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# Axisymmetric bifurcations of thick spherical shells under inflation and compression

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ABSTRACT

Incremental equilibrium equations and corresponding boundary conditions for an isotropic, hyperelastic and incompressible material are summarized and then specialized to a form suitable for the analysis of a spherical shell subject to an internal or an external pressure. A thick-walled spherical shell during inflation is analyzed using four different material models. Specifically, one and two terms in the Ogden energy formulation, the Gent model and an $I_1$ formulation recently proposed by Lopez-Pamies. We investigate the existence of local pressure maxima and minima and the dependence of the corresponding stretches on the material model and on shell thickness. These results are then used to investigate axisymmetric bifurcations of the inflated shell. The analysis is extended to determine the behavior of a thick-walled spherical shell subject to an external pressure. We find that the results of the two terms Ogden formulation, the Gent and the Lopez-Pamies models are very similar, for the one term Ogden material we identify additional critical stretches, which have not been reported in the literature before.


## 1. Introduction

On several occasions the behavior of an inflating spherical *membrane* has been investigated to provide information on the pressure-deformation response. Needleman (1977), Chen and Healey (1991), Müller and Struchtrup (2002), Goriely et al. (2006), Beatty (2011) and Rudykh et al. (2012), among others, have shown that the spherical configuration is maintained during initial inflation up to a pressure maximum at which point the membrane expands rapidly thereby reducing the internal pressure. While the pressure is reducing a sequence of aspherical modes occurs until a local minimum is reached where the spherical symmetry is reestablished. For further inflation, an increase in pressure is always accompanied by an increase in radius. If both, a pressure maximum and minimum exist, bifurcation points of a given mode $n$ appear in pairs, as pointed out by Haughton and Ogden (1978b), Haughton (1980) and Haughton and Kirkinis (2003). As we show numerically, the prediction of a local pressure minimum depends on the constitutive model used in the analysis and on the wall thickness.

The general theory used to investigate stability of spherical shells of *arbitrary thickness* subject to uniform pressure is based on the existence of equilibria for small elastic deformations superimposed upon a finitely deformed configuration, see the important contributions by Green et al. (1952), Wesolowski (1967), Hill (1976), Haughton and Ogden (1978a,b) and Hill (1976) derived closed form solutions to predict the first aspherical buckling mode $n = 1$ for thick-walled spherical shells subject to an uniform external pressure. Specifically, he showed that the $n = 1$ mode does not occur for the neo-Hookean material and that for the Varga material the $n = 1$ mode is only possible for an inflated shell, i.e. negative external pressure. Haughton and Ogden (1978b), in their seminal paper, provide full details of the possible existence of aspherical bifurcation modes for membrane and thick-walled spherical shells subject to internal pressure. They showed that the existence of these modes depends, in addition to the material model, on the thickness of the shell. For a thickness above a critical value no aspherical equilibrium configurations are possible. For shells under compression Haughton and Ogden (1978b) list the critical stretches and the corresponding bifurcation modes $n$. Results are compared using the three terms Ogden formulation, the neo-Hookean and Varga models and a formulation proposed by Hill (1976).

The mechanical and geometrical effects of growth on the stability of a neo-Hookean thick-walled shell is given by Amar et al. (2005). In the context of biological tissues, the effect of strain hardening on the stability of a spherical shell is investigated by Goriely et al. (2006). Specifically, they found that a compressed shell of any thickness made of a Fung material is less stable than a neo-Hookean material.

This paper is organized as follows. Section 2 gives a brief overview of the basic equations in Eulerian and Lagrangian forms describing the stress-stretch response of an elastic materials subject to finite deformations. We derive the incremental equilibrium

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equations and incremental boundary conditions for an isotropic hyperelastic and incompressible material. These equations are well known and have been derived by Haughton and Ogden (1978b), included in the monograph Ogden (1997) and used by, for example, Haughton (1980, 1987), Chaplain and Sleeman (1992, 1993), Haughton and Kirkinis (2003), Amar et al. (2005) and Goriely et al. (2006). The derivations are included in Section 3 because they are needed to introduce proper notations and conventions and are relevant for the solution method adopted inhere. In Section 4 we derive the pressure-deformation response of a spherical shell due to an applied internal pressure. Using spherical coordinates, the incremental governing equations and boundary conditions are specialized to an axisymmetric incremental deformation superimposed upon a known spherically symmetric configuration.

The solution technique is addressed in Section 5, where we provide an outline on the use of the compound matrix method to solve the governing equations. The accuracy of this method has been verified and validated by Haughton (2008), where solutions obtained from the compound matrix method have been compared to the exact values determined by direct computation. The compound matrix method has been used by Haughton and Kirkinis (2003) and Dorfmann and Haughton (2006) to related problems.

In Section 6 we consider four strain energy functions to determine the pressure-stretch response of a spherical shell during inflation and then use the results to analyze aspherical bifurcations associated with modes $n = 1, 2, 3, \ldots$. Specifically, we consider one term and two terms in the formulation proposed by Ogden (1972), the model introduced by Gent (1996) and a generalized $I_1$ model recently published by Lopez-Pamies (2010). We investigate possible local pressure maxima and the corresponding critical stretches for an inflating spherical shell. We emphasize, that the existence of a local pressure minimum depends not only on the material model considered but also on the shell thickness, as pointed out by Haughton and Ogden (1978b). In Section 6.2 we use the compound matrix method to investigate axisymmetric bifurcations of a spherical shell of arbitrary thickness subject to an external pressure. We identify the critical stretches and corresponding aspherical modes and find that the two terms Ogden formulation, the Gent and the Lopez-Pamies models give very similar results. For the one term Ogden model we find additional critical stretches which, to the best of our knowledge, have not been recognized in the literature before.

## 2. Basic equations

### 2.1. Continuum kinematics

Consider a nonlinear elastic solid which is located in the fixed reference configuration $\mathcal{B}_0$ in the absence of any mechanical body forces. To describe the deformation, we denote a generic material point by its position vector $\mathbf{X}$ relative to an arbitrary chosen origin. Application of surface loads deforms the body, so that the point $\mathbf{X}$ occupies the new position $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ in the deformed configuration $\mathcal{B}$. The vector field $\boldsymbol{\chi}$ describes the deformation of the body and assigns to each point $\mathbf{X}$ a unique position $\mathbf{x}$ in $\mathcal{B}$ and viceversa attributes a unique reference position $\mathbf{X}$ in $\mathcal{B}_0$ to each point $\mathbf{x}$. In other words, the deformation function $\boldsymbol{\chi}$ is a one-to-one mapping with suitable regularity properties.

The deformation gradient tensor relative to $\mathcal{B}_0$ is defined by

$$\mathbf{F} = \mathrm{Grad}\,\mathbf{x}, \tag{1}$$

where Grad denotes the gradient operator with respect to $\mathbf{X}$. The Cartesian components are $F_{i\alpha} = \partial x_i / \partial X_\alpha$, where $x_i$ and $X_\alpha$ are the components of $\mathbf{x}$ and $\mathbf{X}$, respectively, with $i, \alpha \in \{1, 2, 3\}$. Roman indices are associated with $\mathcal{B}$ and Greek indices with $\mathcal{B}_0$. We also adopt the standard notation

$$J = \det \mathbf{F} = \frac{\mathrm{d}v}{\mathrm{d}V} > 0, \tag{2}$$

where $\mathrm{d}V$ and $\mathrm{d}v$ are volume elements in $\mathcal{B}_0$ and $\mathcal{B}$, respectively.

The deformation gradient can be decomposed according to the unique polar decomposition

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \tag{3}$$

where $\mathbf{R}$ is a proper orthogonal tensor and $\mathbf{U}$ and $\mathbf{V}$ are positive definite and symmetric, respectively the right and left stretch tensors. These can be expressed in spectral form. For $\mathbf{U}$, for example, we have the spectral decomposition

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \tag{4}$$

where the principal stretches $\lambda_i > 0$, $i \in \{1, 2, 3\}$, are the eigenvalues of $\mathbf{U}$, $\mathbf{u}^{(i)}$ are the corresponding (unit) eigenvectors, and $\otimes$ denotes the tensor product. For a volume preserving (isochoric) deformation, we have

$$J = \det \mathbf{F} = \det \mathbf{U} = \det \mathbf{V} = \lambda_1 \lambda_2 \lambda_3 \equiv 1. \tag{5}$$

Using the polar decomposition (3), we define

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^2, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{V}^2, \tag{6}$$

which denote the right and left Cauchy–Green deformation tensors respectively. The three principal invariants for $\mathbf{C}$, equivalently $\mathbf{B}$, are defined by

$$I_1 = \mathrm{tr}\,\mathbf{C}, \quad I_2 = \frac{1}{2}\left[(\mathrm{tr}\,\mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2)\right], \quad I_3 = \det \mathbf{C} = J^2, \tag{7}$$

where tr is the trace of a second-order tensor. Alternatively, in terms of the principal stretches, the invariants $I_1$, $I_2$, $I_3$ are

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{8}$$

### 2.2. Mechanical balance laws

The equilibrium equation in the current configuration in terms of the Cauchy stress $\boldsymbol{\sigma}$ and in the absence of mechanical body forces may be written in the form

$$\mathrm{div}\,\boldsymbol{\sigma} = \mathbf{0}, \tag{9}$$

where div is the divergence operator with respect to $\mathbf{x}$. Introducing the nominal stress tensor, denoted $\mathbf{T}$, and defined by

$$\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\sigma}, \tag{10}$$

allows the equilibrium Eq. (9) to be written in the alternative form

$$\mathrm{Div}\,\mathbf{T} = \mathbf{0}, \tag{11}$$

where Div is the divergence operator with respect to the reference configuration.

Let $\mathbf{t}_a$ denote the surface force per unit area at a point $\mathbf{x}$ on the surface $\partial\mathcal{B}$ of the current configuration $\mathcal{B}$. Then, the stress $\boldsymbol{\sigma}$ must satisfy

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{t}_a \quad \text{on} \quad \partial\mathcal{B}. \tag{12}$$

The traction boundary condition associated with (11) can be recast as

$$\mathbf{T}^{\mathrm{T}}\mathbf{N} = \mathbf{t}_A \quad \text{on} \quad \partial\mathcal{B}_0, \tag{13}$$

where $\partial\mathcal{B}_0$ denotes the surface of the material body in the reference configuration $\mathcal{B}_0$. The traction vector $\mathbf{t}_A$ is connected to $\mathbf{t}_a$ by

$\mathbf{t}_A dA = \mathbf{t}_a da$ where Nanson's formula $\mathbf{n} da = J\mathbf{F}^{-T}\mathbf{N} dA$ is used to relate the area elements.

### 2.3. Isotropic hyperelasticity

The theory of hyperelasticity characterizes the elastic response of a material by a strain energy function $W$ defined per unit volume in the reference configuration $\mathcal{B}_0$. For a homogeneous material $W$ depends only on the deformation gradient $\mathbf{F}$ and we write $W = W(\mathbf{F})$. The nominal stress tensor is then given by

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{F}}, \tag{14}$$

and, using (10) gives the expression of the corresponding Cauchy stress tensor

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}. \tag{15}$$

The expressions (14) and (15) apply for a material that is not subject to any internal mechanical constraint. If the material is incompressible, the deformation gradient satisfies the constraint

$$\det \mathbf{F} \equiv 1 \tag{16}$$

and the expressions for the nominal and Cauchy stress tensor require modifications. The nominal stress $\mathbf{T}$ and the Cauchy stress $\boldsymbol{\sigma}$ given in terms of $W$ are then amended in the forms

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{F}} - p\mathbf{F}^{-1}, \quad \boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{I}, \tag{17}$$

respectively, where $\mathbf{I}$ is the identity tensor and $p$ is commonly referred to as the *arbitrary hydrostatic pressure*.

For an *isotropic* elastic solid $W$ depends on the deformation only through the principal invariants defined in (7) such that $W = W(I_1, I_2, I_3)$. We now restrict our attention to incompressible material where $I_3 \equiv 1$ and the strain energy function can be written in the form $W = W(I_1, I_2)$. A direct calculation of Eqs. (17), using the derivatives of the strain invariants with respect to $\mathbf{F}$ given in A by Eq. (A.1), leads to

$$\mathbf{T} = 2(W_1 + I_1 W_2)\mathbf{F}^{T} - 2W_2\mathbf{C}\mathbf{F}^{T} - p\mathbf{F}^{-1} \tag{18}$$

and

$$\boldsymbol{\sigma} = 2(W_1 + I_1 W_2)\mathbf{B} - 2W_2\mathbf{B}^2 - p\mathbf{I}, \tag{19}$$

where the notation $W_i = \partial W/\partial I_i$ with $i \in \{1, 2\}$ has been introduced.

## 3. Incremental equations

### 3.1. Increments within the material

We now examine the effect of an incremental deformation superimposed on the current configuration $\mathcal{B}$. Let increments be signified by superposed dots. For example, $\dot{\mathbf{T}}$ denotes the increment in $\mathbf{T}$. This allows to write the incremental form of the equilibrium Eq. (11) as

$$\mathrm{Div}\,\dot{\mathbf{T}} = \mathbf{0}, \tag{20}$$

which requires the (linearized) incremental form of the constitutive Eq. (14). For an unconstrained material this is given by

$$\dot{\mathbf{T}} = A\dot{\mathbf{F}}, \tag{21}$$

where $\dot{\mathbf{F}}$ is the increment in $\mathbf{F}$ and $A$ is a fourth-order tensor. This equation can be written in component form as

$$\dot{T}_{\alpha i} = \mathcal{A}_{\alpha i\beta j}\dot{F}_{j\beta}, \tag{22}$$

where

$$\mathcal{A}_{\alpha i\beta j} = \frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}}. \tag{23}$$

For an isotropic material with no mechanical constraint, $W$ is a function of the three invariants $I_1, I_2, I_3$, and expression (23) can be expanded in the form

$$\mathcal{A}_{\alpha i\beta j} = \sum_{m=1}^{3}\sum_{n=1}^{3} W_{mn}\frac{\partial I_m}{\partial F_{i\alpha}}\frac{\partial I_n}{\partial F_{j\beta}} + \sum_{n=1}^{3} W_n \frac{\partial^2 I_n}{\partial F_{i\alpha}\partial F_{j\beta}},$$

where $W_n = \partial W/\partial I_n$, $W_{mn} = \partial^2 W/\partial I_m \partial I_n$. Expressions for the first and second derivatives of $I_n$, $n \in \{1, 2, 3\}$, with respect to $\mathbf{F}$ are given in A by Eqs. (A.1) and (A.2).

For an *incompressible* material, the nominal stress $\mathbf{T}$ is given by $(17)_1$ and its increment is

$$\dot{\mathbf{T}} = A\dot{\mathbf{F}} - \dot{p}\mathbf{F}^{-1} + p\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}, \tag{24}$$

which replaces (21).

We now update the reference configuration to coincide with the current configuration and denote by $\dot{\mathbf{T}}_0$ the 'push forward' versions of $\dot{\mathbf{T}}$ given by

$$\dot{\mathbf{T}}_0 = J^{-1}\mathbf{F}\dot{\mathbf{T}}, \tag{25}$$

which allows to define the Eulerian counterpart of Eq. (20) as

$$\mathrm{div}\,\dot{\mathbf{T}}_0 = \mathbf{0}. \tag{26}$$

It is convenient now to use the notation $\mathbf{u}$ for the incremental displacement $\dot{\mathbf{x}}$, with $\mathbf{u}$ treated as a function of $\mathbf{x}$, so that $\dot{\mathbf{F}} = (\mathrm{grad}\,\mathbf{u})\mathbf{F}$. Let $\mathbf{d} = \mathrm{grad}\,\mathbf{u}$, with components defined in Cartesian coordinates by $d_{ij} = \partial u_i/\partial x_j$. When the reference configuration is updated to coincide with the current configuration, the incremental constitutive Eq. (21) can be re-cast in the form

$$\dot{\mathbf{T}}_0 = A_0\mathbf{d}, \tag{27}$$

where, in index notation, the tensor $A_0$ is defined by

$$\mathcal{A}_{0jisk} = J^{-1}F_{j\alpha}F_{s\beta}\mathcal{A}_{\alpha i\beta k}. \tag{28}$$

For an incompressible material $J = 1$ in (28), and (27) is replaced by

$$\dot{\mathbf{T}}_0 = A_0\mathbf{d} + p\mathbf{d} - \dot{p}\mathbf{I} \tag{29}$$

and $\mathbf{u}$ satisfies the incremental incompressibility condition

$$\mathrm{div}\,\mathbf{u} = 0. \tag{30}$$

### 3.2. Incremental boundary conditions

In the reference configuration $\mathcal{B}_0$ the material is subject to an applied traction $\mathbf{t}_A$ (defined per unit area of $\partial\mathcal{B}_0$) and the nominal stress $\mathbf{T}$ must satisfy relation (13). On taking the increment of this equation, we obtain

$$\dot{\mathbf{T}}^{T}\mathbf{N} = \dot{\mathbf{t}}_A \quad \text{on} \quad \partial\mathcal{B}_0 \tag{31}$$

and by use of Nanson's formula the incremental boundary conditions in Eulerian form become

$$\dot{\mathbf{T}}_0^{T}\mathbf{n} = \dot{\mathbf{t}}_a \quad \text{on} \quad \partial\mathcal{B}. \tag{32}$$

## 4. Inflation of a spherical shell

### 4.1. Kinematics

Consider a thick-walled spherical shell made of an isotropic, hyperelastic and incompressible material. It is convenient to use spherical polar coordinates $(R, \Theta, \Phi)$ to define the geometric quantities in the reference configuration $\mathcal{B}_0$. Specifically, the geometry is defined as

$$A \leqslant R \leqslant B, \quad 0 \leqslant \Theta \leqslant \pi, \quad 0 \leqslant \Phi \leqslant 2\pi, \tag{33}$$

where $A$ and $B$ denote the inner and outer radii of the sphere, respectively. The shell is now inflated by applying a pressure $P$ on the inner surface $R = A$ to produce a spherical symmetric deformation given by $\mathbf{x} = f(R)\mathbf{X}$, where $R = |\mathbf{X}|$. The deformed geometry is described by

$$a \leqslant r \leqslant b, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi \leqslant 2\pi, \tag{34}$$

where $(r, \theta, \phi)$ are again spherical polar coordinates and $a$ and $b$ are the inner and outer radii of the deformed configuration, respectively. We note that for an incompressible material

$$r^3 = R^3 + a^3 - A^3. \tag{35}$$

The deformation gradient with respect to the spherical polar coordinate axes is diagonal and the associated principal stretches are given by

$$\lambda_1 = \lambda^{-2}, \quad \lambda_2 = \lambda, \quad \lambda_3 = \lambda, \tag{36}$$

with $\lambda = r/R > 1$ being the principal stretch in the $\theta$- and $\phi$-directions. The principal stretch in the radial direction is

$$\lambda_1 = \lambda^{-2} = \frac{\mathrm{d}r}{\mathrm{d}R}. \tag{37}$$

The associated strain invariants are

$$I_1 = 2\lambda^2 + \lambda^{-4}, \quad I_2 = 2\lambda^{-2} + \lambda^4, \quad I_3 \equiv 1. \tag{38}$$

For later use we define the stretches at the inner and outer surfaces of the deformed shell as $\lambda_a = a/A$ and $\lambda_b = b/B$, respectively. It is easy to show that these stretches are not independent and are related by

$$(\lambda_a^3 - 1) = \left(\frac{B}{A}\right)^3 (\lambda_b^3 - 1), \tag{39}$$

with

$$\lambda_a \geqslant \lambda \geqslant \lambda_b \geqslant 1. \tag{40}$$

### 4.2. Equilibrium

For an incompressible material, using Eq. (19), we find the normal components of the Cauchy stress

$$\begin{aligned}
\sigma_{rr} &= 2\left[W_1\lambda^{-4} + 2W_2\lambda^{-2}\right] - p, \\
\sigma_{\theta\theta} &= 2\left[W_1\lambda^2 + W_2(\lambda^4 + \lambda^{-2})\right] - p, \\
\sigma_{\phi\phi} &= \sigma_{\theta\theta},
\end{aligned} \tag{41}$$

with the shear components $\sigma_{r\theta}$, $\sigma_{r\phi}$ and $\sigma_{\theta\phi}$ being identically zero.

Eq. (38) shows that the invariants $I_1$, $I_2$ depend only on the stretch $\lambda$. It is therefore appropriate to define a strain energy function $\hat{W}(\lambda) = W(I_1, I_2)$, with

$$\frac{\mathrm{d}\hat{W}}{\mathrm{d}\lambda} = 4W_1(\lambda - \lambda^{-5}) + 4W_2(\lambda^3 - \lambda^{-3}).$$

In terms of $\hat{W}$, the principal stress difference becomes

$$\sigma_{\theta\theta} - \sigma_{rr} = \frac{1}{2}\lambda\hat{W}_\lambda, \tag{42}$$

where the subscript on $\hat{W}$ indicates differentiation with respect to $\lambda$.

The only component of the equilibrium equation (9) not identically satisfied is given by

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} + \frac{2}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0, \tag{43}$$

which, using Eq. (42) can be written as

$$r\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} = \lambda\hat{W}_\lambda. \tag{44}$$

Integration of (44), using the boundary conditions at the inner and outer surfaces $\sigma_{rr}(a) = -P$ and $\sigma_{rr}(b) = 0$, leads to the pressure-deformation relation

$$P = \int_a^b \lambda\hat{W}_\lambda \frac{\mathrm{d}r}{r}. \tag{45}$$

Using Eq. (35) together with $\lambda = r/R$ results in

$$\lambda\frac{\mathrm{d}r}{r} = -\frac{\mathrm{d}\lambda}{\lambda^3 - 1}, \tag{46}$$

which allows a change of variable in (45). We write

$$P = \int_{\lambda_b}^{\lambda_a} \frac{\hat{W}_\lambda \mathrm{d}\lambda}{(\lambda^3 - 1)}, \tag{47}$$

which gives the internal pressure $P$ as a function of the stretch at the inner surface $\lambda_a$ or equivalently as a function of $\lambda_b$.

### 4.3. Boundary conditions

Eq. (12) provides the stress $\boldsymbol{\sigma}$ when the boundary of the material is subject to an applied traction $\mathbf{t}_a$. It is convenient to write the pressure boundary condition at the interior surface of the sphere in terms of the nominal stress

$$\int_{\partial\mathcal{B}} \mathbf{t}_a \mathrm{d}a = \int_{\partial\mathcal{B}} -P\mathbf{n}\mathrm{d}a = \int_{\partial\mathcal{B}_0} -P\mathbf{F}^{-T}\mathbf{N}\mathrm{d}A = \int_{\partial\mathcal{B}_0} \mathbf{T}^T\mathbf{N}\mathrm{d}A, \tag{48}$$

where we have used Nanson's formula. The connection between applied pressure and nominal stress for an incompressible material is then given by

$$\mathbf{T}^T\mathbf{N} = -P\mathbf{F}^{-T}\mathbf{N}. \tag{49}$$

On taking an increment of (49), we have

$$\dot{\mathbf{T}}^T\mathbf{N} = -\dot{P}\mathbf{F}^{-T}\mathbf{N} + P\mathbf{F}^{-T}\dot{\mathbf{F}}^T\mathbf{F}^{-T}\mathbf{N}, \tag{50}$$

which can be written in the current configuration as

$$\dot{\mathbf{T}}_0^T\mathbf{n} = -\dot{P}\mathbf{n} + P\mathbf{d}^{\mathrm{T}}\mathbf{n}, \tag{51}$$

where we recall that $\mathbf{d} = \operatorname{grad} \mathbf{u}$.

### 4.4. Incremental deformation

Denote the components of the incremental displacement vector $\mathbf{u}$ by $u$, $v$, $w$ along the $r$-, $\theta$- and $\phi$-directions, respectively. For simplicity, we confine attention to axisymmetric modes of deformation with $w = 0$ and with the remaining components being independent of $\phi$. Then, the gradient of the displacement vector $\mathbf{d}$ has the matrix representation

$$\mathrm{d} = \begin{pmatrix} \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} & 0 \\ \frac{\partial v}{\partial r} & \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} & 0 \\ 0 & 0 & \frac{v\cot\theta}{r} + \frac{u}{r} \end{pmatrix}, \tag{52}$$

where the sum of the diagonal terms gives the incremental incompressibility condition (30)

$$r\frac{\partial u}{\partial r} + \frac{\partial v}{\partial \theta} + v\cot\theta + 2u = 0. \tag{53}$$

The equilibrium equation (26), using expression (29) for $\dot{\mathbf{T}}_0$, gives

$$\operatorname{div}\dot{\mathbf{T}}_0 = \operatorname{div}(A_0\mathbf{d}) + (\operatorname{grad}p)\mathbf{d} - \operatorname{grad}\dot{p} = \mathbf{0}. \tag{54}$$

In a spherical polar coordinate system, for the assumed incremental axisymmetric deformation, all components are independent of $\phi$. Therefore, only the $r$- and $\theta$-components of the equilibrium equations are not identically satisfied. These are

$$\frac{\partial \dot{T}_{0rr}}{\partial r}+\frac{2}{r}\dot{T}_{0rr}+\frac{1}{r}\frac{\partial \dot{T}_{0\theta r}}{\partial \theta}+\frac{\cot\theta}{r}\dot{T}_{0\theta r}-\frac{1}{r}\left(\dot{T}_{0\theta\theta}+\dot{T}_{0\phi\phi}\right)=0, \tag{55}$$

$$\frac{\partial \dot{T}_{0r\theta}}{\partial r}+\frac{2}{r}\dot{T}_{0r\theta}+\frac{1}{r}\dot{T}_{0\theta r}+\frac{1}{r}\frac{\partial \dot{T}_{0\theta\theta}}{\partial \theta}+\frac{\cot\theta}{r}\left(\dot{T}_{0\theta\theta}-\dot{T}_{0\phi\phi}\right)=0. \tag{56}$$

Of the three incremental boundary conditions (51) two are not identically satisfied. The boundary condition in the $r$-direction requires that

$$\dot{T}_{0rr}=\begin{cases}-\dot{P}+P\partial u/\partial r & \text{at}\quad r=a,\\ 0 & \text{at}\quad r=b.\end{cases} \tag{57}$$

For the underlying equilibrium configuration the correlations

$$\sigma_{rr}=\lambda_1\frac{\partial W}{\partial\lambda_1}-p,\quad \sigma_{\theta\theta}=\lambda_2\frac{\partial W}{\partial\lambda_2}-p, \tag{58}$$

apply with $\sigma_{rr}=-P$ at the inner surface $r=a$ and $\sigma_{rr}=0$ on $r=b$. Eq. (57), using the incompressibility condition (53), can then alternatively be written as

$$\left(\mathcal{A}_{01111}-\mathcal{A}_{01122}+\lambda_1\frac{\partial W}{\partial\lambda_1}\right)\frac{\partial u}{\partial r}-\dot{p}=\begin{cases}-\dot{P} & \text{on}\quad r=a,\\ 0 & \text{on}\quad r=b.\end{cases} \tag{59}$$

The remaining boundary condition to be satisfied is in the $\theta$-direction and has the form

$$\dot{T}_{0r\theta}=\begin{cases}P(\partial u/\partial\theta-v)/r & \text{at}\quad r=a,\\ 0 & \text{at}\quad r=b.\end{cases} \tag{60}$$

For the development that follows, we recall some correlations between the elastic moduli of incompressible isotropic materials. Specifically, Ogden, 1997 has shown that the following connections are satisfied

$$\mathcal{A}_{02112}=\mathcal{A}_{01212}-\lambda_1\frac{\partial W}{\partial\lambda_1},\quad \mathcal{A}_{01221}=\mathcal{A}_{02121}-\lambda_2\frac{\partial W}{\partial\lambda_2}. \tag{61}$$

Using the explicit expression for $\dot{T}_{0r\theta}$ together with $(61)_1$ we find that the boundary condition (60) can be written alternatively as

$$r\frac{\partial v}{\partial r}+\frac{\partial u}{\partial\theta}-v=0\quad\text{at}\quad r=a,b. \tag{62}$$

Using the equilibrium Eq. (43) and the expressions (58) for $\sigma_{rr}$ and $\sigma_{\theta\theta}$ we find the gradient of $p$, namely

$$r\frac{\mathrm{d}p}{\mathrm{d}r}=r\frac{\mathrm{d}}{\mathrm{d}r}\left(\lambda_1\frac{\partial W}{\partial\lambda_1}\right)+2\left(\lambda_1\frac{\partial W}{\partial\lambda_1}-\lambda_2\frac{\partial W}{\partial\lambda_2}\right). \tag{63}$$

Equilibrium equation (55) can then be rewritten in the alternative form

$$\begin{aligned}&\frac{\partial u}{\partial r}\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\mathcal{A}_{01111}-\mathcal{A}_{01122}+\lambda_1\frac{\partial W}{\partial\lambda_1}\right)\right]\\&+\frac{\partial u}{\partial r}\left[\frac{1}{r}\left(2\mathcal{A}_{01111}-4\mathcal{A}_{01122}-2\mathcal{A}_{01212}+\mathcal{A}_{02222}+\mathcal{A}_{02233}+4\lambda_1\frac{\partial W}{\partial\lambda_1}-\lambda_2\frac{\partial W}{\partial\lambda_2}\right)\right]\\&+u\left[\frac{2}{r^2}\mathcal{A}_{02121}\right]+\frac{\partial^2 u}{\partial r^2}[\mathcal{A}_{01111}-\mathcal{A}_{01122}-\mathcal{A}_{02112}]\\&+\frac{\partial u}{\partial\theta}\left[\frac{\cot\theta}{r^2}\mathcal{A}_{02121}\right]+\frac{\partial^2 u}{\partial\theta^2}\left[\frac{1}{r^2}\mathcal{A}_{02121}\right]=\frac{\partial\dot{p}}{\partial r}.\end{aligned} \tag{64}$$

Similarly, Eq. (56) has the alternative expression

$$\begin{aligned}&\frac{\partial u}{\partial\theta}\left[\frac{1}{r}\frac{\mathrm{d}A_{01212}}{\mathrm{d}r}+\frac{1}{r^2}\left(2A_{01212}-A_{02222}+A_{02233}-\lambda_2\frac{\partial W}{\partial\lambda_2}\right)\right]\\&+v\left[-\frac{1}{r}\frac{\mathrm{d}A_{01221}}{\mathrm{d}r}-\frac{1}{r^2}\left(2A_{01212}-A_{02222}+A_{02233}-\lambda_2\frac{\partial W}{\partial\lambda_2}\right)\right]\\&+\frac{\partial^2 u}{\partial r\partial\theta}\left[\frac{1}{r}\left(A_{01212}+A_{01122}-A_{02222}-\lambda_1\frac{\partial W}{\partial\lambda_1}\right)\right]\\&+\frac{\partial v}{\partial r}\left[\frac{\mathrm{d}A_{01212}}{\mathrm{d}r}+\frac{2}{r}A_{01212}\right]+\frac{\partial^2 v}{\partial r^2}[A_{01212}]\\&=\frac{1}{r}\frac{\partial\dot{p}}{\partial\theta}.\end{aligned} \tag{65}$$

We write the components of the incremental displacement vector $\mathbf{u}$ as the product of a function of $r$ alone and a Legendre polynomial in $\cos(\theta)$ by setting

$$\begin{aligned}&u=f_n(r)P_n(\cos\theta),\quad v=g_n(r)\frac{\mathrm{d}}{\mathrm{d}\theta}P_n(\cos\theta),\\&\dot{p}=h_n(r)P_n(\cos\theta),\end{aligned} \tag{66}$$

for $n=0,1,2,\ldots$ Use of the incompressibility condition (53) and the identity

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}P_n(\cos\theta)+\cot\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_n(\cos\theta)+n(n+1)P_n(cos\theta)=0 \tag{67}$$

provides a correlation between the functions $f_n(r)$ and $g_n(r)$ in the form

$$n(n+1)g_n=rf'_n+2f_n,\quad n=0,1,2,\ldots \tag{68}$$

The equilibrium equation in the $r$-direction (64) can now be written as

$$\begin{aligned}&(A_{01111}-A_{01122}-A_{02112})f''_n\\&+\left\{r\frac{\mathrm{d}}{\mathrm{d}r}(A_{01111}-A_{01122}+A_{01212}-A_{02112})\right.\\&+2(A_{01111}-A_{01122}-A_{02112}+A_{01212})-A_{02121}+A_{02222}\\&\left.+A_{02233}-2A_{01122}-A_{01221}\right\}\frac{1}{r}f'_n+\frac{1}{r^2}A_{02121}(2-m)f_n=h'_n.\end{aligned} \tag{69}$$

where we introduced the notation $m=n(n+1)$. Similarly, Eq. (65) becomes

$$\begin{aligned}&r^3\mathcal{A}_{01212}f'''_n+r^2\left(r\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{A}_{01212}+6\mathcal{A}_{01212}\right)f''_n\\&+\left\{r^2\frac{\mathrm{d}}{\mathrm{d}r}(2\mathcal{A}_{01212}+\sigma_{rr})+mr(\mathcal{A}_{02112}+\mathcal{A}_{01122}-\mathcal{A}_{02222})\right.\\&\left.-r(\mathcal{A}_{01221}+\mathcal{A}_{02121}+\mathcal{A}_{02233}-\mathcal{A}_{02222}-6\mathcal{A}_{01212})\right\}f'_n\\&+(m-2)\left\{r\frac{\mathrm{d}}{\mathrm{d}r}(\mathcal{A}_{01212}-\sigma_{rr})+(\mathcal{A}_{01221}+\mathcal{A}_{02121}+\mathcal{A}_{02233}-\mathcal{A}_{02222})\right\}f_n\\&=mrh_n.\end{aligned} \tag{70}$$

The boundary condition in the $r$-direction given by (59), using Eq. (70), can be expressed in terms of $f_n$ as

$$\begin{aligned}&r^3\mathcal{A}_{01212}f'''_n+r^2\left(r\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{A}_{01212}+6\mathcal{A}_{01212}\right)f''_n\\&+\left\{r\frac{\mathrm{d}}{\mathrm{d}r}(2\mathcal{A}_{01212}+\sigma_{rr})+m(2\mathcal{A}_{01221}+2\mathcal{A}_{01122}-\mathcal{A}_{01111}-\mathcal{A}_{02222}-\mathcal{A}_{01212})\right.\\&\left.-(\mathcal{A}_{01221}+\mathcal{A}_{02121}+\mathcal{A}_{02233}-\mathcal{A}_{02222}-6\mathcal{A}_{01212})\right\}rf'_n\\&+(m-2)\left\{r\frac{\mathrm{d}}{\mathrm{d}r}(\mathcal{A}_{01212}-\sigma_{rr})+(\mathcal{A}_{01221}+\mathcal{A}_{02121}+\mathcal{A}_{02233}-\mathcal{A}_{02222})\right\}f_n=0.\end{aligned} \tag{71}$$

Finally, the boundary condition (62), using Eq. (68), becomes

$$r^2f''_n+2rf'_n+(m-2)f_n=0\quad\text{at}\quad r=a,b. \tag{72}$$

## 5. Numerical solution

We use a numerical computation, known as the compound matrix method, to solve the equilibrium equations (69) and (70) together with the boundary conditions (71) and (72). For a brief overview of the compound matrix method and its use for typical problems in nonlinear elasticity we refer to Haughton and Kirkinis (2003), Haughton (2008) and Dorfmann and Haughton (2006).

It is convenient to rewrite the relevant equations in a compact form. Specifically, the equilibrium equation in the $r$-direction (69) is written as

$$h_n' = A_1 f_n'' + A_2 f_n' + A_3 f_n \tag{73}$$

and the equilibrium equation in the $\theta$-direction (70) becomes

$$f_n''' = B_1 f_n'' + B_2 f_n' + B_3 f_n + B_4 h_n. \tag{74}$$

To apply the compound matrix method we need to replace the third order derivative $f_n'''$ in Eq. (71), which is possible using (74). This allows Eq. (71) to be rewritten as

$$h_n = C_1 f_n'' + C_2 f_n' + C_3 f_n \quad \text{at} \quad r = a, b. \tag{75}$$

Finally, the remaining boundary condition (72) can be written as

$$f_n'' = -\frac{2}{r} f_n' + \frac{2-m}{r^2} f_n \quad \text{at} \quad r = a, b. \tag{76}$$

The coefficients of the equilibrium equations and boundary conditions contain a complex combination of instantaneous moduli and their derivatives, which, in turn, depend on the critical stretch $\lambda$ to be determined. The explicit expressions of all coefficients used in (73)–(75) are detailed in Eqs. (B.1)–(B.3) in Appendix B. In Appendix C the compound matrix method is specialized to the current application and the relevant equations are shown.

## 6. Numerical results

We now apply the numerical method outlined in Section 5 to analyze the behavior of spherical shells made of different materials. Specifically, we consider one term $(N = 1)$ and two terms $(N = 2)$ in the formulation proposed by Ogden (1972)

$$W_{\mathbf{O}} = \sum_{m=1}^{N} \frac{\mu_m}{\alpha_m} \left( \lambda_1^{\alpha_m} + \lambda_2^{\alpha_m} + \lambda_3^{\alpha_m} - 3 \right), \tag{77}$$

which were used respectively by Haughton and Ogden (1978b) and Haughton and Kirkinis, 2003. The model proposed by Gent (1996)

$$W_{\mathrm{G}} = -\frac{\mu}{2} J_{\mathrm{m}} \ln \left( 1 - \frac{I_1 - 3}{J_{\mathrm{m}}} \right) \tag{78}$$

and a generalized $I_1$ model recently published by Lopez-Pamies (2010)

$$W_{\mathrm{L}} = \sum_{m=1}^{N} \frac{3^{1-\alpha_m}}{2} \frac{\mu_m}{\alpha_m} \left( I_1^{\alpha_m} - 3^{\alpha_m} \right). \tag{79}$$

The values of the parameters chosen for the strain energy functions are summarized in Table 1. The magnitude of the parameters used for the one term and two terms Ogden formulation coincide with those used by Haughton and Ogden (1978b) and by Haughton and Kirkinis (2003). Therefore, by comparing results, the implementation of the numerical method can be validated.

The shear moduli, defined in the reference configuration, using one term and two terms in the Ogden formulation are $\mu = 0.25$ MPa and $\mu = 0.253375$ MPa, respectively. For the Gent model (78) we select an identical stiffness $\mu = 0.25$ MPa with $J_{\mathrm{m}} = 97.2$, the latter suggested by Gent (1996) to account for the limited extensibility of molecular chains in rubber. Finally, the magnitude of the parameters for the two terms $(N = 2)$ Lopez-Pamies model (79) were determined by fitting extension and shear data, both generated using two terms in the Ogden strain energy formulation with the values shown in Table 1.

**Table 1**
Summary of model parameters used in Eqs. (77)–(79). The values of $\mu$ and $\mu_i$ are given in MPa.

| Values of material model parameters | | |
|---|---|---|
| One term Ogden model (77) | $\mu_1 = 1$ | $\alpha_1 = 0.5$ |
| Two terms Ogden model (77) | $\mu_1 = 1$ | $\alpha_1 = 0.5$ |
| | $\mu_2 = 0.0001$ | $\alpha_2 = 6.75$ |
| Gent model (78) | $\mu = 0.25$ | $J_m = 97.2$ |
| Two terms Lopez-Pamies | $\mu_1 = 0.25$ | $\alpha_1 = 0.5$ |
| model (79) | $\mu_2 = 0.00064$ | $\alpha_2 = 3.6$ |

### 6.1. Inflation of a spherical shell

We first consider the pressure-stretch response of a spherical shell during inflation and then use these results to analyze aspherical bifurcations associated with modes $n = 1, 2, 3, \ldots$ To investigate the existence of local pressure maxima and minima and the dependence of the corresponding stretches on shell thickness, in addition to the four models listed in the previous subsection, we also consider the neo-Hookean strain energy formulation. Note that the formulation (77) for $N = 1$ with $\mu_1 = \mu$ and $\alpha_1 = 2$ simplifies to the neo-Hookean model. The existence of a pressure maximum or minimum requires that the derivative of Eq. (47) with respect to $\lambda_a$ vanishes for a value of $\lambda_a > 1$. We recall that this criterion has been derived explicitly by Haughton and Ogden (1978b) and is of the form

$$\left( \lambda_a - \lambda_a^{-2} \right) \frac{\mathrm{d}P}{\mathrm{d}\lambda_a} = \frac{\hat{W}_\lambda(\lambda_a)}{\lambda_a^2} - \frac{\hat{W}_\lambda(\lambda_b)}{\lambda_b^2} = 0, \tag{80}$$

which determines the corresponding critical stretches. The values of $\lambda_a$ and $\lambda_b$ are interdependent and are a function of the ratio $B/A$, see Eq. (39). This implies that the solutions of (80) depend, in addition to the particular form of the strain energy function, on the shell thickness.

Fig. 1a depicts the internal pressure $P$ as a function of the stretch $\lambda_a$ for a ratio of $A/B = 0.91$ $(B/A = 1.1)$. It shows local maxima of the internal pressure and the corresponding critical stretches $\lambda_a$. The exact values of $\lambda_a$ are given by Eq. (80) and are 1.410, 1.411, 1.442, 1.322 and 1.426 for the one and two terms Ogden model, for the Gent, for the Lopez-Pamies and for the neo-Hookean formulations, respectively. For comparison, Goriely et al., 2006 have shown that the critical stretch of a spherical neo-Hookean membrane is $\lambda_a = 7^{1/6}$, which coincides with the solution of (80) for $A/B \to 1$. For the two terms Ogden, for the Gent and for the Lopez-Pamies models, a second solution of Eq. (80) exists indicating the stretch $\lambda_a$ corresponding to a local pressure minimum, see Fig. 1a. For these materials, for a continuously increasing pressure, the configuration jumps from the critical value $\lambda_a$ to a larger value, implying a sudden increase in the radius of the spherical configuration. For the remaining two materials, a radial expansion occurs while the pressure decreases. Figs. 1b and 1c show the pressure-stretch responses for thicker shells with $A/B = 0.2$ and 0.1, respectively. Specifically, we see that for an increase in shell thickness an increase in the critical stretch occurs. For example, the results of the Lopez-Pamies model, shown in Fig. 1c, no longer exhibit a snap-through condition and for increasing $\lambda_a$ we have a monotonic increase in pressure. No instability occurs for very thick shells with $A/B = 0.001$, see Fig. 1d.

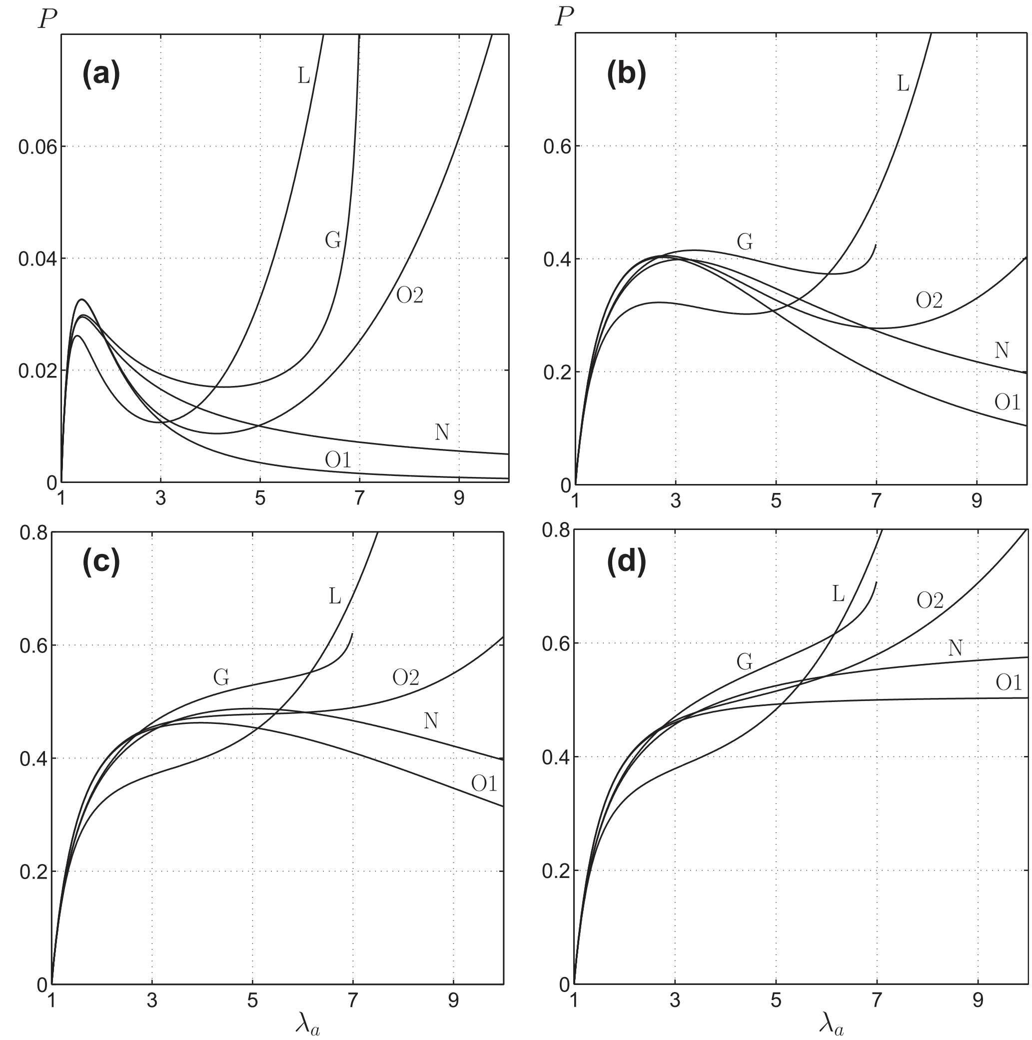


**Fig. 1.** The internal pressure $P$ as a function of the stretch $\lambda_a$ for the Ogden model with one term (O1), two terms (O2), for the Gent (G), for the Lopez-Pamies (L) and for the neo-Hookean (N) formulations. Results of a thin shell with $A/B = 0.91$ are given in (a), shells with intermediate thicknesses $A/B = 0.2$ and 0.1 are analyzed in (b) and (c). The response of a thick shell with $A/B = 0.001$ is given in (d). Note the change is scale of the internal pressure $P$ in going from (a) to (b).

#### 6.1.1. Aspherical bifurcations

In Section 6.1 we discussed snap-through instabilities, which maintain a radially symmetric configuration. Here we are interested in looking at aspherical bifurcations associated with modes $n \geqslant 1$, see Eq. (66).

For a positive internal pressure, we solve the incremental equations as a function of the ratio $A/B$, where we recall that $A/B \to 1$ represents a spherical membrane. Fig. 2a and b show the responses for the one and two terms Ogden models (77), the results in Fig. 2c and d correspond to the Gent (78) and Lopez-Pamies formulations (79), respectively. The results are different and will now be analyzed in detail.

Consider an increase in internal pressure applied to a thin spherical shell with, for example, $A/B = 0.91$. During inflation, for a one term Ogden model, the spherical symmetry is preserved until the critical stretch $\lambda_a = 1.410$ is reached, see Fig. 1a. Information on the existence of aspherical bifurcations can be obtained from Fig. 2a. With $A/B = 0.91$, for increasing values of $\lambda_a$ the snap-through condition is reached when $\lambda_a = 1.410$, i.e. at the intersection with the line representing mode $n = 0$. For further increases in $\lambda_a$ the shell bifurcates into aspherical shapes with mode numbers $n = 1, \ldots, 7$. For each $\lambda_a$ the corresponding internal pressure is given by Eq. (47). The order of mode shapes $n = 0, \ldots, 7$ is independent of shell thickness up to $A/B \approx 0.55$. For smaller values of $A/B$ up to $\approx 0.18$ the mode $n = 0$ is reached first but the subsequent aspherical bifurcations appear in reversed order, i.e. $n = 0, 7, \ldots, 2$. For shells with smaller values of $A/B$ bifurcation modes with $n = 7, \ldots, 2$ occur. Fig. 2a shows additional results, which to the best of our knowledge have not yet been reported in the literature. Specifically, for shells with ratio $A/B$ smaller than 0.525 and for stretches $\lambda_a > 4.95$, additional aspherical bifurcation modes with $n = 7, 6, 5, 4$ occur.

The response of an inflating spherical shell changes completely when a second term is added to the Ogden formulation (77), see Table 1 for the values of the model parameters. The corresponding pressure-stretch response in Fig. 1a shows that, for $A/B = 0.91$ and for increasing pressure, the spherical configuration snaps from the critical value $\lambda_a = 1.411$ to $\lambda_a = 7.55$. The local pressure minimum between these two points, obtained by solving Eq. (80), is at $\lambda_a = 4.13$. We now use this information to interpret results shown in Fig. 2b. Consider a ratio $A/B = 0.91$ and an increasing stretch $\lambda_a$. When $\lambda_a$ reaches the value of 1.411 it intersects the graph corresponding to mode $n = 0$. For further increase in $\lambda_a$, the shell assumes aspherical configurations associated with modes $n = 1, 2, 3, 4$ followed by modes in reversed order $n = 4, 3, 2, 1$. The spherical symmetry $n = 0$ is reestablished for $\lambda_a = 4.13$, which

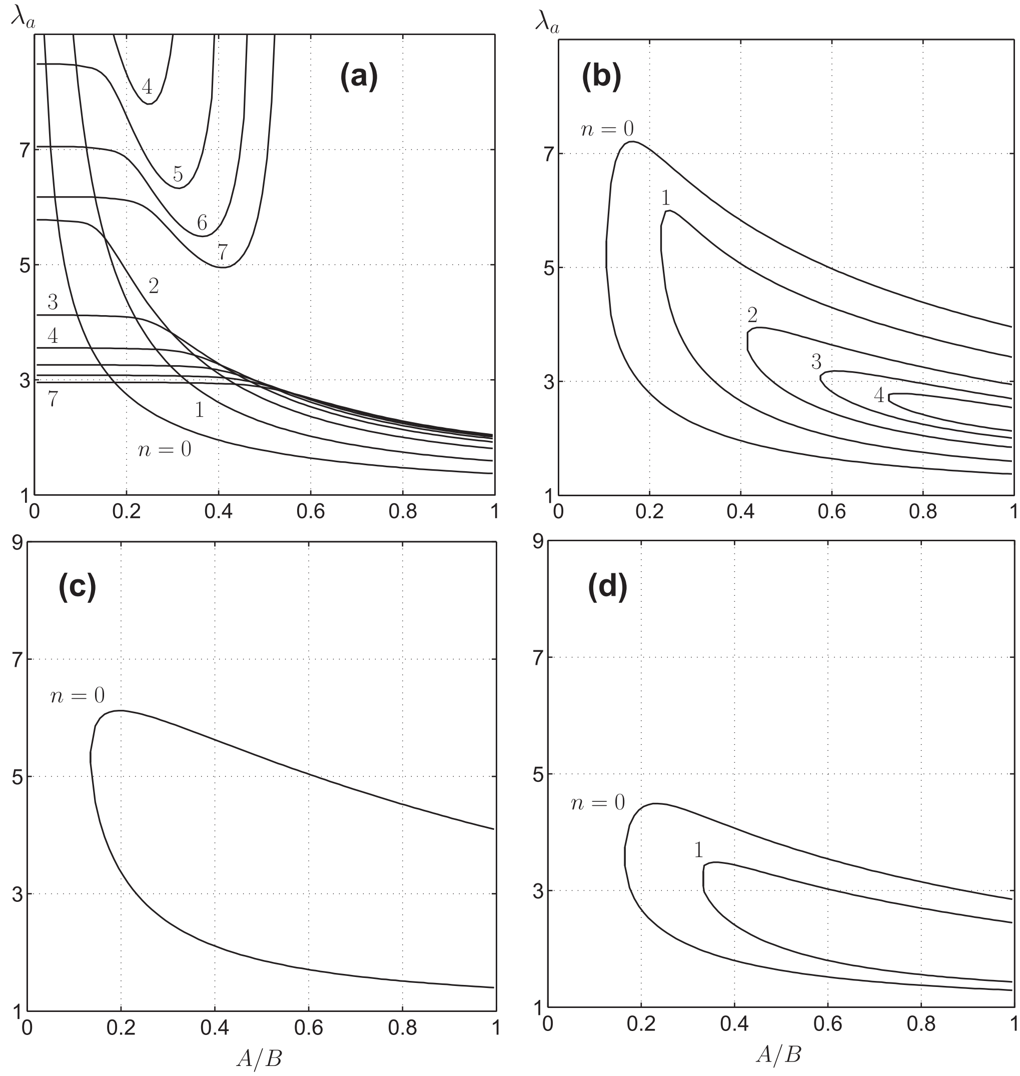


**Fig. 2.** The critical stretches $\lambda_a$ for bifurcation modes $n = 0, 1, 2, \ldots$ are given as functions of the ratio $A/B$, where $A$ ($B$) is the inner (outer) radius. The response of the shell to internal pressure using the one term Ogden formulation is shown in (a), the behavior using two terms in (b). Results for the Gent and Lopez-Pamies models are depicted in (c) and (d).

coincides with the stretch of the local pressure minimum in Fig. 1a. The pressure corresponding to each value of $\lambda_a$ is again given by (47). For increasing shell thickness the number of possible modes becomes smaller until for $A/B$ smaller than 0.227 aspherical bifurcations no longer occur. No local pressure maximum occurs for a thick shell with $A/B \leqslant 0.1$, which is also shown by the results of the two terms Ogden model in Figs. 1c and 1d.

The pressure-stretch response of the Gent model, shown in Fig. 1a, reaches a local maximum when $\lambda_a = 1.442$ and a local minimum when $\lambda_a = 4.27$. In Fig. 2c, these values correspond to the intersection of the line representing mode $n = 0$ with a vertical line through $A/B = 0.91$. Therefore, for any value of $1 < \lambda_a < 1.442$ or $\lambda_a > 4.27$ the deformation consists of a symmetric radial expansion. Independent of shell thickness, the Gent model does not admit any aspherical bifurcations. Fig. 2c further shows that no local pressure maximum and therefore no instability occurs for a sphere with ratio $A/B$ smaller than 0.135.

The Lopez-Pamies model (79) predicts, for a spherical shell with $A/B = 0.91$, a local pressure maximum and minimum for $\lambda_a = 1.322$ and $\lambda_a = 2.97$, respectively. If the pressure is kept at the value corresponding to the local maximum, a sudden radial expansion occurs until the stretch on the inner surface becomes $\lambda_a = 4.71$, see Fig. 1a. On the other hand, if the stretch increases from $\lambda_a = 1.322$ to $\lambda_a = 1.48$ or to $\lambda_a = 2.56$ aspherical bifurcations with $n = 1$ develop. The spherical symmetry is reestablished for a further increase in $\lambda_a$ to 2.97, which is shown in Fig. 2d. The corresponding pressures are determined by equation (47). For a ratio of $A/B$ smaller than 0.169 no local pressure maximum exits and we have $\mathrm{d}P/\mathrm{d}\lambda > 0$, see results in Figs. 1c and 1d. It is interesting to note that in spite of the fact that the Lopez-Pamies model was fitted to the two-terms Ogden model under both extension and shear, the behaviors of spherical shells made out of these two materials are quite different.

### *6.2. A spherical shell under external pressure*

We now use the numerical method outlined in Section 5 to investigate bifurcations of a spherical shell of arbitrary thickness subject to an external pressure. Fig. 3a shows the predicted stretches and corresponding critical modes for the one term Ogden formulation (77) with the values of $\mu_1$, $\alpha_1$ given in Table 1. For thin shells aspherical bifurcations occur with modes $n = 7$ or $n = 20$, both corresponding to a value of $\lambda_a = 0.99$, which can be compared to the results given by Haughton and Ogden (1978b). For smaller

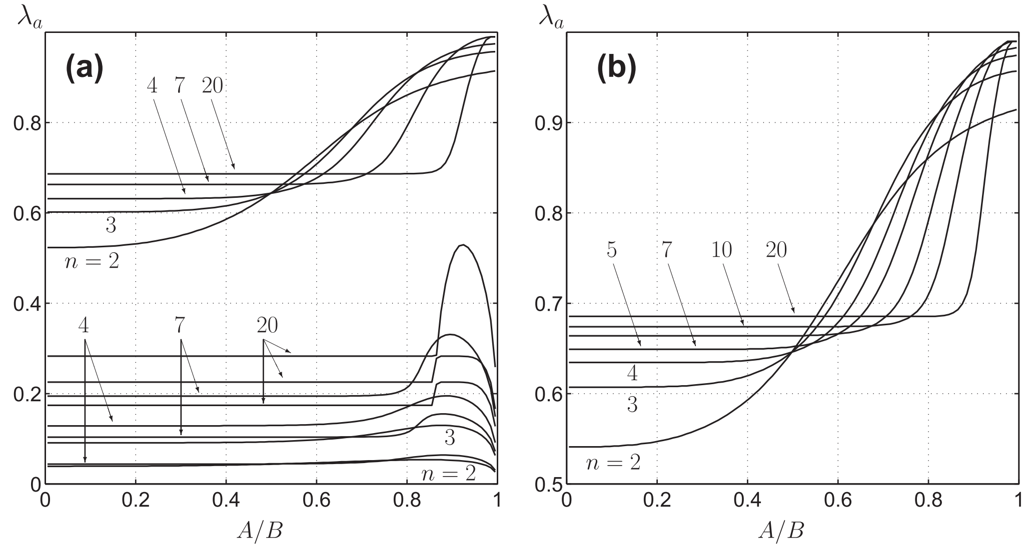


**Fig. 3.** The critical stretches $\lambda_a$ for modes $n = 2, 3, \ldots$ are given as functions of the ratio $A/B$, where $A$ and $B$ are the inner and outer radii of a spherical shell, respectively. The critical values of $\lambda_a$ predicted by the one term Ogden model are shown in (a), for two terms in (b).

values of $A/B$ the shell stabilizes and more compression is needed to induce bifurcations. Also, mode shapes change from initial $n = 7$ and 20 to $n = 4, 3$ and 2. For any value of $A/B$ smaller than $\approx 0.55$ the spherical shell first bifurcates with mode $n = 20$ for a critical value of $\lambda_a = 0.6865$. For thicker shells with, for example $A/B = 0.2$, a further increase in compression induces a sequence of modes $n = 20, 7, 4, 3$ and 2. Fig. 3a provides additional information that has, to the best of our knowledge, not been reported in the literature before. Independent of shell thickness $A/B$, for low values of $\lambda_a$, additional bifurcation modes exist. The results show that mode $n = 20$ occurs repeatedly for different values of $\lambda_a$, as do modes $n = 4$ and $n = 7$. For very small values of $\lambda_a$, modes $n = 2$ and 3 are possible solutions of the incremental equations.

Fig. 3b provides the aspherical bifurcation modes predicted when two terms are used in the Ogden formulation (77). The sequence of mode shapes and critical stretches, for different values of $A/B$, are remarkably similar to the results shown in Fig. 3a when $\lambda_a > 0.5$. As part of this investigation, we also solved numerically the incremental equations using the Gent (78) and Lopez-Pamies (79) models and found those to be reminiscent of the results shown in Fig. 3b. Therefore, these results are not included in this article. For completeness, we refer to Goriely et al. (2006) for the bifurcation analysis using the neo-Hookean material. The corresponding results by Goriely et al. (2006) are very close to the results shown in Fig. 3b.

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## Appendix A

Here we record for reference the expressions for the first derivatives of the three invariants with respect to $\mathbf{F}$. In component form, we have

$$\frac{\partial I_1}{\partial F_{i\alpha}} = 2F_{i\alpha}, \quad \frac{\partial I_2}{\partial F_{i\alpha}} = 2(C_{\gamma\gamma}F_{i\alpha} - C_{\alpha\gamma}F_{i\gamma}), \quad \frac{\partial I_3}{\partial F_{i\alpha}} = 2I_3 F^{-1}_{\alpha i}, \tag{A.1}$$

The second derivatives with respect to $\mathbf{F}$ are

$$\begin{aligned}
&\frac{\partial^2 I_1}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}\delta_{\alpha\beta},\\
&\frac{\partial^2 I_2}{\partial F_{i\alpha}\partial F_{j\beta}} = 2(2F_{i\alpha}F_{j\beta} - F_{i\beta}F_{j\alpha} + c_{\gamma\gamma}\delta_{ij}\delta_{\alpha\beta} - b_{ij}\delta_{\alpha\beta} - c_{\alpha\beta}\delta_{ij}),\\
&\frac{\partial^2 I_3}{\partial F_{i\alpha}\partial F_{j\beta}} = 2I_3(2F^{-1}_{\alpha i}F^{-1}_{\beta j} - F^{-1}_{\alpha j}F^{-1}_{\beta i}).
\end{aligned} \tag{A.2}$$

## Appendix B

The coefficients $A_1,\ A_2,\ A_3$ used in the equilibrium equation in the $r$-direction (73) are given by

$$\begin{aligned}
A_1 &= \mathcal{A}_{01111} - \mathcal{A}_{01122} - \mathcal{A}_{02112},\\
A_2 &= \frac{\mathrm{d}}{\mathrm{d}r}(\mathcal{A}_{01111} - \mathcal{A}_{01122} + \mathcal{A}_{01212} - \mathcal{A}_{02112})\\
&\quad + \frac{2}{r}(\mathcal{A}_{01111} - \mathcal{A}_{01122} - \mathcal{A}_{02112} + \mathcal{A}_{01212})\\
&\quad + \frac{1}{r}(-\mathcal{A}_{02121} + \mathcal{A}_{02222} + \mathcal{A}_{02233} - 2\mathcal{A}_{01122} - \mathcal{A}_{01221}),\\
A_3 &= \frac{2-m}{r^2}\mathcal{A}_{02121}.
\end{aligned} \tag{B.1}$$

Similarly, the expressions of the coefficients $B_1,\ B_2,\ B_3$ and $B_4$ used in the equilibrium equation in the $\theta$-direction (74) are

$$\begin{aligned}
B_1 &= -\frac{1}{\mathcal{A}_{01212}}\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{A}_{01212} - \frac{6}{r},\\
B_2 &= -\frac{1}{\mathcal{A}_{01212}}\left\{\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(2\mathcal{A}_{01212} + \sigma_{rr}) + \frac{m}{r^2}(\mathcal{A}_{02112} + \mathcal{A}_{01122} - \mathcal{A}_{02222})\right.\\
&\quad \left. - \frac{1}{r^2}(\mathcal{A}_{01221} + \mathcal{A}_{02121} + \mathcal{A}_{02233} - \mathcal{A}_{02222} - 6\mathcal{A}_{01212})\right\},\\
B_3 &= \frac{2-m}{\mathcal{A}_{01212}}\left\{\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(\mathcal{A}_{01212} - \sigma_{rr}) + \frac{1}{r^3}(\mathcal{A}_{01221} + \mathcal{A}_{02121} + \mathcal{A}_{02233} - \mathcal{A}_{02222})\right\},\\
B_4 &= \frac{m}{r^2\mathcal{A}_{01212}}.
\end{aligned} \tag{B.2}$$

The explicit expressions of the coefficients $C_1,\ C_2,\ C_3$ used to specify the boundary conditions in Eq. (75) are

$$\begin{aligned}C_1 &= -\frac{B_1}{B_4}-\frac{1}{B_4\mathcal{A}_{01212}}\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{A}_{01212}-\frac{6}{rB_4},\\ C_2 &= -\frac{B_2}{B_4}-\frac{1}{B_4\mathcal{A}_{01212}}\left\{\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(2\mathcal{A}_{01212}+\sigma_{rr})\right.\\ &\quad+\frac{m}{r^2}(2\mathcal{A}_{01221}+2\mathcal{A}_{01122}-\mathcal{A}_{01111}-\mathcal{A}_{02222}-\mathcal{A}_{01212})\\ &\quad\left.-\frac{1}{r^2}(\mathcal{A}_{01221}+\mathcal{A}_{02121}+\mathcal{A}_{02233}-\mathcal{A}_{02222}-6\mathcal{A}_{01212})\right\},\\ C_3 &= -\frac{B_3}{B_4}-\frac{m-2}{r^3B_4\mathcal{A}_{01212}}\left\{r\frac{\mathrm{d}}{\mathrm{d}r}(\mathcal{A}_{01212}-\sigma_{rr})\right.\\ &\quad\left.+(\mathcal{A}_{01221}+\mathcal{A}_{02121}+\mathcal{A}_{02233}-\mathcal{A}_{02222})\right\}.\end{aligned} \tag{B.3}$$

## Appendix C

Here we provide an overview on the use of the compound matrix method to solve the equilibrium equations for $n \geqslant 2$, the case $n = 1$ requires minor modifications and will be outlined in C.1.

The solutions of (73) and (74) need to satisfy the two boundary conditions (75) and (76) at $r = a$, which can be solved in an equivalent manner requiring two linearly independent initial conditions at $r = a$. We write the general solution as a linear combination

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)}, \tag{C.1}$$

where $\mathbf{y} = \mathbf{y}(h, f, f', f'')$ and $\mathbf{y}^{(i)} = \mathbf{y}(h^{(i)}, f^{(i)}, f^{(i)\prime}, f^{(i)\prime\prime})$ with $i \in \{1,2\}$. The compound variables $\phi_i$ are $2\times 2$ determinants such that, for example,

$$\phi_1 = \begin{vmatrix} h^{(1)} & h^{(2)} \\ f^{(1)} & f^{(2)} \end{vmatrix} \equiv \begin{vmatrix} h \\ f \end{vmatrix}, \tag{C.2}$$

where the expression on the right introduces a convenient shorthand notation. For this particular application there are a total of 6 compound variables, which are given by

$$\phi_1 = \begin{vmatrix} h \\ f \end{vmatrix},\quad \phi_2 = \begin{vmatrix} h \\ f' \end{vmatrix},\quad \phi_3 = \begin{vmatrix} h \\ f'' \end{vmatrix},\quad \phi_4 = \begin{vmatrix} f \\ f' \end{vmatrix},\quad \phi_5 = \begin{vmatrix} f \\ f'' \end{vmatrix},\quad \phi_6 = \begin{vmatrix} f' \\ f'' \end{vmatrix}. \tag{C.3}$$

The next step is to determine the derivatives of these variables, where we use Eqs. (73) and (74) to substitute $h_n'$ and $f_n'''$. For $\phi_1$, for example, we have

$$\phi_1' = \begin{vmatrix} h' \\ f \end{vmatrix} + \begin{vmatrix} h \\ f' \end{vmatrix} = A_1\begin{vmatrix} f'' \\ f \end{vmatrix} + A_2\begin{vmatrix} f' \\ f \end{vmatrix} + \begin{vmatrix} h \\ f' \end{vmatrix} = -A_1\phi_5 - A_2\phi_4 + \phi_2,$$

where in the last equality we have made use of the definitions given in (C.3). Similarly substitutions are used for $\phi_2, \ldots, \phi_6$ to obtain

$$\begin{aligned}\phi_1' &= -A_1\phi_5 - A_2\phi_4 + \phi_2, & \phi_4' &= \phi_5,\\ \phi_2' &= -A_1\phi_6 + A_3\phi_4 + \phi_3, & \phi_5' &= \phi_6 + B_1\phi_5 + B_2\phi_4 - B_4\phi_1,\\ \phi_3' &= A_2\phi_6 + A_3\phi_5 + B_1\phi_3 + B_2\phi_2 + B_3\phi_1, & \phi_6' &= B_1\phi_6 - B_3\phi_4 - B_4\phi_2.\end{aligned} \tag{C.4}$$

We require the compound variables to satisfy the boundary condition at $r = a$ and, on the use of Eq. (75) we obtain

$$\begin{aligned}\phi_1 &= \begin{vmatrix} h \\ f \end{vmatrix} = C_1\begin{vmatrix} f'' \\ f \end{vmatrix} + C_2\begin{vmatrix} f' \\ f \end{vmatrix} & \phi_4 &= \begin{vmatrix} f \\ f' \end{vmatrix}\\ \phi_2 &= \begin{vmatrix} h \\ f' \end{vmatrix} = C_1\begin{vmatrix} f'' \\ f' \end{vmatrix} + C_3\begin{vmatrix} f \\ f' \end{vmatrix} & \phi_5 &= \begin{vmatrix} f \\ f'' \end{vmatrix}\\ \phi_3 &= \begin{vmatrix} h \\ f'' \end{vmatrix} = C_2\begin{vmatrix} f' \\ f'' \end{vmatrix} + C_3\begin{vmatrix} f \\ f'' \end{vmatrix} & \phi_6 &= \begin{vmatrix} f' \\ f'' \end{vmatrix}\end{aligned} \tag{C.5}$$

Using $f''$ in Eq. (76) at $r = a$ we obtain the boundary conditions for the differential equation (C.5), namely

$$\begin{aligned}\phi_1 &= \left(\frac{2C_1}{r} - C_2\right)\phi_4, & \phi_4 &= \phi_4 \equiv 1,\\ \phi_2 &= \left(C_3 - \frac{C_1}{r}(m-2)\right)\phi_4, & \phi_5 &= -\frac{2}{r}\phi_4,\\ \phi_3 &= \left(\frac{C_2}{5}(m-2) - \frac{2C_3}{r}\right)\phi_4, & \phi_6 &= \frac{m-2}{r^2}\phi_4.\end{aligned} \tag{C.6}$$

For chosen values of $\lambda_a$ and $n \geqslant 2$ the set of first order differential Eqs. (C.4) can be solved together with the boundary conditions (C.6).

We now rewrite the boundary conditions (75) and (76) as

$$C_1f_n'' + C_2f_n' + C_3f_n - h_n = 0,\quad r^2f_n'' + 2rf_n' + (m-2)f_n = 0. \tag{C.7}$$

By taking the determinant of these two equations, collecting terms with equal coefficients, and making use of Eq. (C.3) we arrive at the target condition at $r = b$, namely

$$\begin{aligned}&C_1[-2r\phi_6 + (2-m)\phi_5] + C_2\left[r^2\phi_6 + (2-m)\phi_4\right]\\ &\quad + C_3\left[r^2\phi_5 + 2r\phi_4\right] - r^2\phi_3 - 2r\phi_2 + (2-m)\phi_1\\ &\quad = 0.\end{aligned} \tag{C.8}$$

For a given $n$, the value of $\lambda_a$ for which Eq. (C.8) is satisfied is the critical value at which there exists an incremental configuration for a small axisymmetric deformation superimposed upon the current configuration at constant load.

### *C.1*

For $n = 1$ the above numerical routine needs to be adjusted as follows. The equilibrium equations (73) and (74) simplify to

$$h_n' = A_1f_n'' + A_2f_n',\quad f_n''' = B_1f_n'' + B_2f_n' + B_4h_n. \tag{C.9}$$

We replace the function $f_n'$ by a function $g_n$ such that

$$h_n' = A_1g_n' + A_2g_n,\quad g_n'' = B_1g_n'' + B_2g_n + B_4h_n, \tag{C.10}$$

and the number of compound variables reduces from 6 to 3

$$\phi_1 = \begin{vmatrix} h \\ g \end{vmatrix},\quad \phi_2 = \begin{vmatrix} h \\ g' \end{vmatrix},\quad \phi_3 = \begin{vmatrix} g \\ g' \end{vmatrix}.$$

The derivatives of the three compound variables become

$$\phi_1' = -A_1\phi_3 + \phi_2,\quad \phi_2' = A_2\phi_3 + B_1\phi_2 + B_2\phi_1,\quad \phi_3' = B_1\phi_3 - B_4\phi_1.$$

Boundary conditions (75) and (76) reduce to

$$h_n = C_1f_n'' + C_2f_n',\quad f_n'' = -\frac{2}{r}f_n' \quad \text{at} \quad r = a, b, \tag{C.11}$$

and using the substitution $g_n = f_n'$ we get

$$h_n = C_1g_n' + C_2g_n,\quad g_n' = -\frac{2}{r}g_n \quad \text{at} \quad r = a, b. \tag{C.12}$$

We now require the compound variables to satisfy the boundary condition at $r = a$

$$\phi_1 = C_1\begin{vmatrix} g' \\ g \end{vmatrix} \equiv -C_1,\quad \phi_2 = C_2\begin{vmatrix} g \\ g' \end{vmatrix} \equiv C_2,\quad \phi_3 = \begin{vmatrix} g \\ g' \end{vmatrix} \equiv 1. \tag{C.13}$$

For the boundary $r = b$ we rewrite the boundary conditions (C.12) as

$$h_n - C_1g_n' - C_2g_n = 0,\quad rg_n' + 2g_n = 0 \tag{C.14}$$

and find

$$r\begin{vmatrix} h \\ g' \end{vmatrix} + 2\begin{vmatrix} h \\ g \end{vmatrix} - 2C_1\begin{vmatrix} g' \\ g \end{vmatrix} - rC_2\begin{vmatrix} g \\ g' \end{vmatrix} = 0. \tag{C.15}$$

Expanding gives the target condition for $r = b$ in compact form

$$r\phi_2 + 2\phi_1 + (2C_1 - rC_2)\phi_3 = 0, \tag{C.16}$$

and its solution determines the critical value of $\lambda_a$ for which there exists an incremental deformation having the form given by (66) with $n = 1$.

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