solution of (1.7) with $R > 0$ given. We observe that to construct such family we can set up the following boundary conditions

$$u_\varepsilon(-L) = (2n_\varepsilon + 1)\pi \qquad \text{and} \qquad u_\varepsilon(L) = (2(n_\varepsilon + m_\varepsilon) + 1)\pi, \tag{1.8}$$

with $n_\varepsilon, m_\varepsilon \in \mathbb{N}$ and $2\pi n_\varepsilon \to R, 2\pi m_\varepsilon \to \int_{-L}^{L}\frac{\mathrm{d}t}{T(E(t))}$. In order to construct such solutions we use Nehari's *broken geodesic* method, in which the solution is constructed by means of finding a critical point of a functional constructed by glueing basic solutions of (1.1) providing one rotation each. The construction of the basic solutions is shown in section 5, and it relies on the fact that the function $T(E)$ defined above is strictly monotone. The existence of the rotating solutions having limiting energy with connected support is shown in theorem 6.1 in section 6, while in theorem 7.1, in section 7 we show the case when E has support consisting of several disjoint intervals. The critical point of the Nehari functional is found using a degree theoretic approach together with a minimization argument. A key ingredient to prove the existence result is to prove that if $\alpha(a) > 0$ and u_ε rotates close to a then it has to continue rotating, moreover, the support of E has to contain an interval $(a, a + \delta)$ with $\delta > 0$. This property is proved in proposition 3.1 and relies on precise expansions of the solutions.

We should point out that, while in this paper we glue clusters of rotating solutions moving counterclockwise and then clockwise, the solutions constructed are such that $\varepsilon u_\varepsilon$ is bounded away from zero in $[-L, L]$ as $\varepsilon \to 0$. The construction of solutions for which $\varepsilon u_\varepsilon$ may converge to zero in a point of $[-L, L]$ remains open. One of the difficulties is that equation (1.7) has some degeneracy in such a point. Another open problem is the construction of clustering solutions of the forced pendulum equation, in which the solution rotates in one cluster and oscillates in another cluster, corresponding to limiting energy E which is negative in one interval (where it oscillates) and positive in another (where it rotates). We should point out that to do that one needs to consider a more general forcing term, since by remark 2.1 if a solution of (1.1) has negative limiting energy E at one point, it is negative at all points of $[-L, L]$ thus rotation cannot occur. These type of clustering solutions may exist for forcing of the form $\varepsilon^2\alpha(t)u_\varepsilon(t) + \varepsilon^2\beta(t)u_\varepsilon'(t)$.

2. Profile and limit equation

In this section we analyse the asymptotic behaviour of a family u_ε of solutions to (1.1) as $\varepsilon \to 0$. Our purpose is to describe this behaviour by means of an asymptotic energy function obtained as a limit of the family E_ε defined in (1.4). Before starting this analysis, we introduce some preliminaries. Consider $v_0(E; t)$, the solution of (1.5) with initial conditions:

$$v_0(E; 0) = 0, \qquad v_0'(E; 0) > 0$$

and energy E, given by (1.6). For equation (1.5) the energy E can take any value in $[-2, \infty)$, however in this article we will only be concerned with $E \geqslant 0$. When $E \in (0, \infty)$, the solution $v_0(E; t)$ is of rotation type and unbounded, while when $E = 0$, the solution $v_0(0; t)$ is the heteroclinic orbit joining $-\pi$ with π. For $E \in [0, \infty)$, we set

$$T(E) = \begin{cases} \text{the time of one rotation} & \text{if } E > 0,\\ \infty & \text{if } E = 0 \end{cases}$$

and we define the area function $A(E)$ by

$$A(E) = \int_{-T(E)/2}^{T(E)/2} v_0'(E; t)^2 \, \mathrm{d}t.$$

We remark that $A(0) = 8$. Some basic properties of these functions are listed in the following proposition, whose proof can be shown using related arguments found in [1–3].

Proposition 2.1.

(i) $T : (0, \infty) \to (0, \infty)$ *is of class* C^1, $T'(E) < 0$ *in* $(0, \infty)$, $T(E) \to \infty$ *as* $E \to 0$ *and* $T(E) \to 0$ *as* $E \to \infty$.

(ii) $A \in C([0, \infty)) \cup C^1(0, \infty)$, $A'(E) = T(E) > 0$ *for* $E \in (0, \infty)$ *and* $A(E) \to \infty$ *as* $E \to \infty$.

Let us consider a family $\{u_\varepsilon\}$ of solutions satisfying (1.3) with $0 < R_1 < R_2$ independent of ε. Then we have

Lemma 2.1. *If* u_ε *satisfies (1.3), then* $\varepsilon u_\varepsilon(t)$ *and* $E_\varepsilon(t)$ *are bounded in* $W^{1,\infty}(-L, L)$, *independent of* ε, *as* $\varepsilon \to 0$.

Proof. We compute

$$E_\varepsilon'(t) = (\varepsilon^2 u_\varepsilon'' + \sin u_\varepsilon)u_\varepsilon' = \varepsilon^2\alpha(t)u_\varepsilon u_\varepsilon'.$$

Thus by (1.3)

$$| E_\varepsilon'(t) | \leqslant R_2\|\alpha\|_\infty \, | \varepsilon u_\varepsilon' | \leqslant \sqrt{2}R_2\|\alpha\|_\infty\sqrt{E_\varepsilon(t) + 2}. \tag{2.1}$$

Thus for any $s, t \in [-L, L]$

$$\left| \sqrt{E_\varepsilon(t) + 2} - \sqrt{E_\varepsilon(s) + 2} \right| = \left| \int_s^t \frac{E_\varepsilon'(\tau)\,\mathrm{d}\tau}{2\sqrt{E_\varepsilon(\tau) + 2}} \right| \leqslant \sqrt{2}R_2\|\alpha\|_\infty L. \tag{2.2}$$

We assume, for contradiction, that $\|E_\varepsilon\|_\infty \to \infty$. Then by (2.2), we have

$$\inf_{t\in[-L,L]} E_\varepsilon(t) \to \infty \qquad \text{as } \varepsilon \to 0,$$

which implies $\varepsilon u_\varepsilon'(t) \neq 0$ in $[-L, L]$ for ε small and

$$\inf_{t\in[-L,L]} \varepsilon u_\varepsilon'(t) \to \infty \quad \text{or} \quad \sup_{t\in[-L,L]} \varepsilon u_\varepsilon'(t) \to -\infty.$$

Thus we have

$$\varepsilon u_\varepsilon(L) - \varepsilon u_\varepsilon(-L) = \int_{-L}^{L} \varepsilon u_\varepsilon'(\tau)\,\mathrm{d}\tau \to \pm\infty,$$

which is in contradiction with (1.3). Thus $\|E_\varepsilon\|_\infty$ is bounded and then $\|\varepsilon u_\varepsilon'\|_\infty$ is bounded and, by (2.1), $\|E_\varepsilon'\|_\infty$ is also bounded. □

By lemma 2.1, there is a subsequence, that we keep calling ε, such that

$$E_\varepsilon(t) \to E(t) \quad \text{and} \quad \varepsilon u_\varepsilon(t) \to 2\pi R(t) \quad \text{weakly* in } W^{1,\infty}(-L, L).$$

Now we remark that to each connected component of $\{t \,/\, E(t) > 0\}$, we can associate + or − as follows. Let $\delta > 0$ be an arbitrary constant and let I_δ be a connected component of $\{t \,/\, E(t) > \delta\}$. Since $E_\varepsilon(t)$ converges uniformly to $E(t)$ in $[-L, L]$, we can see that

$$E_\varepsilon(t) \geqslant \frac{1}{2}\delta \qquad \text{in } I_\delta$$

for ε small. It implies that $(\varepsilon u_\varepsilon'(t))^2 = 2(E_\varepsilon(t) + \cos u_\varepsilon + 1) \geqslant \delta$ in I_δ. In particular, $\varepsilon u_\varepsilon'(t)$ has constant sign in I_δ. Taking a subsequence again if necessary, we can assume that sign $u_\varepsilon'(t)$ is independent of ε and we can associate sign $u_\varepsilon'(t)$ as a sign of I_δ. For a connected component I of $\{t \,/\, E(t) > 0\}$, we can associate a sign as a limit as $\delta \to 0$.

For the purposes of this article we only consider the case $E(t) \geqslant 0$ for all $t \in (-L, L)$ (see remark 2.1 about the case when $E(t)$ takes negative values). It will be convenient to define $\tilde{E}(t) : [-L, L] \to \mathbb{R}$ by

$$\tilde{E}(t) = \begin{cases} E(t) & \text{if } E(t) > 0 \text{ has associated sign '+'}, \\ -E(t) & \text{if } E(t) > 0 \text{ has associated sign '−'}, \end{cases}$$

and extend T as an even function of E. Now we obtain an equation for $\tilde{E}$

Proposition 2.2. *Assume* $E(t) \geqslant 0$, *then*

$$\tilde{E}'(t) = (2\pi)^2 \alpha(t) \frac{R(t)}{T(\tilde{E}(t))} \qquad \text{in} \quad (-L, L).$$

That is,

$$\tilde{E}'(t) = (2\pi)^2 \alpha(t) \frac{1}{T(\tilde{E}(t))} \left(R(-L) + \int_{-L}^{t} \frac{\operatorname{sign}(\tilde{E}(s))}{T(\tilde{E}(s))}\,\mathrm{d}s \right) \qquad \text{in} \quad (-L, L),$$

where we write $1/T(\tilde{E}(t)) = 0$, *whenever* $\tilde{E}(t) = 0$.

Proof. Let us assume $[a, b] \subset supp E$ and that it has associated sign +. Using ideas from [3] we first show that

$$R(b) - R(a) = \int_a^b \frac{1}{T(E(t))}\,\mathrm{d}t. \tag{2.3}$$

Since $\varepsilon u_\varepsilon(b) - \varepsilon u_\varepsilon(a) = \int_a^b \varepsilon u'_\varepsilon(t)\,\mathrm{d}t$ and $\varepsilon u'_\varepsilon(t)$ is bounded in $L^\infty(-L, L)$, (2.3) will follow if for all $\varphi(t) \in C_0^\infty(a, b)$ we have

$$\lim_{\varepsilon\to 0} \int_a^b \varepsilon u'_\varepsilon(t)\varphi(t)\,\mathrm{d}t = 2\pi \int_a^b \frac{\varphi(t)}{T(E(t))}\,\mathrm{d}t. \tag{2.4}$$

For $s > 0$, we consider the function $\rho_s(t) = 1/(2s)$ if $t \in [-s, s]$ and 0 otherwise, then for any $r > 0$ we have $\|\rho_{\varepsilon r} * \varphi - \varphi\|_{L^\infty(a,b)} \to 0$ as $\varepsilon \to 0$. Thus, we have

$$\lim_{\varepsilon\to 0} \int_a^b \varepsilon u'_\varepsilon(t)\varphi(t)\,\mathrm{d}t = \lim_{\varepsilon\to 0} \int_a^b \varepsilon u'_\varepsilon(t)(\rho_{\varepsilon r} * \varphi)(t)\,\mathrm{d}t = \lim_{\varepsilon\to 0} \int_a^b (\rho_{\varepsilon r} * \varepsilon u'_\varepsilon)(t)\varphi(t)\,\mathrm{d}t.$$

Defining $w_\varepsilon(\tau) = u_\varepsilon(\varepsilon\tau + t)$ we have

$$(\rho_{\varepsilon r} * \varepsilon u'_\varepsilon)(t) = \frac{1}{2\varepsilon r} \int_{-\varepsilon r}^{\varepsilon r} \varepsilon u'_\varepsilon(t+s)\,\mathrm{d}s = \frac{1}{2r} \int_{-r}^{r} w'_\varepsilon(\tau)\,\mathrm{d}\tau$$

and we easily see that $v_\varepsilon(\tau) \equiv w_\varepsilon(\tau) - 2\pi[\frac{w_\varepsilon(0)}{2\pi}]$ converges to $v(\tau) = v_0(E(t); \tau + \ell_t)$ for some $\ell_t \in \mathbb{R}$. Since we also have $w'_\varepsilon(\tau) = v'_\varepsilon(\tau) \to v'_0(E(t); \tau + \ell_t)$ uniformly in $[-r, r]$, we obtain that

$$\begin{aligned}\lim_{\varepsilon\to 0}(\rho_{\varepsilon r} * \varepsilon u'_\varepsilon)(t) &= \frac{1}{2r} \int_{-r}^{r} v'_0(E(t); \tau + \ell_t)\,\mathrm{d}\tau \\ &= \frac{1}{2r}\Big(v_0(E(t); r + \ell_t) - v_0(E(t); -r + \ell_t)\Big).\end{aligned}$$

Since $\varepsilon u'_\varepsilon$ is bounded in $L^\infty(-L, L)$, by the Lebesgue's dominant convergence theorem we have

$$\lim_{\varepsilon\to 0} \int_a^b (\rho_{\varepsilon r} * \varepsilon u'_\varepsilon)(t)\varphi(t)\,\mathrm{d}t = \int_a^b \frac{v_0(E(t); r + \ell_t) - v_0(E(t); -r + \ell_t)}{2r}\varphi(t)\,\mathrm{d}t.$$

We note that for any $M > 0$

$$\lim_{r\to\infty} \frac{1}{2r}\Big(v_0(E; r + \ell) - v_0(E; -r + \ell)\Big) = \frac{2\pi}{T(E)},$$

uniformly in $\ell \in \mathbb{R}$ and $E \in [0, M]$, completing the proof of (2.4). Then, it follows from $E'_\varepsilon(t) = \frac{\alpha(t)}{2}\left\{(\varepsilon u_\varepsilon)^2\right\}'$ and (2.3) that

$$E' = (2\pi)^2\alpha(t)R'(t)R(t) = (2\pi)^2\alpha(t)\frac{R(t)}{T(E(t))}. \tag{2.5}$$

When the sign associated to the interval is '$-$', the equation above changes just in the sign in front of E'. Moreover, if the interval $[a, b] \cap \operatorname{supp} E = \emptyset$ then the equation trivially holds. Putting these facts together, we conclude the proof. $\square$

Remark 2.1. Suppose $E(t_0) < 0$ for some $t_0 \in (-L, L)$. Then, in a neighbourhood of t_0, u_ε is bounded and then following the ideas of the proof of proposition 2.2 we can prove that $R' = E' = 0$ there. By a continuation argument, we can prove then that $E(t)$ and $R(t)$ are constant on $[-L, L]$.

Corollary 2.1. *Suppose $E(t) \geqslant 0$ and set $A(t) = A(E(t))$. Then, in each connected component (a, b) of $\{t \,/\, E(t) > 0\}$ we have*

$$A'(t) = \pm(2\pi)^2\alpha(t)R(t) \qquad \text{in } (a, b),$$

where $\pm$ depends on the sign associated to (a, b). That is,

$$A'(t) = \pm(2\pi)^2\alpha(t)\left(R(a) \pm \int_a^t \frac{1}{T(E(s))}\,\mathrm{d}s\right) \qquad \text{in } (a, b).$$

Proof. Recall that, if $E > 0$, then $A(E)$ is differentiable and $\frac{\mathrm{d}A}{\mathrm{d}E} = T(E)$. Thus, for t with $E(t) > 0$, using (2.5) with proper sign we have

$$A'(t) = \frac{\mathrm{d}A}{\mathrm{d}E}(E(t))E'(t) = \pm(2\pi)^2\alpha(t)R(t).$$

□

3. Non-existence of growing solutions when $\alpha(t) > 0$ and $\lim_{\varepsilon\to 0} E_\varepsilon(t) = 0$

This section is devoted to further analysing the behaviour of a sequence of solution u_ε of equation (1.1) as $\varepsilon \to 0$. We are interested in proving the impossibility of having a sequence of increasing solutions, with vanishing energy, in an interval $[a, b] \subset [-L, L]$ where

$$\alpha(t) \geqslant \underline{\alpha} > 0 \qquad \text{in } [a, b]. \tag{3.1}$$

This property will be important in section 6 in the computation of the degree for existence theory. In precise terms we will prove

Proposition 3.1. *Assume (3.1) and suppose that a family of solutions $u_\varepsilon(t)$ of (1.1) in $(-L, L)$ satisfies for $t \in (a, b)$*

$$u_\varepsilon'(t) > 0 \quad \text{if} \qquad u_\varepsilon(t) \in 2\pi\mathbb{Z}, \tag{3.2}$$

$$\max_{t\in[a,b]} |E_\varepsilon(t)| \to 0 \qquad \text{as} \quad \varepsilon \to 0. \tag{3.3}$$

Then for any $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that for $\varepsilon \in (0, \varepsilon_\delta]$

$$u_\varepsilon(t) \notin 2\pi\mathbb{Z} \qquad \text{in } (a, b-\delta]. \tag{3.4}$$

The proof of this proposition requires one to understand the behaviour of solutions with low energy. Roughly speaking, these solutions should have values near odd multiples of π for long periods of time. Thus, in section 3.1 we analyse the behaviour of solutions having values near odd multiples of π, obtaining good estimates on the derivative as in corollary 3.1. On the other hand, the main purpose of section 3.2 is to estimate energy changes of growing solutions having low energy. Finally in section 3.3, we provide a proof of proposition 3.1. Here we use the estimates just obtained in the previous subsections in order to compare the energy changes, which are assumed to be small, with the changes in the value of the function due to rotation.

3.1. Analysis of solutions near $(2n+1)\pi$

We rescale the equation defining $v_\varepsilon(t) = u_\varepsilon(\varepsilon t)$, for $t \in (-L/\varepsilon, L/\varepsilon)$. Thus, if we consider

$$V^\varepsilon(t, v) = -\cos v - 1 - \frac{\varepsilon^2}{2}\alpha(\varepsilon t)v^2,$$

the equation for v_ε becomes

$$v_\varepsilon'' + V_v^\varepsilon(t, v) = 0 \qquad \text{in } (-L/\varepsilon, L/\varepsilon). \tag{3.5}$$

Suppose $n \in \mathbb{N}$ satisfies

$$(2n+1)\pi \in [R_1/\varepsilon, R_2/\varepsilon] \tag{3.6}$$

then, for each $t \in [-L/\varepsilon, L/\varepsilon]$ the equation $V_v^\varepsilon(t, v) = 0$ has a unique solution $\xi_{\varepsilon n}(t)$ in a neighbourhood of $(2n+1)\pi$ for ε small. We will see later that the function $\xi_{\varepsilon n}(t)$ is key to obtaining precise a priori estimates for v_ε when it takes values close to $(2n+1)\pi$.

We have the following on basic properties of $\xi_{\varepsilon n}(t)$.

Lemma 3.1. *The function $\xi_{\varepsilon n}(t)$ is of class C^2 and there are constants C_1 and C_2 such that*

$$\|\xi_{\varepsilon n}(t) - (2n+1)\pi\|_{L^\infty} + \varepsilon^{-1}\|\xi'_{\varepsilon n}(t)\|_{L^\infty} + \varepsilon^{-2}\|\xi''_{\varepsilon n}(t)\|_{L^\infty} \leqslant C_1\varepsilon \tag{3.7}$$

and

$$\|\xi'_{\varepsilon n}(t)\|_{L^2} \leqslant C_2\varepsilon^{3/2}. \tag{3.8}$$

In what follows we set $L^\infty = L^\infty(-L/\varepsilon, L/\varepsilon)$ and $L^2 = L^2(-L/\varepsilon, L/\varepsilon)$.

Proof. We write $\xi_{\varepsilon n}(t) = (2n+1)\pi + h(t)$, so that $h(t)$ satisfies

$$-\sin h(t) = \varepsilon^2\alpha(\varepsilon t)((2n+1)\pi + h(t)). \tag{3.9}$$

By the assumption (3.6) we see that

$$\sin h(t) + \varepsilon^2\alpha(\varepsilon t)h(t) \in [-\varepsilon R_2\alpha(\varepsilon t), -\varepsilon R_1\alpha(\varepsilon t)], \tag{3.10}$$

from which we can deduce the existence of $C_3, C_4 > 0$ such that

$$-C_3\varepsilon\alpha(\varepsilon t) \leqslant \xi_{\varepsilon n}(t) - (2n+1)\pi \leqslant -C_4\varepsilon\alpha(\varepsilon t), \tag{3.11}$$

and the first term in (3.7). Differentiating (3.9) with respect to t, we have

$$-\cos h(t)\cdot h'(t) = \varepsilon^3\alpha'(\varepsilon t)((2n+1)\pi + h(t)) + \varepsilon^2\alpha(\varepsilon t)h'(t),$$

and

$$\begin{aligned}-\cos h(t)\cdot h''(t) + \sin h(t)\cdot h'(t)^2 &= \varepsilon^4\alpha''(\varepsilon t)((2n+1)\pi + h(t))\\ &\quad + 2\varepsilon^3\alpha'(\varepsilon t)h'(t) + \varepsilon^2\alpha(\varepsilon t)h''(t).\end{aligned}$$

From here, (3.9), (3.10) and the assumptions on α, we obtain $C > 0$ and $C' >$ such that

$$|\ h'(t)\ | \leqslant C\varepsilon^2 \tag{3.12}$$

$$|\ h''(t)\ | \leqslant C(\varepsilon^3 + \varepsilon^3\ |\ h'(t)\ | + |\ \sin h(t)\ ||\ h'(t)\ |^2) \leqslant C'\varepsilon^3. \tag{3.13}$$

Inequality (3.8) follows directly from (3.12) and the second and third term in (3.7) follow from (3.12) and (3.13). □

Remark 3.1. $\xi_{\varepsilon n}(t)$ is a better approximation of a solution of (3.5) near $(2n+1)\pi$ than $(2n+1)\pi$ itself, in the following variational sense: setting

$$I_\varepsilon(v) = \int_{-L/\varepsilon}^{L/\varepsilon} \frac{1}{2}\ |\ v'\ |^2 - (V^\varepsilon(t, v) - V^\varepsilon(t, \xi_{\varepsilon n}(t)))\,\mathrm{d}t$$

we see that

$$I_\varepsilon(\xi_{\varepsilon n}(t)) = \frac{1}{2}\|\xi'_{\varepsilon n}(t)\|^2_{L^2(-L/\varepsilon, L/\varepsilon)} \leqslant \frac{1}{2}C_2^2\varepsilon^3$$

and in view of (3.11) we see that

$$\begin{aligned}I_\varepsilon((2n+1)\pi) &= \int_{-L/\varepsilon}^{L/\varepsilon} -V^\varepsilon(t, (2n+1)\pi) + V^\varepsilon(t, \xi_{\varepsilon n}(t))\,\mathrm{d}t\\ &\geqslant C\int_{-L/\varepsilon}^{L/\varepsilon} |\ (2n+1)\pi - \xi_{\varepsilon n}(t)\ |^2\ \mathrm{d}t \geqslant C'\varepsilon.\end{aligned}$$

Continuing with our analysis, we recall the definition of $V^\varepsilon(t,v)$ to get

$$V^\varepsilon_v(t,v) = \sin v - \varepsilon^2\alpha(\varepsilon t)v \qquad \text{and} \qquad V^\varepsilon_{vv}(t,v) = \cos v - \varepsilon^2\alpha(\varepsilon t).$$

Thus, since $V^\varepsilon_v(t,\xi_{\varepsilon n}(t)) = 0$, for any $\nu_0 > 0$ there exists $\delta_0 > 0$ independent of n, ε such that for ε small

$$V^\varepsilon_{vv}(t,v) \in [-1-\nu_0, -1+\nu_0] \tag{3.14}$$

for all $v \in [(2n+1)\pi - 2\delta_0, (2n+1)\pi + 2\delta_0]$, $t \in [-L/\varepsilon, L/\varepsilon]$. And from here

$$|V^\varepsilon_v(t,v) + (v - \xi_{\varepsilon n}(t))| \leqslant \nu_0|v - \xi_{\varepsilon n}(t)| \tag{3.15}$$

for $v \in [(2n+1)\pi - 2\delta_0, (2n+1)\pi + 2\delta_0]$.

In accordance with remark 3.1, next we prove that solutions having values near odd multiples of π are much better approximated by the function $\xi_{\varepsilon n}$. Precisely we have

Proposition 3.2. *There exist a_1, $a_2 > 0$ independent of $\varepsilon > 0$ such that any solution $v_\varepsilon(t)$ of (3.5) in $(s_\varepsilon, t_\varepsilon)$ satisfying*

$$|v_\varepsilon(t) - (2n+1)\pi| \leqslant \delta_0 \qquad \text{for } t \in [s_\varepsilon, t_\varepsilon] \tag{3.16}$$

has the property

$$|\, v_\varepsilon(t) - \xi_{\varepsilon n}(t) \,| \leqslant a_1\varepsilon^3 + a_2 \mathrm{e}^{-\sqrt{1-\nu_0}\min\{|t-s_\varepsilon|,|t-t_\varepsilon|\}}. \tag{3.17}$$

Proof. We set $z_\varepsilon(t) = v_\varepsilon(t) - \xi_{\varepsilon n}(t)$. Then $z_\varepsilon(t)$ satisfies

$$\begin{aligned} &z''_\varepsilon + V^\varepsilon_v(t, z_\varepsilon + \xi_{\varepsilon n}) = -\xi''_{\varepsilon n}(t) \qquad \text{in } (s_\varepsilon, t_\varepsilon), \\ &|\, z_\varepsilon(s_\varepsilon) \,|, \ |\, z_\varepsilon(t_\varepsilon) \,| \leqslant \frac{3}{2}\delta_0. \end{aligned} \tag{3.18}$$

We use the solution of the equation

$$\begin{aligned} &w''_\varepsilon - (1-\nu_0)w_\varepsilon = -\,|\, \xi''_{\varepsilon n}(t) \,| \qquad \text{in } (s_\varepsilon, t_\varepsilon), \\ &w_\varepsilon(s_\varepsilon) = w_\varepsilon(t_\varepsilon) = \frac{3}{2}\delta_0. \end{aligned} \tag{3.19}$$

for a comparison argument. Given such a solution, we claim that

$$z_\varepsilon(t) \leqslant w_\varepsilon(t) \qquad \text{for all } t \in [s_\varepsilon, t_\varepsilon]. \tag{3.20}$$

In fact, assuming the contrary, there exists $t_0 \in (s_\varepsilon, t_\varepsilon)$ such that

$$\begin{aligned} &z_\varepsilon(t_0) - w_\varepsilon(t_0) = \max_{t\in[s_\varepsilon,t_\varepsilon]}(z_\varepsilon(t) - w_\varepsilon(t)) > 0 \qquad \text{and} \\ &z''_\varepsilon(t_0) - w''_\varepsilon(t_0) \leqslant 0. \end{aligned} \tag{3.21}$$

On the other hand, by (3.18)–(3.19), we have

$$\begin{aligned} z''_\varepsilon - w''_\varepsilon &= -V^\varepsilon_v(t, z_\varepsilon + \xi_{\varepsilon n}) - (1-\nu_0)w_\varepsilon - \xi''_{\varepsilon n} + |\, \xi''_{\varepsilon n} \,| \\ &\geqslant -V^\varepsilon_v(t, z_\varepsilon + \xi_{\varepsilon n}) - (1-\nu_0)w_\varepsilon. \end{aligned}$$

Note that $z_\varepsilon(t_0) + \xi_{\varepsilon n}(t_0) > w_\varepsilon(t_0) + \xi_{\varepsilon n}(t_0) > \xi_{\varepsilon n}(t_0)$, since $w_\varepsilon(t) > 0$ in $[s_\varepsilon, t_\varepsilon]$. Thus (3.15) implies

$$z''_\varepsilon(t_0) - w''_\varepsilon(t_0) \geqslant (1-\nu_0)(z_\varepsilon(t_0) - w_\varepsilon(t_0)) > 0. \tag{3.22}$$

(3.21) and (3.22) are in a contradiction and we have (3.20).

To complete the proof, we obtain an estimate for w_ε. For simplicity of notation, we assume $[s_\varepsilon, t_\varepsilon] = [-A_\varepsilon, A_\varepsilon]$ without loss of generality. Setting

$$f_\varepsilon(t) = \frac{1}{2\sqrt{1-\nu_0}} \int_{-A_\varepsilon}^{A_\varepsilon} \mathrm{e}^{-\sqrt{1-\nu_0}|t-s|}\,|\, \xi''_{\varepsilon n}(s) \,|\, \mathrm{d}s,$$

the solution of (3.19) can be written as

$$w_\varepsilon(t) = a\,\mathrm{e}^{\sqrt{1-\nu_0}t} + b\,\mathrm{e}^{-\sqrt{1-\nu_0}t} + f_\varepsilon(t),$$

where a and b can be computed imposing $w_\varepsilon(\pm A_\varepsilon) = \frac{3}{2}\delta_0$. Using that $\|f_\varepsilon(t)\|_{L^\infty} \leqslant \|\xi''_{\varepsilon n}(t)\|_{L^\infty} \leqslant C_1\varepsilon^3$, we find that

$$|a|,\ |b| \leqslant C\mathrm{e}^{-\sqrt{1-\nu_0}A_\varepsilon}(\delta_0 + \varepsilon^3),$$

from where

$$w_\varepsilon(t) \leqslant C\mathrm{e}^{-\sqrt{1-\nu_0}\min\{|t+A_\varepsilon|,|t-A_\varepsilon|\}}(\delta_0 + \varepsilon^3) + C_1\varepsilon^3.$$

This inequality, together with (3.20), gives one side of (3.17). The other side follows using a similar comparison argument from below. □

Using (3.15) we can obtain precise bounds for v'_ε.

Corollary 3.1. *Assume $t_\varepsilon - s_\varepsilon \geqslant 1$. Then, for all $t \in (s_\varepsilon, t_\varepsilon)$,*

$$|\ v'_\varepsilon(t)\ | \leqslant a'_1\varepsilon^2 + a'_2\mathrm{e}^{-\sqrt{1-\nu_0}\min\{|t-s_\varepsilon|,|t-t_\varepsilon|\}}.$$

Proof. We have by (3.18)

$$\begin{aligned}|\ z''_\varepsilon(t)\ | \ &\leqslant |\ V^\varepsilon_v(t, z_\varepsilon(t) + \xi_{\varepsilon n}(t))\ | + |\ \xi''_{\varepsilon n}(t)\ | \leqslant C\ |\ z_\varepsilon(t)\ | + C_2\varepsilon^3 \\ &\leqslant C'\varepsilon^3 + C'\mathrm{e}^{-\sqrt{1-\nu_0}\min\{|t-s_\varepsilon|,|t-t_\varepsilon|\}} \quad \text{in } (s_\varepsilon, t_\varepsilon).\end{aligned}$$

Thus by interpolation, we have

$$|\ z'_\varepsilon(t)\ | \leqslant C''\varepsilon^3 + C''\mathrm{e}^{-\sqrt{1-\nu_0}\min\{|t-s_\varepsilon|,|t-t_\varepsilon|\}} \quad \text{in } (s_\varepsilon, t_\varepsilon).$$

Hence, by (3.7), for all $t \in (s_\varepsilon, t_\varepsilon)$

$$|\ v'_\varepsilon(t)\ | \leqslant |\ z'_\varepsilon(t)\ | + |\ \xi'_{\varepsilon n}(t)\ | \leqslant C'''\varepsilon^2 + C''\mathrm{e}^{-\sqrt{1-\nu_0}\min\{|t-s_\varepsilon|,|t-t_\varepsilon|\}}.$$

□

We end this subsection with two properties of the solutions v_ε of (3.5) near $(2n+1)\pi$, which are a direct consequence of the properties of V^ε, so we omit their proofs.

Lemma 3.2. *Let $v_\varepsilon(t)$ be a solution of (3.5). Then*

$$v''_\varepsilon(t)(v_\varepsilon(t) - \xi_{\varepsilon n}(t)) > 0 \quad \textit{if } 0 < |v_\varepsilon(t) - (2n+1)\pi| \leqslant 2\delta_0.$$

Lemma 3.3. *Let $[a,b] \subset [-L/\varepsilon, L/\varepsilon]$ and suppose that $v_\varepsilon(t)$ is a solution of (3.5) in (a,b) satisfying the inequality*

$$|v(t) - (2n+1)\pi| \leqslant 2\delta_0 \quad \textit{for } t \in [a,b]. \tag{3.23}$$

Then $v_\varepsilon(t)$ is a minimizer of the following variational problem:

$$I_{\varepsilon,[a,b]}(v_\varepsilon) = \inf\{I_{\varepsilon,[a,b]}(v)\ /\ v \textit{ satisfies } (3.23),\ \ v(a) = v_\varepsilon(a),\ v(b) = v_\varepsilon(b)\},$$

where

$$I_{\varepsilon,[a,b]}(v) = \int_a^b \frac{1}{2}\ |\ v'\ |^2 - (V^\varepsilon(t,v) - V^\varepsilon(t, \xi_{\varepsilon n}(t)))\,\mathrm{d}t.$$

3.2. Properties of growing solutions when $\alpha(t) > 0$

We consider the energy function $E_\varepsilon(t)$ for a solution $v_\varepsilon(t)$ of (3.5), where E_ε is given in (1.4) with v'_ε instead of $\varepsilon u'_\varepsilon$. We study the behaviour of these solutions in $[a/\varepsilon, b/\varepsilon]$, under the following assumption: For $\delta_0 > 0$ given in (3.14)–(3.16) and some $\delta_1 \in (0, -\cos\delta_0 + 1)$, the family of solutions v_ε satisfies

$$| E_\varepsilon(t) | \leqslant \delta_1 \qquad \text{in } (a/\varepsilon, b/\varepsilon). \tag{3.24}$$

Under condition (3.24), we see that

$$v'_\varepsilon(t) \neq 0 \qquad \text{if} \quad v_\varepsilon(t) \notin \bigcup_{k\in\mathbb{Z}} [(2k+1)\pi - \delta_0, (2k+1)\pi + \delta_0].$$

Let $C_1 > 0$ be the constant in (3.7). We say that $v_\varepsilon(t)$ is *growing* in $[\hat{s}, \hat{t}]$ if

$$v'_\varepsilon(t) > 0 \quad \text{when} \quad v_\varepsilon(t) \in [\hat{s}, \hat{t}] \setminus \bigcup_{k\in\mathbb{Z}} [(2k+1)\pi - 2C_1\varepsilon, (2k+1)\pi + 2C_1\varepsilon].$$

We remark that

$$v''_\varepsilon(t) > 0 \quad \text{if} \quad v_\varepsilon(t) \in \bigcup_{k\in\mathbb{Z}} [(2k+1)\pi + C_1\varepsilon, (2k+1)\pi + \delta_0],$$

$$v''_\varepsilon(t) < 0 \quad \text{if} \quad v_\varepsilon(t) \in \bigcup_{k\in\mathbb{Z}} [(2k+1)\pi - \delta_0, (2k+1)\pi - C_1\varepsilon].$$

Thus, if for some $n \in \mathbb{N}$, $v_\varepsilon(t)$ satisfies

$$v_\varepsilon(t) \in [2n\pi, 2(n+1)\pi] \qquad \text{for all } t \in [\hat{s}, \hat{t}]$$

and one of the following conditions

(i) $v_\varepsilon(\hat{s}) = 2n\pi$ and $v_\varepsilon(\hat{t}) = 2(n+1)\pi$,
(ii) $|v_\varepsilon(\hat{s}) - (2n+1)\pi| \leqslant 2C_1\varepsilon$ and $v_\varepsilon(\hat{t}) = 2(n+1)\pi$,
(iii) $v_\varepsilon(\hat{s}) = 2n\pi$ and $|v_\varepsilon(\hat{t}) - (2n+1)\pi| \leqslant 2C_1\varepsilon$,

then $v_\varepsilon(t)$ is growing in $[\hat{s}, \hat{t}]$.

In the next lemma we obtain a lower estimate on the energy change in terms of the rotation of the solution, under our basic assumption on the coefficient α (3.1) and assuming that the energy of the solutions is kept small. This estimate is crucial in the proof of proposition 3.1.

Lemma 3.4. *Assume (3.1) and let $v_\varepsilon(t)$ be is a solution of (3.5) for which there exists $[s_\varepsilon, t_\varepsilon] \subset [a/\varepsilon, b/\varepsilon]$ such that $v_\varepsilon(t)$ is growing in $[s_\varepsilon, t_\varepsilon]$ and*

$$| E_\varepsilon(t) | \leqslant \delta_1 \qquad \text{in } [s_\varepsilon, t_\varepsilon]. \tag{3.25}$$

Then there are constants $a_3, a_4 > 0$ independent of ε, v_ε, s_ε, t_ε such that for ε small

$$E_\varepsilon(t_\varepsilon) - E_\varepsilon(s_\varepsilon) \geqslant a_3\varepsilon(v_\varepsilon(t_\varepsilon) - v_\varepsilon(s_\varepsilon)) - a_4\varepsilon^2. \tag{3.26}$$

Proof. First we consider the case where $v_\varepsilon(t)$ satisfies

$$v_\varepsilon(t) \in [2n_\varepsilon\pi, 2(n_\varepsilon+1)\pi] \quad \text{in } (s_\varepsilon, t_\varepsilon) \text{ for some } n_\varepsilon \in \left[\frac{R_1}{2\pi\varepsilon}, \frac{R_2}{2\pi\varepsilon}\right]. \tag{3.27}$$

We compute

$$E_\varepsilon(t_\varepsilon) - E_\varepsilon(s_\varepsilon) = \int_{s_\varepsilon}^{t_\varepsilon} \frac{\mathrm{d}}{\mathrm{d}\tau} E_\varepsilon(\tau)\,\mathrm{d}\tau = \int_{s_\varepsilon}^{t_\varepsilon} \varepsilon^2\alpha(\varepsilon\tau) v_\varepsilon v'_\varepsilon\,\mathrm{d}\tau. \tag{3.28}$$

Let $N_\varepsilon = \{\tau \in [s_\varepsilon, t_\varepsilon] /\ | v_\varepsilon(\tau) - (2n_\varepsilon+1)\pi | < 2C_1\varepsilon\}$, where $C_1 > 0$ is given in (3.7). Then, since v_ε is growing and satisfies (3.25), we see that

$$v'_\varepsilon > 0 \qquad \text{in } [s_\varepsilon, t_\varepsilon] \setminus N_\varepsilon$$

and N_ε is an interval if $N_\varepsilon \neq \emptyset$. Denoting $N_\varepsilon = [\tau_\varepsilon^1, \tau_\varepsilon^2]$ in this case, we have

$$\begin{aligned}\int_{s_\varepsilon}^{\tau_\varepsilon^1} \varepsilon^2\alpha(\varepsilon\tau)v_\varepsilon v_\varepsilon' \,\mathrm{d}\tau &\geqslant \frac{\varepsilon^2}{2}\underline{\alpha}\int_{s_\varepsilon}^{\tau_\varepsilon^1}\frac{\mathrm{d}}{\mathrm{d}\tau}(v_\varepsilon^2)\,\mathrm{d}\tau = \frac{\varepsilon^2}{2}\underline{\alpha}(v_\varepsilon(\tau_\varepsilon^1)^2 - v_\varepsilon(s_\varepsilon)^2)\\ &= \frac{\varepsilon^2}{2}\underline{\alpha}(v_\varepsilon(\tau_\varepsilon^1) + v_\varepsilon(s_\varepsilon))(v_\varepsilon(\tau_\varepsilon^1) - v_\varepsilon(s_\varepsilon))\\ &\geqslant \varepsilon\underline{\alpha}R_1(v_\varepsilon(\tau_\varepsilon^1) - v_\varepsilon(s_\varepsilon)),\end{aligned} \tag{3.29}$$

where $\underline{\alpha}$ is given in (3.1). Similarly we have

$$\int_{\tau_\varepsilon^2}^{t_\varepsilon} \varepsilon^2\alpha(\varepsilon\tau)v_\varepsilon v_\varepsilon' \,\mathrm{d}\tau \geqslant \varepsilon\underline{\alpha}R_1(v_\varepsilon(t_\varepsilon) - v_\varepsilon(\tau_\varepsilon^2)). \tag{3.30}$$

Next we look at the integral in $[\tau_\varepsilon^1, \tau_\varepsilon^2]$, considering 2 cases: (a) $\mid \tau_\varepsilon^2 - \tau_\varepsilon^1 \mid \geqslant 2$ and (b) $\mid \tau_\varepsilon^2 - \tau_\varepsilon^1 \mid \leqslant 2$. If case (a) holds, by lemma 3.3, we have

$$\frac{1}{2}\|v_\varepsilon'\|^2_{L^2(\tau_\varepsilon^1,\tau_\varepsilon^2)} \leqslant I_{\varepsilon,[\tau_\varepsilon^1,\tau_\varepsilon^2]}(v_\varepsilon) \leqslant I_{\varepsilon,[\tau_\varepsilon^1,\tau_\varepsilon^2]}(\zeta_\varepsilon) = O(\varepsilon^2),$$

where ζ_ε interpolates linearly $(2n_\varepsilon+1)\pi - 2C_1\varepsilon$ with $\xi_{\varepsilon n}(\tau_\varepsilon^1+1)$ in $[\tau_\varepsilon^1, \tau_\varepsilon^1+1]$, $\zeta_\varepsilon(\tau) = \xi_{\varepsilon n}(\tau)$ in $[\tau_\varepsilon^1+1, \tau_\varepsilon^2-1]$ and it interpolates linearly $\xi_{\varepsilon n}(\tau_\varepsilon^2-1)$ with $(2n_\varepsilon+1)\pi+2C_1\varepsilon$ and in $[\tau_\varepsilon^2-1, \tau_\varepsilon^2]$. Thus

$$\left|\int_{\tau_\varepsilon^1}^{\tau_\varepsilon^2} \varepsilon^2\alpha(\varepsilon\tau)v_\varepsilon v_\varepsilon' \,\mathrm{d}\tau\right| \leqslant ((2n_\varepsilon+1)\pi + 2C_1\varepsilon)\varepsilon^2\|\alpha\|_\infty\|\zeta_\varepsilon'\|_{L^2(\tau_\varepsilon^1,\tau_\varepsilon^2)} = O(\varepsilon^2). \tag{3.31}$$

If case (b) holds, we also show (3.31). Letting $\tau(s) = (\tau_\varepsilon^2 - \tau_\varepsilon^1)s + \tau_\varepsilon^1$ we define $w_\varepsilon(s) = \frac{1}{\varepsilon}(v_\varepsilon(\tau(s)) - (2n_\varepsilon+1)\pi)$. Then $w_\varepsilon(s)$ satisfies

$$w_\varepsilon'' - \frac{(\tau_\varepsilon^2 - \tau_\varepsilon^1)^2}{\varepsilon}\sin(\varepsilon w_\varepsilon) = \varepsilon(\tau_\varepsilon^2 - \tau_\varepsilon^1)^2\alpha(\varepsilon\tau(s))(\varepsilon w_\varepsilon + (2n_\varepsilon+1)\pi)$$

and $-C_1 \leqslant w_\varepsilon(0) \leqslant w_\varepsilon(1) \leqslant C_1$. Then, up to a subsequence, we have $\tau_\varepsilon^2 - \tau_\varepsilon^1 \to h_0 \in [0, 2]$, $\varepsilon\tau_\varepsilon^1 \to t_0$ and $w_\varepsilon(s) \to w_0(s)$, where $w_0(s)$ satisfies

$$w_0'' - h_0^2 w_0 = Rh_0^2\alpha(t_0),$$

for some $R > 0$ and also $-C_1 \leqslant w_0(0) \leqslant w_0(1) \leqslant C_1$. Thus we have

$$\begin{aligned}\left|\int_{\tau_\varepsilon^1}^{\tau_\varepsilon^2} \varepsilon^2\alpha(\varepsilon\tau)v_\varepsilon v_\varepsilon' \,\mathrm{d}\tau\right| &= \int_0^1 \varepsilon^3\alpha(\varepsilon s)(\varepsilon w_\varepsilon(s) + (2n_\varepsilon+1)\pi)|w_\varepsilon'(s)|\,\mathrm{d}s\\ &\leqslant 3n_\varepsilon\pi\varepsilon^3\int_0^1(|w_0'(s)|+1)\,\mathrm{d}s = O(\varepsilon^2).\end{aligned}$$

Thus we also have (3.31) and now (3.26) follows from (3.29)–(3.31). The general case can be argued similarly by adding integrals like (3.26) over intervals satisfying (3.27), and noticing that by our general assumptions the number of terms to add cannot be larger that m_ε, where $\varepsilon m_\varepsilon$ is bounded. □

To end this subsection we obtain an estimate on the energy for solutions that do not rotate for long periods of time. Precisely we have

Lemma 3.5. Assume (3.1) and suppose that there exist $[s_\varepsilon, t_\varepsilon] \subset [a/\varepsilon, b/\varepsilon]$ with $\mid t_\varepsilon - s_\varepsilon \mid \geqslant 1$ such that (3.25) and for some $n_\varepsilon \in [R_1/(2\pi\varepsilon), R_2/(2\pi\varepsilon)] \cap \mathbb{Z}$

$$v_\varepsilon(t) \in [2n_\varepsilon\pi, 2(n_\varepsilon+1)\pi], \qquad \text{in } (s_\varepsilon, t_\varepsilon), \tag{3.32}$$

Then there exist constants $a_5, a_6 > 0$ such that for all $t \in (s_\varepsilon, t_\varepsilon)$

$$| v_\varepsilon(t) - (2n+1)\pi | \leqslant \frac{3}{2} C_1 \varepsilon + a_6 \mathrm{e}^{-\sqrt{1-\nu_0}\min\{|t-s_\varepsilon|, |t-t_\varepsilon|\}}$$

and

$$| E_\varepsilon(t) | \leqslant a_5 \varepsilon^2 + a_6 \mathrm{e}^{-2\sqrt{1-\nu_0}\min\{|t-s_\varepsilon|, |t-t_\varepsilon|\}}.$$

Proof. Let $\tilde{N}_\varepsilon = \{\tau \in [s_\varepsilon, t_\varepsilon] / \ | v_\varepsilon(\tau) - (2n+1)\pi | < \delta_0\}$. As in the proof of lemma 3.4, if $\tilde{N}_\varepsilon$ is not empty, we may write $\tilde{N}_\varepsilon = [s_\varepsilon^1, t_\varepsilon^1]$. Since we assume (3.25), there exists $\ell_0 > 0$ independent of ε such that

$$t_\varepsilon - t_\varepsilon^1, \ s_\varepsilon^1 - s_\varepsilon \leqslant \ell_0 \qquad \text{for } \varepsilon \text{ small.}$$

In particular, we have

$$\min\{| t - s_\varepsilon^1 |, | t - t_\varepsilon^1 |\} \geqslant \min\{|t - s_\varepsilon|, |t - t_\varepsilon|\} - \ell_0. \tag{3.33}$$

We may also assume $| t_\varepsilon^1 - s_\varepsilon^1 | \geqslant 1$, since on the contrary we have $1 \leqslant | t_\varepsilon - s_\varepsilon | \leqslant 2\ell_0 + 1$ and $\min\{|t - s_\varepsilon|, |t - t_\varepsilon|\} \leqslant \frac{1}{2} | t_\varepsilon - s_\varepsilon | \leqslant \frac{1}{2}(2\ell_0 + 1)$, which means the conclusion of lemma 3.5 holds trivially.

Applying proposition 3.2 and corollary 3.1 in $[s_\varepsilon^1, t_\varepsilon^1]$, we have

$$| v_\varepsilon(t) - \xi_{\varepsilon n}(t) | \leqslant a_1 \varepsilon^3 + a_2 \mathrm{e}^{-\sqrt{1-\nu_0}\min\{|t-s_\varepsilon^1|, |t-t_\varepsilon^1|\}},$$
$$| v_\varepsilon'(t) | \leqslant a_1' \varepsilon^2 + a_2' \mathrm{e}^{-\sqrt{1-\nu_0}\min\{|t-s_\varepsilon^1|, |t-t_\varepsilon^1|\}}.$$

We also remark that by lemma 3.1

$$\begin{aligned} | v_\varepsilon(t) - (2n+1)\pi | &\leqslant | v_\varepsilon(t) - \xi_{\varepsilon n}(t) | + | \xi_{\varepsilon n}(t) - (2n+1)\pi | \\ &\leqslant a_1'' \varepsilon + a_2' \mathrm{e}^{-\sqrt{1-\nu_0}\min\{|t-s_\varepsilon^1|, |t-t_\varepsilon^1|\}}, \end{aligned}$$

so that

$$\cos v_\varepsilon(t) + 1 \leqslant a_1''' \varepsilon^2 + a_2'' \mathrm{e}^{-2\sqrt{1-\nu_0}\min\{|t-s_\varepsilon^1|, |t-t_\varepsilon^1|\}}.$$

Therefore

$$| E_\varepsilon(t) | \leqslant C \varepsilon^2 + C' \mathrm{e}^{-2\sqrt{1-\nu_0}\min\{|t-s_\varepsilon^1|, |t-t_\varepsilon^1|\}}$$

and by (3.33) we conclude that

$$| E_\varepsilon(t) | \leqslant C \varepsilon^2 + C'' \mathrm{e}^{-2\sqrt{1-\nu_0}\min\{|t-s_\varepsilon|, |t-t_\varepsilon|\}}.$$

□

3.3. *Proof of proposition 3.1*

To show proposition 3.1, we assume (3.32) and (3.25) in $[s_\varepsilon, t_\varepsilon] \subset [a/\varepsilon, b/\varepsilon]$.

We write $v_\varepsilon(t) = u_\varepsilon(\varepsilon t)$ and $A_0 = \frac{2}{\sqrt{1-\nu_0}}$. We argue indirectly assuming that

$$J_\varepsilon = \{t \in \left(\frac{a}{\varepsilon}, \frac{b - \frac{1}{2}\delta}{\varepsilon}\right) \Big/ v_\varepsilon(t) \in 2\pi\mathbb{Z}\} = \{t_1^\varepsilon, \ldots, t_{m_\varepsilon}^\varepsilon\} \neq \emptyset,$$

where $a/\varepsilon < t_1^\varepsilon < t_2^\varepsilon < \cdots < t_{m_\varepsilon}^\varepsilon < (b - \frac{1}{2}\delta)/\varepsilon$. Writing $v_\varepsilon(t_1^\varepsilon) = 2(n_\varepsilon + 1)\pi$, by (3.2), we have $v_\varepsilon(t_i^\varepsilon) = 2(n_\varepsilon + i)\pi$ for $i = 1, 2, \ldots, m_\varepsilon$. We also denote by t_0^ε the largest $t \in [-\frac{L}{\varepsilon}, t_1^\varepsilon)$ such that $v_\varepsilon(t) = 2n_\varepsilon\pi$ and by $t_{m_\varepsilon+1}^\varepsilon$ the smallest $t \in (t_{m_\varepsilon}^\varepsilon, \frac{L}{\varepsilon}]$ such that $v_\varepsilon(t) = 2(n_\varepsilon + m_\varepsilon + 1)\pi$. In case some of them do not exist, we just set $t_0^\varepsilon = -L/\varepsilon$ or $t_{m_\varepsilon+1}^\varepsilon = L/\varepsilon$. Next we define

$$\tilde{J}_\varepsilon = \{i \in \{0, 1, 2, \ldots, m_\varepsilon\} / \ | t_{i+1}^\varepsilon - t_i^\varepsilon | \geqslant 2A_0 | \log \varepsilon |\}.$$

To start the proof we claim that for ε small

$$\{1, 2, \dots, m_\varepsilon\} \cap \tilde{J}_\varepsilon = \emptyset. \tag{3.34}$$

To prove the claim we argue indirectly, assuming that $i_0^\varepsilon \geqslant 1$ belongs to $\tilde{J}_\varepsilon$. We set $\hat{t}^\varepsilon = t^\varepsilon_{i_0^\varepsilon} + A_0 \mid \log \varepsilon \mid$ and we see that $[t^\varepsilon_{i_0^\varepsilon}, \hat{t}^\varepsilon] \subset [a/\varepsilon, b/\varepsilon]$ and $v_\varepsilon(t)$ is growing in $[t^\varepsilon_{i_0^\varepsilon}, \hat{t}^\varepsilon]$. Thus by lemma 3.4 we have

$$\begin{aligned} E_\varepsilon(\hat{t}^\varepsilon) - E_\varepsilon(t^\varepsilon_{i_0^\varepsilon}) &\geqslant a_3\varepsilon(v_\varepsilon(\hat{t}^\varepsilon) - v_\varepsilon(t^\varepsilon_{i_0^\varepsilon})) - a_4\varepsilon^2 \\ &\geqslant a_3\varepsilon(v_\varepsilon(\hat{t}^\varepsilon) - 2(n_\varepsilon + 1)\pi) - a_4\varepsilon^2. \end{aligned} \tag{3.35}$$

On the other hand, applying lemma 3.5 in $[t^\varepsilon_{i_0^\varepsilon}, t^\varepsilon_{i_0^\varepsilon+1}]$, we find for $t \in [t^\varepsilon_{i_0^\varepsilon}, t^\varepsilon_{i_0^\varepsilon+1}]$

$$\begin{aligned} &\mid v_\varepsilon(\hat{t}^\varepsilon) - (2(n_\varepsilon + i_0^\varepsilon) + 1)\pi \mid \leqslant \frac{2}{3} 2C_1\varepsilon + a_6 \mathrm{e}^{-\frac{2}{A_0}\min\{|t - t^\varepsilon_{i_0^\varepsilon}|, |t - t^\varepsilon_{i_0^\varepsilon+1}|\}}, \\ &\mid E_\varepsilon(\hat{t}^\varepsilon) \mid \leqslant a_5\varepsilon^2 + a_6 \mathrm{e}^{-\frac{4}{A_0}\min\{|t - t^\varepsilon_{i_0^\varepsilon}|, |t - t^\varepsilon_{i_0^\varepsilon+1}|\}}. \end{aligned}$$

Noting $\min\{\mid t - t^\varepsilon_{i_0^\varepsilon} \mid, \mid t - t^\varepsilon_{i_0^\varepsilon+1} \mid\} = A_0 \mid \log \varepsilon \mid$, we have

$$\mid v_\varepsilon(\hat{t}^\varepsilon) - (2(n_\varepsilon + i_0^\varepsilon) + 1)\pi \mid \leqslant 2C_1\varepsilon, \tag{3.36}$$

$$\mid E_\varepsilon(\hat{t}^\varepsilon) \mid \leqslant a_7\varepsilon^2. \tag{3.37}$$

Recalling that $E_\varepsilon(t^\varepsilon_{i_0^\varepsilon}) = \frac{1}{2} \mid v_\varepsilon(t^\varepsilon_{i_0^\varepsilon}) \mid^2 \geqslant 0$, we see that (3.35)–(3.37) are incompatible for ε small, proving the claim (3.34). Next we prove that

$$\lim_{\varepsilon \to 0} \varepsilon t_1^\varepsilon = b - \frac{1}{2}\delta. \tag{3.38}$$

By (3.34), we have $t^\varepsilon_{i+1} - t^\varepsilon_i \leqslant 2A_0 |\log \varepsilon|$ for $i = 1, 2, \dots, m_\varepsilon$. In particular, we have

$$\lim_{\varepsilon \to 0} \varepsilon(t_2^\varepsilon - t_1^\varepsilon) = 0 \qquad \text{and} \qquad \lim_{\varepsilon \to 0} \varepsilon t^\varepsilon_{m_\varepsilon} = b - \frac{1}{2}\delta. \tag{3.39}$$

By (2.3) and since $2\pi\varepsilon(m_\varepsilon - 2) = \varepsilon(v_\varepsilon(t^\varepsilon_{m_\varepsilon}) - v_\varepsilon(t_2^\varepsilon))$, we also have

$$\begin{aligned} \lim_{\varepsilon \to 0} 2\pi\varepsilon(m_\varepsilon - 2) &= R(b - \frac{1}{2}\delta) - R(a) \\ &= \int_a^{b - \frac{1}{2}\delta} \frac{1}{T(\lim E_\varepsilon(t))} \, \mathrm{d}t = 0. \end{aligned} \tag{3.40}$$

Then we apply lemma 3.4 in $[t_1^\varepsilon, \frac{t_i^\varepsilon + t^\varepsilon_{i+1}}{2}]$ to get

$$\begin{aligned} E_\varepsilon\left(\frac{t_i^\varepsilon + t^\varepsilon_{i+1}}{2}\right) &\geqslant E_\varepsilon(t_1^\varepsilon) + a_3\varepsilon\left(v_\varepsilon\left(\frac{t_i^\varepsilon + t^\varepsilon_{i+1}}{2}\right) - v_\varepsilon(t_1^\varepsilon)\right) - a_4\varepsilon^2 \\ &\geqslant a_3\varepsilon(2(n_\varepsilon + i) - 2(n_\varepsilon + 1))\pi - a_4\varepsilon^2 \\ &\geqslant 2\pi a_3\varepsilon(i - 1) - a_4\varepsilon^2 \end{aligned} \tag{3.41}$$

and noting that (3.3) implies $t^\varepsilon_{i+1} - t^\varepsilon_i \to \infty$ as $\varepsilon \to 0$ for all i, we may apply lemma 3.5 in $[t_i^\varepsilon, t^\varepsilon_{i+1}]$ to obtain

$$E_\varepsilon\left(\frac{t_i^\varepsilon + t^\varepsilon_{i+1}}{2}\right) \leqslant a_5\varepsilon^2 + a_6 \mathrm{e}^{-2\sqrt{1-v_0}|t^\varepsilon_{i+1} - t^\varepsilon_i|}. \tag{3.42}$$

It follows from (3.41) and (3.42) that, for some constants $a_7, a_8 > 0$ independent of ε and i

$$t^\varepsilon_{i+1} - t^\varepsilon_i \leqslant -a_7 \log(a_8\varepsilon(i - 1)) \qquad \text{for } i = 2, 3, \dots, m_\varepsilon - 1.$$

Thus

$$\begin{aligned}
\varepsilon(t^{\varepsilon}_{m_\varepsilon} - t^{\varepsilon}_2) &\leqslant \varepsilon \sum_{i=2}^{m_\varepsilon - 1} -a_7 \log(a_8 \varepsilon (i-1)) \\
&\leqslant \sum_{i=2}^{m_\varepsilon - 1} \int_{\varepsilon(i-2)}^{\varepsilon(i-1)} -a_7 \log(a_8 x)\, dx \\
&= a_7 \varepsilon (m_\varepsilon - 2)\,(1 - \log(a_8 \varepsilon(m_\varepsilon - 2)))\,.
\end{aligned}$$

From here and (3.40), we have $\varepsilon(t^{\varepsilon}_{m_\varepsilon} - t^{\varepsilon}_2) \to 0$ as $\varepsilon \to 0$ and then, by (3.39) we get (3.38).

Now we finally see that (3.4) is true, since $\{t \in [\frac{a}{\varepsilon}, t^{\varepsilon}_1) \,/\, u_\varepsilon(t) \in 2\pi\mathbb{Z}\} = \emptyset$ and (3.38) holds. □

Remark 3.2. The conclusion of proposition 3.1 still holds for a family of solutions of (3.5) in $(a/\varepsilon, b/\varepsilon)$ giving suitable boundary conditions at $t = a/\varepsilon, b/\varepsilon$.

4. The Area Equation and Properties

In this section we study the basic properties of the profile equation. We prefer to study the area equation, as given in corollary 2.1, rather than the energy equation, since it is much simpler. We consider only the + since the other case is analogous, so for a given $R > 0$ we study

$$A'(t) = (2\pi)^2 \alpha(t) \left(R + \int_a^t \frac{1}{T(A(s))}\, \mathrm{d}s \right) \qquad \text{in } (a, b), \tag{4.1}$$

where, for the sake of simplicity of notation, we denote by $T(A)$ the function $T(E(A))$. Before starting the analysis, we need to make more precise the notion of the solution to this equation. We will always assume that $R > 0$ and we extend the function $1/T(A)$ to $A = 8$, simply as 0.

We say that a continuous function $A : [-L, L] \to [8, \infty)$ is a solution of (4.1) in $[-L, L]$, if, for every $t \in [-L, L]$ such that $A(t) > 8$, the function A satisfies the differential equation (4.1) at t. We define the support of a solution as

$$\mathrm{supp}(A) = \overline{\{t \in [-L, L] / A(t) > 8\}}.$$

We start with a basic existence result

Proposition 4.1. *Let $a, b \in [-L, L]$, with $a < b$, and assume that $\alpha(a) > 0$. Then, for every $R > 0$ and $R' > 0$ there is a solution of equation (4.1) in the interval $[a, b]$ such that*

$$\int_a^b \frac{\mathrm{d}s}{T(A(s))} = R'. \tag{4.2}$$

Proof. Given $A_0 > 8$ we consider the equation (4.1) for $t > a$, with initial condition $A(a) = A_0$. We continue the solution while $A(t) > 8$ and we define $t_f = \sup\{t \in (a, b) / A(\tau) > 8 \text{ for } \tau \in [a, t)\}$. If $t_f < b$, we define $A(t) = 8$, for all $t \in [t_f, b]$. Since $\lim_{A \to \infty} T(A) = 0$ we see that, by choosing A_0 large enough, we have that

$$\int_a^b \frac{\mathrm{d}s}{T(A(s))} > R'.$$

On the other hand, let $t_0 = \sup\{t \in (a, b] / \alpha(\tau) > 0 \text{ for } \tau \in [a, t)\}$ and consider $t_i \in [a, t_0)$. Then we solve equation (4.1) for $t > t_i$, with $A(t_i) = 8$.

Before continuing, we need to say a word about the existence and uniqueness of the solution in this case. We may differentiate (4.1) to obtain the equivalent problem

$$A'' = \frac{\alpha'(t)}{\alpha(t)} A' + (2\pi)^2 \frac{\alpha(t)}{T(A)},$$

with initial conditions $A(t_i) = 8$ and $A'(t_i) = (2\pi)^2\alpha(t_i)R > 0$. Because of this last condition we can write the equation for $B(s)$, where $A(B(s)) = s$, for $s \geqslant 8$. For this equation, the classical existence and uniqueness theorem for ordinary differential equations applies since T is a continuous function and α is of class C^2.

Now we continue with our analysis of the solution A of (4.1) extending the solution while $A(t) > 8$ and we define t_f as above. Then we set $A(t) = 8$ for $t \in [a, t_i) \cup (t_f, b]$. We observe that, by choosing t_i close enough to t_0, we have

$$\int_a^b \frac{\mathrm{d}s}{T(A(s))} < R'.$$

To complete the argument, we see that the solution depends continuously on $A_0 \in [8, \infty)$ and $t_i \in [a, t_0)$, so that there must exist $t_i^* \in [a, t_0]$ or $A_0^* \in [8, \infty)$ such that the corresponding solution satisfies (4.2). □

Next we prove a uniqueness result.

Proposition 4.2. *Let $a, b \in [-L, L]$, with $a < b$, and assume that there is t_z such that $\alpha(t) > 0$ in $[a, t_z)$, $\alpha(t_z) = 0$ and $\alpha'(t) < 0$ in $[t_z, b]$. Then, for every $R > 0$ and $R' > 0$ the equation (4.1) possesses at most one solution with support in (a, b) and satisfying (4.2).*

Proof. Assume we have two different solutions A_1 and A_2 satisfying (4.2), with support $[t_i^1, t_f^1]$ and $[t_i^2, t_f^2]$, respectively. We claim that for all $A > 8$ for which there are

$$(t_z <)\, t_A^1 < t_A^2 \qquad \text{with} \quad A_1(t_A^1) = A = A_2(t_A^2), \tag{4.3}$$

then we have

$$A_2'(t_A^2) < A_1'(t_A^1) < 0, \tag{4.4}$$

and that the same is true if we reverse the role of A_1 and A_2.

To prove the claim we first observe that, since A_1 and A_2 satisfy (4.2) and α is decreasing in $[t_z, b]$, we have that (4.4) is satisfied for $A = 8$. Let $\bar{A} > 8$ and t_A^1 and t_A^2 such that (4.3) holds and so that (4.4) holds for all $A \in [8, \bar{A})$. Then, using the monotonicity of the period in terms of A we have that

$$\int_{t_A^1}^{t_f^1} \frac{\mathrm{d}s}{T(A_1(s))} > \int_{t_A^2}^{t_f^2} \frac{\mathrm{d}s}{T(A_2(s))}$$

and from then, since $\alpha(t)$ is decreasing in $[t_z, b]$, we find that

$$\begin{aligned} 0 > A_1'(t_A^1) &= (2\pi)^2\alpha(t_A^1)\left(R + R' - \int_{t_A^1}^{t_f^1} \frac{\mathrm{d}s}{T(A_1(s))}\right) \\ &> (2\pi)^2\alpha(t_A^2)\left(R + R' - \int_{t_A^2}^{t_f^2} \frac{\mathrm{d}s}{T(A_2(s))}\right) = A_2'(t_A^2), \end{aligned}$$

that is, (4.4) is satisfied.

Now with the aid of the claim, we prove the proposition. We may assume that

$$A_1(t_z) < A_2(t_z) \tag{4.5}$$

and consequently they satisfy $A_2(t) > A_1(t)$ for all $t \in (t_i^2, t_z]$.

First, assume that $t_f^1 < t_f^2$. Then, by our hypothesis on α we have $\alpha(t_f^2) < \alpha(t_f^1) < 0$ and by the claim, we see then that $A_1(t) < A_2(t)$, for all $t \in [t_i^1, t_f^1]$, contradicting that both A_1 and A_2 satisfy (4.2). On the other hand, if we assume that $t_f^2 < t_f^1$, then, by the claim we have that $A_2(t_z) < A_1(t_z)$, contradicting (4.5). □

Remark 4.1. The uniqueness result is also true if one or both solutions satisfies $A(a) > 8$, but it may be false if one or both solutions satisfy $A(b) > 8$. Thus, in order to have a complete uniqueness result we would need to know that all solutions with R and R' satisfy $A(b) = 8$.

Given $R_0 > 0$, it is possible to give an estimate on R_0' so that for all $0 < R \leqslant R_0$ and $0 < R' \leqslant R_0'$ the solutions of (4.1) with (R, R') satisfy $A(b) = 8$. We let $h : [a, b] \to \mathbb{R}$ such that $h(t) = 1$ if $t < t_z$ and $h(t) = 0$ if $t \geqslant t_z$ and we consider the solution of the equation

$$A'(t) = (2\pi)^2\alpha(t)\left(R + h(t)\int_a^t \frac{ds}{T(A(s))}\right) \tag{4.6}$$

with initial value $A(t_i) = 8$, for $t_i \in [a, t_z)$ and denote it as $A(t, t_i)$. We observe that the solutions of this equation are ordered by t_i and by R and we have that either

(i) $A(t, t_i) > 8$ for all $t \in [t_z, b]$ or else,
(ii) there is $t_f \in [t_z, b]$ such that $A(t_f, t_i) = 8$.

We let t^* be the infimum all t_i such that (ii) holds and we define

$$R_0' = \int_{t^*}^{t_z} \frac{\mathrm{d}s}{T(A(s, t^*))}.$$

By definition of R_0' and since the solutions of (4.6) are ordered, it is easy to see that given $(R, R') \in (0, R_0] \times (0, R_0']$, all solution of (4.1) with (R, R') satisfy $A(b) = 8$.

Next we prove a compactness property we will use later.

Proposition 4.3. *Given $R > 0$ and $R' > 0$. Let $\mathcal{A}$ be the set of all solutions of (4.1) with (4.2) in $[a, b]$. Then $\mathcal{A}$ is compact in the topology of the norm $\|u\|_\infty = \sup\{|u(t)|/t \in [-L, L]\}$.*

Proof. First we see that there is a constant $c > 0$ so that $|A'(t)| \leqslant c(R+R')$ for all $t \in [-L, L]$. This, together with (4.2), implies that there is $C > 0$ such that $A(t) \leqslant C$ for all $t \in [-L, L]$. Then, using the Arzela–Ascoli theorem, we find that any sequence $\{A_n\} \subset \mathcal{A}$ has a convergent subsequence, say converging to A. Since A_n satisfies (4.1) and (4.2) we see also that A satisfies (4.1) and (4.2). □

In the construction of solutions via the Nehari method, it would be useful to assume uniqueness of the solution, given R and R'. However, when this is not possible it is a useful and suitable notion of isolatedness. Assuming that $\alpha : [a, b] \to \mathbb{R}$, with $[a, b] \subset (-L, L)$, satisfies

$$\alpha(t) > 0 \text{ in } [a, t_z^a) \text{ and } \alpha(t) < 0 \text{ in } (t_z^b, b], \text{ with } \alpha(t_z^a) = \alpha(t_z^b) = 0$$

and that R and R' are given, we say that the set $\mathcal{A}$ of all solutions of (4.1) satisfying (4.2) is *isolated* if $\mathcal{A}$ is non-empty and there is $\delta > 0$, so that for all $A \in \mathcal{A}$ we have $supp A \subset [a + \delta, b - \delta]$. This implies that even though the solution may not be unique any solution in $[a, b]$ will have its support located in an strict subinterval. This localization property is key in the existence result.

Remark 4.2. Under the assumptions of Proposition 4.2, for any $R > 0$ there exists $c_R > 0$ such that for $R' \in (0, c_R)$, the set of solutions of (4.1)–(4.2) in $[a, b]$ with R, R' is unique and thus isolated.

5. The Basic Solutions

In this section we describe the basic solutions, which are the building blocks of approximate solutions of (1.1) and will allow us to set up the Nehari Method. We consider

$$\varepsilon^2 u'' + \sin u = \varepsilon^2 \alpha(t) u \qquad \text{in} \quad (\ell_0, \ell_1), \tag{5.1}$$

with various boundary conditions.

(1) *Solutions at endpoints*: we define the left increasing solution at n as the solution $w_L^i(t)$ of (5.1) that satisfies

$$u(\ell_0) = (2n-1)\pi, \qquad u(\ell_1) = 2n\pi \tag{5.2}$$

and the right increasing solution at n as the solution $w_R^i(t)$ of equation (5.1) that satisfies:

$$u(\ell_0) = 2n\pi, \qquad u(\ell_1) = (2n+1)\pi. \tag{5.3}$$

In a similar way we may define w_L^d (w_R^d) the left decreasing solutions.

(2) *Interior solutions*: we define increasing interior solutions at n as the solution w_I^i of (5.1) that satisfies

$$u(\ell_0) = 2n\pi, \qquad u(\ell_1) = (2n+2)\pi. \tag{5.4}$$

Similarly we may define w_I^d the decreasing interior solutions at n.

(3) *Glueing solutions*: these basic solutions, denoted as w_g^{ii}, are defined in the same way as the interior solutions, but we distinguish them because they play a different role in the construction. In the first case we consider $\ell_1 - \ell_0$ when small, while in the second case $\ell_1 - \ell_0$ is large. Similarly we define w_g^{dd} as the decreasing glueing solution.
Using the basic properties of solutions of the equation (1.5) and the monotonicity of the period $T(E)$ it can be proved that there exists $\varepsilon_0 > 0$ such that for any $-L \leqslant \ell_0 < \ell_1 \leqslant L$, all of the above basic solutions are uniquely defined for all $\varepsilon \in (0, \varepsilon_0]$.

(4) *Homotopy glueing solutions.* We define the increasing-decreasing glueing solution at n as the solution w_g^{id} of (5.1) that satisfies

$$u(\ell_0) = 2n\pi, \qquad u(\ell_1) = 2n\pi, \tag{5.5}$$

with $u'(\ell_0) > 0$. Similarly we define w_g^{di} as the decreasing-increasing glueing solution at n.

In what follows we analyse the existence and uniqueness of solutions w_g^{id} and w_g^{di} and a homotopy of solutions that will be used in section 7 for a homotopy argument in glueing clusters. We will construct these solutions using a variational approach with an appropriate penalization. We will give the details only for the increasing-decreasing function, that is for w_g^{id}, since the other one is analogous. Let $-L < a < c_0 < c_1 < b < L$ and $\ell_0 \in [-L, a]$, $\ell_1 \in [b, L]$, we consider

$$\begin{aligned} &\varepsilon^2 u'' + \sin u = \varepsilon^2 \alpha(t) u + \lambda \chi(t)(u - (2n_\varepsilon + 1)\pi) \quad \text{in } (\ell_0, \ell_1), \\ &u(\ell_0) = u(\ell_1) = 2n_\varepsilon \pi. \end{aligned} \tag{5.6}$$

Here $n_\varepsilon \in \mathbb{N}$, $\lambda \in [0, \infty)$ and $\chi(t)$ denotes the characteristic function of the interval $[c_0, c_1]$, that is, it takes the value 1 in $[c_0, c_1]$ and 0 elsewhere. Let us consider the spaces

$$H_0 = \{u \in H^1(\ell_0, c_0) \;/\; u(\ell_0) = 2n_\varepsilon\pi \text{ and } u(c_0) = (2n_\varepsilon + 1)\pi\},$$

$$H_1 = \{u \in H^1(c_1, \ell_1) \;/\; u(c_1) = (2n_\varepsilon + 1)\pi \text{ and } u(\ell_1) = 2n_\varepsilon\pi\}$$

and

$$\begin{aligned} H_{01} = \{u \in H^1(\ell_0, \ell_1)/u(\ell_0) = u(\ell_1) = 2n_\varepsilon\pi \qquad &\text{and} \\ u(t) \in [2n_\varepsilon\pi, (2n_\varepsilon + 1)\pi + \delta_0] \text{ for } t \in [\ell_0, \ell_1]\}&. \end{aligned}$$

Let J_ε^0 and J_ε^1 be the functionals associated to equation (5.1) in the corresponding intervals, over H_0 and H_1, respectively. Let $J_{\varepsilon,\lambda}$ be a functional corresponding to (5.6) on H_{01}. Then we have

Lemma 5.1. *(i) There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$, $\lambda \in [0, \infty)$, $\ell_0 \in [-L, a]$, $\ell_1 \in [b, L]$, J_ε^0, J_ε^1 and $J_{\varepsilon,\lambda}$ have unique minimizers $u^0_{\varepsilon,\infty}(\ell_0; t)$, $u^1_{\varepsilon,\infty}(\ell_1; t)$ and $u_{\varepsilon,\lambda}(\ell_0, \ell_1; t)$, respectively. (ii) Moreover, the functions defined on $[-L, a]\times[b, -L]\times[0, \infty] \to \mathbb{R}$, given by*

$$(\ell_0, \ell_1, \lambda) \mapsto \begin{cases} u'_{\varepsilon,\lambda}(\ell_0, \ell_1; \ell_0) & \text{if } \lambda \in [0, \infty), \\ (u^0_{\varepsilon,\infty})'(\ell_0; \ell_0) & \text{if } \lambda = \infty \end{cases}$$

and

$$(\ell_0, \ell_1, \lambda) \mapsto \begin{cases} u'_{\varepsilon,\lambda}(\ell_0, \ell_1; \ell_1) & \text{if } \lambda \in [0, \infty), \\ (u^1_{\varepsilon,\infty})'(\ell_1; \ell_1) & \text{if } \lambda = \infty \end{cases}$$

are continuous.

Proof. (i) Existence and uniqueness for minimizers for J_ε^0 and J_ε^1 is obtained in a similar way to other basic solutions. The existence of a minimizer $u_{\varepsilon,\lambda}(t)$ for $J_{\varepsilon,\lambda}$, for small $\varepsilon > 0$ is also rather standard, so we only deal with uniqueness. Let us consider the rescaled function $\tilde u_{\varepsilon,\lambda}(t) = u_{\varepsilon,\lambda}(\varepsilon t) : [\ell_0/\varepsilon, \ell_1/\varepsilon] \to \mathbb{R}$, which satisfies

$$\tilde u'' + \sin\tilde u = \varepsilon^2\alpha(\varepsilon t)\tilde u + \lambda\chi(\varepsilon t)(\tilde u - (2n_\varepsilon + 1)\pi) \qquad \text{in } (\ell_0/\varepsilon, \ell_1/\varepsilon),$$

$\tilde u(\ell_0/\varepsilon) = \tilde u(\ell_1/\varepsilon) = 2n_\varepsilon\pi$ and $\tilde u(t) \in [2n_\varepsilon\pi, (2n_\varepsilon + 1)\pi + \delta_0]$. It is easily observed that as $\varepsilon \to 0$

$$\tilde u_{\varepsilon,\lambda}(t + \frac{\ell_0}{\varepsilon}) - 2n_\varepsilon\pi \to \zeta_0(t) \qquad \text{in } C^1_{loc}([0, \infty), \mathbb{R}),$$
$$\tilde u_{\varepsilon,\lambda}(t + \frac{\ell_1}{\varepsilon}) - 2n_\varepsilon\pi \to \zeta_0(-t) \qquad \text{in } C^1_{loc}((-\infty, 0], \mathbb{R}),$$

where $\zeta_0(t)$ is the heteroclinic solution of (1.5), joining $-\pi$ and π and satisfying $\zeta(0) = 0$, $\zeta'(0) = 2$. Then, for any $\delta > 0$ there exists $r_\delta > 0$ such that

$$\tilde u_{\varepsilon,\lambda}\left(r_\delta + \frac{\ell_0}{\varepsilon}\right), \ \tilde u_{\varepsilon,\lambda}\left(-r_\delta + \frac{\ell_1}{\varepsilon}\right) \in [(2n_\varepsilon + 1)\pi - \delta, (2n_\varepsilon + 1)\pi + \delta],$$

for ε small. Comparing $\tilde u_{\varepsilon,\lambda}(t)$ with a solution of

$$w'' - (1 \mp \nu_0)w = \mp\alpha(\varepsilon t)(2n_\varepsilon + 1)\pi,$$
$$w\left(r_\delta + \frac{\ell_0}{\varepsilon}\right) = w\left(-r_\delta + \frac{\ell_1}{\varepsilon}\right) = (2n_\varepsilon + 1)\pi \pm \delta,$$

which is independent of λ, as in section 3, we have for all $\lambda \in [0, \infty)$

$$|\ \tilde u_{\varepsilon,\lambda}(t) - (2n_\varepsilon + 1)\pi\ | \leqslant a_1\varepsilon^3 + a_2 \mathrm{e}^{-\sqrt{1-\nu_0}\min\{|t-(r_\delta+\frac{\ell_0}{\varepsilon})|, |t-(-r_\delta+\frac{\ell_1}{\varepsilon})|\}} \tag{5.7}$$

for $t \in [r_\delta + \ell_0/\varepsilon, -r_\delta + \ell_1/\varepsilon]$, where a_1, a_2 are independent of ε and λ and satisfies $|a_1|$, $|a_2| \leqslant C\delta$. Thus we see that

$$\|\tilde u_{\varepsilon,\lambda}(t) - \left(\zeta_0\left(t - \frac{\ell_0}{\varepsilon}\right) + \zeta_0\left(-\left(t - \frac{\ell_1}{\varepsilon}\right)\right) + 2n_\varepsilon\pi\right)\|_{L^\infty\left(r_\delta+\frac{\ell_0}{\varepsilon}, -r_\delta+\frac{\ell_1}{\varepsilon}\right)} \to 0 \tag{5.8}$$

as $\varepsilon \to 0$ uniformly in $\lambda \in [0, \infty)$. Next we observe that there exists $C_5 > 0$ such that

$$\int_0^\infty |\ h'\ |^2 - (\cos\zeta_0(t))h^2\,\mathrm{d}t \geqslant C_5\|h\|^2_{H^1(0,\infty)}, \qquad \text{for all } h \in H^1_0(0, \infty),$$

as follows from uniqueness of the heteroclinic orbit for (1.5). It also follows the existence of $C_5' > 0$ independent of a, b such that for $|\ b - a\ | \gg 1$

$$\int_a^b |\ h'\ |^2 - (\cos(\zeta_0(t - a) + \zeta_0(-(t - b))))h^2\,\mathrm{d}t \geqslant C_5'\|h\|^2_{H^1(a,b)}. \tag{5.9}$$

Now we show uniqueness of a minimizer for $J_{\varepsilon,\lambda}$. Arguing indirectly, we assume it has two minimizers $u^1_{\varepsilon,\lambda}(t)$ and $u^2_{\varepsilon,\lambda}(t)$. Setting $\tilde{u}^i_{\varepsilon,\lambda}(t) = u^i_{\varepsilon,\lambda}(\varepsilon t)$ $(i = 1, 2)$, we see that $h(t) = \tilde{u}^1_{\varepsilon,\lambda}(t) - \tilde{u}^2_{\varepsilon,\lambda}(t)$ satisfies

$$h'' - \cos(\theta(t)\tilde{u}^1_{\varepsilon,\lambda}(t) + (1-\theta(t))\tilde{u}^2_{\varepsilon,\lambda}(t))h = \varepsilon^2\alpha(\varepsilon t)h + \lambda\chi(\varepsilon t)h$$

for some function $\theta(t) : [\ell_0/\varepsilon, \ell_1/\varepsilon] \to [0, 1]$. Multiplying by h and integrating on $[\ell_0/\varepsilon, \ell_1/\varepsilon]$, we have

$$\begin{aligned}&\int_{\ell_0/\varepsilon}^{\ell_1/\varepsilon} |h'|^2 - \cos(\theta(t)\tilde{u}^1_{\varepsilon,\lambda}(t) + (1-\theta(t))\tilde{u}^2_{\varepsilon,\lambda}(t))h^2\,\mathrm{d}t\\ &+\varepsilon^2\int_{\ell_0/\varepsilon}^{\ell_1/\varepsilon} \alpha(\varepsilon t)h^2\,\mathrm{d}t + \lambda\int_{\ell_0/\varepsilon}^{\ell_1/\varepsilon} \chi(\varepsilon t)h^2\,\mathrm{d}t = 0.\end{aligned}$$

Since both of $\tilde{u}^i_{\varepsilon,\lambda}(t)$ $(i = 1, 2)$ satisfy (5.8), we find that $h(t) \equiv 0$ for all $\lambda \in [0, \infty)$ for ε small, hence uniqueness of a minimizer.

(ii) Now we consider the behaviour of $\tilde{u}_{\varepsilon,\lambda}(t)$ as $\lambda \to \infty$, for ε fixed. Since $\tilde{u}_{\varepsilon,\lambda}(t)$ minimizes

$$\begin{aligned}\tilde{J}_{\varepsilon,\lambda}(u) = &\int_{\ell_0/\varepsilon}^{\ell_1/\varepsilon} \frac{1}{2} \mid u' \mid^2 + (\cos u + 1) + \frac{\varepsilon^2}{2}\alpha(\varepsilon t)u^2\,\mathrm{d}t\\ &+ \frac{\lambda}{2}\int_{c_0/\varepsilon}^{c_1/\varepsilon} \mid u - (2n_\varepsilon + 1)\pi \mid^2\,\mathrm{d}t,\end{aligned}$$

we have, for some constant M_ε independent of λ, that

$$\int_{c_0/\varepsilon}^{c_1/\varepsilon} |\tilde{u}'_{\varepsilon,\lambda}|^2\,\mathrm{d}t, \quad \lambda\int_{c_0/\varepsilon}^{c_1/\varepsilon} \mid \tilde{u}_{\varepsilon,\lambda} - (2n_\varepsilon + 1)\pi \mid^2\,\mathrm{d}t \leqslant M_\varepsilon. \tag{5.10}$$

Recalling (5.7), it follows from (5.10) that

$$\max_{t\in[c_0/\varepsilon, c_1/\varepsilon]} \mid \tilde{u}_{\varepsilon,\lambda}(t) - (2n_\varepsilon + 1)\pi \mid \to 0$$

as $\lambda \to \infty$, which implies (ii). □

6. Nehari Method and the Existence of single clusters

In this section we will construct a solution of (1.1) exhibiting a single increasing cluster by using the Nehari method whenever we have isolated solutions of (4.1)–(4.2).

We prove the following existence result.

Theorem 6.1. *Let $R > 0$ and $R' > 0$. We assume that the class $\mathcal{A}$ of all solutions of (4.1)–(4.2) in $[a, b]$ with R and R' is non-empty and isolated in $[a, b]$ as defined in section 4. Then there exists a family $\{u_\varepsilon\}$ of solutions of (1.1) in $(-L, L)$ such that after extracting a subsequence*

$$A_\varepsilon(t) \equiv A(E_\varepsilon(t)),$$

where $E_\varepsilon(t)$ is defined in (1.4), converges to some $A(t) \in \mathcal{A}$ as $\varepsilon \to 0$.

To show our theorem 6.1, we consider natural numbers n_ε and m_ε such that

$$\lim_{\varepsilon\to 0} \varepsilon n_\varepsilon = R \qquad \text{and} \qquad \lim_{\varepsilon\to 0} \varepsilon m_\varepsilon = R' \tag{6.1}$$

and, for a positive number τ to be determined later, we define the set

$$\begin{aligned}\Delta_\varepsilon = \{\vec{t} = (t_1, t_2, ..., t_{m_\varepsilon})/t_1 - a \geqslant \tau\varepsilon,\ t_{i+1} - t_i \geqslant \tau\varepsilon, i = 1, ..., m_\varepsilon - 1,\\ b - t_{m_\varepsilon} \geqslant \tau\varepsilon\}.\end{aligned}$$

Given $\vec{t} \in \Delta_\varepsilon$ we consider basic solutions u_i, $i = 0, ..., m_\varepsilon$ in the following way (we refer to section 5):
(i) u_0 is a left increasing solution in the interval $[-L, t_1]$, with boundary values

$$u_0(-L) = (2n_\varepsilon + 1)\pi \qquad \text{and} \qquad u_0(t_1) = 2(n_\varepsilon + 1)\pi,$$

(ii) u_i, for $i = 1, ..., m_\varepsilon - 1$, is an increasing interior solutions in the interval $[t_i, t_{i+1}]$ with boundary values

$$u_i(t_i) = 2(n_\varepsilon + i)\pi \qquad \text{and} \qquad u_i(t_{i+1}) = 2(n_\varepsilon + i + 1)\pi$$

and
(iii) u_{m_ε} is a right increasing solution in the interval $[t_{m_\varepsilon}, L]$ with boundary values

$$u_{m_\varepsilon}(t_{m_\varepsilon}) = 2(n_\varepsilon + m_\varepsilon)\pi \qquad \text{and} \qquad u_{m_\varepsilon}(L) = (2(n_\varepsilon + m_\varepsilon) + 1)\pi.$$

Then we define the function $I_\varepsilon : \Delta_\varepsilon \to \mathbb{R}^{m_\varepsilon}$ as

$$(I_\varepsilon)_i(\vec{t}) = u_i'(t_i) - u_{i-1}'(t_i), \qquad \text{for} \quad i = 1, ..., m_\varepsilon. \tag{6.2}$$

We observe that defining $u_\varepsilon : [-L, L] \to \mathbb{R}$ as $u_\varepsilon(t) = u_i(t)$ if $t \in [t_i, t_{i+1}]$, the function u_ε is a solution of (1.1) provided

$$I_\varepsilon(\vec{t}) = 0. \tag{6.3}$$

In order to find a solution of this equation we will use the degree theory. It will be sufficient to prove that

Proposition 6.1. *There exists* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon \leqslant \varepsilon_0$

$$\deg(I_\varepsilon, \Delta_\varepsilon, 0) = 1.$$

In what follows we prove this proposition. For this purpose we will define an appropriate homotopy in two steps, first we reduce the interval for the equation (1.1) and second we move α to zero.

Accordingly we consider $\lambda \in [0, 1]$ and the equation

$$\varepsilon^2 u'' + \sin u = \min\{1, 2\lambda\}\varepsilon^2\alpha(t)u_\varepsilon \qquad \text{in} \quad t \in [a_\lambda, b_\lambda], \tag{6.4}$$

$$u_\varepsilon(a_\lambda) = (2n_\varepsilon + 1)\pi, \quad u_\varepsilon(b_\lambda) = (2(n_\varepsilon + m_\varepsilon) + 1)\pi, \tag{6.5}$$

where

$$a_\lambda = -(2\lambda - 1)_+ L + (1 - (2\lambda - 1)_+)a$$

and

$$b_\lambda = (2\lambda - 1)_+ L + (1 - (2\lambda - 1)_+)b.$$

We observe that if $\lambda \in [0, 1/2]$ then the equation (6.4) is considered in the interval $[a, b]$ and the right hand side goes from 0 to $\varepsilon^2\alpha u_\varepsilon$, while if $\lambda \in [1/2, 1]$ then equation (6.4) has a fixed right hand side $\varepsilon^2\alpha u_\varepsilon$ and it is considered in an interval going from $[-L, L]$ to $[a, b]$. In both cases the boundary values are the same.

Next, for every $\lambda \in [0, 1]$ we may consider the equation for the area as

$$A_\lambda'(t) = (2\pi)^2\alpha_\lambda(t)\left(R + \int_a^t \frac{\mathrm{d}s}{T(A_\lambda(s))}\right), \qquad t \in (a_\lambda, b_\lambda), \tag{6.6}$$

with the condition

$$\int_a^b \frac{\mathrm{d}s}{T(A_\lambda(s))} = R', \tag{6.7}$$

here we denoted $\alpha_\lambda = \min\{1, 2\lambda\}\alpha$. As in proposition 4.3, we can prove that the class $\mathcal{A}_{[0,1]}$ of all solutions of (6.6) with (6.7) and $\lambda \in [0, 1]$ is compact with the supremum norm. Now we consider the function $E(A_\lambda)$, the energy function associated to $A_\lambda \in \mathcal{A}_{[0,1]}$, then there is E_0 such that

$$E_0 = \sup\{E(A_\lambda(t)) \,/\, t \in [a_\lambda, b_\lambda],\, A_\lambda \in \mathcal{A}_{[0,1]},\ \lambda \in [0,1]\}.$$

Then we choose the number τ used in the definition of Δ_ε as $\tau > 0$ as

$$\frac{2\pi^2}{\tau^2} - 2 > E_0. \tag{6.8}$$

Now we are in a position to start the proof of proposition 6.1. Given $\lambda \in [0,1]$ we consider equation (6.4)–(6.5). We may construct basic solutions as in section 5 and then, given $\vec{t} \in \Delta_\varepsilon$ we may proceed as before to define $H_\varepsilon(\vec{t}, \lambda)$ as in (6.2). Our next lemma allows to define the degree for H and, in particular the degree of I_ε.

Lemma 6.1. *There is $\varepsilon_0 > 0$ so that, for all $0 < \varepsilon \leqslant \varepsilon_0$ and $\lambda \in [0,1]$ the equation*

$$H_\varepsilon(\vec{t}, \lambda) = 0, \qquad \vec{t} \in \partial\Delta_\varepsilon$$

does not have a solution.

Proof. Let us assume that there is a sequence $\varepsilon_n \to 0$, and $\lambda_n \to \bar{\lambda}$ and $\vec{t}_n \in \partial\Delta_{\varepsilon_n}$ such that

$$H_{\varepsilon_n}(\vec{t}_n, \lambda_n) = 0.$$

On one hand we find a sequence of solutions u_{ε_n} that gives rise to a limiting energy function $E_{\bar{\lambda}}$ defined in $[a_{\bar{\lambda}}, b_{\bar{\lambda}}]$ according to proposition 2.2 in section 2. Since the solutions u_{ε_n} were constructed with n_ε and m_ε satisfying (6.1), we see that

$$E_{\bar{\lambda}}(t) \leqslant E_0, \quad \text{for all} \quad t \in [a_{\bar{\lambda}}, b_{\bar{\lambda}}]. \tag{6.9}$$

On the other hand, since $\vec{t}^n \in \partial\Delta_{\varepsilon_n}$, there exists a sequence $t^n_{i_n} \to \bar{t}$ such that:

(1) $t^n_{i_n+1} - t^n_{i_n} = \varepsilon_n\tau$, with $1 \leqslant i_n \leqslant m_{\varepsilon_n} - 1$,
(2) $t^n_1 - a = \varepsilon_n\tau$ or
(3) $b - t^n_{m_{\varepsilon_n}} = \varepsilon_n\tau$.

If (1) holds then there exists $\tilde{t}_n \in (t^n_{i_n}, t^n_{i_n+1})$ such that

$$u'_{\varepsilon_n}(\tilde{t}_n) = \frac{2\pi}{\varepsilon_n\tau},$$

from where

$$E_{\varepsilon_n,\lambda_n}(\tilde{t}_n) = \frac{1}{2}(\varepsilon_n u'_{\varepsilon_n}(\tilde{t}_n))^2 - (\cos u_{\varepsilon_n}(\tilde{t}_n) + 1) \geqslant \frac{2\pi^2}{\tau^2} - 2.$$

From here we obtain $E_{\bar{\lambda}}(\bar{t}) \geqslant \frac{2\pi^2}{\tau^2} - 2$, which contradicts (6.8).

If (2) holds, then we argue depending on the value of $\bar{\lambda}$. In case $\bar{\lambda} \in [0, 1/2)$, then we may use the argument given above, with slight changes, to get a contradiction. In case $\bar{\lambda} \in [1/2, 1]$, we observe that $A_{\bar{\lambda}}(t) = A(E_{\bar{\lambda}}(t))$ is a solution of (4.1)–(4.2) in $[a, b]$. By the isolatedness of $\mathcal{A}$, we have $A_{\bar{\lambda}}(t) = 8$ in a neighbourhood $[a, a+\sigma]$ of a, which implies $E_{\bar{\lambda}}(t) = 0$ in $[a, a+\sigma]$. By the construction, $u_\varepsilon(t)$ is growing in $[a, b]$. Thus by proposition 3.1, $u_\varepsilon(t) \notin 2\pi\mathbb{Z}$ in $(a, a+\frac{\delta}{2}]$. Therefore case (2) cannot take a place for ε small.

Case (3) is similar to (2). □

Proof of proposition 6.1. By lemma 6.1 we only need to compute the degree of $H_\varepsilon(\cdot, 0)$, which is associated to the autonomous equation (1.5). Let $T > 0$ and $u(t; T)$ be the solution of (1.5) of rotation type satisfying $u(0; T) = -\pi$, $u(T; T) = \pi$ and denote $v(T) = u'(0; T)$. Then we have

$$T = \int_{-\pi}^{\pi} \frac{\mathrm{d}s}{\sqrt{v(T)^2 + 2(\cos s + 1)}},$$

so that

$$\frac{\mathrm{d}v}{\mathrm{d}T} = -\left(\int_{-\pi}^{\pi} \frac{v(T)\,\mathrm{d}s}{(v(T)^2+2(\cos s+1))^{3/2}}\right)^{-1} < 0.$$

Thus, at the unique solution equation

$$H_\varepsilon(\vec{t},0)=0, \qquad \vec{t}\in\Delta_\varepsilon,$$

given by $\vec{t}_\varepsilon = a(1,1,\cdots,1)+T_\varepsilon(1,2,\cdots,m_\varepsilon)$, with $T_\varepsilon=(b-a)/(m_\varepsilon+1)$, we have that the derivative is

$$DH_\varepsilon(\vec{t}_\varepsilon,0)=\frac{1}{\varepsilon^2}\frac{\mathrm{d}v}{\mathrm{d}T}(\varepsilon^{-1}T_\varepsilon)M,$$

where M is the matrix with -2 in the diagonal and 1 in the upper and lower diagonal. Then we find that

$$\deg(H_\varepsilon(\vec{t},0),\Delta_\varepsilon,0)=1,$$

which completes the proof. □

7. Nehari Method and the Existence of multiple clusters

In this section we will construct a solution of (1.1) exhibiting a single increasing cluster and a single decreasing cluster. Construction solutions exhibiting with more than 2 clusters can be done essentially in a similar way.

We deal with the following situation: let $R_1,\ R_1'>0$, $R_2=R_1+R_1'$, $R_2'>0$ with $R_1+R_1'>R_2'$ and $-L<a_1<b_1<c_1<c_2<a_2<b_2<L$. We consider a class $\mathcal{A}^1$ of all solutions of (4.1)–(4.2) with $R=R_1$, $R'=R_1'$ in $[a_1,b_1]$ and a class $\mathcal{A}^2$ of all solutions in $[a_2,b_2]$ of

$$A'(t)=-(2\pi)^2\alpha(t)\left(R_2-\int_{a_2}^{t}\frac{1}{T(A(s))}\,\mathrm{d}s\right) \qquad \text{in} \quad [a_2,b_2],$$

$$\int_{a_2}^{b_2}\frac{1}{T(A(s))}\,\mathrm{d}s=R_2'.$$

We show the following

Theorem 7.1. *Assume that $\mathcal{A}^i$ is non-empty and isolated in $[a_i,b_i]$ for $i=1,2$. Then there exists a family $\{u_\varepsilon(t)\}$ of solutions of (1.1) in $(-L,L)$ such that after extracting a subsequence*

$$A_\varepsilon(t)=A(E_\varepsilon(t)),$$

where $E_\varepsilon(t)$ is defined in (1.4), converges to some $A(t)$ as $\varepsilon\to 0$. Moreover $\operatorname{supp} A\subset(a_1,b_1)\cup(a_2,b_2)$ *and* $A\big|_{[a_i,b_i]}\in\mathcal{A}^i$ *for* $i=1,2$.

To prove theorem 7.1 we consider natural numbers $n_\varepsilon^1, m_\varepsilon^1, n_\varepsilon^2, m_\varepsilon^2$ so that

$$\lim_{\varepsilon\to 0}\varepsilon n_\varepsilon^1=R_1, \qquad \lim_{\varepsilon\to 0}\varepsilon m_\varepsilon^1=R_1',$$
$$\lim_{\varepsilon\to 0}\varepsilon n_\varepsilon^2=R_2=R_1+R_1', \qquad \lim_{\varepsilon\to 0}\varepsilon m_\varepsilon^2=R_2'.$$

We choose $\tau>0$ so that

$$\frac{2\pi^2}{\tau^2}-2\geqslant \sup\{E(A_i(t))\,/\,A_i\in\mathcal{A}^i,\ t\in[a_i,b_i],\ i=1,2\}$$

and we define

$$\begin{aligned}
\Delta_\varepsilon^1 = \{\vec{t}^1 = (t_1^1, \dots, t_{m_\varepsilon^1}^1) \,/\, & t_1^1 - a_1 \geqslant \tau\varepsilon,\ t_{i+1}^1 - t_i^1 \geqslant \tau\varepsilon\ (i = 1, 2, \dots, m_\varepsilon^1), \\
& b_1 - t_{m_\varepsilon^1}^1 \geqslant \tau\varepsilon\}, \\
\Delta_\varepsilon^2 = \{\vec{t}^2 = (t_1^2, \dots, t_{m_\varepsilon^2}^2) \,/\, & t_1^2 - a_2 \geqslant \tau\varepsilon,\ t_{i+1}^2 - t_i^2 \geqslant \tau\varepsilon\ (i = 1, 2, \dots, m_\varepsilon^2), \\
& b_2 - t_{m_\varepsilon^2}^2 \geqslant \tau\varepsilon\}.
\end{aligned}$$

For a given $(\vec{t}^1, \vec{t}^2) \in \Delta_\varepsilon^1 \times \Delta_\varepsilon^2$ we consider basic solutions $u_i(t)$ $(i = 0, 1, \dots, m_\varepsilon^1 + m_\varepsilon^2)$ in the following way. For the homotopy argument, our basic solution in the interval $[t_{m_\varepsilon^1}^1, t_1^2]$ depends on the parameter $\lambda \in [0, \infty]$ which is due to introduction of a penalty term $\lambda\chi(t)(u - (2(n_\varepsilon^1 + m_\varepsilon^1) + 1)\pi)$.

(i) $u_0(t)$ is a left increasing solution in $[-L, t_1^1]$ with $u_0(-L) = (2n_\varepsilon^1 + 1)\pi$, $u_0(t_1^1) = 2(n_\varepsilon^1 + 1)\pi$.

(ii) $u_i(t)$ $(i = 1, 2, \dots, m_\varepsilon^1 - 1)$ is an increasing interior solution in $[t_i^1, t_{i+1}^1]$ with $u_i(t_i^1) = 2(n_\varepsilon^1 + i)\pi$, $u_i(t_{i+1}^1) = 2(n_\varepsilon^1 + i + 1)\pi$.

(iii) For $\lambda \in [0, \infty)$, $u_{m_\varepsilon^1}^\lambda(t)$ is an increasing-decreasing glueing solution of the following equation with a penalty:

$$\begin{aligned}
&\varepsilon^2 u'' + \sin u = \varepsilon^2\alpha(t)u + \lambda\chi(t)(u - (2(n_\varepsilon^1 + m_\varepsilon^1) + 1)\pi), \\
&u(t_{m_\varepsilon^1}^1) = 2(n_\varepsilon^1 + m_\varepsilon^1)\pi, \\
&u(t_1^2) = 2(n_\varepsilon^1 + m_\varepsilon^1)\pi, \\
&u'(t_{m_\varepsilon^1}^1) > 0.
\end{aligned}$$

Here $\chi(t)$ is defined by $\chi(t) = 1$ if $t \in [c_1, c_2]$, $\chi(t) = 0$ otherwise.

For $\lambda = \infty$, we set

$$u_{m_\varepsilon^1}^\infty(t) = \begin{cases}
\text{a left increasing solution in } [t_{m_\varepsilon}^1, t_1^2] \text{ with} & \\
\quad u_{m_\varepsilon^1}^\infty(t_{m_\varepsilon}^1) = 2(n_\varepsilon^1 + m_\varepsilon^1)\pi,\ u_{m_\varepsilon^1}^\infty(c_1) = (2(n_\varepsilon^1 + m_\varepsilon^1) + 1)\pi, & \text{in } [t_{m_\varepsilon}^1, c_1], \\
(2(n_\varepsilon^1 + m_\varepsilon^1) + 1)\pi & \text{in } [c_1, c_2], \\
\text{a right decreasing solution in } [c_2, t_1^2] \text{ with} & \\
\quad u_{m_\varepsilon^1}^\infty(c_2) = (2(n_\varepsilon^1 + m_\varepsilon^1) + 1)\pi,\ u_{m_\varepsilon^1}^\infty(t_1^2) = 2(n_\varepsilon^1 + m_\varepsilon^1)\pi & \text{in } [c_2, t_1^2].
\end{cases}$$

(iv) $u_{m_\varepsilon^1+i}(t)$ $(i = 1, 2, \dots, m_\varepsilon^2 - 1)$ is a decreasing interior solution in $[t_i^2, t_{i+1}^2]$ with $u_{m_\varepsilon^1+i}(t_i^2) = 2(n_\varepsilon^1 + m_\varepsilon^1 + 1 - i)\pi$, $u_{m_\varepsilon^1+i}(t_{i+1}^2) = 2(n_\varepsilon^1 + m_\varepsilon^1 - i)\pi$.

(v) $u_{m_\varepsilon^1+m_\varepsilon^2}(t)$ is a right decreasing solution in $[t_{m_\varepsilon^2}^2, L]$ with $u_{m_\varepsilon^1+m_\varepsilon^2}(t_{m_\varepsilon^2}^2) = 2(n_\varepsilon^1 + m_\varepsilon^1 - m_\varepsilon^2 + 1)\pi$, $u_{m_\varepsilon^1+m_\varepsilon^2}(L) = (2(n_\varepsilon^1 + m_\varepsilon^1 - m_\varepsilon^2) + 1)\pi$.

Then for $\lambda \in [0, \infty]$ we define the function $I_\varepsilon^\lambda : \Delta_\varepsilon^1 \times \Delta_\varepsilon^2 \to \mathbb{R}^{m_\varepsilon^1 + m_\varepsilon^2}$ by

$$(I_\varepsilon^\lambda)_i(\vec{t}^1, \vec{t}^2) = \begin{cases}
u_i'(t_i) - u_{i-1}'(t) & \\
\qquad \text{if } i \in \{1, 2, \dots, m_\varepsilon^1 + m_\varepsilon^2\} \setminus \{m_\varepsilon^1, m_\varepsilon^1 + 1\}, & \\
(u_{m_\varepsilon^1}^\lambda)'(t_{m_\varepsilon^1}) - u_{m_\varepsilon^1 - 1}'(t_{m_\varepsilon^1}) & \text{if } i = m_\varepsilon^1, \\
u_{m_\varepsilon^1+1}'(t_{m_\varepsilon^1+1}) - (u_{m_\varepsilon^1}^\lambda)'(t_{m_\varepsilon^1+1}) & \text{if } i = m_\varepsilon^1 + 1.
\end{cases}$$

Here we use convention $t_i = t_i^1$ for $i = 1, 2, \dots, m_\varepsilon^1$ and $t_{m_\varepsilon^1+i} = t_i^2$ for $i = 1, 2, \dots, m_\varepsilon^2$.
We remark that when $\lambda = \infty$, $(I_\varepsilon^\infty)_i(\vec{t}^1, \vec{t}^2)$, $i = 1, \dots, m_\varepsilon^1$, (respectively $(I_\varepsilon^\infty)_{m_\varepsilon^1+i}(\vec{t}^1, \vec{t}^2)$, $i = 1, \dots, m_\varepsilon^2$) does not depend on $\vec{t}^2$ (respectively $\vec{t}^1$). Thus we can write

$$I_\varepsilon^\infty(\vec{t}^1, \vec{t}^2) = (I_{\varepsilon,1}(\vec{t}^1), I_{\varepsilon,2}(\vec{t}^2)). \tag{7.1}$$

Here $I_{\varepsilon,1}(\vec{t}^1)$ (respectively $I_{\varepsilon,2}(\vec{t}^2)$) is corresponding to a solution with a single cluster $[a_1, b_1] \subset (-L, c_1)$ (respectively $[a_2, b_2] \subset (c_2, L)$). By the result in the previous section, we have for ε small

$$\deg(I_\varepsilon^1, \Delta_\varepsilon^1, 0) = 1, \tag{7.2}$$
$$\deg(I_\varepsilon^2, \Delta_\varepsilon^2, 0) = 1. \tag{7.3}$$

For $(\vec{t}^1, \vec{t}^2) \in \Delta_\varepsilon^1 \times \Delta_\varepsilon^2$, we observe that defining $u_\varepsilon(t) = u_i(t)$ in $[t_i, t_{i+1}]$, $u_\varepsilon(t)$ solves

$$\varepsilon^2 u'' + \sin u = \varepsilon^2 \alpha(t) u + \lambda \chi(t)(u - (2(n_\varepsilon^1 + m_\varepsilon^1) + 1)\pi) \quad \text{in } (-L, L) \tag{7.4}$$

if and only if $I_\varepsilon^\lambda(\vec{t}^1, \vec{t}^2) = 0$. In particular, $u_\varepsilon(t)$ satisfies (1.1) if and only if $I_\varepsilon^0(\vec{t}^1, \vec{t}^2) = 0$.

To find a solution of $I_\varepsilon^0(\vec{t}^1, \vec{t}^2) = 0$, we use the Brouwer degree. We show

Proposition 7.1. *There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$*

$$\deg(I_\varepsilon^0, \Delta_\varepsilon^1 \times \Delta_\varepsilon^2, 0) = (-1)^{m_\varepsilon^1 + m_\varepsilon^2}. \tag{7.5}$$

Proof. To show (7.5), first we show for ε small

$$I_\varepsilon^\lambda(\vec{t}^1, \vec{t}^2) \neq 0 \quad \text{for all } \lambda \in [0, \infty] \text{ and } (\vec{t}^1, \vec{t}^2) \in \partial(\Delta_\varepsilon^1 \times \Delta_\varepsilon^2). \tag{7.6}$$

Suppose that there exists sequences $(\vec{t}^1_{\varepsilon_n}, \vec{t}^2_{\varepsilon_n}) \in \Delta_\varepsilon^1 \times \Delta_\varepsilon^2$ with $\varepsilon_n \to 0$ and $\lambda_n \in [0, \infty]$ such that $I_{\varepsilon_n}^{\lambda_n}(\vec{t}^1_{\varepsilon_n}, \vec{t}^2_{\varepsilon_n}) = 0$. Then the corresponding $u_{\varepsilon_n}(t)$ solves (7.4). We set $E_{\varepsilon_n}(t) = \frac{1}{2}(\varepsilon_n u'_{\varepsilon_n})^2 - (\cos u_{\varepsilon_n} + 1)$ and $E(t) = \lim_{n\to\infty} E_{\varepsilon_n}(t)$ (if necessary, we take a subsequence). We can see that $\mathrm{supp} E(t) \subset [a_1, b_1] \cup [a_2, b_2]$. We define $A_i(t) = A(E(t))|_{[a_i, b_i]}$ $(i = 1, 2)$. Then we can see that $A_i \in \mathcal{A}^i$ $(i = 1, 2)$. Thus, by the isolatedness of $\mathcal{A}^i$, we can see as in the proof of lemma 6.1 that $\vec{t}^i \in \partial \Delta_\varepsilon^i$ cannot take a place for large n for $i = 1, 2$. Thus (7.6) holds and we have

$$\deg(I_\varepsilon^0, \Delta_\varepsilon^1 \times \Delta_\varepsilon^2, 0) = \deg(I_\varepsilon^\infty, \Delta_\varepsilon^1 \times \Delta_\varepsilon^2, 0).$$

By (7.1), (7.2), (7.3), we have

$$\begin{aligned}
\deg(I_\varepsilon^\infty, \Delta_\varepsilon^1 \times \Delta_\varepsilon^2, 0) &= \deg((I_{\varepsilon,1}, I_{\varepsilon,2}), \Delta_\varepsilon^1 \times \Delta_\varepsilon^2, 0) \\
&= \deg(I_{\varepsilon,1}, \Delta_\varepsilon^1, 0) \cdot \deg(I_{\varepsilon,2}, \Delta_\varepsilon^2, 0) \\
&= 1.
\end{aligned}$$

□

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