SIAM J. OPTIM.
Vol. 23, No. 1, pp. 54–73

LOWER SEMICONTINUOUS CONVEX RELAXATION IN OPTIMIZATION*

RAFAEL CORREA† AND ABDERRAHIM HANTOUTE†

Abstract. We relate the argmin sets of a given function, not necessarily convex or lower semicontinuous, and its lower semicontinuous convex hull by means of explicit characterizations involving an appropriate concept of asymptotic functions. This question is connected to the subdifferential calculus of the Legendre–Fenchel conjugate function. The final expressions, which also involve a useful extension of the Fenchel subdifferential introduced in [R. Correa and A. Hantoute, *Set-Valued Var. Anal.*, 18 (2010), pp. 405–422], are then written exclusively by means of primal objects relying on the initial function. This work extends to the infinite-dimensional setting of some related results given in [J. Benoist and J.-B. Hiriart-Urruty, *SIAM J. Math. Anal.*, 27 (1996), pp. 1661–1679].

Key words. Legendre–Fenchel conjugate function, lower semicontinuous convex hull, Fenchel subdifferential, asymptotic function, argmin set

AMS subject classifications. 26B05, 26J25, 49H05

DOI. 10.1137/100818091

1. Introduction.

It is our aim in this paper to provide explicit formulas for the Fenchel subdifferential and the argmin sets of the successive Legendre–Fenchel conjugates of a given extended real-valued function, which is defined on an infinite-dimensional real locally convex space. Such formulas will only involve the initial data given by means of the subdifferential and the argmin sets of the initial function. This approach would then avoid the systematic requirement for the explicit calculus on the successive conjugates, which is often a difficult task even when dealing with simple functions. In order to get formulas which are valid in general, avoiding continuity and coercivity restrictions on the initial function or its corresponding conjugate, we introduce an appropriate concept of asymptotic functions. This new object turns out to be very useful within our analysis since it beneficially extends the usual notion of recession functions in the sense of a convex framework. Indeed, it will be shown that it inherits from the recession analysis many useful properties so that it fully characterizes the behavior at infinity of the initial function. Roughly speaking, this concept of asymptotic function will allow us to establish the desired formulas for either the subdifferential of the conjugate or the argmin set of the relaxed problems only by means of the primal data.

Given a function $f : X \to \overline{\mathbb{R}}$, defined on a real locally convex space X, we consider the associated optimization problem and its lower semicontinuous (lsc) convex relaxed problem given, respectively, by

$$\inf_X f, \quad \inf_X \overline{\text{co}} f,$$

where $\overline{\text{co}} f$ denotes the lsc convex hull, that is, the greatest lsc convex function maximized by the function f on X. This kind of relaxation is very useful in practice,

*Received by the editors December 14, 2010; accepted for publication (in revised form) October 24, 2012; published electronically January 24, 2013. Research supported by Fondecyt project 1110019 and ECOS-Conicyt project C10E08.
http://www.siam.org/journals/siopt/23-1/81809.html

†Departamento de Ingeniería Matemática, Universidad de Chile, Centro de Modelamiento Matemático (CMM), Santiago, Chile (rcorrea@dim.uchile.cl, ahantoute@dim.uchile.cl).

namely, in calculus of variations, in mathematical programming problems, as well as in many other theoretical and numerical purposes; see, e.g., [9, 14]. The evident feature of the relaxed problem is that it enters into the convex optimization framework and has the same optimal value as the initial problem. Moreover, the corresponding argmin sets verify the straightforward relationship

$$\overline{\text{co}}(\text{argmin}(f)) \subset \text{argmin}(\overline{\text{co}}f), \tag{1}$$

where $\overline{\text{co}}$ also stands for the closed convex hull of subsets. Because one is mainly interested in solving the initial problem, the most interesting question is whether the converse inclusion in (1) (or some variants of it) holds. This information would avoid serious difficulties in computing the closed convexified function, which requires one to make several algebraic and topological operations on the initial function. But, unfortunately, it readily follows from simple examples that the inclusion in (1) may be strict, and so some objects need to be added in order to fill the gap.

This fundamental question has been considered by many researchers in recent years. First, in the finite-dimensional setting, the approach undertaken in [3] uses concepts from asymptotic analysis traced out from [6, 8]. Indeed, assuming that the function f is lsc and asymptotically epi-pointed, which is equivalent to saying that the interior of the domain of the conjugate is nonempty, or that this conjugate is finite and continuous at some point (in the finite-dimensional setting), it follows from [3, Theorem 4.6] that

$$\text{argmin}(\overline{\text{co}}f) = \text{co}(\text{argmin}(f)) + \text{co}(\text{argmin}(g)), \tag{2}$$

where g represents the asymptotic function of f, which has been introduced independently in [6, 8]. Moreover, in this same setting, the authors derived this last formula from a general rule dealing with the subdifferential mapping of $\overline{\text{co}}f$. One can apply the recent results in [12], dealing with calculus rules of the pointwise suprema, on the Legendre–Fenchel conjugate f^* to give first variants of (2) using the ε-minima of f, ε-argmin(f). Namely, under standard arguments guaranteeing the relationship $\overline{\text{co}}f = f^{**}$ (X^* is endowed with a topology compatible with the duality pair (X, X^*)), it easily follows from [12, Theorem 4] that (X being infinite-dimensional)

$$\text{argmin}(\overline{\text{co}}f) = \partial f^*(\theta) = \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}(f,\theta)}} \overline{\text{co}}(\varepsilon\text{-}\,\text{argmin}(f) + \text{N}_{L\cap \text{dom}\, f^*}(\theta)), \tag{3}$$

where $\mathcal{F}(f,\theta)$ denotes a suitable subset of 2^{X^*}; for instance, the family of finite-dimensional subspaces of X, and ∂, N are, respectively, the Fenchel subdifferential and the normal cone in the sense of convex analysis. In this way, it is clear there that the problem of formulating the argmin set of $\overline{\text{co}}f$ is easier through subdifferential calculus rules dealing with the conjugate function since (assuming that the conjugate function is proper)

$$\text{argmin}(\overline{\text{co}}f) = \partial f^*(\theta).$$

The other issue, followed in [3], would be to evoke the analysis of the subdifferentiation of $\overline{\text{co}}f$ itself and next use the relationship

$$\text{argmin}(\overline{\text{co}}f) = (\partial(\overline{\text{co}}f))^{-1}(\theta).$$

Obviously, the last two expressions above are equivalent, and both give the desired formula for the argmin set of $\overline{\text{co}}f$, but, from the point of view of technicality, the two issues are completely different. However, the approach using the conjugate instead of $\overline{\text{co}}f$ has benefited from the discovery of new subdifferential calculus rules; see, for instance, [11, 12, 13]. In this respect, it was shown in a recent work [5, Theorem 4] that

$$\partial f^*(\theta) = \bigcap_{L\in\mathcal{F}(f,\theta)} \overline{\text{co}}\left((\partial_L^r f)^{-1}(\theta) + \mathrm{N}_{L\cap \operatorname{dom} f^*}(\theta)\right) \tag{4}$$

for an appropriate enlargement ∂_L^r of the Fenchel subdifferential. This last formula can be seen as a duality result in the sense that it allows us to know the impact on the conjugate of given properties on the initial function and vice versa. For instance, one can derive from (4) many characterizations for the Gâteaux or Fréchet differentiability of f^* in terms of the behavior of the linear-perturbed optimization problems $\inf(f + x^*)$; for related results see [1, 4, 16].

Now, provided that the function f is bounded from below by a continuous affine mapping, we obtain from (4) another characterization of the argmin set of $\overline{\text{co}}f$, given according to [5, Corollary 8] by

$$\operatorname{argmin}(\overline{\text{co}}f) = \bigcap_{L\in\mathcal{F}(f,\theta)} \overline{\text{co}}\left((\partial_L^r f)^{-1}(\theta) + \mathrm{N}_{L\cap \operatorname{dom} f^*}(\theta)\right). \tag{5}$$

In particular, under quite general continuity and coercivity assumptions, namely, the weak lower semicontinuity of f and the continuity of the conjugate function at a point of its domain (that is, f is asymptotically epi-pointed; see, e.g., [3], [10], [21]), it is established in [5, Corollary 8(iii)] that the formula above reduces to

$$\operatorname{argmin}(\overline{\text{co}}f) = \overline{\text{co}}\left(\operatorname{argmin}(f)\right) + \mathrm{N}_{\operatorname{dom} f^*}(\theta). \tag{6}$$

Moreover, in the finite-dimensional setting, according to [5, Corollary 8(iv)] this last formula still holds if we omit the closure operation from the right-hand side. It is also worth observing that in many circumstances, the domain of the conjugate function, or its normal cone, can be explicitly determined by investigating the behavior of the initial function [3, 21]; see also Proposition 13. However, despite this fact and the general applicability of the previous formulas, the presence of the dual term involving the domain of the conjugate would nevertheless suggest an inappropriate preponderance of the role of the conjugate function within the above-presented formulas.

At this moment, the question we address in the current work is to what extent should the formulas in (4)–(6) be written only by means of primal objects without requiring calculus on the conjugate function? To give an idea of what the desired characterizations would look like and what material should be used, we discuss here two typical and important situations. First, if f is a convex lsc proper function, then its asymptotic (also called recession) function in the sense of convex analysis denoted by g is proper, lsc, and convex so that

$$\mathrm{N}_{\operatorname{dom} f^*}(\theta) = (\partial\sigma_{\operatorname{dom} f^*})^{-1}(\theta) = (\partial g)^{-1}(\theta) = \operatorname{argmin}(g). \tag{7}$$

In other words, the normal cone to the domain of the conjugate is fully characterized by means of calculus on f via its recession function. Now, if the function f is only lsc and asymptotically epi-pointed, the space X is finite-dimensional, and g is the asymptotic function of f in the sense of [6, 8], then according to [3] the asymptotic (actually recession) function of $\overline{\text{co}}g$ is $\overline{\text{co}}g$ itself, and we have that

$$\text{argmin}(\overline{\text{co}}g) = \text{co}\,(\text{argmin}(g))\,. \tag{8}$$

Consequently, similarly as in (7) we derive that

$$\text{N}_{\text{dom}\, f^*}(\theta) = \text{co}\,(\text{argmin}(g))\,, \tag{9}$$

and hence the formulas in (4)–(6) can be rewritten so that only direct calculus on f is involved.

At this stage, our objective in this paper is twofold. First, we introduce and study an appropriate concept of asymptotic functions which obeys calculus rules like those in (8) and (9) and beneficially extends the classical recession function of convex analysis and of [6, 8] (see also [2, 7, 18, 19, 22]). Second, we give formulas for the subdifferential of the conjugate function only by means of the primal data and so extend (4)–(6) among other results to the infinite-dimensional setting without making any continuity or coercivity assumptions.

The organization of the rest of the paper is as follows: in section 2, we fix the notation and notions which will be used throughout the paper. In section 3 we recall and comment on the main results of [5], giving the formulas in (5)–(6) among others. In the same section, we establish and adapt some of these results to the class of positively homogeneous functions. In section 4, we introduce and study the concept of asymptotic functions. Finally, in section 5, we give the desired formulas for the Fenchel subdifferential operator of the conjugate function and the argmin set of the lsc convex hull. We discuss there many special situations regarding the topology of the domain of the conjugate function, the continuity assumptions on the initial function and its conjugate, the topology of the underlying space, etc.

2. Notation.

In this paper, X and X^* are two real locally convex (lc) separated spaces paired in duality by the bilinear form $(x^*, x) \in X^* \times X \to \langle x^*, x\rangle = \langle x, x^*\rangle = x^*(x)$. When not mentioned explicitly, the topologies on X and X^* are compatible with the pairing (in such a way that the dual of X is X^* and the dual of X^* is X). By $\sigma(X, X^*)$ and $\sigma(X^*, X)$ we denote the weak and weak* topologies defined on X and X^*, respectively; we use the symbol $\rightharpoonup$ for the corresponding convergence in both topologies. The null vectors in the involved spaces are all denoted by θ, and the convex symmetric neighborhoods of θ are called θ-neighborhoods. We use the notation $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$, and

$$\Delta_k := \{(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k \mid \lambda_1, \ldots, \lambda_k \geq 0,\ \lambda_1 + \cdots + \lambda_k = 1\}, \quad k \in \mathbb{N}^*.$$

The following notation and preliminary results are standard; see, e.g., the books [14, 20, 23] (for instance, for the indicator function we follow the notation in [14]). Given two nonempty sets A and B in X (or in X^*) and $\Lambda \subset \mathbb{R}$, we define

$$A + B := \{a + b \mid a \in A,\ b \in B\},\ \Lambda A := \{\lambda a \mid \lambda \in \Lambda,\ a \in A\},$$
$$A + \emptyset = \emptyset + A := \emptyset,\ \Lambda\emptyset = \emptyset A := \emptyset.$$

By $\text{co}(A)$, $\text{cone}(A)$, and $\text{aff}(A)$ we denote the *convex hull*, the *conic hull*, and the *affine hull* of A, respectively. By $\text{par}(A)$ we denote the *parallel subspace to* $\text{aff}(A)$; for instance, $\text{par}(A) = \text{aff}(A) - a$ for $a \in A$, and so $\text{par}(A) = \text{aff}(A)$ when $\theta \in A$. We use $\text{int}(A)$, $\text{cl}^w(A)$, and $\text{cl}(A)$ (or, indistinctly, $\overline{A}$ and $\overline{A}^w$) to respectively denote the *interior*, the *weak closure*, and the *closure* of A. Hence, $\overline{\text{co}}(A) := \text{cl}(\text{co}(A))$, $\overline{\text{aff}}(A) := \text{cl}(\text{aff}(A))$, $\overline{\text{cone}}(A) := \text{cl}(\text{cone}(A))$, and $\overline{\text{par}}(A) := \text{cl}(\text{par}(A))$. We use $\text{ri}(A)$ to denote the (topological) *relative interior* of A (i.e., the interior of A relative to $\text{aff}(A)$ if $\text{aff}(A)$ is closed, and the empty set otherwise [23]); hence, if $\text{ri}(A) \neq \emptyset$, then by definition $\text{aff}(A)$ and $\text{par}(A)$ are closed subsets. By A° and A^- we respectively denote the (one-sided) *polar* and the *negative dual cone* of A given by $A^\circ := \{x^* \in X^* \mid \langle x^*, x\rangle \leq 1$ for all $x \in A\}$ and $A^- := (\text{cone}(A))^\circ$. The *normal cone* to A at x is defined as

$$\text{N}_A(x) := (A - x)^- \text{ if } \ x \in A; \ \emptyset \text{ if } x \in X \setminus A.$$

The *support* and the *indicator* functions of A are, respectively, the functions $\sigma_A : X^* \to \overline{\mathbb{R}}$ and $\text{I}_A : X \to \overline{\mathbb{R}}_+$ defined by

$$\sigma_A(x^*) := \sup\{\langle x^*, a\rangle \mid a \in A\}, \ \text{I}_A(x) := 0 \text{ if } x \in A; \ +\infty \text{ if } x \in X \setminus A,$$

with the convention $\sigma_\emptyset = -\infty$.

In what follows we shall use the convention $(+\infty) + \alpha = \alpha + (+\infty) := +\infty$ for every $\alpha \in \overline{\mathbb{R}}$. If $f : X \to \overline{\mathbb{R}}$ (or $f : X^* \to \overline{\mathbb{R}}$) is a given function, we use $\text{dom} f$ and $\text{epi} f$ to respectively denote the *(effective) domain* and the *epigraph* of f,

$$\text{dom} f := \{x \in X \mid f(x) < +\infty\}, \ \text{epi} f := \{(x, \lambda) \in X \times \mathbb{R} \mid f(x) \leq \lambda\}.$$

We use $f_{|A}$ to denote the restriction of f on the set A, with the convention that $f_{|A} \equiv +\infty$ when $A = \emptyset$. We say that f is *proper* if $\text{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$ and *positively homogeneous* if $f(\lambda x) = \lambda f(x)$ for every $\lambda \geq 0$ and $x \in X$ with the convention $0.(\pm\infty) = 0$. We denote by $\text{cl} f$ the *lsc hull* of f; similarly, $\text{cl}^w f$ denotes the *weak lsc hull* of f (the epigraph of which is the closure of $\text{epi} f$ with respect to the weak topology of $X \times \mathbb{R}$). The *lsc convex hull* (also called *lsc convex envelope*) of f is the function $\overline{\text{co}} f : X \longrightarrow \overline{\mathbb{R}}$ such that

$$\text{epi}(\overline{\text{co}} f) = \overline{\text{co}}(\text{epi} f).$$

We denote by $\Gamma_0(X)$ the set of the lsc convex proper functions defined on X.

The (Legendre–Fenchel) *conjugate* of $f : X \to \overline{\mathbb{R}}$ is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined as

$$f^*(x^*) := \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\},$$

while $f^{**} : X \to \overline{\mathbb{R}}$, given by

$$f^{**}(x) := \sup_{x^* \in X^*}\{\langle x, x^*\rangle - f^*(x^*)\},$$

is the *biconjugate* of f. It is known that f^{**} is the supremum of all continuous affine mappings maximized by the function f on X, and so $f^{**} = \overline{\text{co}} f$ whenever this last function is proper. For $\varepsilon \geq 0$, the *ε-subdifferential* of f is the set-valued mapping $\partial_\varepsilon f : X \rightrightarrows X^*$ which assigns to $x \in X$ the (possibly empty) set

$$\partial_\varepsilon f(x) := \{x^* \in X^* \mid f^*(x^*) + f(x) \leq \langle x^*, x\rangle + \varepsilon\}$$

(in particular, this implies that $\partial_\varepsilon f(x) = \emptyset$ whenever $f(x) \notin \mathbb{R}$). If $\varepsilon = 0$, we recover the usual *Fenchel subdifferential* of f at x, $\partial f(x) := \partial_0 f(x)$. The ε-subdifferential of f^*, $\partial_\varepsilon f^* : X^* \rightrightarrows X$, is defined similarly: $\partial_\varepsilon f^*(x^*) := \{x \in X \mid f^*(x^*) + f^{**}(x) \leq$

$\langle x^*, x\rangle + \varepsilon\}$. When $\inf_X f \in \mathbb{R}$, we denote the set of *global ε-minima* of f by ε-argmin f; if $\varepsilon = 0$, we will simply write argmin f. Hence, provided that f^* is proper it holds that $\operatorname{argmin}(\overline{\operatorname{co}} f) = (\partial f^*)^{-1}(\theta)$.

Finally, given another function g, the inf-convolution of f and g is the function $f\Box g : X \to \overline{\mathbb{R}}$ given by $f\Box g(x) := \inf\{f(x_1)+g(x_2) \mid x_1+x_2 = x\}$. If $M : X \rightrightarrows X^*$ (or $X^* \rightrightarrows X$) is a set-valued operator, then $M^{-1} : X^* \rightrightarrows X$ denotes its inverse set-valued mapping, $M^{-1}(x^*) := \{x \in X \mid x^* \in Mx\}$.

3. Subdifferential of the conjugate via normal cones. In this section, the proposed formulas are of geometric nature and involve the normal cone to the domain of the conjugate function. We shall use the following enlargement of the Fenchel subdifferential introduced and studied in [5] (with the notation $\partial_L^* f$). We recall that the (lc) topologies on X and X^* are compatible with the duality pairing (X, X^*).

Definition 1. *Given a function $f : X \to \overline{\mathbb{R}}$ and a subset $L \subset X^*$, a vector $x^* \in X^*$ is said to be a relative subgradient of f at $x \in X$ with respect to L if $x^* \in L$, $f^*(x^*) \in \mathbb{R}$, and there exists a net $(x_\gamma) \subset X$ with $x_\gamma \underset{\widetilde{L}}{\rightharpoonup} x$ such that*

$$\lim(f(x_\gamma) - \langle x_\gamma, x^*\rangle) = -f^*(x^*), \tag{10}$$

where $x_\gamma \underset{\widetilde{L}}{\rightharpoonup} x$ means that $\lim\langle x_\gamma - x, y^*\rangle = 0$ for all $y^* \in \widetilde{L} := \overline{\operatorname{par}}(L \cap \operatorname{dom} f^*)$. The set of such vectors, denoted by $\partial_L^r f(x)$, is called the relative subdifferential of f at x with respect to L.

It is worth recalling that when $f \in \Gamma_0(X)$, the operator $\partial^r f$ (i.e., when $\operatorname{dom} f^* \subset L$) coincides with the usual Fenchel subdifferential [5, Proposition 2]. More generally, we always have that (see [5, Proposition 1(i)])

$$L \cap \partial f(x) \subset L \cap \partial(\operatorname{cl}^w f)(x) \subset \partial_L^r f(x). \tag{11}$$

If $\operatorname{dom} f^* \subset L$, then we write $\partial^r f(x) := \partial_L^r f(x)$ since this last set does not depend on L. Also, it is clear in view of the Fenchel inequality that (10) is equivalent to

$$f^*(x^*) \le \liminf(\langle x_\gamma, x^*\rangle - f(x_\gamma)). \tag{12}$$

Following the terminology in [15], provided that f is convex, the last inequality above means that x^* is an infinitesimal subgradient of f at the generalized point $\{x_\gamma\}$ [17]. In this respect, we added the term "relative" to our definition in order to refer to the way the involved net (x_γ) converges to x relatively to $\operatorname{dom} f^*$.

Example 1. If $f = e^{-|x|} + \mathrm{I}_{\mathbb{R}\setminus(-1,+1)}$, then for every subset $L \subset \mathbb{R}$ containing 0 it holds that

$$\partial_L^r f(x) = \{0\} \quad \text{for all } x \in \mathbb{R}.$$

Proof. Indeed, we have that $f^*(0) = 0 \in \mathbb{R}$ and

$$\lim_{k\to\infty} e^{-|k|} = 0 = -f^*(0).$$

Thus, since $\operatorname{par}(L \cap \operatorname{dom} f^*) = \{0\}$, given any $x \in \mathbb{R}$ we infer that $0 \in \partial_L^r f(x)$. The converse inclusion $\partial_L^r f(x) \subset \{0\}$ holds in view of the relationship $\partial_L^r f(x) \subset \operatorname{dom} f^* = \{0\}$. ∎

The coincidence of $\partial^r f$ and ∂f evoked above in the convex case may also occur for not necessarily convex functions.

PROPOSITION 1. *Let function $f : X \to \overline{\mathbb{R}}$, $x \in X$, and $L \subset X^*$ be given. Then, $\partial_L^r f(x) = L \cap \partial f(x)$ in each case of the following*:

(i) *for some $z^* \in \mathrm{cl}\,(L \cap \mathrm{dom}\, f^*)$ it holds that*

$$f(x) \leq \liminf_{\substack{y \rightharpoonup x \\ \widetilde{L}}} (f(y) - \langle y - x, z^* \rangle);$$

(ii) $L = X^*$, $\mathrm{int}(\mathrm{dom}\, f^*) \neq \emptyset$, *and f is weakly lsc.*

Proof. Assertion (ii) is known (see [5]). To prove (i) we need only, in view of (11), show that $\partial_L^r f(x) \subset L \cap \partial f(x)$. We pick $x^* \in \partial_L^r f(x)$ $(\subset L \cap (f^*)^{-1}(\mathbb{R}))$ and a net $x_\gamma \underset{\widetilde{L}}{\rightharpoonup} x$ such that $\lim(f(x_\gamma) - \langle x_\gamma, x^* \rangle) = -f^*(x^*)$. Thus, by the current assumption we obtain that

$$\begin{aligned} f(x) &\leq \liminf(f(x_\gamma) - \langle x_\gamma - x, z^* \rangle) \\ &= \liminf(f(x_\gamma) - \langle x_\gamma - x, x^* \rangle) = -f^*(x^*) + \langle x, x^* \rangle, \end{aligned}$$

showing that $x^* \in \partial f(x)$; hence, $x^* \in L \cap \partial f(x)$. ∎

In what follows, for function $f : X \to \overline{\mathbb{R}}$ and $x^* \in X^*$ we use the notation

$$\mathcal{F}(f) := \{L \subset X^* \text{ closed and convex} \mid f^*_{|\mathrm{ri}(L \cap \mathrm{dom}\, f^*)} \text{ is finite and continuous}\}, \tag{13}$$

$$\mathcal{F}(f, x^*) := \{L \in \mathcal{F}(f) \mid x^* \in L\}. \tag{14}$$

Observe that, according to our current convention (that is, $f^*_{|A} \equiv +\infty$ when $A = \emptyset$), the last set above can be equivalently written as

$$\begin{aligned} \mathcal{F}(f, x^*) = \{L \subset X^* \text{ closed and convex} \mid x^* \in L, \mathrm{ri}(L \cap \mathrm{dom}\, f^*) \neq \emptyset, \\ f^*_{|\mathrm{ri}(L \cap \mathrm{dom}\, f^*)} \text{ is continuous}\}. \end{aligned}$$

This set is slightly different from the one introduced in [5], given by

$$\widehat{\mathcal{F}}_{x^*} := \{L \subset X^* \text{ convex} \mid x^* \in L,\ \mathrm{ri}(L \cap \mathrm{dom}\, f^*) \neq \emptyset,\ f^*_{|\mathrm{ri}(L \cap \mathrm{dom}\, f^*)} \text{ is continuous}\}$$

and used to get the formula (see [5, Theorem 4])

$$\partial f^*(x^*) = \bigcap_{L \in \widehat{\mathcal{F}}_{x^*}} \overline{\mathrm{co}} \left((\partial_L^r f)^{-1}(x^*) + \mathrm{N}_{L \cap \mathrm{dom}\, f^*}(x^*) \right).$$

Consequently, since we can easily check that $\mathcal{F}(f, x^*) \subset \widehat{\mathcal{F}}_{x^*}$ and

$$\bigcap_{L \in \mathcal{F}(f, x^*)} \overline{\mathrm{co}} \left((\partial_L^r f)^{-1}(x^*) + \mathrm{N}_{L \cap \mathrm{dom}\, f^*}(x^*) \right) \subset \partial f^*(x^*),$$

we obtain the following proposition.

PROPOSITION 2. *Given a function $f : X \to \overline{\mathbb{R}}$, for every $x^* \in X^*$ we have the formula*

$$\partial f^*(x^*) = \bigcap_{L \in \mathcal{F}(f, x^*)} \overline{\mathrm{co}} \left((\partial_L^r f)^{-1}(x^*) + \mathrm{N}_{L \cap \mathrm{dom}\, f^*}(x^*) \right).$$

Hence, provided that $X^* \in \mathcal{F}(f)$, the following assertions hold true:

(i) $\partial f^*(x^*) = \overline{\mathrm{co}}((\partial^r f)^{-1}(x^*) + \mathrm{N}_{\mathrm{dom}\, f^*}(x^*))$;

(ii) [5, *Corollary* 6] *if* $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$, *then*

$$\partial f^*(x^*) = \mathrm{N}_{\operatorname{dom} f^*}(x^*) + \overline{\operatorname{co}}\left((\partial^r f)^{-1}(x^*)\right)$$
$$\left(= \mathrm{N}_{\operatorname{dom} f^*}(x^*) + \overline{\operatorname{co}}\left((\partial f)^{-1}(x^*)\right) \text{ when } f \text{ is weakly lsc}\right);$$

(iii) [5, *Corollary* 7] *if* $X = \mathbb{R}^n$, f *is lsc, and* $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$, *then*

$$\partial f^*(x^*) = \mathrm{N}_{\operatorname{dom} f^*}(x^*) + \operatorname{co}\left((\partial f)^{-1}(x^*)\right).$$

The next result concerns the important case of positively homogeneous functions, where the formulas of the subdifferential of the conjugate are given without the term including the normal cone. It is worth observing that, when $f : X \to \overline{\mathbb{R}}$ is a positively homogeneous function bounded from below, then $\operatorname{argmin} f = \{x \in X \mid f(x) = 0\}$ and $f^* = \mathrm{I}_{\partial f(\theta)}$; hence, $\operatorname{dom} f^* = \partial f(\theta)$, and so $\partial f^*(\theta) = \mathrm{N}_{\operatorname{dom} f^*}(\theta) \neq \emptyset$. In particular, a closed convex set $L \subset X^*$ belongs to $\mathcal{F}(f)$ iff $\operatorname{ri}(L \cap \operatorname{dom} f^*) \neq \emptyset$. In other words, under the positive homogeneity and boundedness from below of f, $\mathcal{F}(f)$ is equivalently written as

$$\mathcal{F}(f) = \{L \subset X^* \text{ closed and convex} \mid \operatorname{ri}(L \cap \operatorname{dom} f^*) \neq \emptyset\}. \tag{15}$$

THEOREM 3. *Let* $f : X \to \overline{\mathbb{R}}$ *be a positively homogeneous function such that* $X^* \in \mathcal{F}(f)$. *Then, for every* $x^* \in X^*$ *the following statements hold*:

(i) $\partial f^*(x^*) = \overline{\operatorname{co}}((\partial^r f)^{-1}(x^*))$;

(ii) *if* $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$ *and* f *is weakly lsc, then* $\partial f^*(x^*) = \overline{\operatorname{co}}((\partial f)^{-1}(x^*))$;

(iii) *if in* (ii) *we assume* $X = \mathbb{R}^n$, *then* $\partial f^*(x^*) = \operatorname{co}((\partial f)^{-1}(x^*))$.

Proof. Since $\partial f^*(x^*) = \partial(f - x^*)^*(\theta)$ and $f - x^*$ is also a positively homogeneous function satisfying $\operatorname{dom}(f - x^*)^* = (\operatorname{dom} f^*) + x^*$, we may assume that $x^* = \theta$. Also, observe that assertion (ii) is an immediate consequence of (i) together with Proposition 1(ii). Assertion (iii) comes from (ii) as follows: if $(\operatorname{argmin} f =)$ $(\partial f)^{-1}(\theta) = \emptyset$, then we are done (by (ii)). Otherwise, we have that $\operatorname{argmin} f = \{x \in X \mid f(x) = 0\}$. To conclude, in view of (ii) it suffices to establish that $\operatorname{co}(\operatorname{argmin} f)$ is closed. Indeed, taking into account that $\operatorname{argmin} f$ is a cone together with Carathéodory's theorem, we pick a sequence of the form $\left(\sum_{i=1}^n u_{k,i}\right)_k$, $u_{k,i} \in \operatorname{argmin} f$, which converges to some $u \in \mathbb{R}^n$. But, by the current assumption (epi-pointedness and positive homogeneity of f), together with Proposition 13, there are $w \in \mathbb{R}^n$ and $\mu > 0$ such that

$$f(x) + \langle x, w \rangle \geq \mu \|x\| \quad \text{for all } x \in \mathbb{R}^n.$$

Then, since $u_{k,i} \in \operatorname{argmin} f$, we get $\langle u_{k,i}, w \rangle = f(u_{k,i}) + \langle u_{k,i}, w \rangle \geq \mu \|u_{k,i}\|$ for all $k \in \mathbb{N}$ and $i \in \{1, \ldots, k\}$ so that, summing up over i and using the Cauchy–Schwarz inequality,

$$\mu \sum_{i=1}^k \|u_{k,i}\| \leq \left\langle \sum_{i=1}^k u_{k,i}, w \right\rangle \leq \left\| \sum_{i=1}^k u_{k,i} \right\| \|w\|.$$

But $\left(\sum_{i=1}^n u_{k,i}\right)_k$ converges to u, and so, without loss of generality, for each $i \in \{1, \ldots, k\}$ we obtain that

$$\sup_k \|u_{k,i}\| \leq \mu^{-1}\left((\|u\| + 1)\|w\|\right) < +\infty;$$

that is, $(u_{k,i})_i$ is bounded, and so it has an accumulation point u_i which belongs to $\operatorname{argmin} f$ (invoking the lower semicontinuity of f). Therefore, $v = \sum_{i=1}^n u_i \in$

co(argmin f), as we wanted to prove. To prove (i) we recall that $f^* = \mathrm{I}_{\partial f(\theta)}$, $\mathrm{dom}\, f^* = \partial f(\theta)$, and $\partial f^*(\theta) = \mathrm{N}_{\mathrm{dom}\, f^*}(\theta) \neq \emptyset$. Thus, since $\overline{\mathrm{co}} f$ is also a (proper lsc convex) positively homogeneous function and $\mathrm{argmin}(\overline{\mathrm{co}} f) = \partial f^*(\theta)$ $(\neq \emptyset)$, we infer that $\overline{\mathrm{co}} f = \sigma_{\mathrm{dom}\, f^*}(\cdot)$ at the same time as

$$\inf_X f = \inf_X \overline{\mathrm{co}} f = -f^*(\theta) = 0,\ f(\theta) = f^{**}(\theta) = 0,\, \theta \in \partial f(\theta) \subset \partial^r f(\theta) \tag{16}$$

(see (11) for the last inclusion.) We choose, invoking the assumption $X^* \in \mathcal{F}(f)$, vector $\bar{z}^* \in \mathrm{ri}(\mathrm{dom}\, f^*)$ and θ-neighborhood $V \subset X^*$ such that

$$f^*(\bar{z}^* + z^*) = \mathrm{I}_{\partial f(\theta)}(\bar{z}^* + z^*) = 0 \quad \text{for all } z^* \in V \cap \mathrm{par}(\mathrm{dom}\, f^*) \tag{17}$$

(recall (15) together with the facts that $\theta \in \mathrm{dom}\, f^*$ and $\overline{\mathrm{par}}(\mathrm{dom}\, f^*) = \mathrm{par}(\mathrm{dom}\, f^*)$). We also recall, as a consequence of the condition $X^* \in \mathcal{F}(f)$, that (see Proposition 2(i))

$$\partial f^*(\theta) = \overline{\mathrm{co}} \left((\partial^r f)^{-1}(\theta) + \mathrm{N}_{\mathrm{dom}\, f^*}(\theta) \right); \tag{18}$$

hence, the first inclusion $\overline{\mathrm{co}} \left((\partial^r f)^{-1}(\theta) \right) \subset \partial f^*(\theta)$ follows. To establish the converse one we need only show that $(\partial f^*(\theta) =)\ \mathrm{N}_{\mathrm{dom}\, f^*}(\theta) \subset \overline{\mathrm{co}} \left((\partial^r f)^{-1}(\theta) \right)$, or, equivalently,

$$\sigma_{\mathrm{N}_{\mathrm{dom}\, f^*}(\theta)}(v^*) \leq \sigma_{(\partial^r f)^{-1}(\theta)}(v^*) \tag{19}$$

for every given vector $v^* \in X^*$. We begin by studying the case $v^* \notin \mathrm{par}(\mathrm{dom}\, f^*)$. For we first verify that

$$(\mathrm{par}(\mathrm{dom}\, f^*))^{\perp} \subset (\partial^r f)^{-1}(\theta). \tag{20}$$

Indeed, given $y \in (\mathrm{par}(\mathrm{dom}\, f^*))^{\perp}$, from the relationship $f(\theta) = -f^*(\theta) = 0$ (see (16)) together with the obvious fact that $\langle \theta - y, y^* \rangle = 0$, for all $y^* \in \mathrm{par}(\mathrm{dom}\, f^*)$, by Definition 1 it follows that $\theta \in \partial^r f(y)$, and so $y \in (\partial^r f)^{-1}(\theta)$; that is, (20) holds. Now, going back to (19), if $\sigma_{(\partial^r f)^{-1}(\theta)}(v^*) < +\infty$, by (20) we can write $\sigma_{(\mathrm{par}(\mathrm{dom}\, f^*))^{\perp}}(v^*) < \sigma_{(\partial^r f)^{-1}(\theta)}(v^*) < +\infty$, which yields the contradiction $v^* \in (\mathrm{par}(\mathrm{dom}\, f^*))^{\perp\perp} = \mathrm{par}(\mathrm{dom}\, f^*)$. Hence, $\sigma_{(\partial^r f)^{-1}(\theta)}(v^*) = +\infty$, and (19) obviously holds.

The rest of the proof is devoted to the case when $v^* \in \mathrm{par}(\mathrm{dom}\, f^*)$. To proceed we pick $v \in \mathrm{N}_{\mathrm{dom}\, f^*}(\theta)$ so that $0 = \inf \overline{\mathrm{co}} f \leq (\overline{\mathrm{co}} f)(v) = (\overline{\mathrm{co}} f)^{\infty}(v) = \sigma_{\mathrm{dom}\, f^*}(v) \leq 0$; that is, $(v, 0) \in \mathrm{epi}(\overline{\mathrm{co}} f) = \overline{\mathrm{co}}(\mathrm{epi}\, f)$. Let $U \subset X$ be a given (symmetric) θ-neighborhood satisfying

$$\langle z, v^* \rangle \leq 1,\ \langle z, \bar{z}^* \rangle \leq 1 \quad \text{for all } z \in U. \tag{21}$$

Then, for each $m \in \mathbb{N}^*$ there exists $(v_m, \alpha_m) \in \mathrm{co}(\mathrm{epi}\, f)$ such that $(v_m - v, \alpha_m) \in m^{-1}(U \times [-1, 1])$. Let $T : X \times \mathbb{R} \to \mathbb{R}^3$ be the (continuous) linear operator defined by

$$T(x, t) := (\langle x, v^* \rangle, \langle x, \bar{z}^* \rangle, t)$$

so that $T(\mathrm{co}(\mathrm{epi}\, f)) = \mathrm{co}(T(\mathrm{epi}\, f))$. Then, by taking into account Carathéodory's theorem, it follows that

$$(\langle v_m, v^* \rangle, \langle v_m, \bar{z}^* \rangle, \alpha_m) = \sum_{1 \leq j \leq 4} \lambda_{j,m}(\langle v_{j,m}, v^* \rangle, \langle v_{j,m}, \bar{z}^* \rangle, \alpha_{j,m}), \tag{22}$$

where $(\lambda_{1,m}, \ldots, \lambda_{4,m}) \in \Delta_4$ and $(v_{j,m}, \alpha_{j,m}) \in \mathrm{epi}\, f$ for all $m \geq 1$ and $1 \leq j \leq 4$.

Now, for fixed $i \in \{1, \ldots, 4\}$ we choose any vector $z^* \in V \cap \operatorname{par}(\operatorname{dom} f^*)$. By (17) together with the Fenchel inequality it follows that

$$\langle \lambda_{i,m} v_{i,m}, \bar{z}^* + z^* \rangle \leq \lambda_{i,m} f(v_{i,m}) \leq \lambda_{i,m} \alpha_{i,m}; \tag{23}$$

hence, in particular, when $z^* = \theta$ it holds that

$$\langle \lambda_{i,m} v_{i,m}, \bar{z}^* \rangle \leq \lambda_{i,m} f(v_{i,m}) \leq \lambda_{i,m} \alpha_{i,m}. \tag{24}$$

So, we write

$$\begin{aligned} \langle \lambda_{i,m} v_{i,m}, \bar{z}^* \rangle \quad & \geq -\left\langle \textstyle\sum_{1 \leq j \leq 4,\ j \neq i} \lambda_{j,m} v_{j,m}, \bar{z}^* \right\rangle + \langle v, \bar{z}^* \rangle - m^{-1} && \text{(by (21)–(22))} \\ & \geq -\textstyle\sum_{1 \leq j \leq 4,\ j \neq i} \lambda_{j,m} \alpha_{j,m} + \langle v, \bar{z}^* \rangle - m^{-1} && \text{(by (24))} \\ & \geq -\textstyle\sum_{1 \leq j \leq 4} \lambda_{j,m} \alpha_{j,m} + \langle v, \bar{z}^* \rangle - m^{-1} && \text{(as } \alpha_{j,m} \geq 0) \\ & = -\alpha_m + \langle v, \bar{z}^* \rangle - m^{-1} \geq \langle v, \bar{z}^* \rangle - 2m^{-1}. && \text{(by (22))} \end{aligned}$$

Therefore, (23) leads us to

$$\begin{aligned} \langle \lambda_{i,m} v_{i,m}, z^* \rangle & \leq \lambda_{i,m} \alpha_{i,m} - \langle \lambda_{i,m} v_{i,m}, \bar{z}^* \rangle \\ & \leq \alpha_m - \langle v, \bar{z}^* \rangle + 2m^{-1} \leq -\langle v, \bar{z}^* \rangle + 3m^{-1} \ \leq 3 - \langle v, \bar{z}^* \rangle \leq r, \end{aligned}$$

where $r := \max\{1, 3 - \langle v, \bar{z}^* \rangle\}$. Set $Y := \operatorname{lin}(\operatorname{dom} f^*)$. Interpreting $v_{i,m}$ as a linear functional on X^* (i is fixed), and taking $y_{i,m} := \lambda_{i,m} v_{i,m|Y}$ (the restriction on Y), we obtain that $(y_{i,m})_m \subset r(V \cap \operatorname{par}(\operatorname{dom} f^*))^\circ \subset Y^*$. Using the Alaoglu–Bourbaki theorem, by taking a subnet if necessary we may suppose that $(y_{i,m})_m$ (as a net) weak*-converges to some $y_i \in Y^*$. Let $x_i \in X$ be such that $y_i = x_{i|Y}$. Then $\langle \lambda_{i,m} v_{i,m} - x_i, y^* \rangle \to 0$ for all $y^* \in Y$. Because $0 \leq f(\lambda_{i,m} v_{i,m}) = \lambda_{i,m} f(v_{i,m}) \leq \lambda_{i,m} \alpha_{i,m} \leq m^{-1}$, we obtain that $x_i \in (\partial^r f)^{-1}(\theta)$. On the other hand, we have (as $v^* \in \operatorname{par}(\operatorname{dom} f^*)$)

$$\langle v, v^* \rangle = \lim \sum_{1 \leq i \leq 4} \langle \lambda_{i,m} v_{i,m}, v^* \rangle = \sum_{1 \leq i \leq 4} \langle x_i, v^* \rangle;$$

here, the first equality comes from (21) and (22). Consequently, using the conic structure of $(\partial_r f)^{-1}(\theta)$, we get $\langle v, v^* \rangle \leq \sigma_{(\partial_r f)^{-1}(\theta)}(v^*)$. But v was arbitrarily chosen in $\mathrm{N}_{\operatorname{dom} f^*}(\theta)$ so that $\sigma_{\mathrm{N}_{\operatorname{dom} f^*}(\theta)}(v^*) \leq \sigma_{(\partial_r f)^{-1}(\theta)}(v^*)$. Thus, the proof of (19) is finished. ▯

We close this section by giving the following result, which is the counterpart of Proposition 2 and Theorem 3 when the ε-subdifferential is evoked. We recall that for any function $f : X \to \overline{\mathbb{R}}$ having a proper conjugate, according to (3), for every $x^* \in X^*$ we have that

$$\partial f^*(x^*) = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(f, x^*)}} \overline{\operatorname{co}} \left((\partial_\varepsilon f)^{-1}(x^*) + \mathrm{N}_{L \cap \operatorname{dom} f^*}(x^*) \right). \tag{25}$$

Proposition 4. *Let $f : X \to \overline{\mathbb{R}}$ be such that $X^* \in \mathcal{F}(f)$. If $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$, then for every $x^* \in X^*$ we have the formula*

$$\partial f^*(x^*) = \mathrm{N}_{\operatorname{dom} f^*}(x^*) + \bigcap_{\varepsilon > 0} \overline{\operatorname{co}} \left((\partial_\varepsilon f)^{-1}(x^*) \right).$$

In particular, if f is positively homogeneous, then

$$\partial f^*(x^*) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\left((\partial_\varepsilon f)^{-1}(x^*)\right).$$

Proof. We fix $x^* \in X^*$. The inclusion "$\supset$" is straightforward and follows directly from the definition of $\partial_\varepsilon f$ and $\mathrm{N}_{\mathrm{dom}\, f^*}(x^*)$. To prove the converse inclusion, according to Proposition 2(ii), we need only show that, for every given $\varepsilon > 0$,

$$(\partial^r f)^{-1}(x^*) \subset \overline{\mathrm{co}}\left((\partial_\varepsilon f)^{-1}(x^*)\right).$$

Indeed, if $x \in (\partial^r f)^{-1}(x^*)$, then by the current assumption, $\mathrm{int}(\mathrm{dom}\, f^*) \neq \emptyset$, there exists a net $(x_\gamma)_{\gamma\in D}$ which converges to x in $(X, \sigma(X, X^*))$ so that

$$\lim(f(x_\gamma) - \langle x_\gamma, x^*\rangle) = -\langle x, x^*\rangle + \lim f(x_\gamma) = -f^*(x^*);$$

hence, $x_\gamma \in (\partial_\varepsilon f)^{-1}(x^*) \subset \overline{\mathrm{co}}\left((\partial_\varepsilon f)^{-1}(x^*)\right)$ for all big enough γ. Whence, the conclusion follows by taking limits on γ. Finally, if f is positively homogeneous, then the last formula follows in a similar way by using Theorem 3(i) instead of Proposition 2(ii). □

The first conclusion of Proposition 4 may not be true if $\mathrm{int}(\mathrm{dom}\, f^*) = \emptyset$, as we show in the following example.

Example 2. We consider the set, in $\mathbb{R}^2$,

$$A := \left\{\left(a, (a-1)^{-1}\right),\ a > 1;\ \left(b, -b^{-1}\right),\ b > 0\right\}$$

and the corresponding indicator function I_A of A so that $f^* = \sigma_A$. In this case, we have that $\mathrm{cl}(\mathrm{dom}\, f^*) = \mathbb{R}_-\times\{0\}$, and so $\mathrm{int}(\mathrm{dom}\, f^*) = \mathrm{int}(\mathrm{cl}(\mathrm{dom}\, f^*)) = \emptyset$. By direct calculation it follows that

$$\partial f^*(-1, 0) = \{(u, \mu) \in \overline{\mathrm{co}}\,(A) = \mathbb{R}_+\times\mathbb{R} \mid -u = f^*(-1, 0) = 0\} = \{0\} \times \mathbb{R},$$

and so

$$\emptyset = (\{0\} \times \mathbb{R}) + \emptyset = \mathrm{N}_{\mathrm{dom}\, f^*}(-1, 0) + \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\left(\left(b, -b^{-1}\right),\ 0 < b \leq \varepsilon\right) \subsetneqq \partial f(-1, 0).$$

4. Asymptotic analysis.

We introduce and study in this section the promised concept of asymptotic functions. We show that it inherits many of the characteristics of the usual recession functions in the sense of the convex analysis setting, namely, the property investigated in Theorem 7.

DEFINITION 2. *Let $f : X \to \overline{\mathbb{R}}$ be a given function, and take $z^* \in \mathrm{dom}\, f^*$ (assumed nonempty). We call the relative asymptotic function of f the function $f^\infty : X \to \overline{\mathbb{R}}$ defined by*

$$f^\infty(x) := \liminf_{s\to 0^+,\ y \underset{\widetilde{M}}{\rightharpoonup} x} \left(sf\left(s^{-1}y\right) - \langle y - x, z^*\rangle\right),$$

where $y \underset{\widetilde{M}}{\rightharpoonup} x$ means that $\langle y - x, y^*\rangle \to 0$ for all $y^* \in \widetilde{M} := \overline{\mathrm{par}}(\mathrm{dom}\, f^*)$.

It is clear in view of the convergence $\underset{\widetilde{M}}{\rightharpoonup}$ that the function f^∞ does not depend on the choice of z^* in $\mathrm{dom}\, f^*$. Also, the condition $\mathrm{dom}\, f^* \neq \emptyset$ is not restrictive regarding our current objective aiming at calculating the subdifferential of f^*; otherwise, such a subdifferential is always empty. In this respect, if we denote $g := f - z^*$, $z^* \in \operatorname{dom} f^*$, then $\theta \in \operatorname{dom} g^*$, and it can be easily verified that

$$f^\infty = g^\infty + z^*. \tag{26}$$

Consequently, by invoking the Fenchel inequality we get

$$g^\infty(x) + \sigma_{\operatorname{dom} f^*}(x) = f^\infty(x) \geq \sigma_{\operatorname{dom} f^*}(x) \quad \text{for every } x \in X, \tag{27}$$

entailing that $f^\infty > -\infty$ and $g^\infty \geq 0$. On another hand, if $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$, then for every $x \in X$ we obtain that

$$f^\infty(x) = \liminf_{s \to 0^+,\ y \rightharpoonup x} sf\left(s^{-1}y\right), \tag{28}$$

showing that f^∞ coincides with the usual asymptotic function in the sense of [6, 7]. In this case, the epigraph of f^∞ is given by

$$\operatorname{epi} f^\infty = \bigcap_{\varepsilon > 0} \operatorname{cl}^w\left((0, \varepsilon] \operatorname{epi} f\right).$$

In the convex setting, when f is proper, lsc, and convex, f^∞ is nothing else but the usual corresponding recession function.

PROPOSITION 5. *Let $f \in \Gamma_0(X)$ be given, and fix $x_0 \in \operatorname{dom} f$. Then $f^\infty \in \Gamma_0(X)$, and for every $x \in X$ we have that*

$$f^\infty(x) = \sup_{s>0} s^{-1}(f(x_0 + sx) - f(x_0)).$$

Proof. Let us first observe that $f^* \in \Gamma_0(X^*)$, and so $\operatorname{dom} f^* \neq \emptyset$, and for every given $x, x_0 \in X$ we have

$$\sup_{s>0} s^{-1}(f(x_0 + sx) - f(x_0)) = \sigma_{\operatorname{dom} f^*}(x).$$

Hence, the first inequality "$\geq$" follows in view of (27). To show the converse one, it suffices to define $x_k := x + k^{-1}x_0$, $k \geq 1$, and observe that

$$\begin{aligned} f^\infty(x) \leq \liminf_{k\to\infty}(k^{-1}f\left(k(x + k^{-1}x_0)\right)) &= \lim_{k\to+\infty} k^{-1}(f\left(k(x + k^{-1}x_0)\right) - f(x_0)) \\ &= \sup_{s>0} s^{-1}(f(x_0 + sx) - f(x_0)) = \sigma_{\operatorname{dom} f^*}(x). \qquad \square \end{aligned}$$

The following lemma gives some other properties of the relative asymptotic functions, which will be used later on.

LEMMA 6. *Let $f : X \to \overline{\mathbb{R}}$ have a proper conjugate. Then, f^∞ is positively homogeneous and satisfies $f^\infty(\theta) = 0$ together with*

$$[\operatorname{aff}(\operatorname{dom} f^*)]^\perp \times \{0\} \subset \operatorname{epi} f^\infty \cap \{X \times \{0\}\}.$$

Proof. The positive homogeneity of f^∞ is immediate from Definition 2. To show that $f^\infty(\theta) = 0$ we pick $z^* \in \operatorname{dom} f^*$ and $x_0 \in \operatorname{dom} f$. Then, by taking $y = sx_0$ it follows that

$$f^\infty(\theta) \leq \liminf_{s\to 0^+}(sf\left(x_0\right) - s\langle x_0, z^*\rangle) = 0.$$

Hence, the desired equality follows because $f^\infty(\theta) \geq \sigma_{\operatorname{dom} f^*}(\theta) = 0$, according to (27). Finally, if $v \in [\operatorname{aff}(\operatorname{dom} f^*)]^\perp$ is given, then the sequence given by $x_k := k^{-1}x_0$ (recall that $f(x_0) \in \mathbb{R}$) satisfies, for all $y^* \in \overline{\operatorname{par}}(\operatorname{dom} f^*)$,

$$\lim_{k\to+\infty} \langle k^{-1}x_0 - v, y^*\rangle = \lim_{k\to+\infty} k^{-1}\langle x_0, y^*\rangle = 0,$$

so that $f^\infty(v) \leq \liminf_{k\to\infty}(k^{-1}f(x_0) - \langle z^*, k^{-1}x_0 - v\rangle) = 0$. ∎

The following example illustrates the concept of relative asymptoticity on some elementary functions.

Example 3. (i) For $f(x,y) = \sqrt{\exp(-x^2) + y^2}$, we have $\operatorname{dom} f^* = \{0\} \times [-1,1]$ and $f^\infty(u,v) = |v|$.

(ii) For $f(x,y) = \sqrt{\exp(-x^2) + y^2} + x$, we have $\operatorname{dom} f^* = \{1\} \times [-1,1]$ and $f^\infty(u,v) = |v| + u$.

(iii) For $f(x,y) = \sqrt{\exp(-|x|) + |y|}$, we have $\operatorname{dom} f^* = \{(0,0)\}$ and $f^\infty \equiv 0$.

(iv) For $f(x,y) = \max\{0, -|x|+|y|\}+x$, we have $\operatorname{dom} f^* = \{(1,0)\}$ and $f^\infty(u,v) = u$.

Remark 1. (i) While Example 3(i) shows that (28) may hold in higher dimensions even if $\operatorname{int}(\operatorname{dom} f^*) = \emptyset$, Example 3(iv) gives us, for $(u,v) \in \mathbb{R}^2$ with $|v| > |u|$,

$$\liminf_{\substack{s\to 0^+ \\ (x,y)\to(u,v)}} sf\left(s^{-1}(x,y)\right) = \max\{0, -|u| + |v|\} + u > f^\infty(u,v);$$

that is, f^∞ may differ from the asymptotic function used in [6, 8].

(ii) If in Definition 2 we assume that $\theta \in \operatorname{dom} f^*$, then the expression of f^∞ simplifies to

$$f^\infty(x) = \liminf_{s\to 0^+,\ y \underset{\widetilde{M}}{\to} x} sf\left(s^{-1}y\right);$$

observe that this relationship is not correct in general, also according to Example 3(iv).

(iii) Finally, it follows from Example 3(iv) that, in general, a positively homogeneous function and its relative asymptotic function may be different from each other.

The following theorem provides us with a fundamental property for our analysis.

THEOREM 7. *Let $f : X \to \overline{\mathbb{R}}$ be a function such that $X^* \in \mathcal{F}(f)$. Then, we have that*

$$\overline{\operatorname{co}}(f^\infty) = (\overline{\operatorname{co}} f)^\infty.$$

Consequently, for any given given $y \in \operatorname{dom} f$ it holds that

$$\overline{\operatorname{co}}\left(f^\infty\right)(x) = \sup_{t>0} t^{-1}(\overline{\operatorname{co}} f(y+tx) - \overline{\operatorname{co}} f(y)) = \sigma_{\operatorname{dom} f^*}(x) \quad \textit{for every } x \in X.$$

Proof. In view of (26) we may suppose that $\theta \in \operatorname{dom} f^*$ so that, by the current assumption, $\overline{\operatorname{par}}(\operatorname{dom} f^*) = \operatorname{par}(\operatorname{dom} f^*) = \operatorname{aff}(\operatorname{dom} f^*)$. Also, observe that the functions f, f^*, and $\overline{\operatorname{co}} f$ are proper so that $(\theta, 0) \in \operatorname{epi} f^\infty$ (see Lemma 6) and the last equality in the second conclusion of the theorem holds (see, e.g., [23]). Hence, by invoking (27) we get $\overline{\operatorname{co}}(f^\infty) \geq \sigma_{\operatorname{dom} f^*} = (\overline{\operatorname{co}} f)^\infty$. To establish the second inequality it suffices to prove that for any given $(u^*, \mu) \in X^* \times \mathbb{R}$ we have that

$$\sigma_{\operatorname{epi}((\overline{\operatorname{co}} f)^\infty)}(u^*, \mu) \leq \sigma_{\operatorname{epi} f^\infty}(u^*, \mu). \tag{29}$$

To this aim, invoking Lemma 6, we observe that $\sigma_{\mathrm{epi}\, f^\infty}(u^*,\mu) = +\infty$ whenever $\sigma_{\mathrm{epi}\, f^\infty}(u^*,\mu) > 0$ or $\mu > 0$; hence, (29) trivially holds in this case. This last fact ($\sigma_{\mathrm{epi}\, f^\infty}(u^*,\mu) = +\infty$) also occurs when $u^* \notin \mathrm{par}(\mathrm{dom}\, f^*)$ and $\mu \le 0$. Indeed, otherwise, by Lemma 6 we get $\sigma_{[\mathrm{par}(\mathrm{dom}\, f^*)]^\perp}(u^*) \le \sigma_{\mathrm{epi}\, f^\infty}(u^*,\mu) < +\infty$, implying the contradiction $u^* \in [\mathrm{par}(\mathrm{dom}\, f^*)]^{\perp\perp} = \mathrm{par}(\mathrm{dom}\, f^*)$. In view of the argument evoked above, in order to establish (29) we need only show that for fixed $(x,\alpha) \in \mathrm{epi}((\overline{\mathrm{co}} f)^\infty)$ and $(u^*,\mu) \in \mathrm{par}(\mathrm{dom}\, f^*) \times [-\frac{1}{2}, 0]$, satisfying $\sigma_{\mathrm{epi}\, f^\infty}(u^*,\mu) = 0$, we have that

$$\langle (x,\alpha),(u^*,\mu)\rangle \le 0. \tag{30}$$

To proceed, by taking into account the assumption $X^* \in \mathcal{F}(f)$ we fix $\bar z^* \in \mathrm{ri}(\mathrm{dom}\, f^*)$ and θ-neighborhood $V \subset X^*$ such that

$$f^*(\bar z^* + z^*) \le f^*(\bar z^*) + 1 \quad \text{for all } z^* \in V \cap \mathrm{par}(\mathrm{dom}\, f^*). \tag{31}$$

We also choose a θ-neighborhood $U \subset X$ such that

$$\sup_{u\in U, \beta\in[-1,1]} \langle (u,\beta),(u^*,\mu)\rangle \le 1, \ \ \sup_{u\in U} \langle u, \bar z^*\rangle \le 1. \tag{32}$$

Now, since $(x,\alpha) \in \mathrm{epi}((\overline{\mathrm{co}} f)^\infty) = \bigcap_{\varepsilon>0} \overline{(0,\varepsilon]\overline{\mathrm{co}}(\mathrm{epi}\, f)} = \bigcap_{\varepsilon>0} \overline{(0,\varepsilon]\, \mathrm{co}(\mathrm{epi}\, f)}$ (see, e.g., [23]), for each $m \ge 1$ we find $\gamma_m \in (0, m^{-1})$, $k_m \in \mathbb{N}^*$, $(\lambda_{1,m},\dots,\lambda_{k_m,m}) \in \Delta_{k_m}$, and $(x_{1,m},\alpha_{1,m}),\dots,(x_{k_m,m},\alpha_{k_m,m}) \in \mathrm{epi}\, f$ such that

$$(x,\alpha) - \sum_{1\le i\le k_m} \gamma_m \lambda_{i,m}(x_{i,m},\alpha_{i,m}) \in (m^{-1}U) \times [-m^{-1}, m^{-1}]. \tag{33}$$

Then, arguing as in the proof of Theorem 3, we find $(\tilde\lambda_{1,m},\tilde\lambda_{2,m},\tilde\lambda_{3,m},\tilde\lambda_{4,m}) \in \Delta_4$ all positive (without loss of generality) such that

$$\sum_{1\le i\le k_m} \lambda_{i,m} \begin{pmatrix} \gamma_m\langle x_{i,m},u^*\rangle \\ \gamma_m\langle x_{i,m},\bar z^*\rangle \\ \gamma_m\alpha_{i,m} \end{pmatrix} = \sum_{1\le i\le 4} \tilde\lambda_{i,m} \begin{pmatrix} \langle \gamma_m x_{i,m},u^*\rangle \\ \langle \gamma_m x_{i,m},\bar z^*\rangle \\ \gamma_m\alpha_{i,m} \end{pmatrix}. \tag{34}$$

Aiming at showing the convergence properties of the sequences $(\gamma_m\tilde\lambda_{i,m}x_{i,m})_m$ and $(\gamma_m\tilde\lambda_{i,m}\alpha_{i,m})_m$, $1 \le i \le 4$, we first observe from the inequalities $-\infty < \inf_X f \le \overline{\mathrm{co}} f(x_{i,m}) \le f(x_{i,m}) \le \alpha_{i,m}$ that $(\gamma_m\tilde\lambda_{i,m}\alpha_{i,m})_{i,m}$ are uniformly bounded from below. Then, since (recall (33) and (34))

$$\sum_{1\le i\le 4} \gamma_m\tilde\lambda_{i,m}\alpha_{i,m} = \sum_{1\le i\le k_m} \gamma_m\lambda_{i,m}\alpha_{i,m} \in [\alpha - m^{-1}, \alpha + m^{-1}],$$

we infer that some $\delta > 0$ exists so that

$$-\delta \le \gamma_m\tilde\lambda_{i,m}\alpha_{i,m} \le \delta \quad \text{for all } i = 1,\dots,4 \text{ and } m \ge 1. \tag{35}$$

Next, by (31) together with the Fenchel inequality, for each m, $i \in \{1,\dots,4\}$ and $z^* \in V \cap \mathrm{par}(\mathrm{dom}\, f^*)$ we obtain that

(36)
$$\langle \tilde\lambda_{i,m}x_{i,m}, \bar z^* + z^*\rangle \le \tilde\lambda_{i,m}(f(x_{i,m}) + f^*(\bar z^* + z^*)) \le \tilde\lambda_{i,m}\alpha_{i,m} + \tilde\lambda_{i,m}f^*(\bar z^*) + \tilde\lambda_{i,m},$$
$$\langle \tilde\lambda_{i,m}x_{i,m}, \bar z^*\rangle \le \tilde\lambda_{i,m}(f(x_{i,m}) + f^*(\bar z^*)) \le \tilde\lambda_{i,m}(\alpha_{i,m} + f^*(\bar z^*)).$$

Multiplying the last inequality by γ_m (>0) and next summing up over $j \in \{1,\dots,4\}\setminus\{i\}$, we get

$$\sum_{j\neq i,\ 1\leq j\leq 4} \langle \gamma_m \tilde\lambda_{j,m} x_{j,m}, \bar z^*\rangle \leq \sum_{j\neq i,\ 1\leq j\leq 4} \gamma_m \tilde\lambda_{j,m}\alpha_{j,m} + \sum_{j\neq i,\ 1\leq j\leq 4} \gamma_m \tilde\lambda_{j,m} f^*(\bar z^*),$$

so that, invoking (32) and (33),

$$\begin{aligned}\langle \gamma_m \tilde\lambda_{i,m} x_{i,m}, \bar z^*\rangle &\geq -\sum_{j\neq i,\ 1\leq j\leq 4} \langle \gamma_m \tilde\lambda_{j,m} x_{j,m}, \bar z^*\rangle + \langle x, \bar z^*\rangle - m^{-1}\\ &\geq -\sum_{j\neq i,\ 1\leq j\leq 4} \gamma_m \tilde\lambda_{j,m}\alpha_{j,m} - \sum_{j\neq i,\ 1\leq j\leq 4} \gamma_m \tilde\lambda_{j,m} f^*(\bar z^*) + \langle x, \bar z^*\rangle - \frac{1}{m}\\ &\geq \gamma_m \tilde\lambda_{i,m}\alpha_{i,m} - \alpha - \sum_{j\neq i,\ 1\leq j\leq 4} \gamma_m \tilde\lambda_{j,m} f^*(\bar z^*) + \langle x, \bar z^*\rangle - 2m^{-1}.\end{aligned}$$

Consequently, using (36), for all $z^* \in V \cap \operatorname{par}(\operatorname{dom} f^*)$ we obtain that (for all m)

$$\begin{aligned}\langle \gamma_m \tilde\lambda_{i,m} x_{i,m}, z^*\rangle &\leq -\langle \gamma_m \tilde\lambda_{i,m} x_{i,m}\ , \bar z^*\rangle + \gamma_m \tilde\lambda_{i,m}\alpha_{i,m} + \gamma_m \tilde\lambda_{i,m} f^*(\bar z^*) + \gamma_m \tilde\lambda_{i,m}\\ &= \gamma_m f^*(\bar z^*) + \alpha + 2m^{-1} - \langle x, \bar z^*\rangle + \tilde\lambda_{i,m}\gamma_m \leq r\end{aligned}$$

for some positive constant r (independent of m and i). Therefore, as in the proof of Theorem 3, we show the existence of some $x_i \in X$ such that (taking a subnet of $(\gamma_m \tilde\lambda_{i,m} x_{i,m})$ if necessary)

$$\lim \langle \gamma_m \tilde\lambda_{i,m} x_{i,m} - x_i, y^*\rangle = 0 \quad \text{for every } y^* \in \operatorname{par}(\operatorname{dom} f^*). \tag{37}$$

Whence, since we also may suppose that the corresponding net $(\gamma_m \tilde\lambda_{i,m}\alpha_{i,m})_m$ also converges to some $\alpha_i \in \mathbb{R}$ (recall (35)), we deduce that

$$\begin{aligned}f^\infty(x_i) &\leq \lim \gamma_m \tilde\lambda_{i,m} f\left(\frac{1}{\gamma_m \tilde\lambda_{i,m}} \gamma_m \tilde\lambda_{i,m} x_{i,m}\right)\\ &= \lim \gamma_m \tilde\lambda_{i,m} f(x_{i,m}) \leq \lim \gamma_m \tilde\lambda_{i,m}\alpha_{i,m} = \alpha_i;\end{aligned}$$

that is, $(x_i, \alpha_i) \in \operatorname{epi} f^\infty$. Therefore, multiplying (33) by (u^*, μ) and using (34) and (37) together with (32), we obtain that

$$\begin{aligned}\langle (x,\alpha), (u^*,\mu)\rangle &\leq \lim_m \sum_{1\leq i\leq 4} \langle \gamma_m \tilde\lambda_{i,m} x_{i,m}, u^*\rangle + \sum_{1\leq i\leq 4} \gamma_m \tilde\lambda_{i,m}\alpha_{i,m}\mu + 2m^{-1}\\ &= \sum_{1\leq i\leq 4} \langle (x_i,\alpha_i), (u^*,\mu)\rangle \leq \sigma_{\operatorname{epi} f^\infty}(u^*,\mu) = 0;\end{aligned}$$

in other words, (30) holds. This completes the proof of the theorem. ∎

We deduce from Theorem 7 other useful properties of the relative asymptotic function.

COROLLARY 8. *Let function $f : X \to \overline{\mathbb{R}}$ be given, and set $L \in \mathcal{F}(f)$. Then the following statements hold:*

(i) $\overline{\operatorname{co}}\,((f\square\sigma_L)^\infty) = (\overline{\operatorname{co}}(f\square\sigma_L))^\infty$.

(ii) $((f\square\sigma_L)^\infty)^* = \mathrm{I}_{\operatorname{cl}(L\cap\operatorname{dom} f^*))}$, *and so* $\operatorname{dom}((f\square\sigma_L)^\infty)^* = \operatorname{cl}(L\cap\operatorname{dom} f^*)$.

(iii) $X^* \in \mathcal{F}((f\square\sigma_L)^\infty)$.

(iv) $((f\Box\sigma_L)^\infty)^\infty = (f\Box\sigma_L)^\infty$.

(v) $\partial^r(f\Box\sigma_L)^\infty = \partial(f\Box\sigma_L)^\infty$.

Proof. Assertion (i) is direct from Theorem 7 because $X^* \in \mathcal{F}(f\Box\sigma_L)$, as a consequence of the relationship $(f\Box\sigma_L)^* = f^* + \sigma_L{}^*$. Assertion (ii) follows from (i) since $\overline{\text{co}}(f\Box\sigma_L)$ is proper and $(\overline{\text{co}}(f\Box\sigma_L))^\infty = \sigma_{\text{dom}(f\Box\sigma_L)^*} = \sigma_{L\cap\text{dom} f^*}$. Assertion (iii) is also immediate from (ii).

Let us verify assertion (iv): we may assume (without loss of generality) that $L = X^*$, so that $f\Box\sigma_L = f$, and $\theta \in \text{dom} f^*$ (recall (26)). We consider the lc space $Y := \text{par}(\text{dom} f^*)$ ($\subset X^*$) and its dual Y^*, endowed with the weak topology $(Y^*, \sigma(Y^*, Y))$ and in which X is identified as a subset. For fixed $x \in X$ and $\lambda \in \mathbb{R}$, we pick $m > 0$ and an open neighborhood $V \subset Y^*$ of x. If $(f^\infty)^\infty(x) < \lambda$, then since $\text{par}(\text{dom}(f^\infty)^*) = \text{par}(\text{dom} f^*)$ (by (ii)), in view of the positive homogeneity of f^∞ (Lemma 6), there exists $z \in V \cap X$ such that $f^\infty(z) < \lambda$. Hence, we find $y \in V$ and $s \in (0, m)$ such that $sf(s^{-1}y) < \lambda$, showing that $f^\infty(x) \le \lambda$. Therefore, as λ goes to $(f^\infty)^\infty(x)$ we infer that $f^\infty(x) \le (f^\infty)^\infty(x)$. Conversely, if $f^\infty(x) < \lambda$, then invoking again the positive homogeneity of f^∞, for every sequence $(s_k) \subset \mathbb{R}_+^*$ we have that $s_k f^\infty(s_k^{-1}x) < \lambda$. Hence, since $\theta \in \text{dom} f^* \subset \text{dom}(f^\infty)^*$ (by (ii)) we obtain that $(f^\infty)^\infty(x) \le \lambda$. Consequently, as λ goes to $f^\infty(x)$ we deduce that $(f^\infty)^\infty(x) \le f^\infty(x)$, completing the proof of (iv).

Assertion (v): let $x \in X$ be fixed, and assume, without loss of generality, that $X^* \in \mathcal{F}(f)$. Then, from (11) we get $\partial f^\infty(x) \subset \partial^r f^\infty(x)$. To prove the converse inclusion, we pick $x^* \in \partial^r f^\infty(x)$ so that $(f^\infty)^*(x^*) \in \mathbb{R}$, and, taking into account (ii), there exists a net $(x_\gamma) \subset X$ such that $\lim\langle x_\gamma - x, y^*\rangle = 0$ for all $y^* \in \text{par}(\text{dom} f^*)$ and $\lim(f^\infty(x_\gamma) - \langle x_\gamma, x^*\rangle) = -(f^\infty)^*(x^*)$. Therefore, by invoking (ii) and (iv), the positive homogeneity of f^∞, and the fact that $x^* \in \text{dom}(f^\infty)^*$, we obtain that

$$\begin{aligned} f^\infty(x) - \langle x, x^*\rangle = -\langle x, x^*\rangle + (f^\infty)^\infty(x) &\le -\langle x, x^*\rangle + \lim(f^\infty(x_\gamma) - \langle x_\gamma - x, x^*\rangle) \\ &= \lim(f^\infty(x_\gamma) - \langle x_\gamma, x^*\rangle) = -(f^\infty)^*(x^*), \end{aligned}$$

showing that $x^* \in \partial f^\infty(x)$. ∎

5. Final formulas.

In this section, we give the promised formulas for the Fenchel subdifferential of the conjugate function and the argmin set of the closed convexified function by means exclusively of the primal data, namely, the initial function.

THEOREM 9. *Given a function $f : X \to \overline{\mathbb{R}}$, for every $x^* \in X^*$ we have the formula*

$$\partial f^*(x^*) = \bigcap_{L\in\mathcal{F}(f,x^*)} \overline{\text{co}}\left((\partial_L^r f)^{-1}(x^*) + (\partial(f\Box\sigma_L)^\infty)^{-1}(x^*)\right).$$

Moreover, provided that $X^* \in \mathcal{F}(f)$, the following hold true:

(i) $\partial f^*(x^*) = \overline{\text{co}}\left((\partial^r f)^{-1}(x^*) + (\partial f^\infty)^{-1}(x^*)\right)$.

(ii) *If* $\text{int}(\text{dom} f^*) \neq \emptyset$, *then*

$$\partial f^*(x^*) = \overline{\text{co}}\left((\partial^r f)^{-1}(x^*)\right) + \overline{\text{co}}\left((\partial f^\infty)^{-1}(x^*)\right).$$

In addition, if f is weakly lsc, then $\partial f^*(x^*) = \overline{\text{co}}\left((\partial f)^{-1}(x^*)\right) + \overline{\text{co}}\left((\partial f^\infty)^{-1}(x^*)\right)$.

(iii) *If $X = \mathbb{R}^n$ and* $\text{int}(\text{dom} f^*) \neq \emptyset$, *then*

$$\begin{aligned} \partial f^*(x^*) &= \text{co}\left((\partial^r f)^{-1}(x^*)\right) + \text{co}\left((\partial f^\infty)^{-1}(x^*)\right) \\ &\left(= \text{co}\left((\partial f)^{-1}(x^*)\right) + \text{co}\left((\partial f^\infty)^{-1}(x^*)\right) \text{ when } f \text{ is lsc}\right). \end{aligned}$$

Proof. Fix $x^* \in X^*$ and $L \in \mathcal{F}(f, x^*)$ so that $x^* \in L$ by definition. If $x^* \notin \operatorname{dom} f^*$, then $\partial f^*(x^*) = (\partial_L^r f)^{-1}(x^*) = \emptyset$, and so all of the involved formulas hold (with $L = X^*$ in case of (i)–(iii)). Hence, in the remainder of the proof we suppose that $x^* \in \operatorname{dom} f^*$, which entails that $\overline{\mathrm{co}}(f \square \sigma_L) \in \Gamma_0(X)$ and, thus,

$$\mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*) = \mathrm{N}_{\overline{\operatorname{dom}(f\square\sigma_L)^*}}(x^*) = \partial \mathrm{I}_{\overline{\operatorname{dom}(\overline{\mathrm{co}}(f\square\sigma_L))^*}}(x^*) = \partial\left[(\overline{\mathrm{co}}(f\square\sigma_L))^{\infty}\right]^*(x^*).$$

But, according to Theorem 7, we have that $(\overline{\mathrm{co}}(f\square\sigma_L))^{\infty} = \overline{\mathrm{co}}(f\square\sigma_L)^{\infty}$, and so

$$\mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*) = \partial\left[\overline{\mathrm{co}}\left([f\square\sigma_L]^{\infty}\right)\right]^*(x^*) = \partial((f\square\sigma_L)^{\infty})^*(x^*).$$

Moreover, since $(f\square\sigma_L)^{\infty}$ is positively homogeneous and $X^* \in \mathcal{F}((f\square\sigma_L)^{\infty})$, according to Corollaries 6 and 8(iii), respectively, from Theorem 3(i) and Corollary 8(v) we obtain that

$$\partial((f\square\sigma_L)^{\infty})^*(x^*) = \overline{\mathrm{co}}\left((\partial^r(f\square\sigma_L)^{\infty})^{-1}(x^*)\right) = \overline{\mathrm{co}}\left((\partial(f\square\sigma_L)^{\infty})^{-1}(x^*)\right),$$

which leads us to $\mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*) = \overline{\mathrm{co}}\left((\partial(f\square\sigma_L)^{\infty})^{-1}(x^*)\right)$. Consequently, the desired formula follows by applying Proposition 2. Finally, the statements (i)–(iii) follow in a similar way (with $L = X^*$) by using Proposition 2. ∎

Similarly, as in the previous theorem, based on the formulas in (25), together with its variants corresponding to the cases $X^* \in \mathcal{F}(f)$ and/or $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$ (see [5]), and Proposition 4, we give in the following an alternative to Theorem 9 by using the ε-subdifferential.

THEOREM 10. *Given a function $f : X \to \overline{\mathbb{R}}$, for every $x^* \in X^*$ we have the formula*

$$\partial f^*(x^*) = \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}(f,x^*)}} \overline{\mathrm{co}}\left((\partial_\varepsilon f)^{-1}(x^*) + (\partial(f\square\sigma_L)^{\infty})^{-1}(x^*)\right).$$

Moreover, provided that $X^* \in \mathcal{F}(f)$, the following hold true:

(i) $\partial f^*(x^*) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}((\partial_\varepsilon f)^{-1}(x^*) + (\partial f^{\infty})^{-1}(x^*))$;

(ii) *if* $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$, *then*

$$\partial f^*(x^*) = \overline{\mathrm{co}}\left((\partial f^{\infty})^{-1}(x^*)\right) + \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\left((\partial_\varepsilon f)^{-1}(x^*)\right);$$

(iii) *if* $X = \mathbb{R}^n$ *in* (ii), *then*

$$\partial f^*(x^*) = \mathrm{co}\left((\partial f^{\infty})^{-1}(x^*)\right) + \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\left((\partial_\varepsilon f)^{-1}(x^*)\right).$$

Next, we give formulas which express $\operatorname{argmin} \overline{\mathrm{co}} f$ by means of $\operatorname{argmin} f$ and $\operatorname{argmin} f^{\infty}$. The proof is immediate and follows by using the relationship $\operatorname{argmin} \overline{\mathrm{co}} f = \partial f^*(\theta)$, which holds for functions having a proper conjugate. To put the final formulas into a coherent picture we use the following notation: for $L \in \mathcal{F}(f, \theta)$ we set

$$L\text{-rel-argmin}\, f := (\partial_L^r f)^{-1}(\theta), \quad \text{rel-argmin}\, f := (\partial^r f)^{-1}(\theta);$$

that is, $x \in L$-rel-argmin f iff $\inf f \in \mathbb{R}$ and there exists a net $(x_\gamma) \subset X$ such that $\lim\langle x_\gamma - x, y^*\rangle = 0$ for all $y^* \in \overline{\mathrm{par}}(L \cap \operatorname{dom} f^*)$, and $\lim f(x_\gamma) = \inf f$. In particular, if $X^* \in \mathcal{F}(f)$, $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$, and f is weakly lsc, then rel-argmin $f = \operatorname{argmin} f$.

THEOREM 11. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ have a proper conjugate. Then, we have the formula*

$$\operatorname{argmin} \overline{\operatorname{co}} f = \bigcap_{L \in \mathcal{F}(f,\theta)} \overline{\operatorname{co}} \left(L\text{-rel-argmin}\, f + \operatorname{argmin}(f \square \sigma_L)^\infty \right).$$

Moreover, provided that $X^* \in \mathcal{F}(f)$, the following hold true:

(i) $\operatorname{argmin} \overline{\operatorname{co}} f = \overline{\operatorname{co}} \left(\text{rel-argmin}\, f + \operatorname{argmin} f^\infty\right)$;

(ii) *if* $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$, *then*

$$\operatorname{argmin} \overline{\operatorname{co}} f = \overline{\operatorname{co}} \left(\text{rel-argmin}\, f\right) + \overline{\operatorname{co}} \left(\operatorname{argmin} f^\infty\right).$$

In addition, if f is weakly lsc, then $\operatorname{argmin} \overline{\operatorname{co}} f = \overline{\operatorname{co}} \left(\operatorname{argmin} f\right) + \overline{\operatorname{co}} \left(\operatorname{argmin} f^\infty\right)$;

(iii) *if $X = \mathbb{R}^n$ and* $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$, *then*

$$\operatorname{argmin} \overline{\operatorname{co}} f = \operatorname{co} \left(\text{rel-argmin}\, f\right) + \operatorname{co} \left(\operatorname{argmin} f^\infty\right)$$
$$\left(= \operatorname{co} \left(\operatorname{argmin} f\right) + \operatorname{co} \left(\operatorname{argmin} f^\infty\right) \text{ when } f \text{ is lsc}\right).$$

The following theorem gives other formulas for the argmin set of the lsc convex hull by using the ε-subdifferential of the initial function. Its proof immediately follows from Theorem 10.

THEOREM 12. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ have a proper conjugate. Then, we have the formula*

$$\operatorname{argmin} \overline{\operatorname{co}} f = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(f,\theta)}} \overline{\operatorname{co}} \left(\varepsilon\text{-} \operatorname{argmin} f + \operatorname{argmin}(f \square \sigma_L)^\infty \right).$$

Moreover, provided that $X^* \in \mathcal{F}(f)$, the following hold true:

(i) $\operatorname{argmin} \overline{\operatorname{co}} f = \bigcap_{\varepsilon > 0} \overline{\operatorname{co}} \left(\varepsilon\text{-} \operatorname{argmin} f + \operatorname{argmin} f^\infty\right)$;

(ii) *if* $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$, *then*

$$\operatorname{argmin} \overline{\operatorname{co}} f = \overline{\operatorname{co}} \left(\operatorname{argmin} f^\infty\right) + \bigcap_{\varepsilon > 0} \overline{\operatorname{co}} \left(\varepsilon\text{-} \operatorname{argmin} f\right);$$

(iii) *if $X = \mathbb{R}^n$ in* (ii), *then*

$$\operatorname{argmin} \overline{\operatorname{co}} f = \operatorname{co} \left(\operatorname{argmin} f^\infty\right) + \bigcap_{\varepsilon > 0} \overline{\operatorname{co}} \left(\varepsilon\text{-} \operatorname{argmin} f\right).$$

The conditions in Theorems 9–12 relying on the behavior of f^* can be naturally expressed by means of primal objects. For instance, we have the following proposition, which can be proved using the same arguments as in its finite-dimensional version given in [3]; see also [10].

PROPOSITION 13. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be such that* $\operatorname{dom} f^* \neq \emptyset$. *Then, the following are equivalent:*

(i) $X^* \in \mathcal{F}(f)$ *and* $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$;

(ii) *there exists a θ-neighborhood $U \subset X^*$ together with $x^* \in X^*$ and constant $r \in \mathbb{R}$ such that*

$$f(x) \geq \langle x^*, x \rangle + \sigma_U(x) - r \quad \text{for all } x \in X.$$

In particular, if X is a reflexive Banach space with a norm denoted by $\|\cdot\|$, then each of the assertions (i)–(ii) *is equivalent to*

(iii) *there exists* $x^* \in X^*$ *such that*

$$\liminf_{\|x\| \to +\infty} \frac{f(x) - \langle x^*, x \rangle}{\|x\|} > 0.$$

Remark 2. (i) Theorem 11(iii) has been established in [3] using a different approach.

(ii) Behind the formulas established in Theorem 11 for the argmin set of the closed convex hull $\overline{\text{co}} f$ are results concerning the existence theory of the optimal solutions of the optimization problem $\inf_X f$. For instance, Theorem 11(iii) shows that under the conditions that f^* is continuous on $\text{int}(\text{dom}\, f^*)$ $(\neq \emptyset)$ and f is weakly lsc, if the relaxed problem $\inf_X \overline{\text{co}} f$ has optimal solutions, then the initial problem also does.

Conclusion. In this paper we gave formulas for the Fenchel subdifferential of the conjugate of any function by means of the primal initial data. As a consequence, we obtained formulas for the argmin set of the lsc convex hull, also expressed in terms of primal objects. This was possible thanks to some appropriate concepts of relative subdifferential and relative asymptotic functions, which can be considered as natural extensions of the Fenchel subdifferential and the recession function, respectively.

Acknowledgments. We are grateful to the referees for many very helpful suggestions and constructive comments that have substantially improved the paper. We also would like to thank Professor C. Zălinescu for making valuable observations and suggestions, which allowed us to improve the exposition of our results.

REFERENCES

[1] E. Asplund and R. T. Rockafellar, *Gradients of convex functions*, Trans. Amer. Math. Soc., 139 (1969), pp. 443–467.
[2] A. Auslender and M. Teboulle, *Asymptotic Cones and Functions in Optimization and Variational Inequalities*, Springer, New York, 2003.
[3] J. Benoist and J.-B. Hiriart-Urruty, *What is the subdifferential of the closed convex hull of a function?*, SIAM J. Math. Anal., 27 (1996), pp. 1661–1679.
[4] J. M. Borwein and J. Vanderwerff, *Differentiability of conjugate functions and perturbed minimization principles*, J. Convex Anal., 16 (2009), pp. 707–711.
[5] R. Correa and A. Hantoute, *New formulas for the Fenchel subdifferential of the conjugate function*, Set-Valued Var. Anal., 18 (2010), pp. 405–422.
[6] G. Debreu, *Theory of Values*, John Wiley, New York, 1959.
[7] J.-P. Dedieu, *Critères de fermeture pour l'image d'un fermé non convexe par une multiapplication*, C. R. Acad. Sci. Paris Sér. A-B, 287 (1978), pp. A491–A493.
[8] J.-P. Dedieu, *Cônes asymptotes d'ensembles non convexes*, Bull. Soc. Math. France Mém., 60 (1979), pp. 31–44.
[9] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North–Holland, Amsterdam, 1976.
[10] A. Fougères, *Coercivité, convexité, relaxation: Une extension naturelle du théorème d'inf-équicontinuité de J.-J. Moreau*, C. R. Acad. Sci. Paris, Sér. A-B, 285 (1977), pp. A711–A713.
[11] A. Hantoute, *Subdifferential set of the supremum of lower semi-continuous convex functions and the conical hull intersection property*, Top, 14 (2006), pp. 355–374.
[12] A. Hantoute and M. A. López, *A complete characterization of the subdifferential set of the supremum of an arbitrary family of convex functions*, J. Convex Anal., 15 (2008), pp. 831–858.
[13] A. Hantoute, M. A. López, and C. Zălinescu, *Subdifferential calculus rules in convex analysis: A unifying approach via pointwise supremum functions*, SIAM J. Optim., 19 (2008), pp. 863–882.
[14] J.-B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms* I, II, Springer, Berlin, 1993.

[15] S. S. Kutateladze, *On the subdifferential of a convex operator at a generalized point*, Sibirsk. Mat. Zh., 38 (1997), pp. 591–597 (in Russian); translation in Siberian Math. J., 38 (1997), pp. 507–512.
[16] M. Lassonde, *Asplund spaces, Stegall variational principle and the RNP*, Set-Valued Var. Anal., 17 (2009), pp. 183–193.
[17] V. L. Levin, *Convex Analysis in Spaces of Measurable Functions and Its Application in Mathematics and Economics*, Nauka, Moscow, 1985 (in Russian).
[18] D. T. Luc, *Recession maps and applications*, Optimization, 27 (1993), pp. 1–15.
[19] J.-P. Penot and C. Zălinescu, *Convex analysis can be helpful for the asymptotic analysis of monotone operators: Asymptotic analysis of monotone operators*, Math. Program. Ser. B, 116 (2009), pp. 481–498.
[20] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.
[21] A. Seeger, *Degree of pointedness of a convex function*, Bull. Austral. Math. Soc., 53 (1996), pp. 159–167.
[22] C. Zălinescu, *Recession cones and asymptotically compact sets*, J. Optim. Theory Appl., 77 (1993), pp. 209–220.
[23] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, River Edge, NJ, 2002.