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Supremum of the Airy$_2$ Process Minus a Parabola on a Half Line

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Abstract Let $\mathcal{A}_2(t)$ be the Airy$_2$ process. We show that the random variable

$$\sup_{t\leq\alpha}\big\{\mathcal{A}_2(t)-t^2\big\}+\min\{0,\alpha\}^2$$

has the same distribution as the one-point marginal of the Airy$_{2\to 1}$ process at time α. These marginals form a family of distributions crossing over from the GUE Tracy-Widom distribution $F_{\mathrm{GUE}}(x)$ for the Gaussian Unitary Ensemble of random matrices, to a rescaled version of the GOE Tracy-Widom distribution $F_{\mathrm{GOE}}(4^{1/3}x)$ for the Gaussian Orthogonal Ensemble. Furthermore, we show that for every α the distribution has the same right tail decay $e^{-\frac{4}{3}x^{3/2}}$.

Keywords Airy processes · Last passage percolation · KPZ universality

1 Introduction

The Airy processes are a collection of stochastic processes which are expected to govern the spatial fluctuations of random growth models in the one dimensional KPZ universality class for wide classes of initial data. They are defined through their finite dimensional distributions, which are given by Fredholm determinants. The three basic processes are Airy$_2$ [23], corresponding to curved, or droplet initial data; Airy$_1$ [4, 6, 26], corresponding to flat initial data; and Airy$_{\mathrm{stat}}$ [5], corresponding to equilibrium initial data.

The KPZ class is identified at the roughest level by the unusual $t^{1/3}$ scale of fluctuations. It is expected to contain a large class of random growth processes, including the Kardar-Parisi-Zhang equation itself, as well as randomly stirred one dimensional fluids, polymer

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chains directed in one dimension and fluctuating transversally in the other due to a random potential (with applications to domain interfaces in disordered crystals), driven lattice gas models, reaction-diffusion models in two-dimensional random media (including biological models such as bacterial colonies), randomly forced Hamilton-Jacobi equations, etc. A combination of non-rigorous methods (renormalization, mode-coupling, replicas) and mathematical breakthroughs on a few special models has led to very precise predictions of universal scaling exponents and exact statistical distributions describing the long time properties. These predictions have been repeatedly confirmed through Monte Carlo simulation as well as experiments; in particular, recent spectacular experiments on turbulent liquid crystals by Takeuchi and Sano [28, 29] have been able to even confirm some of the predicted fluctuation statistics.

The conjectural picture that has developed is that the universality class is divided into subuniversality classes which depend on the initial data class, but not on other details of the particular models. Because of their self-similarity properties, the three basic initial data are, at the level of continuum partition functions (taking logarithms gives free energies or height functions): Dirac δ_0, corresponding to curved, or droplet type initial data; 0, corresponding to growth off a flat substrate; and $e^{B(x)}$ where $B(x)$ is a two sided Brownian motion, corresponding to growth in equilibrium. Of course, in discrete models of various types one is dealing with discrete approximations of such initial data. There are also three additional nonhomogeneous subuniversality classes corresponding to starting with one of the basic three on one side of the origin, and another on the other side. The spatial fluctuations in these six basic classes of initial data are supposed to be given asymptotically by the six known Airy processes: the three basic Airy processes, Airy$_2$, Airy$_1$ and Airy$_{\text{stat}}$, and the crossover Airy processes Airy$_{2\to 1}$ [7], Airy$_{2\to\text{BM}}$ [11, 19] and Airy$_{1\to\text{BM}}$ [8].

However, since all initial data are superpositions of Dirac masses, there is a sense is which the Airy$_2$ process is the most basic. Although the various microscopic models are not linear in the initial data, this is the case for the stochastic heat equation, whose logarithm is the solution of the KPZ equation. And for other models, the linearity should hold asymptotically. In the limit, the logarithm of the superpositions of exponentials of Airy$_2$ processes becomes a variational problem.

The conclusion is a conjecture that the one-point marginals of the other Airy processes should be obtained through certain variational problems involving the Airy$_2$ process. The first example of this was the celebrated result of Johansson [20] (see also [14]) that the supremum of the Airy$_2$ process minus a parabola has the same distribution as a rescaled version of the one-dimensional marginal of the Airy$_1$ process, i.e., the GOE Tracy-Widom distribution:

$$\mathbb{P}\Big(\sup_{x\in\mathbb{R}}\big\{\mathcal{A}_2(x)-x^2\big\}\le m\Big)=\mathbb{P}\big(\mathcal{A}_1(0)\le 2^{-1/3}m\big)=F_{\text{GOE}}\big(4^{1/3}m\big). \tag{1.1}$$

The general conjecture is based on heuristics which we describe next in the context of the stochastic heat equation.

1.1 Heuristics

We will explain the heuristics first for the case of the Airy$_{2\to 1}$ process. Let $z(t,x)$ denote the solution of the one-dimensional stochastic heat equation

$$\partial_t z=\frac{1}{2}\partial_x^2 z - z\xi$$

where $\xi(t,x)$ is space-time white noise. The solution at position x and time t starting from a Dirac mass at y at time 0 can be written as

$$z(0,y;t,x)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}-\frac{t}{24}+2^{-1/3}t^{1/3}A_t(2^{-1/3}t^{-2/3}(x-y))},\tag{1.2}$$

where A_t is conjectured to converge to the Airy$_2$ process, $A_t(x)\to\mathcal{A}_2(x)$ (see Conjecture 1.5 in [1] for a precise statement, and [24] for a non-rigorous derivation). Starting from the step initial data $z(0,x)=\mathbf{1}_{x>0}$ the prediction is

$$-\log z(t,x)\approx\frac{1}{2t}x^2\mathbf{1}_{x<0}+\frac{1}{24}t+\log(\sqrt{2\pi t})-2^{-1/3}t^{1/3}\mathcal{A}_{2\to1}\big(2^{-1/3}t^{-2/3}x\big).\tag{1.3}$$

On the other hand, by linearity we have for each fixed x, in distribution,

$$\begin{aligned}z(t,x)&=\int_0^\infty dy\,z(0,y;t,x)\\&=\int_0^\infty dy\,\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}-\frac{t}{24}+2^{-1/3}t^{1/3}A_t(2^{-1/3}t^{-2/3}(x-y))}.\end{aligned}\tag{1.4}$$

Note however that as written the equality can only hold in distribution for each t and x. If one wants a stronger statement, for fixed t but multiple x, one has to replace $A_t(2^{-1/3}t^{-2/3}(x-y))$ in (1.2) by a two parameter process $\tilde{A}_t(2^{-1/3}t^{-2/3}x,2^{-1/3}t^{-2/3}y)$, keeping track of the statistical dependence on the initial y. For fixed y, it is distributionally, as a process in x, equal to $A_t(2^{-1/3}t^{-2/3}(x-y))$. And, by symmetry, the same is true for fixed x, as a process in y. However, they are not equal in distribution in the sense of two parameter processes in both x and y. The limit of $\tilde{A}_t(x,y)$ is unknown at this time, so one is stuck at the level of one-dimensional marginals.

Calling $\tilde{x}=2^{-1/3}t^{-2/3}x$ and $\tilde{y}=2^{-1/3}t^{-2/3}y$ we can rewrite the exponent in (1.4) as

$$2^{-1/3}t^{1/3}\big[A_t(\tilde{x}-\tilde{y})-(\tilde{x}-\tilde{y})^2\big]-\frac{1}{24}t$$

so that for large t the fluctuation field $2^{1/3}t^{-1/3}[\log z(t,x)+\frac{1}{24}t+\log(\sqrt{2\pi t})]$ is well approximated by

$$\sup_{\tilde{y}\ge0}\big(\mathcal{A}_2(\tilde{x}-\tilde{y})-(\tilde{x}-\tilde{y})^2\big).$$

Comparing with (1.3) we deduce that the processes $\sup_{y\ge0}(\mathcal{A}_2(x-y)-(x-y)^2)$ and $\mathcal{A}_{2\to1}(x)-x^2\mathbf{1}_{x<0}$ should have the same one-dimensional distribution or, equivalently, that

$$\mathcal{A}_{2\to1}(x)-x^2\mathbf{1}_{x<0}\overset{\text{(d)}}{=}\sup_{y\le x}\big\{\mathcal{A}_2(y)-y^2\big\}\tag{1.5}$$

for each fixed $x\in\mathbb{R}$.

The same argument works for the other two crossover cases. If we let $z(0,x)=e^{B(x)}\mathbf{1}_{x\ge0}$, where $B(x)$ is a standard Brownian motion, then (1.3) and (1.4) are replaced respectively by

$$-\log z(t,x)\approx\frac{1}{2t}x^2\mathbf{1}_{x<0}+\frac{1}{24}t+\log(\sqrt{2\pi t})-2^{-1/3}t^{1/3}\mathcal{A}_{2\to\mathrm{BM}}\big(2^{-1/3}t^{-2/3}x\big)$$

and

$$z(t,x)=\int_0^\infty dy\,z(0,y;t,x)=\int_0^\infty dy\,\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}-\frac{t}{24}+B(y)+2^{-1/3}t^{1/3}A_t(2^{-1/3}t^{-2/3}(x-y))},$$

and now the same scaling argument allows to conjecture that

$$\mathcal{A}_{2\to\mathrm{BM}}(x) - x^2\mathbf{1}_{x<0} \stackrel{(\mathrm{d})}{=} \sup_{y\le x}\big(\mathcal{A}_2(y) + \widetilde{B}(x-y) - y^2\big)$$

for each fixed $x\in\mathbb{R}$, where now $\widetilde{B}(y)$ is a Brownian motion with diffusion coefficient 2. An analogous argument with $z(0,x) = \mathbf{1}_{x\le 0} + e^{B(x)}\mathbf{1}_{x\ge 0}$ translates into conjecturing that

$$\mathcal{A}_{1\to\mathrm{BM}}(x) \stackrel{(\mathrm{d})}{=} \sup_{y\in\mathbb{R}}\big(\mathcal{A}_2(y) + \widetilde{B}(x-y)\mathbf{1}_{y\le x} - y^2\big)$$

for each fixed $x\in\mathbb{R}$.

In this article we prove the conjecture (1.5) for $\mathcal{A}_{2\to 1}$, which connects the three Airy processes with non-random initial data. To state the result precisely, we now recall the exact definitions of the Airy$_2$, Airy$_1$, and Airy$_{2\to 1}$ processes, together with some additional background.

1.2 Statement of Main Results

The Airy$_2$ process $\mathcal{A}_2$, introduced by Prähofer and Spohn [23], is a stationary process on the real line whose one dimensional marginals are given by the Tracy-Widom largest eigenvalue distribution for the Gaussian Unitary Ensemble (GUE) from random matrix theory [30]. It is expected to govern the asymptotic spatial fluctuations in a wide variety of random growth models on a one dimensional substrate with curved initial conditions, and the point-to-point free energies of directed random polymers in $1+1$ dimensions (the KPZ universality class). It also arises as the scaling limit of the top eigenvalue in Dyson's Brownian motion [16] for GUE (see [2] for more details). It is defined through its finite-dimensional distributions, which are given by a determinantal formula: given $\xi_1,\dots,\xi_m\in\mathbb{R}$ and $t_1<\cdots<t_m$ in $\mathbb{R}$,

$$\mathbb{P}\big(\mathcal{A}_2(t_1)\le\xi_1,\dots,\mathcal{A}_2(t_m)\le\xi_m\big) = \det\big(I - \mathrm{f}^{1/2}K_2^{\mathrm{ext}}\mathrm{f}^{1/2}\big)_{L^2(\{t_1,\dots,t_m\}\times\mathbb{R})},$$

where $\mathbb{P}$ denotes probability, we have counting measure on $\{t_1,\dots,t_m\}$ and Lebesgue measure on $\mathbb{R}$, f is defined on $\{t_1,\dots,t_m\}\times\mathbb{R}$ by

$$\mathrm{f}(t_j,x) = \mathbf{1}_{x\in(\xi_j,\infty)}, \tag{1.6}$$

and the *extended Airy kernel*, [17, 21, 23] is defined by

$$K_2^{\mathrm{ext}}(t,\xi;t',\xi') = \begin{cases} \int_0^\infty d\lambda\, e^{-\lambda(t-t')}\,\mathrm{Ai}(\xi+\lambda)\,\mathrm{Ai}(\xi'+\lambda), & \text{if } t\ge t',\\ -\int_{-\infty}^0 d\lambda\, e^{-\lambda(t-t')}\,\mathrm{Ai}(\xi+\lambda)\,\mathrm{Ai}(\xi'+\lambda), & \text{if } t<t', \end{cases}$$

where $\mathrm{Ai}(\cdot)$ is the Airy function.

The Airy$_1$ process, introduced by Sasamoto [26], is another stationary process, whose one-point distribution is now given by the Tracy-Widom largest eigenvalue distribution for the Gaussian Orthogonal Ensemble (GOE) from random matrix theory [31]. It is defined through its finite-dimensional distributions,

$$\mathbb{P}\big(\mathcal{A}_1(t_1)\le\xi_1,\dots,\mathcal{A}_1(t_n)\le\xi_n\big) = \det\big(I - \mathrm{f}^{1/2}K_1^{\mathrm{ext}}\mathrm{f}^{1/2}\big)_{L^2(\{t_1,\dots,t_n\}\times\mathbb{R})},$$

with f as in (1.6) and

$$\begin{aligned} K_1^{\mathrm{ext}}(t,\xi;t',\xi') &= -\frac{1}{\sqrt{4\pi(t'-t)}}\exp\left(-\frac{(\xi'-\xi)^2}{4(t'-t)}\right)\mathbf{1}_{t'>t}\\ &\quad + \mathrm{Ai}\big(\xi+\xi'+(t'-t)^2\big)\exp\left(\big(t'-t\big)\big(\xi+\xi'\big)+\frac{2}{3}\big(t'-t\big)^3\right). \end{aligned}$$

It is expected to govern the asymptotic spatial fluctuations in random growth models with flat initial conditions, and the point-to-line free energies of directed random polymers.

The Airy$_{2\to 1}$ process $\mathcal{A}_{2\to 1}$, introduced by Borodin, Ferrari and Sasamoto [7], is given by

$$\mathbb{P}\big(\mathcal{A}_{2\to 1}(t_1)\le \xi_1,\dots,\mathcal{A}_{2\to 1}(t_m)\le \xi_m\big)=\det\big(I-\mathrm{f}^{1/2}K_\infty \mathrm{f}^{1/2}\big)_{L^2(\{t_1,\dots,t_m\}\times\mathbb{R})},$$

with f as in (1.6) and

$$K_\infty(s,x;t,y)=-\frac{1}{\sqrt{4\pi(t-s)}}\exp\biggl(-\frac{(\tilde y-\tilde x)^2}{4(t-s)}\biggr)\mathbf{1}_{t>s} + \frac{1}{(2\pi\mathrm{i})^2}\int_{\gamma_+}dw\int_{\gamma_-}dz\,\frac{e^{w^3/3+tw^2-\tilde y w}}{e^{z^3/3+sz^2-\tilde x z}}\frac{2w}{(z-w)(z+w)},$$

where $\tilde x = x - s^2\mathbf{1}_{s\le 0}$, $\tilde y = y - t^2\mathbf{1}_{t\le 0}$ and the paths γ_+,γ_- satisfy $-\gamma_+\subseteq\gamma_-$ with $\gamma_+ : e^{\mathrm{i}\phi_+}\infty\to e^{-\mathrm{i}\phi_+}\infty$, $\gamma_- : e^{-\mathrm{i}\phi_-}\infty\to e^{\mathrm{i}\phi_-}\infty$ for some $\phi_+\in(\pi/3,\pi/2)$, $\phi_-\in(\pi/2,\pi-\phi_+)$. The Airy$_{2\to 1}$ process crosses over between the Airy$_2$ and the Airy$_1$ processes in the sense that $\mathcal{A}_{2\to 1}(t+\tau)$ converges to $2^{1/3}\mathcal{A}_1(2^{-2/3}\tau)$ as $t\to\infty$ and $\mathcal{A}_2(\tau)$ when $t\to-\infty$. It is expected to govern the asymptotic spatial fluctuations in random growth models when the initial conditions are *half flat*. In particular, it is shown in [7] that it governs the asymptotic fluctuations for the totally asymmetric (to the left) simple exclusion process starting with particles only at the even positive integers.

Define the *crossover distributions* $G_\alpha^{2\to 1}$, for $\alpha\in\mathbb{R}$, as follows:

$$G_\alpha^{2\to 1}(m)=\det(I-P_mK_\alpha P_m) \tag{1.7}$$

where P_m denotes the projection onto the interval $[m,\infty)$, $K_\alpha = K_\alpha^1 + K_\alpha^2$ and the kernels K_α^1 and K_λ^2 are given by

$$K_\alpha^1(x,y)=\int_0^\infty d\lambda\, e^{2\alpha\lambda}\,\mathrm{Ai}\big(x-\lambda+\max\{0,\alpha\}^2\big)\,\mathrm{Ai}\big(y+\lambda+\max\{0,\alpha\}^2\big)$$

and

$$K_\alpha^2(x,y)=\int_0^\infty d\lambda\,\mathrm{Ai}\big(x+\lambda+\max\{0,\alpha\}^2\big)\,\mathrm{Ai}\big(y+\lambda+\max\{0,\alpha\}^2\big).$$

Here, and in everything that follows, the determinant means the Fredholm determinant in the Hilbert space $L^2(\mathbb{R})$. As noted in Appendix A of [7], the kernel K_∞ can be expressed in terms of Airy functions:[1]

$$K_\infty(s,t;x,y)=L_0(s,x;t,y)+e^{2t^3/3-2s^3/3+t\tilde y-s\tilde x}[L_1+L_2](s,x;t,y), \tag{1.8}$$

where

$$L_0(s,x;t,y)=-e^{(s-t)\Delta}(\tilde x,\tilde y)=-\frac{1}{\sqrt{4\pi(t-s)}}e^{-(\tilde x-\tilde y)^2/4(t-s)},$$

$$L_1(s,x;t,y)=\int_0^\infty d\lambda\, e^{\lambda(s+t)}\,\mathrm{Ai}(\hat x-\lambda)\,\mathrm{Ai}(\hat y+\lambda),$$

$$L_2(s,x,t,y)=\int_0^\infty d\lambda\, e^{\lambda(t-s)}\,\mathrm{Ai}(\hat x+\lambda)\,\mathrm{Ai}(\hat y+\lambda)$$

[1] This corresponds to a minor correction of the formula appearing in [7], where the exponential prefactor appears in front of L_0 instead of L_1+L_2.

with $\tilde{x} = x - s^2\mathbf{1}_{s\le 0}$, $\tilde{y} = y - t^2\mathbf{1}_{t\le 0}$, $\hat{x} = x + s^2\mathbf{1}_{s\ge 0}$ and $\hat{y} = y + t^2\mathbf{1}_{t\ge 0}$. Using this for $s = t = \alpha$ it is straightforward to check that $K_\infty(t,\cdot;t,\cdot)$ is just a similarity transform of the kernel K_α, and therefore

$$G_\alpha^{2\to 1}(m) = \mathbb{P}\big(\mathcal{A}_{2\to 1}(\alpha) \le m\big).$$

Remark 1.1 The operator $P_m K_\alpha P_m$ appearing inside the determinant defining $G_\alpha^{2\to 1}$ in (1.7) is trace class (this follows from (2.12) together with a similar bound for K_α^2). This should be compared with the fact that the extended kernel given in [7] for the higher dimensional joint distributions of $\mathcal{A}_{2\to 1}$ is not trace class (but, as shown in Appendix B of [7], there is a conjugate kernel which is).

The following result confirms the conjecture (1.5):

Theorem 1 *Fix $\alpha \in \mathbb{R}$. For every $m \in \mathbb{R}$,*

$$\mathbb{P}\Big(\sup_{t\le\alpha}\big(\mathcal{A}_2(t) - t^2\big) \le m - \min\{0,\alpha\}^2\Big) = G_\alpha^{2\to 1}(m).$$

We remark that the equality is easy to obtain in the limits $\alpha \to \infty$ and $\alpha \to -\infty$. Since the Airy$_2$ process is stationary one expects that, as $\alpha \to -\infty$, $\sup_{t\le\alpha}(\mathcal{A}_2(t) - t^2)$ is attained at $t \approx \alpha$, and thus $\sup_{t\le\alpha}(\mathcal{A}_2(t) - t^2) + \min\{0,\alpha\}^2 \approx \mathcal{A}_2(\alpha)$, which has distribution $F_{\mathrm{GUE}}(m)$. For the right hand side, observe that for $\alpha < 0$ the kernel K_α^2 equals the *Airy kernel* K_{Ai}, which is given explicitly by

$$K_{\mathrm{Ai}}(x,y) = \int_0^\infty d\lambda\, \mathrm{Ai}(x+\lambda)\,\mathrm{Ai}(y+\lambda). \tag{1.9}$$

In addition, one can check that, due to the exponential factor $e^{2\alpha\lambda}$, $P_m K_\alpha^1 P_m$ goes to 0 in trace norm as $\alpha \to -\infty$ (see (2.16)). Consequently,

$$\lim_{\alpha\to-\infty} G_\alpha^{2\to 1}(m) = \det(I - P_m K_{\mathrm{Ai}} P_m) \tag{1.10}$$

which is also $F_{\mathrm{GUE}}(m)$.

On the other hand, as $\alpha \to \infty$, the left hand side becomes $\mathbb{P}(\sup_{t\in\mathbb{R}}(\mathcal{A}_2(t) - t^2) \le m) = F_{\mathrm{GOE}}(4^{1/3}m)$. For the right hand side, it is not hard to check that $P_m K_\alpha^2 P_m$ goes to 0 in trace norm as $\alpha \to \infty$. In addition, using (1.8) and (A.6) of [7] with $\tau_1 = \tau_2 = \alpha$ we get for $\alpha > 0$

$$K_\alpha^1(x,y) = 2^{-1/3}\,\mathrm{Ai}\big(2^{-1/3}(x+y)\big) - \overline{K}_\alpha^1(x,y)$$

with $\overline{K}_\alpha^1(x,y) = \int_{-\infty}^0 d\lambda\, e^{2\alpha\lambda}\,\mathrm{Ai}(x+\alpha^2+\lambda)\,\mathrm{Ai}(y+\alpha^2-\lambda)$, and one can check that $P_m\overline{K}_\alpha^1 P_m$ goes to 0 in trace norm as $\alpha \to \infty$ (see the comment following (2.16)). The first term on the right hand side above corresponds to the kernel $\widetilde{B}(x,y) = 2^{-1/3}\,\mathrm{Ai}(2^{-1/3}(x+y))$, and we deduce after changing variables $x \mapsto 2^{-2/3}x$, $y \mapsto 2^{-2/3}y$ in the resulting determinant that

$$\lim_{\alpha\to\infty} G_\alpha^{2\to 1}(m) = \det(I - P_{4^{1/3}m} B P_{4^{1/3}m}), \tag{1.11}$$

where $B(x,y) = \frac{1}{2}\mathrm{Ai}(\frac{1}{2}(x+y))$, which is also equal to $F_{\mathrm{GOE}}(4^{1/3}m)$ [18].

The fact that $G_\alpha^{2\to 1}$ crosses over between the GUE and GOE distributions is of course a particular case of the crossover property of the Airy$_{2\to 1}$ process. Note that the scaling by

$4^{1/3}$ in the GOE end of this interpolation implies that both ends satisfy the same asymptotics $\log(1-G^{2\to1}_{\pm\infty}(m)) \sim -\frac{4}{3}m^{3/2}$ as $m\to\infty$.[2] In fact, the same upper bound holds for all $\alpha\in\mathbb{R}$:

Proposition 1.2 *For every $\alpha\in\mathbb{R}$, there is a $c>0$ such that,*

$$1-G^{2\to1}_{\alpha}(m) \le cm\, e^{-\frac{4}{3}m^{3/2}+\alpha m} \quad \textit{as } m\to\infty.$$

The proof of Theorem 1 is based on a continuum statistics formula for the Airy$_2$ process, developed in [14], which is well adapted to such variational problems. Fix a function $g\in H^1([\ell,r])$ and introduce an operator $\Theta^g_{[\ell,r]}$ which acts on $L^2(\mathbb{R})$ as follows: $\Theta^g_{[\ell,r]}f(\cdot)=u(r,\cdot)$, where $u(r,\cdot)$ is the solution at time r of the boundary value problem

$$\begin{aligned} \partial_t u + Hu &= 0 \quad \text{for } x<g(t),\ t\in(\ell,r)\\ u(\alpha,x) &= f(x)\mathbf{1}_{x<g(\alpha)} \\ u(t,x) &= 0 \quad \text{for } x\ge g(t) \end{aligned} \tag{1.12}$$

for the *Airy Hamiltonian*,

$$H=-\partial_x^2+x.$$

The formula reads

$$\mathbb{P}\big(\mathcal{A}_2(t)\le g(t) \text{ for } t\in[\ell,r]\big)=\det\big(I-K_{\mathrm{Ai}}+\Theta^g_{[\ell,r]}e^{(r-\ell)H}K_{\mathrm{Ai}}\big). \tag{1.13}$$

Choosing $g(t)=m-\min\{0,\alpha\}^2+t^2$ and $r=\alpha$, (1.13) gives an explicit formula for the probability in Theorem 1 in the limit $\ell\to-\infty$. The proof will consist on computing this limit and showing that it coincides with $G^{2\to1}_{\alpha}(m)$.

Note that this strategy is considerably more difficult to implement for the Airy$_{\mathrm{stat}}$, Airy$_{2\to\mathrm{BM}}$ and Airy$_{1\to\mathrm{BM}}$ processes, because it involves computing an expectation of the Fredholm determinant in (1.13) with respect to Brownian motion paths. For example, in the stationary case, for which the one-point distribution is given by $\mathbb{P}(\mathcal{A}_{\mathrm{stat}}(t)\le m)=F_0(m)$ with F_0 the Baik-Rains distribution [9], we would need to check the formula

$$F_0(m)=\lim_{L\to\infty}\mathbb{E}\Big(\det\big(I-K_{\mathrm{Ai}}+\Theta^{B(\cdot)+(\cdot)^2+m}_{[-L,L]}e^{2LH}K_{\mathrm{Ai}}\big)\Big), \tag{1.14}$$

where $\Theta^{B(\cdot)+(\cdot)^2+m}_{[-L,L]}$ has an explicit kernel, which can be derived from Theorem 3 of [14], and is given by

$$\begin{aligned}&\Theta^{B(\cdot)+(\cdot)^2+m}_{[-L,L]}(x,y)\\ &\quad = e^{-L(x+y)+2L^3/3}\frac{e^{-(x-y)^2/8L}}{\sqrt{8\pi L}}\widehat{\mathbb{P}}_{\substack{\hat B(-L)=x-L^2\\ \hat B(L)=y-L^2}}\big(\hat B(s)\le B(s)+m \text{ on } [-L,L]\big),\end{aligned} \tag{1.15}$$

where $\hat B$ a Brownian bridge from $x-L^2$ at time $-L$ to $y-L^2$ at time L, B is an independent two sided Brownian motion with $B(0)=0$ (both with diffusion coefficient 2), and the expectation $\mathbb{E}$ in (1.14) is with respect to B while the probability $\widehat{\mathbb{P}}$ in (1.15) is with respect to $\hat B$.

[2] This is due to the known asymptotics $\log(1-F_{\mathrm{GOE}}(m))\sim-\frac{2}{3}m^{3/2}$ and $\log(1-F_{\mathrm{GUE}}(m))\sim-\frac{4}{3}m^{3/2}$, which follow from the formulas for these distributions in terms of the Painlevé II function [30, 31].

1.3 Connection with Last Passage Percolation

It is worth remarking that the general picture we have described holds essentially exactly at the discrete level in the case of last passage percolation. Here one considers a family $\{w(i,j)\}_{i,j\in\mathbb{Z}^+}$ of independent identically distributed random variables and lets Π_n be the collection of up-right paths of length n, that is, paths $\pi=(\pi_0,\dots,\pi_n)$ such that $\pi_i-\pi_{i-1}\in\{(1,0),(0,1)\}$. The *point-to-point last passage time* is defined, for $m,n\in\mathbb{Z}^+$, by

$$L^{\text{point}}(m,n)=\max_{\pi\in\Pi_{m+n}:(0,0)\to(m,n)}\sum_{i=0}^{m+n}w\big(\pi(i)\big),$$

where the notation in the subscript in the maximum means all up-right paths connecting the origin to (m,n). Similarly, the *point-to-line last passage time* is defined by

$$L^{\text{line}}(n)=\max_{k=-n,\dots,n}L^{\text{point}}(n-k,n+k).$$

Next one defines the process $t\mapsto H_n^{\text{point}}(t)$ by linearly interpolating the values given by scaling $L^{\text{point}}(m,n)$ through the relation

$$L^{\text{point}}(n+y,n-y)=c_1n+c_2n^{1/3}H_n^{\text{point}}\big(c_3n^{-2/3}y\big),$$

where the constants c_i depend only on the distribution of the $w(i,j)$. In the special case where the $w(i,j)$ have a geometric distribution, Johansson [20] showed that

$$H_n^{\text{point}}(t)\to\mathcal{A}_2(t)-t^2 \tag{1.16}$$

in distribution, in the topology of uniform convergence on compact sets, where $\mathcal{A}_2$ is the Airy$_2$ process. One can also define a rescaled version H_n^{line} of $L^{\text{line}}(n)$, obtaining the relation

$$H_n^{\text{line}}=\sup_{t\in\mathbb{R}}\big\{H^{\text{point}}(t)\big\}$$

(here we are setting $H_n^{\text{point}}(t)=0$ for $|t|>c_3n^{1/3}$). It is known [10] that H_n^{line} converges in distribution to a GOE Tracy-Widom random variable, and hence (1.16) allows to take $n\to\infty$ in the last equality to recover (1.1) (this was Johansson's original proof of (1.1), the proof in [14] is based on (1.12)).

In principle, this idea can be extended to the obtain variational formulas for the other Airy processes. For example, one could attempt to replace point-to-line last passage times by point-to-half-line last passage times to recover (1.5). Unfortunately, the connection between Airy$_{2\to1}$ (and the other Airy processes) and last passage percolation is made through translating the corresponding results for the totally asymmetric exclusion process, and by doing this the boundary conditions end up away from the line $\{(n-k,n+k),\ k=-n,\dots,n\}$, so (1.16) is not directly applicable. In work in progress Corwin, Liu and Wang [13] obtain an improved version of the slow decorrelation result proved in [12], which would lead to a general version of formulas for last passage times in last passage percolation in terms of variational problems for the Airy$_2$ process. In particular, such a result would give a proof of the conjectures made in Sect. 1.1 and, as a consequence, would show that (1.14), and similar formulas for the other Airy processes, hold.

2 Derivation of the Formula

As in [14, 22] we will first give an expression for the distribution of the supremum over a finite interval and then take a limit. Thus we choose an $L>-\alpha$, which will later be taken to infinity, and work on the interval $[-L,\alpha]$. For notational simplicity we will write

$$\overline{m}=m-\min\{0,\alpha\}^2.$$

We recall that the shifted Airy functions $\phi_\lambda(x) = \mathrm{Ai}(x-\lambda)$ are the generalized eigenfunctions of the Airy Hamiltonian, as $H\phi_\lambda = \lambda\phi_\lambda$, and the Airy kernel K_{Ai} is the projection of H onto its negative generalized eigenspace (see Remark 1.1 of [14]). Therefore $e^{(\alpha+L)H}K_{\mathrm{Ai}}$ has integral kernel

$$e^{(\alpha+L)H}K_{\mathrm{Ai}}(x,y) = \int_{-\infty}^{0} d\lambda\, e^{(\alpha+L)\lambda}\,\mathrm{Ai}(x-\lambda)\,\mathrm{Ai}(y-\lambda). \tag{2.1}$$

We also deduce that $e^{(\alpha+L)H}K_{\mathrm{Ai}} = K_{\mathrm{Ai}}e^{(\alpha+L)H}K_{\mathrm{Ai}}$, so we may use the cyclic property of determinants to rewrite (1.13) as

$$\mathbb{P}\big(\mathcal{A}_2(t)\le g(t) \text{ for } t\in[-L,\alpha]\big) = \det\big(I - K_{\mathrm{Ai}} + e^{(\alpha+L)H}K_{\mathrm{Ai}}\Theta^g_{[-L,\alpha]}K_{\mathrm{Ai}}\big). \tag{2.2}$$

We will apply the above for $g(t) = t^2+\overline{m}$, and for this choice of g we will write $\Theta_{\alpha,L}$ for $\Theta^g_{[-L,\alpha]}$. In this case the resulting kernel can be computed explicitly, and a minor variation of (1.4) in [14] gives

$$\Theta_{\alpha,L}(x,y) = \bar{P}_{\overline{m}+L^2}e^{-(\alpha+L)H}\bar{P}_{\overline{m}+\alpha^2} - \bar{P}_{\overline{m}+L^2}R_{\alpha,L}\bar{P}_{\overline{m}+\alpha^2},$$

where $\bar{P}_m = I - P_m$ denotes the projection onto the interval $(-\infty,m]$ and $R_{\alpha,L}$ is given by

$$R_{\alpha,L}(x,y) = \frac{1}{\sqrt{4\pi(\alpha+L)}}e^{-Lx-\alpha y+(\alpha^3+L^3)/3-\frac{(x-L^2+y-\alpha^2-2\overline{m})^2}{4(\alpha+L)}}. \tag{2.3}$$

Following [14] we decompose $\Theta_{\alpha,L}$ as

$$\Theta_{\alpha,L} = e^{-(\alpha+L)H}\bar{P}_{\overline{m}+\alpha^2} - R_{\alpha,L}\bar{P}_{\overline{m}+\alpha^2} - \Omega_{\alpha,L},$$

where $\Omega_{\alpha,L} = P_{\overline{m}+L^2}(e^{-(\alpha+L)H} - R_{\alpha,L})\bar{P}_{\overline{m}+\alpha^2}$. Using this in (2.2) we can write

$$\begin{aligned}&\mathbb{P}\big(\mathcal{A}_2(t)\le t^2 \text{ for } t\in[-L,\alpha]\big)\\ &\quad= \det\big(I - K_{\mathrm{Ai}}P_{\overline{m}+\alpha^2}K_{\mathrm{Ai}} - e^{(\alpha+L)H}K_{\mathrm{Ai}}R_{\alpha,L}\bar{P}_{\overline{m}+\alpha^2}K_{\mathrm{Ai}} - e^{(\alpha+L)H}K_{\mathrm{Ai}}\Omega_{\alpha,L}K_{\mathrm{Ai}}\big).\end{aligned} \tag{2.4}$$

We will show below that

$$\widetilde{\Omega}_{\alpha,L} := e^{(\alpha+L)H}K_{\mathrm{Ai}}\Omega_{\alpha,L}K_{\mathrm{Ai}} \xrightarrow[L\to\infty]{} 0 \tag{2.5}$$

in trace norm. Then since the mapping $A\mapsto\det(I+A)$ is continuous in the space of trace class operators (see (2.11) below), all that is left to do in order to take $L\to\infty$ in (2.4) is to compute the limit of the operator $e^{(\alpha+L)H}K_{\mathrm{Ai}}R_{\alpha,L}$.

We will first proceed formally to identify the limit, and then verify it in Lemma 2.1. Since K_{Ai} is a projection and H leaves K_{Ai} invariant, we will pretend that $e^{(\alpha+L)H}$ and K_{Ai} commute, so we have to compute the limit of $e^{(\alpha+L)H}R_{\alpha,L}$. Define the *reflection operator* ϱ_m by

$$\varrho_m f(x) = f(2m-x).$$

Then the operator $R_{\alpha,L}$ defined in (2.3) can be rewritten as

$$R_{\alpha,L} = e^{(\alpha^3+L^3)/3}e^{-L\xi}e^{(\alpha+L)\Delta}\varrho_{\overline{m}+(L^2+\alpha^2)/2}e^{-\alpha\xi}. \tag{2.6}$$

Here $e^{r\xi}$ (ξ stands for a generic variable) denotes the multiplication operator $(e^{r\xi}f)(x) = e^{rx}f(x)$ and Δ is the Laplacian, so $e^{r\Delta}$ is the heat kernel as in the introduction. The advantage of this form for $R_{\alpha,L}$ is that it will allow us to use the Baker-Campbell-Hausdorff formula to compute formally $e^{(\alpha+L)H}R_{\alpha,L}$.

We will use the following identities, where $[\cdot,\cdot]$ denotes commutator:

$$[H,\Delta]=[\xi,\Delta]=-2\nabla,\qquad [H,\nabla]=[\xi,\nabla]=-I,\qquad [H,\xi]=-2\nabla.$$

If A and B are two operators such that $[A,[A,B]]=c_1I$ and $[B,[A,B]]=c_2I$ for some $c_1,c_2\in\mathbb{R}$, then the Baker-Campbell-Hausdorff formula[3] reads

$$e^Ae^B=e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]}. \tag{2.7}$$

In particular, if $[A,B]=cI$ then

$$e^{A+B}=e^Ae^Be^{-\frac{1}{2}[A,B]}. \tag{2.8}$$

Using (2.7) we have

$$e^{-L\xi}e^{(\alpha+L)\Delta}=e^{L^2(L+\alpha)/6}e^{(\alpha+L)\Delta+L(\alpha+L)\nabla-L\xi}.$$

Using (2.7) again we deduce that

$$\begin{aligned}e^{(\alpha+L)H}e^{-L\xi}e^{(\alpha+L)\Delta}&=e^{L^2(\alpha+L)/6}e^{(\alpha+L)H}e^{(\alpha+L)\Delta+L(\alpha+L)\nabla-L\xi}\\&=e^{\alpha^3/6-\alpha L^2/2-L^3/3}e^{\alpha\xi+(L^2-\alpha^2)\nabla}.\end{aligned}$$

By (2.8) we have $e^{\alpha\xi+(L^2-\alpha^2)\nabla}=e^{\alpha(L^2-\alpha^2)/2}e^{\alpha\xi}e^{(L^2-\alpha^2)\nabla}$, so the above identity gives

$$e^{(\alpha+L)H}e^{-L\xi}e^{(\alpha+L)\Delta}=e^{-(\alpha^3+L^3)/3}e^{\alpha\xi}e^{(L^2-\alpha^2)\nabla}.$$

Using this in (2.6) we deduce that

$$e^{(\alpha+L)H}R_{\alpha,L}=e^{\alpha\xi}e^{(L^2-\alpha^2)\nabla}\varrho_{\overline{m}+(L^2+\alpha^2)/2}e^{-\alpha\xi}.$$

Since $e^{r\nabla}$ is the shift operator $(e^{r\nabla}f)(x)=f(x+r)$, we have $e^{r\nabla}\varrho_m=\varrho_{m-r/2}$, and we obtain

$$e^{(\alpha+L)H}R_{\alpha,L}=e^{\alpha\xi}\varrho_{\overline{m}+\alpha^2}e^{-\alpha\xi}.$$

The conclusion from the above is the following

Lemma 2.1

$$e^{(\alpha+L)H}K_{\mathrm{Ai}}R_{\alpha,L}=K_{\mathrm{Ai}}e^{\alpha\xi}\varrho_{\overline{m}+\alpha^2}e^{-\alpha\xi}.$$

We postpone the proof of this lemma until the end of this section. Putting this formula and (2.5) in (2.4) and using Lemma 3.1 of [14] gives

$$\begin{aligned}&\mathbb{P}\Big(\sup_{t\le\alpha}\big(\mathcal{A}_2(t)-t^2\big)\le\overline{m}\Big)\\&=\lim_{L\to\infty}\mathbb{P}\big(\mathcal{A}_2(t)\le t^2\text{ for }t\in[-L,\alpha]\big)\\&=\det\big(I-K_{\mathrm{Ai}}P_{\overline{m}+\alpha^2}K_{\mathrm{Ai}}-K_{\mathrm{Ai}}e^{\alpha\xi}\varrho_{\overline{m}+\alpha^2}e^{-\alpha\xi}\bar{P}_{\overline{m}+\alpha^2}K_{\mathrm{Ai}}\big).\end{aligned} \tag{2.9}$$

Having obtained an expression for the probability we are interested in, all that remains to show is that it coincides with our definition of $G_\alpha^{2\to 1}$ (1.7). We recall (as can be seen directly

[3]The Baker-Campbell-Hausdorff formula can be found in most introductory books on Lie groups and algebras. A general version can be found in [15]. However, in this very simple context, it is more readily computed by hand.

from its definition (1.9)) that the Airy kernel can be expressed as $K_{\mathrm{Ai}} = A\bar{P}_0 A^*$, where A is the *Airy transform* which acts on $f \in L^2(\mathbb{R})$ as

$$Af(x) = \int_{-\infty}^{\infty} dz\, \mathrm{Ai}(x-z) f(z).$$

Since $A^* = A^{-1}$ we have by the cyclic property of determinants that the right hand side of (2.9) equals

$$\det\big(I - \bar{P}_0 A^* P_{\overline{m}+\alpha^2} A\bar{P}_0 - \bar{P}_0 A^* e^{\alpha\xi} \varrho_{\overline{m}+\alpha^2} e^{-\alpha\xi} \bar{P}_{\overline{m}+\alpha^2} A\bar{P}_0\big).$$

Recalling that $\varrho_0 f(x) = f(-x)$ we have similarly that the last determinant equals

$$\begin{aligned}&\det\big(I - \varrho_0\bar{P}_0 A^* P_{\overline{m}+\alpha^2} A\bar{P}_0\varrho_0 - \varrho_0\bar{P}_0 A^* e^{\alpha\xi} \varrho_{\overline{m}+\alpha^2} e^{-\alpha\xi} \bar{P}_{\overline{m}+\alpha^2} A\bar{P}_0\varrho_0\big)\\ &\quad= \det\big(I - P_0\varrho_0 A^* P_{\overline{m}+\alpha^2} A\varrho_0 P_0 - P_0\varrho_0 A^* e^{\alpha\xi} \varrho_{\overline{m}+\alpha^2} e^{-\alpha\xi} \bar{P}_{\overline{m}+\alpha^2} A\varrho_0 P_0\big).\end{aligned}$$

Shifting the variables in the last determinant by $-m$ we deduce that

$$\mathbb{P}\Big(\sup_{t\geq\alpha}\big(\mathcal{A}_2(t) - t^2\big) \leq \overline{m}\Big) = \det(I - P_m E_1 P_m - P_m E_2 P_m),$$

where

$$E_1(x,y) = \int_{-\infty}^{\overline{m}+\alpha^2} d\lambda\, \mathrm{Ai}\big(x - m + 2\overline{m} + 2\alpha^2 - \lambda\big) e^{-2(\lambda-\overline{m}-\alpha^2)\alpha}\, \mathrm{Ai}(y - m + \lambda)$$

and

$$E_2(x,y) = \int_{\overline{m}+\alpha^2}^{\infty} d\lambda\, \mathrm{Ai}(x - m + \lambda)\, \mathrm{Ai}(y - m + \lambda).$$

Shifting λ by $\overline{m}+\alpha^2$ in both integrals and changing λ to $-\lambda$ shows that $E_1(x,y) = K^1_\alpha(y,x)$ and $E_2 = K^2_\alpha$, whence the equality in Theorem 1 follows since $E_1^* = K^1_\alpha$ and $E_2^* = K^2_\alpha$.

All we have left is to prove (2.5). We will denote by $\|\cdot\|_{\mathrm{op}}$, $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively the operator, trace class and Hilbert-Schmidt norms of operators on $L^2(\mathbb{R})$ (see Sect. 3 of [14] for the definitions or [27] for a complete treatment). Recall that

$$\|AB\|_1 \leq \|A\|_2\|B\|_2 \quad\text{and}\quad \|AB\|_2 \leq \|A\|_2\|B\|_{\mathrm{op}}. \tag{2.10}$$

Proof of (2.5) Let $\varphi(x) = (1+x^2)^{1/2}$ and define the multiplication operator $Mf(x) = \varphi(x)f(x)$. Then by (2.10) we have that

$$\|\widetilde{\Omega}_{\alpha,L}\|_1 \leq \big\|e^{(\alpha+L)H} K_{\mathrm{Ai}} M^{-1}\big\|_2 \big\|MP_{\overline{m}+L^2}\big(e^{-(\alpha+L)H} - R_{\alpha,L}\big)\bar{P}_{\overline{m}+\alpha^2} K_{\mathrm{Ai}}\big\|_2.$$

Now $\|e^{(\alpha+L)H} K_{\mathrm{Ai}} M^{-1}\|_2 = \|M^{-1} e^{(\alpha+L)H} K_{\mathrm{Ai}}\|_2$ by the symmetry of $e^{(\alpha+L)H} K_{\mathrm{Ai}}$, and then (3.4) in [14] gives $\|e^{(\alpha+L)H} K_{\mathrm{Ai}} M^{-1}\|_2 \leq c(\alpha+L)^{-1/2}$. Then to finish the proof it will be enough to estimate $\|MP_{\overline{m}+L^2} e^{-(\alpha+L)H}\bar{P}_{\overline{m}+\alpha^2} K_{\mathrm{Ai}}\|_2$ and $\|MP_{\overline{m}+L^2} R_{\alpha,L}\bar{P}_{\overline{m}+\alpha^2} K_{\mathrm{Ai}}\|_2$.

We start with the second norm. By (2.3), (2.10) and the fact that K_{Ai} is a projection we have

$$\begin{aligned}&\|MP_{\overline{m}+L^2} R_{\alpha,L}\bar{P}_{\overline{m}+\alpha^2} K_{\mathrm{Ai}}\|_2^2\\ &\quad\leq \|MP_{\overline{m}+L^2} R_{\alpha,L}\bar{P}_{\overline{m}+\alpha^2}\|_2^2 \|K_{\mathrm{Ai}}\|_{\mathrm{op}}^2\\ &\quad\leq \int_{\overline{m}}^{\infty} dx\,\varphi(x)^2 \int_{-\infty}^{\overline{m}+\alpha^2} dy\, \frac{1}{4\pi(\alpha+L)} e^{-4L^3/3+2\alpha^3/3-2Lx-2\alpha y-\frac{(x+y-2\overline{m}-\alpha^2)^2}{4(\alpha+L)}},\end{aligned}$$

where we have performed the change of variables $x \mapsto x + L^2$. The inner Gaussian integral gives

$$\frac{1}{4\sqrt{\pi(\alpha+L)}} e^{-4L^3/3+2(\alpha-L)x-4\overline{m}\alpha+4L\alpha^2+8\alpha^3/3} \times \left[1+\operatorname{erf}\left(\frac{1}{2}(\alpha+L)^{-1/2}\big[4\alpha(\alpha+L)+x-\overline{m}\big]\right)\right],$$

where $\operatorname{erf}(z) = 2\pi^{-1/2}\int_0^z dt\, e^{-t^2} \le 1$. Then

$$\|P_{\overline{m}+L^2} R_{\alpha,L} \bar{P}_{\overline{m}+\alpha^2}\|_2^2 \le C e^{-4L^3/3+4L\alpha^2} \int_{\overline{m}}^{\infty} dx\, \varphi(x)^2 e^{2(\alpha-L)x},$$

which clearly goes to 0 as $L \to \infty$.

On the other hand one can check that $e^{-(\alpha+L)H}$ has integral kernel given by

$$e^{-(\alpha+L)H}(x,y) = \frac{1}{\sqrt{4\pi(\alpha+L)}} e^{-Lx-\alpha y+(\alpha^3+L^3)/3-\frac{(x-y)^2}{4(\alpha+L)}}$$

(this is done either by applying the Feynman-Kac and Cameron-Martin-Girsanov formulas as in [14], or directly by integrating this kernel against the kernel of $e^{(\alpha+L)H}$, which is given, similarly to (2.1), by $\int_{-\infty}^{\infty} d\lambda\, e^{-(\alpha+L)\lambda} \operatorname{Ai}(x-\lambda)\operatorname{Ai}(y-\lambda)$). Note that this kernel is the same as the one given in (2.3) only without the reflection in the Gaussian term. It is easy to check then that the same calculation as the one in the above paragraph shows that $\|M P_{\overline{m}+L^2} e^{-(\alpha+L)H} \bar{P}_{\overline{m}+\alpha^2} K_{\mathrm{Ai}}\|_2 \longrightarrow 0$ as $L \to \infty$. This finishes the proof of (2.5). □

Proof of Lemma 2.1 By (2.1) and (2.3), the kernel of $e^{(\alpha+L)H} K_{\mathrm{Ai}} R_{\alpha,L}$ is given by

$$e^{(\alpha+L)H} K_{\mathrm{Ai}} R_{\alpha,L}(x,y) = \int_{-\infty}^{0} d\lambda \int_{-\infty}^{\infty} dz\, e^{(\alpha+L)\lambda} \operatorname{Ai}(x-\lambda)\operatorname{Ai}(z-\lambda) \cdot \frac{1}{\sqrt{4\pi(\alpha+L)}} e^{-(z+y-\alpha^2-2\overline{m}^2-L^2)^2/4(\alpha+L)+\alpha^3/3+L^3/3-Lz-\alpha y}.$$

By completing the square in z in the exponential, the z integral can be seen as a heat kernel applied to an Airy function. Using the formula $e^{t\Delta}\operatorname{Ai}(x) = e^{2t^3/3+tx}\operatorname{Ai}(x+t^2)$ (see for instance Proposition 1.2 in [25]) we obtain after some manipulations

$$e^{(\alpha+L)H} K_{\mathrm{Ai}} R_{\alpha,L}(x,y) = \int_{-\infty}^{0} d\lambda\, e^{2\alpha^3+2\alpha\overline{m}^2-2\alpha y} \operatorname{Ai}(x-\lambda)\operatorname{Ai}\big(2\overline{m}^2+2\alpha^2-y-\lambda\big),$$

which corresponds to the claimed formula. □

We turn finally to the proof of Proposition 1.2. It relies on the following simple but very useful observation:

Lemma 2.2 *If A is a trace class operator on a Hilbert space $\mathcal{H}$,*

$$\det(I+A) - 1 - \operatorname{tr}(A) \le \frac{1}{2}\|A\|_1^2 e^{\|A\|_1}.$$

Proof Let $\Lambda^n(A) = A \otimes \cdots \otimes A$ (n times, where $\otimes$ denotes the tensor product of operators), which is an operator in the Hilbert space $\Lambda^n(\mathcal{H})$ known as the alternating product, see [27] for more details. Then

$$\det(I+A) = \sum_{k=0}^{\infty} \operatorname{tr}\big(\Lambda^k(A)\big)$$

(this equality can be taken as the definition of the Fredholm determinant, see for instance (3.5) in [27]). Of course $\Lambda^0(A) = I$ and $\Lambda^1(A) = A$, so all we need to show is that

$$\sum_{k=2}^{\infty} \operatorname{tr}\big(\Lambda^k(A)\big) \le \frac{1}{2}\|A\|_1^2 e^{\|A\|_1}.$$

But this follows directly from the inequality $|\operatorname{tr}(\Lambda^k(A))| \le \|\Lambda^k(A)\|_1 \le (k!)^{-1}\|A\|_1^k$ (see (3.1) and (3.4) in [27]). □

Proof of Proposition 1.2 Using the inequality (Theorem 3.4 in [27])

$$\big|\det(I + A) - \det(I + B)\big| \le \|A - B\|_1 e^{1+\|A\|_1+\|B\|_1} \tag{2.11}$$

for any two trace class operators A and B, we can write

$$\begin{aligned} 1 - G_\alpha^{2\to1}(m) &= 1 - \det\big(I - P_m K_\alpha^1 P_m - P_m K_\alpha^2 P_m\big) \\ &\le 1 - \det\big(I - P_m K_\alpha^1 P_m\big) + \big\|P_m K_\alpha^2 P_m\big\|_1 e^{1+2\|P_m K_\alpha^1 P_m\|_1+\|P_m K_\alpha^2 P_m\|_1}. \end{aligned}$$

We will show below that

$$\big\|P_m K_\alpha^1 P_m\big\|_1 \le c\, e^{-\frac{2}{3}m^{3/2}}, \tag{2.12}$$

while a similar (and simpler) calculation gives the estimate $\|P_m K_\alpha^2 P_m\|_1 \le c e^{-\frac{4}{3}m^{3/2}}$ (this is in fact it is the same standard calculation based on (2.11) that gives the estimate $1 - F_{\mathrm{GUE}}(m) \le c\, e^{-\frac{4}{3}m^{3/2}}$). Therefore all we need to show is that

$$1 - \det\big(I - P_m K_\alpha^1 P_m\big) \le c\, e^{-\frac{4}{3}m^{3/2}}$$

and thus, in view of Lemma 2.2, the proof it will be enough to show that

$$\operatorname{tr}\big(P_m K_\alpha^1 P_m\big) \le cm\, e^{-\frac{4}{3}m^{3/2}+\alpha m} \quad \text{and} \quad \big\|P_m K_\alpha^1 P_m\big\|_1 \le c\, e^{-\frac{2}{3}m^{3/2}}. \tag{2.13}$$

The trace can be computed directly: letting $\bar\alpha = \max\{0, \alpha\}$ we have

$$\operatorname{tr}\big(P_m K_\alpha^1 P_m\big) = \int_m^\infty dx \int_0^\infty d\lambda\, e^{\alpha\lambda} \operatorname{Ai}(x - \lambda + \bar\alpha) \operatorname{Ai}(x + \lambda + \bar\alpha).$$

We split the integral according on the regions $\{\lambda < x\}$ and $\{\lambda \ge x\}$. On the first region, changing variables $\lambda \mapsto x\gamma$, and since the Airy function is decreasing and positive on the positive axis and we may assume $m > 0$, the integral is bounded by

$$\begin{aligned} &\int_m^\infty dx \int_0^1 d\gamma\, x e^{\alpha x\gamma} \operatorname{Ai}\big((1-\gamma)x\big) \operatorname{Ai}\big((1+\gamma)x\big) \\ &\quad \le c \int_m^\infty dx \int_0^1 d\gamma\, x e^{\alpha x\gamma - \frac{2}{3}[(1-\gamma)^{3/2}+(1+\gamma)^{3/2}]x^{3/2}}, \end{aligned}$$

where we used the first of the estimates

$$\big|\operatorname{Ai}(x)\big| \le c\, e^{-\frac{2}{3}x^{3/2}} \quad \text{for } x > 0, \qquad \big|\operatorname{Ai}(x)\big| \le c \quad \text{for } x \le 0 \tag{2.14}$$

(see (10.4.59–60) in [3]). The term in brackets in the above estimate is larger than 2, and thus the whole integral is bounded by $cme^{\alpha m - \frac{4}{3}m^{3/2}}$. On the other region we have similarly, using (2.14), that the integral is bounded by

$$\int_m^\infty dx \int_1^\infty d\gamma\, x e^{\alpha x\gamma} \operatorname{Ai}\big((1-\gamma)x+\bar{\alpha}\big) \operatorname{Ai}\big((1+\gamma)x\big)$$
$$\le c \int_m^\infty dx \int_1^\infty d\gamma\, x e^{\alpha x\gamma - \frac{2}{3}(1+\gamma)^{3/2} x^{3/2}},$$

which again can be seen to be bounded by $cme^{\alpha m - \frac{4}{3}m^{3/2}}$. This gives the first bound in (2.13).

For the second one write $K_\alpha^1 = (B^1 e^{-\xi} P_0)(P_0 e^{(1+\alpha)\xi} B^2)$, where $B^1(x,\lambda) = \operatorname{Ai}(x-\lambda+\bar{\alpha})$ and $B^2(\lambda, y) = \operatorname{Ai}(y+\lambda+\bar{\alpha})$. Then

$$\big\| P_m K_\alpha^1 P_m \big\|_1 \le \big\| P_m B^1 e^{-\xi} P_0 \big\|_2 \big\| P_0 e^{(1+\alpha)\xi} B^2 P_m \big\|_2. \tag{2.15}$$

The square of the first norm equals

$$\int_m^\infty dx \int_0^\infty d\lambda \operatorname{Ai}(x-\lambda+\bar{\alpha})^2 e^{-2\lambda} \le \int_m^\infty dx\, e^{-2(x+\bar{\alpha})} \int_{-\infty}^\infty d\lambda \operatorname{Ai}(-\lambda)^2 e^{-2\lambda} \le c\, e^{-2m},$$

thanks to (2.14). Similarly, the square of the second norm equals

$$\int_m^\infty dy \int_0^\infty d\lambda \operatorname{Ai}(y+\lambda+\bar{\alpha})^2 e^{2(1+\alpha)\lambda}$$
$$\le c \int_m^\infty dy \int_0^\infty d\lambda\, e^{-\frac{4}{3}(y+\lambda+\bar{\alpha})^{3/2} + 2(1+\alpha)\lambda} \le c\, e^{-\frac{4}{3}m^{3/2}},$$

again thanks to (2.14). Using these two bounds in (2.15) gives the second estimate in (2.13) and finishes the proof. □

Observe that for $\alpha < -1$ the last bound in the proof above can be upgraded to $ce^{2(1+\alpha) - \frac{4}{3}m^{3/2}}$, giving

$$\big\| P_m K_\alpha^1 P_m \big\|_1 \le c\, e^{\alpha - m - \frac{2}{3}m^{3/2}} \underset{\alpha\to-\infty}{\longrightarrow} 0, \tag{2.16}$$

which we used in justifying (1.10). A similar (although slightly more complicated) estimate gives $\| P_m \overline{K}_\alpha^1 P_m \|_1 \longrightarrow_{\alpha\to\infty} 0$, which was used in the derivation of (1.11).

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